BIFURCATION AND EXACT TRAVELING WAVE SOLUTIONS FOR THE GENERALIZED NONLINEAR DISPERSIVE MK(M,N) EQUATION*

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Abstract This paper investigated the generalized nonlinear dispersive mK(m,n) equation by the planar dynamical systems method, the bifurcations of the system with different parameter region of this equation are presented. Moreover, we find different kinds of exact explicit solutions like peak type solutions, periodic wave solutions and valley type solutions.

Keywords Generalized nonlinear dispersive mK(m,n) equation, dynamical system method, exact travelling wave solutions.

MSC(2010) 35Q53, 35C07, 35Q92.

1. Introduction

In the formation of patterns in liquid drops, nonlinear dispersive played an important role, Rosenau and Hyman [19] proposed the K(m,n) equation:

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, m > 0, 1 < n \le 3.$$
(1.1)

Wazwaz [21] introduced and studied the exact solutions of several generalized form of the mK(n,n) equation, the so-called mK(m,n) equation in one-, two-, and three-dimensional spatial spaces.

Recently, Yan [22] extended these equations to more general forms by making index of u in each term different and obtained compacton solutons, solitary wave solutions and periodic wave solutions. He et al. [10] considered the mK(n,n) equation by the method of planar dynamical systems and derived exact explicit solutions. Lai, He and Qing [12] studied the equation in a higher spaces and obtained explicit traveling wave solution in terms of sin, cos, sec and csc profiles.

In this paper, we apply bifurcation method [1-3, 6-9, 18] to study the following generalized nonlinear dispersive mK(m,n) equations:

$$u^{n-1}u_t + a(u^m)_x + (u^n)_{xxx} = 0, (1.2)$$

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^{*}The authors were supported by National Natural Science Foundation of China (11001115, 11201473), Natural Science Foundation of Shandong (ZR2015AL008) and Natural Science Foundation of Liaocheng University (318012025).

in which a, m, n are constants and $m \ge 1, n \ge 1$. The bifurcation method is extensively used by researchers [4, 5, 11, 13-17, 21, 23].

Introducing the traveling wave transformation $\xi = x - ct$, let $u(x, t) = u(\xi)$ and substituting it into Eq.(1.2), it follows that

$$-cu^{n-1}u' + a(u^m)' + (u^n)''' = 0, (1.3)$$

in which $c \neq 0$ is a constant and the prime represents $\frac{d}{d\xi}$. Integrating Eq.(1.3) once, we obtain

$$g - \frac{c}{n}u^n + au^m + n(n-1)u^{n-2}u'^2 + nu^{n-1}u'' = 0,$$
(1.4)

in which g is an integral constant.

Introducing y = u', then Eq.(1.4) is rewritten by the following planar system

$$\begin{cases} \frac{du}{d\xi} = y, \\ \frac{dy}{d\xi} = -\frac{n(n-1)u^{n-2}y^2 + au^m - \frac{c}{n}u^n + g}{nu^{n-1}}, \end{cases}$$
(1.5)

which has the first integral of

$$H(u,y) = n^2 u^{2n-2} y^2 + \frac{2an}{m+n} u^{m+n} - \frac{c}{n} u^{2n} + 2gu^n = h.$$
(1.6)

Since all traveling wave solutions of (1.2) are determined by the phase orbits defined by the vector fields of system (1.5), we will analyze the bifurcations conditions and phase portraits of (1.5).

The rest of the paper is organized as follows. In Section 2, we seek for the equilibrium points of (1.5). In Section 3, we discuss the bifurcation of phase portraits of system (1.5) and obtain its traveling wave solutions. The paper is ended with the conclusion.

2. The analysis for generalized nonlinear dispersive mK(m,n) equation

Notice that the line l: u = 0 is a singular line, which made smooth system (1.5) have non-smooth traveling wave solutions. Therefore, we let $d\xi = nu^{n-1}d\tau$, then the system

$$\begin{cases} \frac{du}{d\tau} = nu^{n-1}y, \\ \frac{dy}{d\tau} = -n(n-1)u^{n-2}y^2 - au^m + \frac{c}{n}u^n - g, \end{cases}$$
(2.1)

has the same phase portraits with system (1.5) except on the singular line u = 0.

For studying the singular points of system (2.1), let $f(u) = \frac{c}{n}u^n - au^m - g$, $f'(u) = cu^{n-1} - amu^{m-1}$. We can find out zero points of f(u) easily. Suppose that m > n(m < n) has the similar results with this situation).

(I) g = 0. When m - n = 2k + 1 is odd, f'(u) has one zero point $\widetilde{u_0}$, $\widetilde{u_0} = \frac{2k+\sqrt{\frac{C}{ma}}}{\sqrt{\frac{C}{ma}}}$. When m - n = 2k is even and ac > 0, f'(u) has two zero points $\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_1} = \frac{2k\sqrt{\frac{C}{ma}}}{\sqrt{\frac{C}{ma}}}, \widetilde{u_2} = -\frac{2k\sqrt{\frac{C}{ma}}}{\sqrt{\frac{C}{ma}}}$. As a result, we get conclusions as follows. (i) When m - n = 2k + 1 is odd and a > 0, $f(\widetilde{u_0}) > 0(a < 0, f(\widetilde{u_0}) < 0)$, f(u)

(i) When m - n = 2k + 1 is odd and $a > 0, f(\widetilde{u}_0) > 0 (a < 0, f(\widetilde{u}_0) < 0), f(u)$ has two singular points O(0,0) and $A_1(u_1,0), u_1 = {}^{2k+1}\sqrt{\frac{c}{na}}$. (ii) When m - n = 2k is even and $ac > 0, f(\widetilde{u}_1) > 0, f(\widetilde{u}_2) < 0(f(\widetilde{u}_1) < 0)$.

(ii) When m - n = 2k is even and ac > 0, $f(\tilde{u}_1) > 0$, $f(\tilde{u}_2) < 0(f(\tilde{u}_1) < 0, f(\tilde{u}_2) > 0)$, f(u) has three singular points O(0,0), $A_2(u_2,0)$, $A_3(u_3,0)$, where $u_2 = \frac{2k}{na} \frac{c}{na} u_3 = -\frac{2k}{na} \frac{c}{na}$.

 $\begin{array}{l} u_2 = \sqrt[2k]{\frac{c}{na}}, u_3 = -\sqrt[2k]{\frac{c}{na}}. \\ (\mathrm{II}) \ g \neq 0. \ \text{Support that} \ a > 0(a < 0 \ \text{has the similar result}). \ \text{When} \ m-n = 2k+1 \ \text{is odd, we have} \ f'(\widetilde{u}_0) = 0, \ f(\widetilde{u}_0) = c(\frac{1}{m} - \frac{1}{n})(\frac{c}{ma})^{\frac{n}{m-n}} - g, \ \text{where the curve} \ \mathrm{L}(1): g = c(\frac{1}{n} - \frac{1}{m})(\frac{c}{ma})^{\frac{m}{m-n}}. \ \text{devide the } (c,g) \ \text{parametric plane into six regions.} \ \text{When} \ m-n = 2k+2 \ \text{is even and} \ ac > 0, \ \text{the curve} \ \mathrm{L}(2): g = c(\frac{1}{n} - \frac{1}{m})(\frac{c}{ma})^{\frac{n}{m-n}} \ \text{and} \ \mathrm{L}(3): g = c(\frac{1}{n} + \frac{1}{m})(\frac{c}{ma})^{\frac{n}{m-n}} \ \text{devide the } (c,g) \ \text{parameter plane into three regions.} (\text{see Fig.1}) \end{array}$



Figure 1. The bifurcation set of system(1.2).(a)m - n = 2k + 1 is odd;(b)m - n = 2k + 2 is even

Letting (u_i, y_i) be any equilibrium point of system (2.1), the coefficient matrix of linearized system (2.1) can be presented

$$M(u_i, y_i) = \begin{pmatrix} n(n-1)u_i^{n-2}y_i & nu_i^{n-1} \\ -n(n-1)(n-2)u_i^{n-3}y_i^2 + f'(u_i) - 2n(n-1)u_i^{n-2}y_i \end{pmatrix}$$

we can obtain that

$$J(u_i, 0) = \begin{vmatrix} 0 & nu_i^{n-1} \\ f'(u_i) & 0 \end{vmatrix} = -nu_i^{n-1}f'(u_i).$$
(2.2)

Substitute an equilibrium point of system (2.1) into (2.2) by the theory of planar dynamical system [13], if J < 0, the equilibrium point is a saddle point; if J > 0, the equilibrium point is a center point; If J = 0 and the index of equilibrium point is 0, then it is a cusp.

Thus, we can know that when g = 0, (i) if m - n = 2k + 1 and c > 0, equilibrium point O(0,0) is a saddle point, $A_1(u_1,0)$ is a center point; if m - n = 2k + 1 and

c < 0, equilibrium point $A_1(u_1, 0)$ is a saddle point. (ii) if m - n = 2k + 2 and a > 0, c > 0, equilibrium point O(0, 0) is a saddle point, $A_2(u_2, 0)$ and $A_3(u_3, 0)$ are two center points; if m - n = 2k + 2 and a < 0, c < 0, equilibrium point $A_2(u_2, 0)$ and $A_3(u_3, 0)$ are two saddle points.

3. Bifurcation sets and exact solutions of system (2.1)

In this section, we study the bifurcation set and exact solutions of the planar Hamiltonian system(2.1).

Theorem 3.1. When g = 0, we have,

(i) When m - n = 2k + 1 and a > 0, c > 0, system (2.1) has a periodic family orbit and a homoclinic orbit, then Eq.(1.2) has a periodic wave solution and a peak type solitary wave solution. When a < 0, c > 0, system (2.1) has a family periodic orbit and a homoclinic orbit, then Eq.(1.2) has a periodic wave solution and a valley type solitary wave solution. The solitary wave solution has the expression

$$u(\xi) = \frac{5c}{8a} \operatorname{sech}^2\left(\sqrt{\frac{c}{32}}\xi\right).$$
(3.1)

(ii) when m - n = 2k + 1 and a > 0, c < 0, system (2.1) has a homoclinic orbit, then Eq.(1.2) has a valley type solitary wave solution. When a < 0, c < 0, system (2.1) has a homoclinic orbit, then Eq.(1.2) a peak type solitary wave solution. The solution is given by

$$u(\xi) = \frac{5c}{8a}\sec^2\left(\sqrt{\frac{-c}{32}}\xi\right).$$
(3.2)

(iii) when m - n = 2k + 2 and a > 0, c > 0, system (2.1) has two family periodic orbits and two homoclinic orbit, then Eq.(1.2) has two periodic wave solutions and a solitary wave solution. The expression of the solitary wave solution is

$$u(\xi) = \pm \sqrt{\frac{3c}{4a}} \operatorname{sech}\left(\sqrt{\frac{c}{32}}\xi\right).$$
(3.3)

Proof. (i) When m - n = 2k + 1 and a > 0, c > 0 or a < 0, c > 0 (see Figure 2(a) and Figure 2(b)), corresponding to the homoclinic orbit defined by H(u, y) = 0, we can obtain

$$y^2 = -\frac{a}{5}u^3 + \frac{c}{8}u^2. \tag{3.4}$$

By (3.4) and $\frac{du}{d\xi} = y$, the solitary wave solution (3.1) is derived.

(*ii*) Similarly, when m - n = 2k + 1 and a > 0, c < 0 or a < 0, c < 0 (see Figure 2(c) and Figure 2(d)), corresponding to H(u, y) = 0, we can obtain

$$y^2 = -\frac{a}{5}u^3 + \frac{c}{8}u^2.$$
(3.5)

In terms of (3.5) and $\frac{du}{d\xi} = y$, the solitary wave solution (3.2) is obtained.



Figure 2. The phase portraits of system (2.5) with m - n = 2k + 1. (a) a > 0, c > 0; (b) a < 0, c > 0; (c) a > 0, c < 0; (d) a < 0, c < 0

(*iii*) In this situation, the phase portraits are performed in Figure 3(a), corresponding to homoclinic orbit which is defined by H(u, y) = 0, we can obtain

$$y^2 = -\frac{a}{24}u^4 + \frac{c}{32}u^2. \tag{3.6}$$

Substituting (3.6) into $\frac{du}{d\xi} = y$, we derive the solitary wave solution (3.3).

Theorem 3.2. When $g \neq 0$, we have

(i) when $(c,g) \in A_3$, system (2.1) has three family periodic orbits. Thus, Eq.(1.2) has three periodic wave solutions. A periodic wave solution has the form

$$u(\xi) = u_1 - (u_1 - u_2) sn^2 \left(\frac{1}{2}\sqrt{\frac{a}{5}(u_1 - u_3)}\xi, \sqrt{\frac{u_1 - u_2}{u_1 - u_3}}\right),$$
(3.7)



Figure 3. The phase portraits of system (2.5) with m - n = 2k. (a) a > 0, c > 0; (b) a < 0, c > 0; (c) a > 0, c < 0; (d) a < 0, c < 0

in which u_1, u_2 and u_3 are three real roots of equation $\frac{c}{n}u^n - au^m - g = 0$ and $u_1 > u_2 > 0 > u_3$.

(ii) when $(c,g) \in B_1$, system (2.1) has two family periodic orbits. Then, Eq.(1.2) has two periodic wave solutions. A periodic wave solution has the form

$$u(\xi) = u_2 sn \left(u_1 \sqrt{\frac{2a}{3}} \xi, \frac{u_2}{u_1} \right), \tag{3.8}$$

in which u_1 and u_2 are two real roots of equation $\frac{c}{n}u^n - au^m - g = 0$ and $u_1 > u_2 > 0$. **Proof.** (i) When $(c,g) \in A_3$ (see Figure 4(c)), corresponding to H(u,y) = 0, we have

$$y^{2} = -\frac{a}{5}u^{3} + \frac{c}{8}u^{2} + \frac{g}{4}$$

= $\frac{a}{5}(u_{1} - u)(u - u_{2})(u - u_{3}).$ (3.9)



Figure 4. The phase portraits of system (2.5) with m - n = 2k + 1. (a) $(c, g) \in A(1)$; (b) $(c, g) \in A(2)$; (c) $(c, g) \in A(3)$; (d) $(c, g) \in A(4)$; (e) $(c, g) \in A(5)$; (f) $(c, g) \in A(6)$



Figure 5. The phase portraits of system (2.5) with m - n = 2k + 2. (a) $(c, g) \in B(1)$; (b) $(c, g) \in B(2)$; (c) $(c, g) \in B(3)$

Substituting Eq.(3.9) into $\frac{du}{d\xi} = y$, we can obtain

$$\frac{du}{\sqrt{(u_1 - u)(u - u_2)(u - u_3)}} = \pm \sqrt{\frac{a}{5}} d\xi, \qquad (3.10)$$

integrating along the periodic orbit, we can obtain the periodic wave solution (3.7).

(*ii*) When $(c,g) \in B_1$ (see Figure 5(a)), corresponding to $H(u,y) = -\frac{14}{3}$, we

have

$$y^{2} = -\frac{2c}{3}u^{4} + \frac{c}{8}u^{2} - \frac{g}{2}$$

= $-\frac{2c}{3}(u_{1}^{2} - u^{2})(u_{2}^{2} - u^{2}).$ (3.11)

By Eq.(3.11) and $\frac{du}{d\xi} = y$, we obtain

$$\frac{du}{\sqrt{(u_1^2 - u^2)(u_2^2 - u^2)}} = \pm \sqrt{\frac{-2c}{3}} d\xi, \qquad (3.12)$$

integrating along the periodic orbit, we can obtain the periodic wave solution (3.8). $\hfill\square$



Figure 6. (a) solution (3.1) when a > 0; (b) solution (3.1) when a < 0; (c) solution (3.2) when a > 0; (d) solution (3.2) when a < 0; (e) solution (3.3) when $u(\xi) > 0$; (f) solution (3.3) when $u(\xi) < 0$

According to the above analysis, we have the conclusion as follows:

(1) If m - n = 2k + 1 and c > 0, Eq.(1.2) has a family periodic wave solution and a solitary wave solution. As a > 0 (see Figure 2(a)), the solitary wave solution is peak type; as a < 0 (see Figure 2(b)), the solitary solution is valley type.

(2) If m - n = 2k + 1 and c < 0, Eq.(1.2) has a solitary wave solution. If a > 0 (see Figure 2(c)), it is a valley type solitary wave solution; when a < 0 (see Figure 2(d)), it is a peak type solution.

(3) If m - n = 2k and a > 0, c > 0 (see Figure 3(a)), Eq.(1.2) has two family periodic wave solutions, a peak type and a valley type solitary wave solution.

(4) If m - n = 2k and a < 0, c < 0 (see Figure 3(d)), Eq.(1.2) has a peak type and a valley type solitary wave solution.

(5) If (c,g) ∈ A₃ (see Figure 4(c)), Eq.(1.2) has three periodic wave solutions.
(6) If (c,g) ∈ B₁ (see Figure 5(a)), Eq.(1.2) has two family periodic wave solutions.

(7) If $(c,g) \in B_2$ (see Figure 5(b)), Eq.(1.2) has two compacton solutions.

4. Conclusion

In this paper, we use the bifurcation method to study exact traveling wave solutions of generalized nonlinear dispersive mK(m,n) equation. We derive exact solitary wave solutions, periodic wave solutions and compactons solutions. The bifurcation and phase portraits under different parameters are also given, furthermore, we can get the types of solutions easily.

Acknowledgements

The authors are grateful to the referees for their comments and suggestions which have led to a significant improvement of the paper.

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