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SOLVABILITY FOR A COUPLED SYSTEM OF PERTURBED IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH PERIODIC AND ANTI-PERIODIC BOUNDARY CONDITIONS*

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Abstract In this paper, we first provide a refinement result for the abstract continuation theorem for k-set contractions. The new version of the theorem is equivalent to the usual one and it better adapts to study the existence of solutions for nonlinear differential equations. Then we discuss a new class of coupled system of implicit fractional boundary value problem. The nonlinear terms of equations involving perturbations, and the boundary conditions are constituted by periodic and anti-periodic boundary conditions. Based on the abstract continuation theorem for k-set contractions, an interesting existence result is obtained. Finally, an example is constructed for illustrating the application of our main results.

Keywords Implicit fractional differential equation, periodic and anti-periodic boundary conditions, abstract continuation theorem, k-set contraction, coupled system.

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1. Introduction

This paper is concerned with the existence of solutions for the following coupled system of nonlinear implicit fractional differential equations with periodic and anti-periodic boundary conditions (BCs):

$$\begin{cases} {}^CD^{\alpha}_{0+}x(t)=f(t,x(t),{}^CD^{\beta}_{0+}y(t))+e_1(t),\ t\in(0,1),\\ {}^CD^{\beta}_{0+}y(t)=g(t,y(t),{}^CD^{\alpha}_{0+}x(t))+e_2(t),\ t\in(0,1),\\ x(0)=-x(1),\ y(0)=y(1), \end{cases} \tag{1.1}$$

where $0<\alpha,\beta<1$, ${}^CD^{q}_{0+}$ denotes the Caputo fractional derivative of order q ($q=\alpha,\beta$); f,g and perturbation terms e_1,e_2 are continuous functions on $[0,1]$.

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As is well known, the ordinary differential equation (ODE) is an equation which is composed of a function involving a dependent variable, say x, only a single independent variable, say t, and at least one derivative of x with respect to t. For example, an nth-order differential equation can be expressed in the following form:

$$F(t,x,x',x'',\cdots,x^{(n)})=0. \tag{1.2}$$

From the differential equation (1.2), if $x^{(n)}$ can be solved by

$$x^{(n)}=f(t,x,x',x'',\cdots,x^{(n-1)}),$$

then (1.2) is called an explicit ODE. Conversely, if (1.2) cannot be expressed in the explicit from, it is said to be an implicit ODE. It should be pointed out that implicit differential equations are an important part of ODEs and discussing the existence of solutions for such differential equations is one of the interesting topic in the theoretical research of ODEs. A great deal of mathematical effort has been devoted to the study of integer-order implicit differential equations (system) with various boundary conditions (see [19, 20, 23, 38]). For example, by using continuation theorem, Petryshyn [38] considered the existence of solutions to the following second-order implicit differential equation

$$x''(t)=f(t,x,x',x'')-y(t),\ t\in(0,T),$$

subject to either Dirichlet, Neumann, periodic, Sturm-Liouville, or anti-periodic boundary conditions, where $f\in C([0,T]\times\mathbb{R}^3,\mathbb{R})$.

Nowadays, fractional differential equations (FDEs) occur naturally in the various fields of science and engineering. There numerous literature reveals that fractional model has been very successful in providing a more reliable, realistic, and compact model for a variety of systems in diverse fields such as physics, bioengineering, finance, control theory, signal processing, etc. For details, see [17, 24, 34, 42, 47] and the references therein. In recent years, more attention has been given to investigating the existence of solutions of initial and boundary value problems for implicit fractional differential equations, see [3–7, 10–14, 32, 35, 37, 40, 41, 43–46, 48] and the references therein. For example, by using Banach's contraction principle, Sousa and de Oliveira [43] studied the existence and uniqueness of solution for the following nonlinear implicit fractional Cauchy problem:

$$\begin{cases} {}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}y(t)=f(t,y(t),{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}y(t)),\ t\in(a,T),\\ I_{a+}^{1-\gamma;\psi}y(a)=y_a,\ y_a\in\mathbb{R}, \end{cases}$$

where $T>a$, $\alpha\in(0,1]$, $\beta\in[0,1]$, ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative of order α and type β; $I_{a+}^{1-\gamma;\psi}$ is the Riemann-Liouville fractional integral of order $1-\gamma$, with respect to function ψ; $f:[a,T]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function.

Coupled systems of FDEs are of great significance due to such systems appear in a variety of problems of interdisciplinary fields such as synchronization phenomena, nonlocal thermoelasticity, bioengineering, ecological models, disease models, anomalous diffusion, etc. For details, the reader is referred to the papers [1, 2] and the references therein. Recently, many researchers began to focus on the existence of solutions for the coupled systems of implicit fractional boundary value problems (BVPs), for instance see [3, 5–7, 32, 40, 46, 48] and the references therein. For example, by using Schauder's and Banach fixed point theorem, Samina et al. [40]

investigated the existence and uniqueness of solutions for the following coupled system of implicit FDEs with anti-periodic BCs:

$$\begin{cases} {}^CD_{0+}^{\alpha}x(t)=\vartheta(t,y(t),{}^CD_{0+}^{\alpha}x(t)),\ t\in(0,\eta),\\ {}^CD_{0+}^{\beta}y(t)=\theta(t,x(t),{}^CD_{0+}^{\beta}y(t)),\ t\in(0,\eta),\\ x(0)=-x(\eta),\ \ x'(0)=-x'(\eta),\\ y(0)=-y(\eta),\ \ y'(0)=-y'(\eta), \end{cases}$$

where $\alpha,\beta\in(1,2]$, $\eta\in(0,+\infty)$, ${}^CD_{0+}^{\alpha}$ and ${}^CD_{0+}^{\beta}$ are the Caputo fractional derivatives of order α and β, respectively; $\vartheta,\theta:[0,\eta]\times\mathbb{R}^2\to\mathbb{R}$ are nonlinear continuous functions. After that, by using Schaefer's and Banach fixed point theorem, Wang, Zada and Waheed [46] discussed the existence and uniqueness of solutions for the following implicit coupled system with anti-periodic BCs:

$$\begin{cases} {}^CD_{0+}^{p}u(t)-\alpha(t,y(t),{}^CD_{0+}^{p}u(t))-\frac{\Gamma(\sigma)}{\Gamma(\delta)}I_{0+}^{\sigma}g(t,y(t),{}^CD_{0+}^{p}u(t))=0,\ t\in(0,T),\\ {}^CD_{0+}^{q}y(t)-\chi(t,u(t),{}^CD_{0+}^{q}y(t))-\frac{\Gamma(\sigma)}{\Gamma(\delta)}I_{0+}^{\sigma}f(t,u(t),{}^CD_{0+}^{p}y(t))=0,\ t\in(0,T),\\ u(t)|_{t=0}=-u(t)|_{t=T},\ {}^CD_{0+}^{r}u(t)|_{t=0}=-{}^CD_{0+}^{r}u(t)|_{t=T},\\ y(t)|_{t=0}=-y(t)|_{t=T},\ {}^CD_{0+}^{\omega}y(t)|_{t=0}=-{}^CD_{0+}^{\omega}y(t)|_{t=T}, \end{cases}$$

where $1<p,q\le2$, $0\le r,\omega\le2$, $\sigma,\delta>0$, $T>0$, ${}^CD_{0+}^{(\cdot)}$ denotes the Caputo fractional derivative, I_{0+}^{σ} is the Riemann-Liouville fractional integral of order σ; $\alpha,\chi,g,f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Many papers dealing with implicit fractional boundary value problems. However only a few papers [11, 13, 48] consider implicit fractional boundary value problems at resonance (we say that a BVP is called at resonance if the corresponding homogeneous BVP has a nontrivial solution). In [13], Benchohra, Bouriah and Nieto considered the following nonlinear implicit FDE with periodic BCs:

$$\begin{cases} {}^HD_{1+}^{\alpha}y(t)=f(t,y(t),{}^HD_{1+}^{\alpha}y(t)),\ t\in J,\ 0<\alpha\le1,\\ y(1)=y(T),\ T>1, \end{cases}$$

where $J=[1,T]$, ${}^HD_{1+}^{\alpha}$ is the Hadamard fractional derivative of order α, $f:J\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function. The existence of solution is proved by coincidence degree theory. In [48], Zhang, Liu and Xue discussed the existence and uniqueness of solutions for the following coupled system of implicit FDEs with periodic BCs:

$$\begin{cases} D_{0+}^{\alpha}x(t)=f(t,t^{1-\beta}y(t),D_{0+}^{\beta}y(t)),\ t\in(0,1),\\ D_{0+}^{\beta}y(t)=g(t,t^{1-\alpha}x(t),D_{0+}^{\alpha}x(t)),\ t\in(0,1),\\ \lim\limits_{t\to0^+}t^{1-\alpha}x(t)=x(1),\ \lim\limits_{t\to0^+}t^{1-\beta}y(t)=y(1), \end{cases}$$

where $0<\alpha,\beta\le1$, D_{0+}^{q} is the Riemann-Liouville fractional derivative of order q ($q=\alpha,\beta$); $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are two nonlinear continuous functions. The results are proved by Mawhin's continuation theorem.

Motivated by the aforementioned works, an interesting question arises: whether it is possible to consider an implicit coupled system which contains both resonance and non-resonance problems. To this end, in present paper we aim to study the existence of solutions for system (1.1). From system (1.1), it is easy to see that the BVP formed by the first differential equation with the anti-periodic boundary condition $x(0)=-x(1)$ is a non-resonance problem, and the BVP consisting of the second differential equation with the periodic boundary condition $y(0)=y(1)$ is a resonance problem. The contributions and novelties of this article are as follows:

- we provide an improved result for the abstract continuation theorem for k-set contractions (see Theorem 3.1). It should be noted that Theorem 3.1 is equivalent to the Theorem 2.1 and it better adapts to study the existence of solutions for ordinary differential equations;
- Most of the existing studies on implicit fractional boundary value problems were studied by using fixed point theorem. In the current work, we propose to study such problems by means of the abstract continuation theorem for k-set contractions;
- We study the existence of solutions for nonlinear coupled system of fractional differential equations under periodic and anti-periodic boundary conditions, which is untreated in the previous works.
- Note that implicit coupled system (1.1) happens to be at resonance because its associated homogeneous system BVP has a semi-nontrivial solution $(x(t), y(t)) =(0, c)\not\equiv(0, 0),\ c\in\mathbb{R}$. We point out that the resonance coupled system problems, which corresponding homogeneous BVPs have a semi-nontrivial solution, can be studied by the method presented in this paper. Furthermore, some resonance BVPs of p-Laplacian differential equations can be equivalently transformed into such coupled system problems (see [15, 16, 26, 27]). For example, let $y(t)=\phi_p({}^CD^{\alpha}_{0+}x(t))$, then the BVP considered in [16] is equivalent to the following system:

$$\begin{cases} {}^CD^{\alpha}_{0+}x(t)=\phi_q(y(t)),\ t\in(0,1),\\ {}^CD^{\alpha}_{0+}y(t)=f(t,x(t),\phi_q(y(t))),\ t\in(0,1),\\ x(0)=0,\ y(0)=y(1)\text{ or }x(1)=0,\ y(0)=y(1), \end{cases} \tag{1.3}$$

  where $\phi_p(s)=|s|^{p-2}s(p>1)$, $\phi_q=(\phi_p)^{-1}$. By Lemma 2.1 (see section 2), we can easily verify that the corresponding homogeneous BVPs of systems (1.3) have semi-nontrivial solution $(x(t), y(t))=(0, c)\not\equiv(0, 0),\ c\in\mathbb{R}$.

The remainder of the paper is organized as follows. In Sect. 2, some basic definitions and lemmas on Caputo fractional calculus and the (Kuratovski) measure of non-compactness are presented. Moreover, the classical result of the abstract continuation theorem for k-set contractions (Theorem 2.1) is recalled. In Sect. 3, an improved abstract continuation theorem for k-set contractions (Theorem 3.1) is proved. In Sect. 4, applying the Theorem 3.1, the existence result for problem (1.1) is established. In Sect. 5, an interesting example is constructed to illustrate our main results. The last section is devoted to the conclusions.

2. Preliminaries

In this section, we present some definitions and lemmas which are useful in the rest of the paper.

Definition 2.1 ([8, 28]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $x:[0,+\infty)\to\mathbb{R}$ is defined by

$$I_{0+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,$$

provided that the right-hand side integral is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([8, 28]). The Caputo fractional derivative of order $\alpha>0$ of a $(n-1)$-times absolutely continuous function $x:[0,+\infty)\to\mathbb{R}$ is defined by

$${}^{C}D_{0+}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds,\ n-1<\alpha<n,\ n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 ([8, 28]). *Let $\alpha>0$. If $x, {}^{C}D_{0+}^{\alpha}x\in L([0,1],\mathbb{R})$, then*

$$I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x(t)=x(t)+c_0+c_1t+\cdots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}$, $i=0,1,\ldots,n-1$, and $n=[\alpha]+1$.

Definition 2.3 ([9, 22]). Let X be a real Banach space and $S\subset X$ be a bounded subset. The (Kuratovski) measure of non-compactness of S is defined by

$$\alpha_X(S)=\inf\Big\{\delta>0:\text{there is a finite number of subsets } S_i\subset S \\ \text{such that } S=\cup_{i=1}^{m}S_i \text{and } \operatorname{diam}(S_i)\le\delta\Big\},$$

where $\operatorname{diam}(S_i)$ denotes the diameter of set S_i.

Remark 2.1. According to the definition of $\alpha_X(S)$, it follows that $0\le\alpha_X(S)<+\infty$.

Definition 2.4 ([9, 22]). Let X, Z be two real Banach spaces and $\Omega\subset X$ be a bounded open subset. A continuous and bounded operator $N:\Omega\to Z$ is called k-set contractive (k is a nonnegative constant) if for any bounded set $S\subset\Omega$, there is $\alpha_Z(N(S))\le k\alpha_X(S)$.

Lemma 2.2 ([9, 22]). *Let S,T be two bounded subsets in X, then the measure of non-compactness has the following properties:*

(i) $\alpha_X(S)=0\Leftrightarrow S$ *is a relatively-compact subset;*

(ii) $S\subset T\Rightarrow\alpha_X(S)\le\alpha_X(T)$*;*

(iii) $\alpha_X(\bar{S})=\alpha_X(S)$*;*

(iv) $\alpha_X(S\cup T)=\max\{\alpha_X(S),\alpha_X(T)\}$.

Definition 2.5 ([36]). Let X,Z be two real Banach spaces and consider the linear operator $L:\operatorname{dom}L\subset X\to Z$. Then L is called a Fredholm operator with index zero if the following conditions are satisfied:

(i) $\operatorname{Im}L$ is a closed subset of Z;

(ii) $\dim \mathrm{Ker} L = \mathrm{codim}\, \mathrm{Im}\, L < +\infty$.

Suppose $L : \mathrm{dom} L \subset X \to Z$ is a Fredholm operator with index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that

$$\mathrm{Im}\, P = \mathrm{Ker} L, \ \mathrm{Im}\, L = \mathrm{Ker} Q, \ X = \mathrm{Ker} L \oplus \mathrm{Ker} P, \ Z = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q,$$

and $L|_{\mathrm{dom} L \cap \mathrm{Ker} P} : \mathrm{dom} L \to \mathrm{Im} L$ is invertible.

Theorem 2.1 (Abstract continuation theorem for k-set contractions [39]). *Let $L : domL \subset X \to Z$ be a Fredholm operator of index zero, $y \in Z$ be a fixed point, and $N : \bar{\Omega} \to Z$ is k-set contractive with $k < l(L)$, where $\Omega \subset X$ is bounded, open and symmetric about $\theta (\theta \in \Omega)$. Assume that the following conditions are satisfied:*

(*a*) $Lx \neq \lambda Nx + \lambda y$, *for all* $(x, \lambda) \in (domL \cap \partial\Omega) \times (0, 1)$*;*

(*b*) $[QN(x) + Qy, x] \cdot [QN(-x) + Qy, x] < 0$, *for all* $x \in KerL \cap \partial\Omega$,

where θ is the zero element in X, $[\cdot, \cdot]$ is some bounded bilinear form on $Z \times X$. Then there exists $x \in \bar{\Omega}$ such that $Lx - Nx = y$.

Remark 2.2. In Theorem 2.1, if we take the fixed point $y = \theta \in Z$, then the operator equation $Lx - Nx = y$ will degenerate into $Lx = Nx$, which happens to be the abstract equation studied in Mawhin's continuation theorem (see [36]); Compared with Mawhin's continuation theorem, the advantage of applying Theorem 2.1 is that it is unnecessary to verify that the operator N is L-compact.

Remark 2.3. Abstract continuation theorem for k-set contractions has important theoretical significance, which is widely used to study the existence of solutions for differential equations, see [18, 31, 33] and the references therein.

3. Improved result for Theorem 2.1

In this section, an improved abstract continuation theorem for k-set contractions will be given.

Lemma 3.1. *Let X, Z be two linear normed spaces and $[\cdot, \cdot]$ be a bounded bilinear form on $Z \times X$. If, for each $(\alpha, \beta) \in Z \times X$, there is $\alpha = 0$ or $\beta = 0$, then $[\alpha, \beta] = 0$.*

Proof. If fact, let $(\alpha, \beta) \in Y \times X$, without loss of generality, we may assume $\alpha = 0$. Now that $[\cdot, \cdot]$ is a bilinear functional, it follows that

$$[\alpha, \beta] = [0, \beta] = [\lambda 0, \beta] = \lambda [0, \beta], \ \text{for all } \lambda \in \mathbb{R}, \ \beta \in X.$$

Then, by the arbitrariness of λ, we conclude that $[0, \beta] = 0$. This ends the proof. □

A direct application of the Lemma 3.1 is equivalent the Theorem 2.1 to the following form:

Theorem 3.1. *Let $L : domL \subset X \to Z$ be a Fredholm operator of index zero, $y \in Z$ be a fixed point, and $N : \bar{\Omega} \to Z$ is k-set contractive with $k < l(L)$, where $\Omega \subset X$ is bounded, open and symmetric about $\theta (\theta \in \Omega)$. Assume that the following conditions are satisfied:*

(i) $Lx{\neq}\lambda Nx{+}\lambda y,$ *for all* $(x,\lambda){\in}[(domL\backslash KerL){\cap}\partial\Omega]{\times}(0,1)$*;*

(ii) $[QN(x){+}Qy,x]\cdot[QN(-x){+}Qy,x]{<}0,$ *for all* $x{\in}KerL{\cap}\partial\Omega,$

where θ is the zero element in X, $[\cdot,\cdot]$ is some bounded bilinear form on $Z{\times}X$. Then there exists $x{\in}\bar{\Omega}$ such that $Lx{-}Nx{=}y$.

Proof. By Theorem 2.1, the theorem will be proved if we can show the hypothesis (a) in Theorem 2.1 is satisfied. Since $L:\Omega{\subset}X{\to}Z$ is a Fredholm operator of index zero, then $\mathrm{Im}\,L$ is a closed subset in Z, and there exist closed subspace $X_1{\subset}X$ and $Z_2{\subset}Z$ such that $X{=}\mathrm{Ker}L\oplus X_1$, $Z{=}\,\mathrm{Im}\,L\oplus Z_2$, $\dim Z_2{=}\dim\mathrm{Ker}L$. Because $Q:Z{\to}Z_2$ is a projection , it follows that $\mathrm{Ker}Q{=}\,\mathrm{Im}\,L$, $\mathrm{Im}\,Q{=}Z_2$. We now claim that the hypothesis (a) in theorem 2.1 holds. If otherwise, there exist $x{\in}\mathrm{dom}L{\cap}\partial\Omega$ and $\lambda_0{\in}(0,1)$ such that

$$Lx{=}\lambda_0 Nx{+}\lambda_0 y. \tag{3.1}$$

Applying the operators Q and $I{-}Q$ on both sides of Eq. (3.1), respectively, we obtain

$$QNx{+}Qy{=}0, \tag{3.2}$$

$$Lx{=}\lambda_0 Nx{+}\lambda_0 y,\ \ \lambda_0{\in}(0,1). \tag{3.3}$$

By using hypothesis (ii), we have $QNx{+}Qy{\neq}0$, for all $x{\in}\mathrm{Ker}L{\cap}\partial\Omega$. Then we obtain from Eq. (3.2) that

$$x{\in}(\mathrm{dom}L\backslash\mathrm{Ker}L){\cap}\partial\Omega,$$

which, together with the Eq. (3.3), yields

$$Lx{=}\lambda_0 Nx{+}\lambda_0 y,\ x{\in}(\mathrm{dom}L\backslash\mathrm{Ker}L){\cap}\partial\Omega,\ \lambda_0{\in}(0,1).$$

This is contrary to hypothesis (i). Then the proof is ended. □

Remark 3.1. Now, we have improved the condition (a) in Theorem 2.1 to condition (i) in Theorem 3.1. The advantage of the improvement is that operator L is invertible in the domain $\mathrm{dom}L\backslash\mathrm{Ker}L$. Let $K_P{=}\,(L|_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$ and $P:X{\to}X$ be a projection with $\mathrm{Im}\,P{=}\mathrm{Ker}L$. Then, for any $x{\in}X$, it follows that $x{=}Px{+}(I{-}P)x$ and $(I{-}P)x{=}K_PL(I{-}P)x{=}K_PLx$. This means that the prior estimate of condition (i) in Theorem 3.1 can be converted to verify the boundedness of $||Px||$ and $||K_PLx||$. Therefore, using Theorem 3.1 can simplify the process of prior estimation, and hence, the Theorem 3.1 is more practical than Theorem 2.1.

4. The application of Theorem 3.1

In this section, we will use Theorem 3.1 to study the existence of solutions for problem (1.1). To state our result, we need to introduce two Banach spaces. Define

$$X_1{=}\left\{x(t):x(t),{}^{C}D^{\alpha}_{0+}x(t){\in}C[0,1]\right\},\ X_2{=}\left\{y(t):y(t),{}^{C}D^{\beta}_{0+}y(t){\in}C[0,1]\right\},$$

endowed with the norms

$$||x||_{X_1}{=}\max\left\{||x||_\infty,||{}^{C}D^{\alpha}_{0+}x||_\infty\right\},\ ||y||_{X_2}{=}\max\left\{||y||_\infty,||{}^{C}D^{\beta}_{0+}y||_\infty\right\},$$

respectively, where $||\cdot||_\infty=\max_{t\in[0,1]}|\cdot|$. Proceeding as in the proof of ([25], Lemma 2.3), we can obtain $(X_1,||\cdot||_{X_1})$ and $(X_2,||\cdot||_{X_2})$ are Banach spaces. Take $Z_1=C[0,1]$ with the maximum norm $||z||_{Z_1}=||z||_\infty=\max_{t\in[0,1]}|z(t)|$. We now consider two Banach spaces $X=X_1\times X_2$, $Z=Z_1\times Z_1$, respectively, with the norms

$$||(x,y)||_X=\max\{||x||_{X_1},||y||_{X_2}\},\quad ||(u,v)||_Z=\max\{||u||_\infty,||v||_\infty\}.$$

Define the linear operators $L_i:\mathrm{dom}L_i\subset X_i\to Z_1$ $(i=1,2)$ and the nonlinear operators $N_i:X\to Z_1$ $(i=1,2)$ as follows:

$$L_1x(t)={}^CD^\alpha_{0+}x(t),\ x(t)\in\mathrm{dom}L_1,\ N_1(x,y)=f(t,x(t),{}^CD^\beta_{0+}y(t)),\ (x,y)\in X,$$
$$L_2y(t)={}^CD^\beta_{0+}y(t),\ y(t)\in\mathrm{dom}L_2,\ N_2(x,y)=g(t,y(t),{}^CD^\alpha_{0+}x(t)),\ (x,y)\in X,$$

where

$$\mathrm{dom}L_1=\{x\in X_1:x(0)=-x(1)\},\ \mathrm{dom}L_2=\{y\in X_2:y(0)=y(1)\}.$$

Define the linear operator $L:\mathrm{dom}L\subset X\to Z$ and the nonlinear operator $N:X\to Z$ as follows:

$$L(x,y)=(L_1x,L_2y),(x,y)\in\mathrm{dom}L,\ N(x,y)=(N_1(x,y),N_2(x,y)),(x,y)\in X,\quad(4.1)$$

where

$$\mathrm{dom}L=\{(x,y)\in X:x\in\mathrm{dom}L_1,\ y\in\mathrm{dom}L_2\}.$$

Let $e(t)=(e_1(t),e_2(t))\in C[0,1]\times C[0,1]$, then the coupled system of BVP (1.1) can be rewritten in the form

$$L(x,y)=N(x,y)+e.$$

Theorem 4.1. *Suppose that following conditions are satisfied:*

(H_1) *There exist constants $k_1,k_2\in[0,1)$ such that for any $x,y,v_i,u_i\in\mathbb{R}$ $(i=1,2)$ and all $t\in[0,1]$,*

$$|f(t,x,u_1)-f(t,x,v_1)|\le k_1|u_1-v_1|,\ |g(t,y,u_2)-g(t,y,v_2)|\le k_2|u_2-v_2|.$$

(H_2) *There exist nonnegative functions $p_i(t),q_i(t),r_i(t)$ $(i=1,2)$ such that*

$$|f(t,u_{11},v_{11})|\le p_1(t)|u_{11}|+q_1(t)|v_{11}|+r_1(t),\ \textit{for all}\ t\in[0,1],\ (u_{11},v_{11})\in\mathbb{R}^2;$$
$$|g(t,u_{12},v_{12})|\le p_2(t)|u_{12}|+q_2(t)|v_{12}|+r_2(t),\ \textit{for all}\ t\in[0,1],\ (u_{12},v_{12})\in\mathbb{R}^2.$$

(H_3) *For $u\in\mathbb{R}$, there exist constants $a,A>0$ and $b,B\ge0$ such that if $|u|>A$, then*

$$|g(t,u,v)|\ge a|u|-b|v|-B,\ \textit{for all}\ t\in[0,1],\ v\in\mathbb{R}.$$

(H_4) *For $c\in\mathbb{R}$, there exists constant $G>0$ such that if $|c|>G$, then*

$$\int_0^1(1-s)^{\beta-1}[g(s,c,0)+e_2(s)]ds\cdot\int_0^1(1-s)^{\beta-1}[g(s,-c,0)+e_2(s)]ds<0.$$

Then BVP (1.1) has at least one solution in X, provided that

$$\left(\frac{b}{a}+\frac{1}{\Gamma(\beta+1)}+\Lambda+\frac{4^{-\beta}}{a\beta}\right)m_0<1,$$

where

$$p_i^0{=}||p_i||_\infty,\ q_i^0{=}||q_i||_\infty,\ r_i^0{=}||r_i||_\infty,\ e_i^0{=}||e_i||_\infty\ (i{=}1,2),$$
$$m_0{=}\max\left\{p_1^0{+}q_1^0, p_2^0{+}q_2^0\right\},\ \Lambda{=}\max\left\{\frac{3}{2\Gamma(\alpha+1)}, \frac{1}{\Gamma(\beta+1)}\right\}.$$

Before we prove Theorem 4.1, let's first prove some auxiliary lemmas.

Lemma 4.1. *X_i $(i{=}1,2)$ is compactly embedded into Z_1.*

Proof. Obviously, we have $X_i{\subset}Z_1(i{=}1,2)$. Define the embedded mapping

$$i: X_1{\to}Z_1,\ ix(t){=}x(t),\ x(t){\in}X_1.$$

Then

$$||ix||_{Z_1}{=}\max_{t\in[0,1]}|x(t)|{\le}||x||_{X_1}{=}\max\left\{\max_{t\in[0,1]}|x(t)|,\ \max_{t\in[0,1]}|{}^CD_{0+}^\alpha x(t)|\right\},$$

that is, X_1 is continuously embedded into Z_1. Let $\Omega{\subset}X_1$ be any bounded subset. Now we prove that $i(\Omega){\subset}Z_1$ is a relatively-compact set. In fact, since Ω is bounded, there exists a constant $M{>}0$ such that $||x||_{X_1}{\le}M$ for all $x(t){\in}\Omega$. On the one hand, by using the definition of i, we obtain

$$||ix||_{Z_1}{=}||x||_{Z_1}{\le}||x||_{X_1}{\le}M,$$

that is, $i(\Omega)$ is bounded uniformly. On the other hand, since $||x||_{X_1}{\le}M$, we obtain $||{}^CD_{0+}^\alpha x||_\infty{\le}M$. Then, for any $t_1, t_2{\in}[0,1]$, without loss of generality we can assume that $0{\le}t_1{<}\ t_2{\le}1$, by Lemma 2.1, one has

$$\begin{aligned}|ix(t_2){-}ix(t_1)|&{=}|x(t_2){-}x(t_1)|{=}|I_{0+}^\alpha{}^CD_{0+}^\alpha x(t_1){-}I_{0+}^\alpha{}^CD_{0+}^\alpha x(t_2)|\\
&{=}\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1{-}s)^{\alpha-1}{}^CD_{0+}^\alpha x(s)ds{-}\int_0^{t_2}(t_2{-}s)^{\alpha-1}{}^CD_{0+}^\alpha x(s)ds\right|\\
&{\le}\frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_1}\left[(t_1{-}s)^{\alpha-1}{-}(t_2{-}s)^{\alpha-1}\right]\left|{}^CD_{0+}^\alpha x(s)\right|ds\right.\\
&\quad\left.+\int_{t_1}^{t_2}(t_2{-}s)^{\alpha-1}\left|{}^CD_{0+}^\alpha x(s)\right|ds\right\}\\
&{\le}\frac{M}{\Gamma(\alpha{+}1)}\left[2(t_2{-}t_1)^\alpha{-}(t_2^\alpha{-}t_1^\alpha)\right],\end{aligned}$$

which tends to zero as $t_1{\to}t_2$ independent of x, that is, $i(\Omega)$ is equicontinuous. Hence, by the Arzelà-Ascoli theorem, i is compact on X_1. Therefore, X_1 is compactly embedded into Z_1. Likewise, we can prove that X_2 is compactly embedded into Z_1. The proof is completed. □

Lemma 4.2. *Let the operators L and N be as in (4.1). Then*

(*i*) $KerL{=}\{(x,y){\in}domL : x(t){=}0,\ y(t){\equiv}c{\in}\mathbb{R},\ t{\in}[0,1]\}$;

(*ii*) $ImL{=}\left\{(u,v){\in}Z : \int_0^1(1{-}s)^{\beta-1}v(s)ds{=}0\right\}$;

(*iii*) *L is a Fredholm operator with index zero;*

(*iv*) $l(L){\geq}1$, *where*

$$l(L){=}sup\left\{r{>}0: r\alpha_X(\Omega){\leq}\alpha_Z(L(\Omega)),\ for\ all\ bounded\ subset\ \Omega{\subset}domL\right\};$$

(*v*) N *is a* k*-set contractive map, where* $k{=}\max\{k_1,k_2\}$.

Proof. Define the projection operator $Q: Z{\to}Z$ as

$$Q(u,v){=}\left(0,\beta\int_0^1(1{-}s)^{\beta-1}v(s)ds\right),\ (u,v){\in}Z. \tag{4.2}$$

Then the proofs of properties (i), (ii) and (iii) are trivial, which can be proved by the similar process as in the proof of ([48], Lemma 3.1). We now show that the properties (iv) and (v) are also hold. In fact, for any bounded set $\Omega{\subset}X$, let $\sigma{=}\alpha_Z(L(\Omega))$, then $\sigma{\geq}0$. By Lemma 4.1, we obtain $\Omega{\subset}Z$ is a relatively-compact set. Combining this with Lemma 2.2, we have $\alpha_Z(\Omega){=}0$. Using the definition of the measure of noncompactness: for any $\varepsilon{>}0$, there is a finite number of subsets $\Omega_i{\subset}X$ $(i{=}1,2,\cdots,n)$ such that $\Omega{=}\cup_{i=1}^n\Omega_i$ and $\mathrm{diam}_Z(\Omega_i){<}\varepsilon$ $(i{=}1,2,\cdots,n)$. In view of $\alpha_Z(L(\Omega)){=}\sigma$, so, $\alpha_Z(L(\Omega_i)){<}\sigma{+}\varepsilon$. Hence, for the above $\varepsilon{>}0$ and for every $i{\in}\{1,2,\cdots,n\}$, there is a finite number of subsets $\sum_i^1,\cdots,\sum_i^{n_i}{\subset}Z$ such that $L(\Omega_i){=}\cup_{k=1}^{n_i}\sum_i^k$ and $\mathrm{diam}_Z(\sum_i^k){\leq}\sigma{+}\varepsilon$. Now we define $\Omega_i^k{=}\left\{(x,y){\in}\Omega_i : L(x,y){\in}\sum_i^k\right\}$, then we must have $\mathrm{diam}_Z(L(\Omega_i^k)){\leq}\sigma{+}\varepsilon$ and $\Omega{=}\cup_{i=1}^n\cup_{k=1}^{n_i}\Omega_i^k$. Thus, for any (x_1,y_1), $(x_2,y_2){\in}\Omega_i^k$, by the fact that

$${}^CD_{0+}^{\alpha}x_1{-}{}^CD_{0+}^{\alpha}x_2{=}L_1x_1{-}L_1x_2,\ {}^CD_{0+}^{\beta}y_1{-}{}^CD_{0+}^{\beta}y_2{=}L_2y_1{-}L_2y_2,$$

it follows

$$\begin{aligned}&\max\left\{\left\|{}^CD_{0+}^{\alpha}x_1{-}{}^CD_{0+}^{\alpha}x_2\right\|_\infty,\left\|{}^CD_{0+}^{\beta}y_1{-}{}^CD_{0+}^{\beta}y_2\right\|_\infty\right\}\\ {=}&\max\left\{\|L_1x_1{-}L_1x_2\|_\infty,\|L_2y_1{-}L_2y_2\|_\infty\right\}\leq\sigma{+}\varepsilon.\end{aligned} \tag{4.3}$$

Note that $\mathrm{diam}_Z(\Omega_i){<}\varepsilon(i{=}1,2,\cdots,n)$, we have

$$\mathrm{diam}_Z(\Omega_i^k){<}\varepsilon. \tag{4.4}$$

Combining formulas (4.3) and (4.4), we obtain

$$\mathrm{diam}_X(\Omega_i^k){\leq}\sigma{+}\varepsilon.$$

Then, from the arbitrariness of ε that $\alpha_X(\Omega){\leq}\sigma{=}\alpha_Z(L(\Omega))$. Consequently, we infer that $l(L){\geq}1$, and hence, (iv) holds.

(v). Let $\Omega{\subset}X$ be any bounded subset. In virtue of $f,g{\in}C([0,1]{\times}\mathbb{R}^2,\mathbb{R})$, we get f,g are uniformly continuous on $\bar{\Omega}$. Then, for $(\gamma,\eta){\in}X$, define the operator $N_{\gamma,\eta}: \bar{\Omega}{\to}Z$ by

$$N_{\gamma,\eta}(x(t),y(t)){=}\left(f(t,y(t),{}^CD_{0+}^{\beta}\eta(t)),g(t,x(t),{}^CD_{0+}^{\alpha}\gamma(t))\right),\ (x,y){\in}\bar{\Omega},$$

we have $\{N_{\gamma,\eta}:(\gamma,\eta){\in}X\}$ is uniformly continuous on $\bar{\Omega}$, that is, for any $\varepsilon{>}0$, there exists $\delta(\varepsilon){>}0$ such that for $(\gamma,\eta){\in}X$, $(x_1,y_1),(x_2,y_2){\in}\bar{\Omega}$, if $||(x_1,y_1){-}(x_2,y_2)||_X{<}\delta$, then $\|N_{\gamma,\eta}(x_1,y_1){-}N_{\gamma,\eta}(x_2,y_2)\|_\infty{<}\varepsilon$. We now show that N is a k-set contractive map. In fact, let $\sigma{=}\alpha_X(\Omega)$, then for any $\varepsilon{>}0$, there is a finite number of subsets $\Omega_1,\cdots,\Omega_m{\subset}\Omega$ such that $\Omega{=}\cup_{i=1}^m\Omega_i$ and

$$\mathrm{diam}_X(\Omega_i){\leq}\sigma{+}\varepsilon. \tag{4.5}$$

Now by Lemma 4.1, we obtain Ω is relatively compact in Z, and hence $\Omega_i(i{=}1,2,\cdots,m)$ is also relatively compact in Z, that is, $\alpha_Z(\Omega_i){=}0(i{=}1,2,\cdots,m)$. Consequently, for every $i{\in}\{1,2,\cdots,m\}$ there is a finite number of subsets $\Omega_i^1,\Omega_i^2,\cdots,\Omega_i^{m_i}{\subset}\Omega_i$ such that $\Omega_i{=}\cup_{j=1}^{m_i}\Omega_i^j$ and $\mathrm{diam}_Z(\Omega_i^j){<}\varepsilon(i{=}1,2,\cdots,m;j{=}1,2,\cdots,m_i)$. Noting that $\{N_{\gamma,\eta}:(\gamma,\eta){\in}X\}$ is uniformly equicontinuous, then for every $i{\in}\{1,2,\cdots,m\}$ and $(\gamma,\eta){\in}X$,

$$\mathrm{diam}_Z\big(N_{\gamma,\eta}(\Omega_i^j)\big){\le}\varepsilon,\ \ j{=}1,2,\cdots,m_i, \tag{4.6}$$

which, further implying that for any $(x,y),(\gamma,\eta){\in}\Omega_i^j$ $(i{=}1,2,\cdots,m;\ j{=}1,2,\cdots,m_i)$,

$$\|N(x,y){-}N(\gamma,\eta)\|_\infty{\le}\|N(x,y){-}N_{\gamma,\eta}(x,y)\|_\infty{+}\|N_{\gamma,\eta}(x,y){-}N(\gamma,\eta)\|_\infty. \tag{4.7}$$

Combining this with condition (H_1) gives

$$\begin{aligned}
&\|N(x,y){-}N_{\gamma,\eta}(x,y)\|_\infty\\
&{=}\max\Big\{\sup_{t\in[0,1]}\big|f(t,x(t),{}^CD_{0+}^\beta y(t)){-}f(t,x(t),{}^CD_{0+}^\beta\eta(t))\big|,\\
&\quad\sup_{t\in[0,1]}\big|g(t,y(t),{}^CD_{0+}^\alpha x(t)){-}g(t,y(t),{}^CD_{0+}^\alpha\gamma(t))\big|\Big\}\\
&{\le}\max\Big\{k_1\big\|{}^CD_{0+}^\beta y(t){-}{}^CD_{0+}^\beta\eta(t)\big\|_\infty,k_2\big\|{}^CD_{0+}^\alpha x(t){-}{}^CD_{0+}^\alpha\gamma(t)\big\|_\infty\Big\}\\
&{\le}k\max\Big\{\big\|{}^CD_{0+}^\beta y(t){-}{}^CD_{0+}^\beta\eta(t)\big\|_\infty,\big\|{}^CD_{0+}^\alpha x(t){-}{}^CD_{0+}^\alpha\gamma(t)\big\|_\infty\Big\}.
\end{aligned} \tag{4.8}$$

On the one hand, the Eqs. (4.5) and (4.8) imply

$$\|N(x,y){-}N_{\gamma,\eta}(x,y)\|_\infty{\le}k(\sigma{+}\varepsilon). \tag{4.9}$$

On the other hand, the Eq. (4.6) imply

$$\|N_{\gamma,\eta}(x,y){-}N(\gamma,\eta)\|_\infty{=}\|N_{\gamma,\eta}(x,y){-}N_{\gamma,\eta}(\gamma,\eta)\|_\infty{\le}\varepsilon, \tag{4.10}$$

for any $(x,y),(\gamma,\eta){\in}\Omega_i^j$ $(i{=}1,2,\cdots,m;\ j{=}1,2,\cdots,m_i)$. By substituting (4.9) and (4.10) into (4.7), we obtain

$$\|N(x,y){-}N(\gamma,\eta)\|_\infty{\le}k\sigma{+}(k{+}1)\varepsilon,$$

for any $(x,y),(\gamma,\eta){\in}\Omega_i^j$ $(i{=}1,2,\cdots,m;j{=}1,2,\cdots,m_i)$. Then, from the arbitrariness of ε that $\alpha_Z(N(\Omega))\ {\le}k\alpha_X(\Omega)$. Thus we have derived that N is a k-set contractive map. The proof is finished. ☐

Lemma 4.3. *Define the linear operator* $K_P:\mathrm{Im}\,L\to domL\cap Y$ *as follows:*

$$K_P(u,v){=}\Big(I_{0+}^\alpha u{-}\Big(\frac{1}{2}I_{0+}^\alpha u|_{t=1}\Big),\ I_{0+}^\beta v\Big),\ \ (u,v){\in}\,\mathrm{Im}\,L,$$

then $K_P{=}(L|_{domL\cap Y})^{-1}$ *and satisfies*

$$||K_P(u,v)||_X{\le}\Lambda||(u,v)||_Z,\ \textit{for all}\ (u,v){\in}\,\mathrm{Im}\,L,$$

where $Y{=}\{(x,y){\in}X:y(0){=}0\}$.

Proof. Define the linear operator $P:X\to X$ by

$$P(x,y)=(0,y(0)),\quad (x,y)\in X.$$

It is easy to verify that P is a projection operator and satisfies $Y=\mathrm{Ker}P$. The remainder of the argument is analogous to that in ([48], Lemma 3.2) and we omit the details here. This finishes the proof. □

With the help of the preceding lemmas we are now turning to the proof of Theorem 4.1.

Proof of Theorem 4.1. In fact, we only need to construct a bounded open set Ω which satisfies the conditions (i) and (ii) of Theorem 3.1. To this end, we denote

$$\Omega_1=\{(x,y)\in\mathrm{dom}L\backslash\mathrm{Ker}L: L(x,y)=\lambda(N(x,y)+e),\ \lambda\in(0,1)\}.$$

We now show that Ω_1 is bounded. In deed, for any $(x,y)\in\Omega_1$, then $N(x,y)+e\in\mathrm{Im}\,L=\mathrm{Ker}Q$, that is, $Q(N(x,y)+e)=0$, namely,

$$\int_0^1(1-s)^{\beta-1}g(s,y(s),{}^CD^{\alpha}_{0+}x(s))ds=-\int_0^1(1-s)^{\beta-1}e_2(s)ds.$$

Hence,

$$\begin{aligned}&\int_0^{3/4}(1-s)^{\beta-1}g(s,y(s),{}^CD^{\alpha}_{0+}x(s))ds\\ =&-\int_{3/4}^1(1-s)^{\beta-1}g(s,y(s),{}^CD^{\alpha}_{0+}x(s))ds-\int_0^1(1-s)^{\beta-1}e_2(s)ds.\end{aligned}$$

It follows from the continuous of $t\mapsto(1-t)^{\beta-1}g(t,y(t),{}^CD^{\alpha}_{0+}x(t)),\ t\in[0,3/4]$ that there exists $\zeta\in(0,3/4)$ such that

$$\begin{aligned}&(1-\zeta)^{\beta-1}g(\zeta,y(\zeta),{}^CD^{\alpha}_{0+}x(\zeta))\\ =&-\int_{3/4}^1(1-s)^{\beta-1}g(s,y(s),{}^CD^{\alpha}_{0+}x(s))ds-\int_0^1(1-s)^{\beta-1}e_2(s)ds.\end{aligned}$$

By using (H_2), we obtain

$$\begin{aligned}&\left|g(\zeta,y(\zeta),{}^CD^{\alpha}_{0+}x(\zeta))\right|\\ \le&\frac{1}{\beta(1-\zeta)^{\beta-1}}\left[\left(p_2^0\|y\|_\infty+q_2^0\left\|{}^CD^{\alpha}_{0+}x\right\|_\infty+r_2^0\right)4^{-\beta}+e_2^0\right].\end{aligned}\tag{4.11}$$

Since $(x,y)\in\mathrm{dom}L\backslash\mathrm{Ker}L$, by Lemma 4.3, we find that

$$\begin{aligned}\|(I-P)(x,y)\|_X&=\|K_PL(I-P)(x,y)\|_X\\ &\le\Lambda\|L(x,y)\|_Z=\Lambda\max\left\{\left\|{}^CD^{\alpha}_{0+}x\right\|_\infty,\left\|{}^CD^{\beta}_{0+}y\right\|_\infty\right\}.\end{aligned}\tag{4.12}$$

On the one hand, if there exists $t_0\in[0,1]$ such that $|y(t_0)|\le A$, then by $y(0)=y(t_0)-I^{\beta}_{0+}{}^CD^{\beta}_{0+}y(t)|_{t=t_0}$, we get

$$|y(0)|\le A+\frac{1}{\Gamma(\beta+1)}\left\|{}^CD^{\beta}_{0+}y\right\|_\infty.\tag{4.13}$$

On the other hand, suppose $|y(t)|>A$ for any $t\in[0,1]$. The hypothesis (H$_3$) implies

$$\left|g(t,y(t),{}^CD_{0+}^{\alpha}x(t))\right|\geq a\,|y(t)|-b\left|{}^CD_{0+}^{\alpha}x\right|-B. \tag{4.14}$$

From the inequalities (4.11) and (4.14), we obtain

$$\begin{aligned}|y(\zeta)|\leq&\frac{B}{a}+\frac{b}{a}\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}\\&+\frac{1}{a(1-\zeta)^{\beta-1}\beta}\left[\left(p_2^0\|y\|_{\infty}+q_2^0\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+r_2^0\right)4^{-\beta}+e_2^0\right].\end{aligned} \tag{4.15}$$

Using the fact that $y(0)=y(\zeta)-I_{0+}^{\beta}\,{}^CD_{0+}^{\beta}y(t)|_{t=\zeta}$, (4.15) leads to the inequality

$$\begin{aligned}|y(0)|\leq&\frac{B}{a}+\frac{b}{a}\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+\frac{1}{\Gamma(\beta+1)}\left\|{}^CD_{0+}^{\beta}y\right\|_{\infty}\\&+\frac{1}{a(1-\zeta)^{\beta-1}\beta}\left[\left(p_2^0\|y\|_{\infty}+q_2^0\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+r_2^0\right)4^{-\beta}+e_2^0\right]\\\leq&\frac{B}{a}+\frac{b}{a}\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+\frac{1}{\Gamma(\beta+1)}\left\|{}^CD_{0+}^{\beta}y\right\|_{\infty}\\&+\frac{1}{a\beta}\left[\left(p_2^0\|y\|_{\infty}+q_2^0\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+r_2^0\right)4^{-\beta}+e_2^0\right].\end{aligned} \tag{4.16}$$

Then, using (4.13) and (4.16), for any $(x,y)\in\Omega_1$, we have

$$\begin{aligned}|y(0)|\leq&\frac{B}{a}+A+\frac{b}{a}\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+\frac{1}{\Gamma(\beta+1)}\left\|{}^CD_{0+}^{\beta}y\right\|_{\infty}\\&+\frac{1}{a\beta}\left[\left(p_2^0\|y\|_{\infty}+q_2^0\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+r_2^0\right)4^{-\beta}+e_2^0\right].\end{aligned} \tag{4.17}$$

We also have,

$$\|P(x,y)\|_X=|y(0)|. \tag{4.18}$$

In view of $(x,y)=P(x,y)+(I-P)(x,y)$, we obtain from the Eqs. (4.12), (4.17) and (4.18) that

$$\begin{aligned}\|(x,y)\|_X\leq&\|P(x,y)\|_X+\|(I-P)(x,y)\|_X\\\leq&|y(0)|+\Lambda\max\left\{\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty},\left\|{}^CD_{0+}^{\beta}y\right\|_{\infty}\right\}\\\leq&\max\left\{\frac{B}{a}+A+\left(\frac{b}{a}+\Lambda\right)\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+\frac{1}{\Gamma(\beta+1)}\left\|{}^CD_{0+}^{\beta}y\right\|_{\infty}\right.\\&+\frac{1}{a\beta}\left[\left(p_2^0\|y\|_{\infty}+q_2^0\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+r_2^0\right)4^{-\beta}+e_2^0\right],\\&\frac{B}{a}+A+\frac{b}{a}\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+\left(\frac{1}{\Gamma(\beta+1)}+\Lambda\right)\left\|{}^CD_{0+}^{\beta}y\right\|_{\infty}\\&\left.+\frac{1}{a\beta}\left[\left(p_2^0\|y\|_{\infty}+q_2^0\left\|{}^CD_{0+}^{\alpha}x\right\|_{\infty}+r_2^0\right)4^{-\beta}+e_2^0\right]\right\}.\end{aligned}$$

Furthermore, it follows from (H$_2$) that

$$\begin{aligned}\left|{}^CD_{0+}^{\alpha}x\right|=&\left|f(t,x(t),{}^CD_{0+}^{\beta}y(t))+e_1(t)\right|\\\leq&p_1^0\|x\|_{\infty}+q_1^0\left\|{}^CD_{0+}^{\beta}y\right\|_{\infty}+r_1^0+e_1^0,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\left|{}^C D_{0+}^{\beta} y\right| =& \left|g(t, y(t), {}^C D_{0+}^{\alpha} x(t)) + e_2(t)\right| \\ \leq& p_2^0 \|y\|_\infty + q_2^0 \left\|{}^C D_{0+}^{\alpha} x\right\|_\infty + r_2^0 + e_2^0.\end{aligned} \tag{4.20}$$

Now we divide the proof into two cases.

Case 1. For any $(x, y) \in \Omega_1$,

$$\begin{aligned}\|(x, y)\|_X \leq& \frac{B}{a} + A + \left(\frac{b}{a} + \Lambda\right) \left\|{}^C D_{0+}^{\alpha} x\right\|_\infty + \frac{1}{\Gamma(\beta+1)} \left\|{}^C D_{0+}^{\beta} y\right\|_\infty \\ &+ \frac{1}{a\beta} \left[\left(p_2^0 \|y\|_\infty + q_2^0 \left\|{}^C D_{0+}^{\alpha} x\right\|_\infty + r_2^0\right) 4^{-\beta} + e_2^0\right].\end{aligned}$$

Combining this with (4.19) and (4.20), we have

$$\begin{aligned}\|(x, y)\|_X \leq& \frac{B}{a} + A + \left(\frac{b}{a} + \Lambda\right) \left[\left(p_1^0 + q_1^0\right) \|(x, y)\|_X + r_1^0 + e_1^0\right] \\ &+ \frac{1}{\Gamma(\beta+1)} \left[\left(p_2^0 + q_2^0\right) \|(x, y)\|_X + r_2^0 + e_2^0\right] \\ &+ \frac{1}{a\beta} \left[\left(\left(p_2^0 + q_2^0\right) \|(x, y)\|_X + r_2^0\right) 4^{-\beta} + e_2^0\right].\end{aligned}$$

Solving the above inequality, we find that

$$\|(x, y)\|_X \leq \frac{\frac{B}{a} + A + \left(\frac{b}{a} + \Lambda\right)\left(r_1^0 + e_1^0\right) + \frac{1}{\Gamma(\beta+1)}\left(r_2^0 + e_2^0\right) + \frac{1}{a\beta}\left(r_2^0 4^{-\beta} + e_2^0\right)}{1 - \left[\left(\frac{b}{a} + \Lambda\right)\left(p_1^0 + q_1^0\right) + \left(\frac{1}{\Gamma(\beta+1)} + \frac{4^{-\beta}}{a\beta}\right)\left(p_2^0 + q_2^0\right)\right]} := M_1.$$

Case 2. For any $(x, y) \in \Omega_1$,

$$\begin{aligned}\|(x, y)\|_X \leq& \frac{B}{a} + A + \frac{b}{a} \left\|{}^C D_{0+}^{\alpha} x\right\|_\infty + \left(\frac{1}{\Gamma(\beta+1)} + \Lambda\right) \left\|{}^C D_{0+}^{\beta} y\right\|_\infty \\ &+ \frac{1}{a\beta} \left[\left(p_2^0 \|y\|_\infty + q_2^0 \left\|{}^C D_{0+}^{\alpha} x\right\|_\infty + r_2^0\right) 4^{-\beta} + e_2^0\right].\end{aligned}$$

By substituting (4.19) and (4.20) into above inequality, one can obtain

$$\begin{aligned}\|(x, y)\|_X \leq& \frac{B}{a} + A + \frac{b}{a} \left[\left(p_1^0 + q_1^0\right) \|(x, y)\|_X + r_1^0 + e_1^0\right] \\ &+ \left(\frac{1}{\Gamma(\beta+1)} + \Lambda\right) \left[\left(p_2^0 + q_2^0\right) \|(x, y)\|_X + r_2^0 + e_2^0\right] \\ &+ \frac{1}{a\beta} \left[\left(\left(p_2^0 + q_2^0\right) \|(x, y)\|_X + r_2^0\right) 4^{-\beta} + e_2^0\right].\end{aligned}$$

A simple computation yields

$$\|(x, y)\|_X \leq \frac{\frac{B}{a} + A + \frac{b}{a}\left(r_1^0 + e_1^0\right) + \left(\frac{1}{\Gamma(\beta+1)} + \Lambda\right)\left(r_2^0 + e_2^0\right) + \frac{1}{a\beta}\left(r_2^0 4^{-\beta} + e_2^0\right)}{1 - \left[\frac{b}{a}\left(p_1^0 + q_1^0\right) + \left(\frac{1}{\Gamma(\beta+1)} + \Lambda + \frac{4^{-\beta}}{a\beta}\right)\left(p_2^0 + q_2^0\right)\right]} := M_2.$$

Thus we arrive at the conclusion that Ω_1 is bounded. Take $\Omega = \{(x, y) \in X : ||(x, y)||_X < G + M_1 + M_2\}$, then the condition (i) in Theorem 3.1 holds. It remains to prove that the condition (ii) in Theorem 3.1 is also satisfied. To show this, we define a bounded bilinear form $[\cdot,\cdot]$ on $Z\times X$ by

$$[(u,v),(x,y)]=\int_0^1 x(t)u(t)dt+\int_0^1 y(t)v(t)dt,\ \text{for all } (u,v)\in Z,\ (x,y)\in X.$$

For $(x,y)\in\mathrm{Ker}L\cap\partial\Omega$, then $(x,y)=(0,c)$, $c\in\mathbb{R}$. And then we can deduce from the expression (4.2)

$$\begin{aligned}&[QN(x,y)+Q(e_1,e_2),(x,y)]\cdot[QN(-x,-y)+Q(e_1,e_2),(x,y)]\\ =&c^2\beta^2\int_0^1(1-s)^{\beta-1}[g(s,c,0)+e_2(s)]ds\cdot\int_0^1(1-s)^{\beta-1}[g(s,-c,0)+e_2(s)]ds\\ <&0,\end{aligned}$$

by using the hypotheses (H_4). Thus all the conditions of Theorem 3.1 are satisfied and the conclusion of the Theorem 3.1 implies that there exists $(x,y)\in\mathrm{dom}L\cap X$ such that $L(x,y)=N(x,y)+e$. This shows that the original problem (1.1) has at least one solution. The proof is completed. □

5. Example

Example 5.1. Consider the following system of fractional boundary value problems:

$$\begin{cases}{}^CD_{0+}^{1/2}x(t)=\frac{1}{21}t^2\sin x(t)+\frac{t}{16+48t}\,{}^CD_{0+}^{3/4}y(t)+\frac{t}{1+t}+\cos t,\ t\in(0,1),\\ {}^CD_{0+}^{3/4}y(t)=\frac{1}{12}h(y)y(t)+\frac{t}{15+45t}\sin{}^CD_{0+}^{1/2}x(t)+\frac{1}{4}t^2+\sin t,\ t\in(0,1),\\ x(0)=-x(1),\ y(0)=y(1).\end{cases}\tag{5.1}$$

Corresponding to problem (1.1), here $\alpha=1/2$, $\beta=3/4$, $e_1(t)=\cos t$, $e_2(t)=\sin t$ and

$$\begin{aligned}&f(t,x(t),{}^CD_{0+}^{\beta}y(t))=\frac{1}{21}t^2\sin x(t)+\frac{t}{16+48t}\,{}^CD_{0+}^{3/4}y(t)+\frac{t}{1+t},\\ &g(t,y(t),{}^CD_{0+}^{\alpha}x(t))=\frac{1}{12}h(y)y(t)+\frac{t}{15+45t}\sin{}^CD_{0+}^{1/2}x(t)+\frac{1}{4}t^2,\\ &h(y)=\begin{cases}0.6, & \text{if } |y|>2,\\ 0.1y+0.4, & \text{if } 1\le y\le 2,\\ 0.5, & \text{if } |y|<1,\\ -0.1y+0.4, & \text{if} -2\le y\le -1.\end{cases}\end{aligned}$$

Choosing,

$$\begin{aligned}&k_1=\frac{1}{64},\ k_2=\frac{1}{60},\ A=1,\ a=\frac{1}{24},\ b=\frac{1}{60},\ B=\frac{1}{4},\ G=30,\\ &p_1(t)=\frac{1}{21}t^2,\ q_1(t)=\frac{t}{16+48t},\ r_1(t)=\frac{t}{1+t},\ e_1(t)=\cos t,\\ &p_2(t)=\frac{1}{20},\ q_2(t)=\frac{t}{15+45t},\ r_2(t)=\frac{1}{4}t^2,\ e_2(t)=\sin t.\end{aligned}$$

It is easy to find that assumptions (H_1)-(H_4) in Theorem 4.1 hold. Furthermore,

$$\begin{aligned}&\left(\frac{b}{a}+\frac{1}{\Gamma(\beta+1)}+\Lambda+\frac{4^{-\beta}}{a\beta}\right)m_0\\ =&\left(\frac{2}{5}+\frac{1}{\Gamma(7/4)}+\frac{3}{\sqrt{\pi}}+8\sqrt{2}\right)\times\frac{1}{15}<\left(\frac{2}{5}+\frac{919}{1000}+\frac{300}{177}+\frac{284}{25}\right)\times\frac{1}{15}<1.\end{aligned}$$

Now all the assumptions in Theorem 4.1 are satisfied and, consequently, its conclusion implies that the BVP (5.1) has at least one solution.

6. Conclusions

In this article, we improve the classical results of the abstract continuation theorem for k-set contractions (Theorem 2.1) to Theorem 3.1 and illustrated that the improved results are better than the previous one. With the aid of Theorem 3.1, we give some sufficient conditions of existence of solutions for problem (1.1); in addition, we also give an example to illustrate the application of the theorem. A point that should be stressed is that the problem (1.1) is different from the implicit coupled system problems studied in previous literatures. The special point of problem (1.1) is not only that the equations contain perturbed terms, but also the BCs are composed of periodic and anti-periodic conditions. BVPs with periodic and anti-periodic conditions are new problems studied in recent work [21,29,30]. In reference [29], the authors called this kind of BCs as the mixed periodic BCs and commented that it is natural to discuss the BVPs involving the mixed periodic BCs instead of the anti-periodic BCs. Last but not least, an interesting extension of our studies would be to consider the fractional p-Laplacian differential equations with mixed periodic BCs.

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