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EXISTENCE AND APPROXIMATE CONTROLLABILITY OF HILFER FRACTIONAL EVOLUTION EQUATIONS IN BANACH SPACES*

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Abstract This paper is concerned with the existence of mild solutions as well as approximate controllability for Hilfer fractional evolution equations in Banach spaces. Firstly, we give an appropriate definition of mild solutions for this type of fractional equations. The definition of mild solutions for studied problem was given based on a cosine family generated by the operator A and probability density function. Secondly, we discuss the existence results of the mild solutions for our concerned problem under the case sine family is compact. Moreover, we establish the approximate controllability when the corresponding linear system is approximately controllable. At last, as an application, two examples are presented to illustrate the abstract results.

Keywords Hilfer fractional evolution equations, mild solution, approximate controllability, cosine family.

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1. Introduction

From the view point of physics, we can consider some more special problems on fractional evolutions which are abstracted from fractional differential equations, fractional differential equations have been applied to various fields successfully, for example, physics, engineering, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and they have been emerging as an important area of investigation in the last few decades; see [1,4,8,9,11]. Some recent papers investigated the problem of the existence of mild solution for abstract differential equations with fractional derivative [12,13,20,28,30,35,49,60–62,66]. Since the mild solution definition in integer order abstract differential equations obtained by variation of constant formulas can not be generalized directly to fractional order abstract differential equations, Zhou and Jiao [65] gave a suit concept on mild solutions by applying laplace transform and probability density functions for evolution equation with Caputo fractional derivative. By using sectorial operator, Shu et al. [50] gave a definition of mild solution for fractional differential equation with order $1 < \alpha < 2$

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and investigated the existence of mild solution.

On the other hand, Hilfer [29, 30] proposed a generalized Riemann-Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. Hilfer fractional derivative $D_{0+}^{\alpha,\beta}, \alpha \in (n-1,n), \beta \in (0,1)$. For $\beta = 0$, $D_{0+}^{\alpha,0}$ corresponds to the Riemman-Liouville fractional derivative. For $\beta = 1$, $D_{0+}^{\alpha,1}$ corresponds to the Caputo fractional derivative. Hilfer fractional derivative is performed in the theoretical simulation of dielectric relaxation in glass forming materials. For more properties and applications of Hilfer fractional derivative, see [29, 30]. In [20], Furati et al. considered an initial value problem for a class of nonlinear fractional differential equations involving Hilfer fractional derivative. Very recently, Gu and Trujillo [25] were concerned to focus on the investigation of the existence of mild solutions of the evolution of fractional equation in the sense of Hilfer fractional derivative, using noncompactness measure in Banach space E.

The concept of controllability, when it was first introduced by Kalman et al. [38] has become an active area of investigation due to its great applications in the field of physics. In recent years, controllability is one of the fundamental concepts in mathematical control theory and widely used in many fields of science and technology. Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Some authors have extended the concept to infinite-dimensional systems in Banach spaces, we refer the reader to see the references [5, 14–17, 21–23, 39, 40, 51, 53, 55, 56, 60, 62, 63, 67, 68].

Controllability theory for abstract semilinear control systems in infinite-dimensional space, with or without the usual impulse effects, has been developed to some extent, see [39,40,45,51–53,55,56,67,68] and the references therein. As we know, it is difficult to realise the exact controllability for abstract semilinear control systems in infinite-dimensional space since the controllability operator is asked to be surjective. Therefore, it is necessary to study the weaker concept of controllability, namely approximate controllability. There are various works on approximate controllability of systems standing for differential equations, integro-differential equations, differential inclusions, neutral functional differential equations, and impulsive differential equations of integer order in Banach spaces.

We note that most of the current research focuses on the order of the fractional differential equations in infinite dimensional spaces is frequently considered between 0 and 1, since the probability density function $M_\xi(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(1-\xi(k+1))}$ is defined only when $\alpha \in (0,1)$. There are only few papers that deal with the fractional differential equations of order $1 < \alpha < 2$. The existence of mild solutions for fractional differential and integro-differential equations of order $\alpha \in (1,2)$ has attracted much attention in recent years. Li et al. [41] considered two fractional evolution problems with Riemann-Liouville derivative by using the concept of resolvent family. Shu [50] studied the existence and uniqueness of mild solutions for nonlocal fractional differential equations based on the concept of sectorial operator. Moreover, the approximate controllability for a class of Hilfer fractional differential equations of order $1 < \alpha < 2$ and type $0 \le \beta \le 1$ is considered by Lv and Yang in [40].

Motivated by the above mentioned arguments and some discussion, in this paper, we discuss the existence and approximate controllability for Hilfer nonlinear fractional evolution equations of the form

$$\begin{cases} D_{0+}^{\alpha,\beta}u(t) = Au(t) + f(t,u(t)) + By(t), \quad t \in (0,b], \quad \alpha \in (1,2), \\ (g_{(1-\beta)(2-\alpha)} * u)(0) = u_0, \quad (g_{(1-\beta)(2-\alpha)} * u)'(0) = u_1, \end{cases} \tag{1.1}$$

where $D_{0+}^{\alpha,\beta}$ is the Hilfer fractional derivative which will be given in next section, $0 \le \beta \le 1, 1 < \alpha < 2$, the state $u(\cdot)$ takes values in a Banach space E. $J = [0,b]$ $(b > 0), J' = (0,b]$, the operator A generates a strongly continuous, exponentially bounded cosine family on E, the control function $y(t)$ takes values in $L^2(J,U)$ of admissible control functions for a Babach spaces U, B is a bounded linear operator from U into E; $f : J \times E \to E$ is given functions satisfying some assumptions and $u_0 \in \mathcal{D}(A), u_1 \in E$.

The symbol $*$ represents a convolution and $g_\alpha(\cdot)$ is Riemann-Liouville kernel given by

$$g_\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

where $\Gamma(\alpha)$ is the Gamma function. Note that $g_0(t) = 0$, because $\Gamma(0)^{-1} = 0$. These functions satisfy the semigroup property $g_\alpha * g_\beta = g_{\alpha+\beta}$.

Our aim is to study the existence and approximate controllability for the problem (1.1). To the best of our knowledge, it is the first investigation of Hilfer fractional evolutions of order $1 < \alpha < 2$ and type $0 \le \beta \le 1$ by using the operator cosine family $\{C(t)\}_{t\ge 0}$ and probability density function, we introduce a suitable concept of mild solution for system (1.1). Existence and approximate controllability for the problem (1.1) is obtained by means of Schauder's fixed point theorem. At last, as an application, we give two examples to illustrate the obtained results.

The motives and highlights in this paper are as follows:

1. For the Eq. (1.1), it generalizes the classical fractional derivatives of Riemann-Liouville and Caputo, respectively. In addition to the integer case, by choosing $\alpha = 1$. Obviously, our results can be applied to the evolution equations with Riemann-Liouville fractional derivative and Caputo fractional derivative.

2. When order $1 < \alpha < 2$, many scholars use the method of solution operator to give the proper definition of mild solution. Especially, in this paper, we use cosine family to combine probability density function and the Laplace transform to give the proper definition of mild solution.

This paper is organized as follows. The second part of the paper demonstrates the space of the weighted functions and their respective norm, as well as the concepts of Hilfer fractional derivative, notations, and definitions. The third part states an existence result of Hilfer fractional evolution equations in the Banach space by means of Schauder's fixed-point theorem. The fourth part establish the approximate controllability. And the last section is provided two examples to illustrate the applications of the obtained results. Concluding part close this article.

2. Preliminaries

In this section, we briefly recall some basic known results which will be used in the sequel. Throughout this work, we set $J = [0,b]$, where $b > 0$ is a constant.

Let E be a Banach space with the norm $\|\cdot\|$. We denote by $\mathcal{L}(E)$ the Banach space of all bounded linear operators from E to E, and denote by $C(J, E)$ the Banach space of all continuous E-valued functions on interval J with the norms $\|u\|_C = \sup_{t\in J}\|u(t)\|$.

On the other hand, the weighted space of functions u on J is defined by

$$C_{\beta,2-\alpha}(J,E) = \Big\{u \in C(J',E),\ \lim_{t\to 0} t^{(1-\beta)(2-\alpha)}u(t) \ \text{ exists and infine}\Big\},$$

where $0 \le \beta \le 1, 1 < \alpha < 2$, with the norm

$$\|u\|_{C_{\beta,2-\alpha}} = \sup_{t\in J'}\Big\|t^{(1-\beta)(2-\alpha)}u(t)\Big\|.$$

Evidently, the space $C_{\beta,2-\alpha}(J,E)$ is a Banach space. We also note that

(i) when $\beta = 1$, then $C_{\beta,2-\alpha}(J,E) = C(J,E)$ and $\|\cdot\|_{C_{\beta,2-\alpha}} = \|\cdot\|$.

(ii) Let $u(t) = t^{(\nu-1)(1-\mu)}v(t)$ for $t \in J', u \in C_{\beta,2-\alpha}(J,E)$ if and only if $v \in C(J,E)$ and $\|u\|_{C_{\beta,2-\alpha}} = \|v\|$.

Let $B_r(J) = \{v \in C(J,E)|\ \|v\| \le r\}$ and $B_r^\alpha(J') = \{u \in C_{\beta,2-\alpha}(J,E)|$ $\|u\|_{C_{\beta,2-\alpha}} \le r\}$, then B_r and B_r^α are two bounded closed and convex subsets of $C(J,E)$ and $C_{\beta,2-\alpha}(J,E)$, respectively.

For completeness we recall the following definitions from fractional calculus.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : [0,\infty) \to E$ is defined as

$$I_{0+}^\alpha f(t) = (g_\alpha * f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,\ t > 0,$$

provided the right side is point-wise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

In the following, we introduce the generalized Mittag-Leffler function $E^n_{\mu,\nu}(\cdot)$ and the Wright-type function $M_\xi(\cdot)$, for more details, see [35].

$$E^n_{\mu,\nu}(z) = \sum_{k=0}^{\infty}\frac{(n)_k z^k}{k!\Gamma(\mu k+\nu)},\ \mu,\nu,n,z \in \mathbb{C},\ Re(\mu) > 0,$$

where $(n)_k$ is the Pochhammer symbol defined by $(n)_0 = 1$ and $(n)_k = n(n+1)\cdots(n+k-1)$ for $k \in \mathbb{N}$, and

$$M_\xi(z) = \sum_{k=0}^{\infty}\frac{(-z)^k}{k!\Gamma(1-\xi(k+1))},\quad \xi \in (0,1),\ z \in \mathbb{C}.$$

Lemma 2.1 ([42]). *For any $t > 0$, the Wright-type function has the following properties*

$$M_\xi \ge 0,\qquad \int_0^\infty \theta^\delta M_\xi(\theta)d\theta = \frac{\Gamma(1+\delta)}{1+\xi\delta},\qquad -1 < \delta < \infty.$$

Definition 2.2 (see [29, 30]). The Hilfer fractional derivative of order $0 \le \beta \le 1$ and $0 < \alpha < 1$ is defined as

$$D_{0+}^{\alpha,\beta} f(t) = I_{0+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0+}^{(1-\beta)(1-\alpha)} f(t).$$

Remark 2.1 ([29,58]). For $\alpha \in (n-1, n], \beta \in [0,1]$, the Laplace transform formula

$$\mathcal{L}[D_{0+}^{\alpha,\beta} f(t)](s) = s^{\alpha}\mathcal{L}[f(t)](s) - \sum_{k=0}^{n-1} s^{n-k-1-\beta(n-\alpha)}\Big[\lim_{t\to 0^+} \frac{d^k}{dt^k}(I_{0+}^{(1-\beta)(n-\alpha)} f)(t)\Big]$$

is valid.

We briefly review the definition and some useful properties of the theory of cosine family. For some detail, see [57].

Definition 2.3 ([57]). A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space E into itself is called a strongly cosine family if and only if

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$,

(ii) $C(0) = I$,

(iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x \in E$.

$(S(t))_{t\in\mathbb{R}}$ is the sine function associated with the strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ which is defined by:

$$S(t)u = \int_0^t C(s)ds, \quad u \in E,\ t \in \mathbb{R}. \tag{2.1}$$

$\mathcal{D}(A)$ be the domain of the operator A which is defined by:

$$\mathcal{D}(A) = \{u \in E : C(t)u \text{ is twice continuously differential in } t\}.$$

$\mathcal{D}(A)$ is the Banach space endowed with the graph norm $\|\cdot\|_A = \|u\| + \|Au\|$ for all $u \in \mathcal{D}(A)$.

Proposition 2.1. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family in E. The following are true:*

(i) $S(t)u$ is continuous in t on $\mathbb{R}$ for each fixed point $u \in E$.

(ii) if $u \in E$, then $S(t)u \in \mathcal{D}(A)$ and $\frac{d}{dt}C(t)u = AS(t)u$;

(iii) if $u \in \mathcal{D}(A)$, then $S(t)u \in \mathcal{D}(A)$ and $AS(t)u = S(t)Au$;

(iv) $S(s+t) + S(s-t) = 2S(s)C(t)$ for all $s, t \in \mathbb{R}$;

(v) there exist constants $M \ge 1$ and $\omega \ge 0$ such that $\|C(t)\| \le Me^{\omega|t|}$ for all $t, s \in \mathbb{R}$;

$$\|S(t) - S(s)\| \le M\Big|\int_s^t e^{\omega|s|}ds\Big|.$$

Proposition 2.2. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family in E satisfying $\|C(t)\| \le Me^{\omega|t|}, t \in \mathbb{R}$, and let A be the infinitesimal generator of $C(t), t \in \mathbb{R}$. Then for $u \in E$ and $\{\lambda^2 \in \mathbb{C} : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda(\lambda^2 I - A)^{-1}u = \int_0^\infty e^{-\lambda t}C(t)udt, \quad (\lambda^2 I - A)^{-1}u = \int_0^\infty e^{-\lambda t}S(t)udt, \tag{2.2}$$

where $\rho(A)$ is the resolvent set of A.

In this sequel, we assume that the operator A generates a strongly continuous cosine family $\{C(t) : t \geq 0\}$ which is exponentially bounded (i.e., there exist $M \geq 0$ and $\omega \geq 0$, such that $\|C(t)\|_{\mathcal{L}(E)} \leq Me^{\omega t}$ for $t \geq 0$) on E.

Lemma 2.2. *Let $u_0 \in \mathcal{D}(A), u_1 \in E$ and $f \in L^1([0,\infty), E)$, if u is a solution of the problem*

$$\begin{cases} D_{0+}^{\alpha,\beta} u(t) = Au(t) + f(t), \quad t \in [0,\infty), \quad \alpha \in (1,2), \\ (g_{(1-\beta)(2-\alpha)} * u)(0) = u_0, \quad (g_{(1-\beta)(2-\alpha)} * u)'(0) = u_1, \end{cases} \tag{2.3}$$

then u satisfies the following equation

$$\begin{aligned} u(t) =& g_{2q-1+\beta(2-\alpha)}(t)u_0 + (g_{2q-1+\beta(2-\alpha)} * AT_q)(t)u_0 \\ &+ (g_{\beta(2-\alpha)} * T_q)(t)u_1 + \int_0^t T_q(t-s)f(s)ds, \ t \in [0,\infty), \end{aligned}$$

where $q = \frac{\alpha}{2}$ and

$$T_q(t) = t^{q-1} \int_0^\infty q\theta M_q(\theta) S(t^q\theta) d\theta.$$

Proof. For any $\omega \geq 0$, it is easy to know that $\{\lambda^\alpha \in \mathbb{C} : Re\lambda > \omega^{\frac{2}{\alpha}}\} \subset \rho(A)$. Let $Re\lambda > \omega$. Applying the Laplace transform

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) dt, \quad \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt$$

to (2.3), we have

$$\begin{aligned} &\lambda^\alpha \widehat{u}(\lambda) - \lambda^{1-\beta(2-\alpha)} (g_{(1-\beta)(2-\alpha)} * u)(0) - \lambda^{-\beta(2-\alpha)} (g_{(1-\beta)(2-\alpha)} * u)'(0) \\ =& A\widehat{u}(\lambda) + \widehat{f}(\lambda). \end{aligned}$$

Then, we have

$$\widehat{u}(\lambda) = \lambda^{1-\beta(2-\alpha)} (\lambda^\alpha I - A)^{-1} u_0 + \lambda^{-\beta(2-\alpha)} (\lambda^\alpha I - A)^{-1} u_1 + (\lambda^\alpha I - A)^{-1} \widehat{f}(\lambda). \tag{2.4}$$

Using (2.2), we have

$$\begin{aligned} \widehat{u}(\lambda) =& \lambda^{(1-\frac{\alpha}{2})-\beta(2-\alpha)} \int_0^\infty e^{-\lambda^{\frac{\alpha}{2}} s} C(s) u_0 ds \\ &+ \lambda^{-\beta(2-\alpha)} \int_0^\infty e^{-\lambda^{\frac{\alpha}{2}} s} S(s) u_1 ds + \int_0^\infty e^{-\lambda^{\frac{\alpha}{2}} s} S(s) \widehat{f}(\lambda) ds \end{aligned} \tag{2.5}$$

provided that the integral in (2.5) exists, where I is the identity operator defined on E.

Let

$$\varpi_\kappa(\theta) = \frac{\kappa}{\theta^{\kappa+1}} M_\kappa(\theta^{-\kappa}),$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\theta} \varpi_\kappa(\theta) d\theta = e^{-\lambda^\kappa}, \quad \kappa \in (0,1). \tag{2.6}$$

Then, by $s = t^q$ and (2.6), we have

$$
\begin{aligned}
\int_0^\infty e^{-\lambda^q s} S(s)u_1 ds &= \int_0^\infty q t^{q-1} e^{-(\lambda t)^q} S(t^q) u_1 dt \\
&= \int_0^\infty \int_0^\infty e^{-(\lambda t\theta)} q t^{q-1} \varpi_q(\theta) S(t^q) u_1 d\theta dt \\
&= q \int_0^\infty \int_0^\infty e^{-\lambda t} \frac{t^{q-1}}{\theta^q} \varpi_q(\theta) S\Big(\frac{t^q}{\theta^q}\Big) u_1 dt d\theta \\
&= \int_0^\infty e^{-\lambda t} \Big[q \int_0^\infty \frac{t^{q-1}}{\theta^q} \varpi_q(\theta) S\Big(\frac{t^q}{\theta^q}\Big) u_1 d\theta \Big] dt \\
&= \int_0^\infty e^{-\lambda t} \Big[\int_0^\infty t^{q-1} q\theta M_q(\theta) S(t^q\theta) u_1 d\theta \Big] dt \\
&= \int_0^\infty e^{-\lambda t} \Big[T_q(t) u_1 \Big] dt
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda^q s} S(s) \widehat{f}(\lambda) ds \\
&= \int_0^\infty q t^{q-1} e^{-(\lambda t)^q} S(t^q) \Big(\int_0^\infty e^{-\lambda s} f(s) ds \Big) dt \\
&= \int_0^\infty \int_0^\infty e^{-(\lambda t\theta)} q t^{q-1} \varpi_q(\theta) S(t^q) \Big(\int_0^\infty e^{-\lambda s} f(s) ds \Big) d\theta dt \\
&= q \int_0^\infty \int_0^\infty e^{-\lambda t} \frac{t^{q-1}}{\theta^q} \varpi_q(\theta) S\Big(\frac{t^q}{\theta^q}\Big) \Big(\int_0^\infty e^{-\lambda s} f(s) ds \Big) dt d\theta \\
&= q \int_0^\infty \Big(\int_0^\infty \int_0^t e^{-\lambda t} \frac{(t-s)^{q-1}}{\theta^q} \varpi_q(\theta) S\Big(\frac{(t-s)^q}{\theta^q}\Big) f(s) ds dt \Big) d\theta \\
&= \int_0^\infty e^{-\lambda t} \Big[q \int_0^t \int_0^\infty (t-s)^{q-1} M_q(\theta) S((t-s)^q\theta) f(s) d\theta ds \Big] dt \\
&= \int_0^\infty e^{-\lambda t} \Big[\int_0^t T_q(t-s) f(s) ds \Big] dt.
\end{aligned} \tag{2.8}
$$

In addition, we have

$$
\begin{aligned}
&\lambda^{1-q} \int_0^\infty e^{-\lambda^q s} C(s) u_0 ds \\
&= \lambda^{1-q} \int_0^\infty e^{-\lambda^q t} C(t) u_0 dt = \lambda^{1-2q} \lambda^q \int_0^\infty e^{-\lambda^q t} C(t) u_0 dt \\
&= \int_0^\infty e^{-\lambda t} g_{2q-1}(t) dt \Big[C(t) e^{-\lambda^q t} u_0 \mid_{t=\infty}^{t=0} + \int_0^\infty e^{-\lambda^q t} AS(t) u_0 dt \Big] \\
&= \int_0^\infty e^{-\lambda t} g_{2q-1}(t) u_0 dt + \int_0^\infty e^{-\lambda t} g_{2q-1}(t) dt \int_0^\infty e^{-\lambda^q t} AS(t) u_0 dt \\
&= \int_0^\infty e^{-\lambda t} g_{2q-1}(t) u_0 dt + \int_0^\infty e^{-\lambda t} g_{2q-1}(t) dt \int_0^\infty e^{-\lambda t} AT_q(t) u_0 dt \\
&= \int_0^\infty e^{-\lambda t} g_{2q-1}(t) u_0 dt + \int_0^\infty e^{-\lambda t} \Big[(g_{2q-1} * AT_q)(t) u_0 \Big] dt.
\end{aligned} \tag{2.9}
$$

Thus, it follows from (2.5), (2.7)-(2.9), for $t \in [0, \infty)$, we get

$$\widehat{u}(\lambda) = \lambda^{-\beta(2-\alpha)} \Big[\int_0^\infty e^{-\lambda t} g_{2q-1}(t) u_0 dt + \int_0^\infty e^{-\lambda t} \Big((g_{2q-1} * AT_q)(t) u_0 \Big) dt \Big] + \lambda^{-\beta(2-\alpha)} \int_0^\infty e^{-\lambda t} \Big[T_q(t) u_1 \Big] dt + \int_0^\infty e^{-\lambda t} \Big[\int_0^t T_q(t-s) f(s) ds \Big] dt. \tag{2.10}$$

Since the Laplace inverse transform of $\lambda^{-\beta(2-\alpha)}$ is

$$\mathcal{L}^{-1}\Big(\lambda^{-\beta(2-\alpha)}\Big) = \begin{cases} \frac{t^{\beta(2-\alpha)-1}}{\Gamma(\beta(2-\beta))}, & 0 < \beta < 1, \\ \delta(t), & \beta = 0, \end{cases}$$

where $\delta(t)$ is the Delta function, we use $C(0) = I$ and $C'(t)u_0 = AS(t)u_0$ and we invert the last Laplace transform to obtain

$$\begin{aligned} u(t) =& \Big(\mathcal{L}^{-1}\Big(\lambda^{-\beta(2-\alpha)}\Big) * \mathcal{L}^{-1} \Big[\int_0^\infty e^{-\lambda t} g_{2q-1}(t) u_0 dt \\ &+ \int_0^\infty e^{-\lambda t} \Big((g_{2q-1} * AT_q)(t) u_0 \Big) dt \Big] \Big) \\ &+ \Big(\mathcal{L}^{-1}\Big(\lambda^{-\beta(2-\alpha)}\Big) * \mathcal{L}^{-1} \Big[\int_0^\infty e^{-\lambda t} \Big[T_q(t) u_1 \Big] dt \Big] \Big) \\ &+ \mathcal{L}^{-1} \Big[\int_0^\infty e^{-\lambda t} \Big[\int_0^t T_q(t-s) f(s, u(s)) ds \Big] dt \Big] \Big) \\ =& g_{2q-1+\beta(2-\alpha)}(t) u_0 + (g_{2q-1+\beta(2-\alpha)} * AT_q)(t) u_0 \\ &+ (g_{\beta(2-\alpha)} * T_q)(t) u_1 + \int_0^t T_q(t-s) f(s) ds. \end{aligned} \tag{2.11}$$

This completes the proof. □

Base on the above Lemma 2.2, we give the following definition of mild solution of the problem (1.1).

Definition 2.4. A function $u \in C_{\beta,2-\alpha}(J, E)$ is called a mild solution of the problem (1.1), if for $u_0 \in \mathcal{D}(A), u_1 \in E$ and each $y \in L^2(J, U)$, which satisfies

$$\begin{aligned} u(t) =& g_{2q-1+\beta(2-\alpha)}(t) u_0 + (g_{2q-1+\beta(2-\alpha)} * AT_q)(t) u_0 \\ &+ (g_{\beta(2-\alpha)} * T_q)(t) u_1 + \int_0^t T_q(t-s) By(s) ds \\ &+ \int_0^t T_q(t-s) f(s, u(s)) ds, \quad t \in J'. \end{aligned} \tag{2.12}$$

Next, we will show that some properties of the operator $T_q(t)$ from the characteristics of cosine family.

Lemma 2.3. *Assume that the operator A generates a strongly continuous cosine family $\{C(t) : t \geq 0\}$ which is exponentially bounded on E, for any fixed $t \geq 0, \omega \geq 0$, the operator $T_q(t)$ is a linear operators, and for any $u \in E$,*

$$\|T_q(t)u\| \leq Mt^{2q-1} E^2_{q,2q}(\omega t^q) \|u\|.$$

Proof. Since $S(t)$ is linear operator for $t \geq 0$, then $T_q(t)$ is also linear operator. For any $u \in E, \omega \geq 0$ and fixed $t \geq 0$, by Lemma 2.1, we have

$$\begin{aligned}
\|T_q(t)u\| &\leq t^{q-1}\int_0^\infty q\theta M_q(\theta)\|S(t^q\theta)u\|d\theta \\
&\leq t^{q-1}\int_0^\infty q\theta M_q(\theta)\Big\|\int_0^{t^q\theta} C(s)uds\Big\|d\theta \\
&= M\|u\|t^{2q-1}\int_0^\infty q\theta^2 M_q(\theta)e^{\omega t^q\theta}d\theta \\
&= M\|u\|t^{2q-1}\frac{1}{\Gamma(2)}\int_0^\infty q\theta^2 M_q(\theta)\sum_{k=0}^\infty \frac{(\omega t^q)^k\theta^k}{k!}d\theta \\
&= M\|u\|t^{2q-1}\frac{q}{\Gamma(2)}\sum_{k=0}^\infty \frac{(\omega t^q)^k}{k!}\int_0^\infty \theta^{2+k}M_q(\theta)d\theta \\
&= M\|u\|t^{2q-1}\frac{q}{\Gamma(2)}\sum_{k=0}^\infty \frac{(\omega t^q)^k}{k!}\frac{\Gamma(1+2+k)}{\Gamma(1+q(2+k))} \\
&= M\|u\|t^{2q-1}\sum_{k=0}^\infty \frac{(\omega t^q)^k}{k!}\frac{\Gamma(2+k)}{\Gamma(2)\Gamma(qk+2q)} \\
&= M\|u\|t^{2q-1}\sum_{k=0}^\infty \frac{(\omega t^q)^k}{k!}\frac{(2)_k}{\Gamma(qk+2q)} \\
&= Mt^{2q-1}E^2_{q,2q}(\omega t^q)\|u\|.
\end{aligned}$$

This completes the proof. □

Lemma 2.4. *Assume that the operator A generates a strongly continuous cosine family $\{C(t) : t \geq 0\}$ which is exponentially bounded on E, for any fixed $t \geq 0, \omega \geq 0$, we have the following estimate*

$$\begin{aligned}
&\|(g_{2q-1+\beta(2-\alpha)} * AT_q)(t)u\| \leq Mt^{2q-1}E^2_{q,2q}(\omega t^q)\|u\|, \ u \in \mathcal{D}(A), \\
&\|(g_{\beta(2-\alpha)} * T_q)(t)u\| \leq Mt^{\beta(2-\alpha)+2p-1}E^2_{q,(\beta(2-\alpha)+2p)}(\omega t^q)\|u\|, \ u \in E.
\end{aligned}$$

Proof. For any $u \in E, \omega \geq 0$ and fixed $t \geq 0$, by Lemma 2.1 and Definition 2.1, and in view of Theorem 3 in [48], we have

$$\begin{aligned}
&\|(g_{\beta(2-\alpha)} * T_q)(t)u\| \\
&\leq \int_0^t g_{\beta(2-\alpha)}(t-s)s^{q-1}\int_0^\infty q\theta M_q(\theta)\|S(s^q\theta)u\|d\theta ds \\
&\leq \int_0^t g_{\beta(2-\alpha)}(t-s)s^{q-1}\int_0^\infty q\theta M_q(\theta)\Big\|\int_0^{s^q\theta} C(s)uds\Big\|d\theta ds\|u\| \\
&\leq M\int_0^t g_{\beta(2-\alpha)}(t-s)s^{2q-1}\int_0^\infty q\theta^2 M_q(\theta)e^{\omega s^q\theta}d\theta ds\|u\| \\
&= M\int_0^t g_{\beta(2-\alpha)}(t-s)s^{2q-1}\frac{1}{\Gamma(2)}\int_0^\infty q\theta^2 M_q(\theta)\sum_{k=0}^\infty \frac{(\omega s^q)^k\theta^k}{k!}d\theta ds\|u\| \\
&= M\int_0^t g_{\beta(2-\alpha)}(t-s)s^{2q-1}\frac{q}{\Gamma(2)}\sum_{k=0}^\infty \frac{(\omega s^q)^k}{k!}\int_0^\infty \theta^{2+k}M_q(\theta)d\theta ds\|u\|
\end{aligned}$$

$$
\begin{aligned}
&= M\int_0^t g_{\beta(2-\alpha)}(t-s)s^{2q-1}\frac{q}{\Gamma(2)}\sum_{k=0}^{\infty}\frac{(\omega s^q)^k}{k!}\frac{\Gamma(1+2+k)}{\Gamma(1+q(2+k))}ds\|u\| \\
&= M\int_0^t g_{\beta(2-\alpha)}(t-s)s^{2q-1}\sum_{k=0}^{\infty}\frac{(\omega s^q)^k}{k!}\frac{\Gamma(2+k)}{\Gamma(2)\Gamma(qk+2q)}ds\|u\| \\
&= M\int_0^t g_{\beta(2-\alpha)}(t-s)s^{2q-1}\sum_{k=0}^{\infty}\frac{(\omega s^q)^k}{k!}\frac{(2)_k}{\Gamma(qk+2q)}ds\|u\| \\
&= M\int_0^t g_{\beta(2-\alpha)}(t-s)s^{2q-1}E^2_{q,2q}(\omega s^q)ds\|u\| \\
&= Mt^{\beta(2-\alpha)+2p-1}E^2_{q,\beta(2-\alpha)+2p}(\omega t^q)\|u\|.
\end{aligned}
$$

For any $\omega \geq 0$ and fixed $t \geq 0$, by Proposition 2.1 (iii) and by Lemma 2.1, and in view of Theorem 3 in [48], we have

$$
\begin{aligned}
&\|(g_{2q-1+\beta(2-\alpha)} * AT_q)(t)u\| \\
&\leq \int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{q-1}\int_0^{\infty} q\theta M_q(\theta)\|S(s^q\theta)Au\|d\theta ds \\
&\leq \int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{q-1}\int_0^{\infty} q\theta M_q(\theta)\left\|\int_0^{s^q\theta} C(s)uds\right\|d\theta ds\|u\|_A \\
&\leq M\int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{2q-1}\int_0^{\infty} q\theta^2 M_q(\theta)e^{\omega s^q\theta}d\theta ds\|u\|_A \\
&= M\int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{2q-1}\frac{1}{\Gamma(2)}\int_0^{\infty} q\theta^2 M_q(\theta)\sum_{k=0}^{\infty}\frac{(\omega s^q)^k\theta^k}{k!}d\theta ds\|u\|_A \\
&= M\int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{2q-1}\frac{q}{\Gamma(2)}\sum_{k=0}^{\infty}\frac{(\omega s^q)^k}{k!}\int_0^{\infty}\theta^{2+k}M_q(\theta)d\theta ds\|u\|_A \\
&= M\int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{2q-1}\frac{q}{\Gamma(2)}\sum_{k=0}^{\infty}\frac{(\omega s^q)^k}{k!}\frac{\Gamma(1+2+k)}{\Gamma(1+q(2+k))}ds\|u\|_A \\
&= M\int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{2q-1}\sum_{k=0}^{\infty}\frac{(\omega s^q)^k}{k!}\frac{\Gamma(2+k)}{\Gamma(2)\Gamma(qk+2q)}ds\|u\|_A \\
&= M\int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{2q-1}\sum_{k=0}^{\infty}\frac{(\omega s^q)^k}{k!}\frac{(2)_k}{\Gamma(qk+2q)}ds\|u\|_A \\
&= M\int_0^t g_{2q-1+\beta(2-\alpha)}(t-s)s^{2q-1}E^2_{q,2q}(\omega s^q)ds\|u\|_A \\
&= Mt^{\beta(2-\alpha)+4p-2}E^2_{q,\beta(2-\alpha)+4p-1}(\omega t^q)\|u\|_A.
\end{aligned}
$$

□

Lemma 2.5. *Assume that the operator A generates a strongly continuous cosine family $\{C(t) : t \geq 0\}$ which is exponentially bounded on E, for any fixed $t \geq 0, \omega \geq 0$, and for any $u \in E$ and for any $t_2, t_1 \geq 0$, we have*

$$\|T_q(t_2)u - T_q(t_1)u\| \to 0, \quad \textit{as } t_2 \to t_1.$$

Proof. For any $u \in E$ and $0 \le t_1 < t_2 \le b$, in view of Proposition 2.1 (v) and Lemma 2.7, we have

$$\begin{aligned}
&\|T_q(t_2)u - T_q(t_1)u\| \\
\le &\left\| \int_0^\infty q\sigma M_q(\theta)[t_2^{q-1}S(t_2^q\theta) - t_1^{q-1}S(t_1^q\theta)]ud\theta \right\| \\
\le &\int_0^\infty q\theta M_q(\theta)d\theta \Big[t_2^{q-1}\|S(t_2^q\theta) - S(t_1^q\theta)\| + \|t_2^{q-1} - t_1^{q-1}\|S(t_1^q\theta)\Big] \cdot \|u\| \\
\le &t_2^{q-1}(t_2^q - t_1^q)ME_{q,2q}^2(\omega t_2^q)\|u\| + (t_2^{1-q} - t_1^{1-q})t_2^{3q-2}ME_{q,2q}^2(\omega t_1^q)\|u\|.
\end{aligned}$$

Thus, we have

$$\|T_q(t_2)u - T_q(t_1)u\| \to 0, \quad \text{as } t_2 \to t_1.$$

This completes the proof. □

Lemma 2.6 ([31], Proposition 2.1)**.** *Let X, Y be Banach spaces, let $\mathbb{S} : [0, \infty) \to \mathcal{L}(X, Y)$ be stongly continuous, and let $a \in L^1_{loc}[0, \infty)$ be a scalar function, both a and $\mathbb{S}$ of finite exponential type. Then for every $\omega > \omega_0(\mathbb{S}), \omega_0(a)$ one has*

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t}(\widehat{a * \mathbb{S}})(\lambda)d\lambda = a * \mathbb{S}$$

in $\mathcal{L}(X, Y)$, uniformly in t from compact subsets of $[0, \infty)$.

Lemma 2.7. *Assume that the operator A generates a strongly continuous cosine family $\{C(t) : t \ge 0\}$ which is exponentially bounded on E, for any fixed $t \ge 0, \omega \ge 0$, then $S(t)$ is compact on $\mathcal{L}(E)$.*

Proof. Let $t > 0$ be fixed, it follows that $g_1 \in L^1_{loc}[0, \infty)$ and therefore, by Lemma 2.10 we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega_i N} e^{\lambda t}(\widehat{g_1 * C})(\lambda)d\lambda = (g_1 * C)(t) = \int_0^t C(s)ds = S(t)$$

in $\mathcal{L}(E)$, by Lemma 2.6, and we conclude that $S(t)$ is compact for all $t > 0$. □

Lemma 2.8. *Assume that $S(t)$ is compact on $\mathcal{L}(E)$ for every $t > 0$. Then $T_q(t)$ is compact on $\mathcal{L}(E)$ for every $t > 0$.*

Proof. The proof process is similar to the proof of Lemma 3.5 in paper [32], here we omit it. □

Definition 2.5. Let $u(t; f, y)$ be a mild solution of the system (1.1) associated with nonlinear term f and control $y \in L^2(J, U)$ at the time t. Then

$$K_b(f) = \{u(b, f, y) : y \in L^2(J, U)\} \subset E,$$

the nonempty subset $K_b(f)$ in E consisting of all terminal states of (1.1) is called the reachable set at the time a of the system (1.1).

Definition 2.6. The system (1.1) is said to be approximately controllable on the interval J if $K_b(f)$ is dense in E, means $\overline{K_b(f)} = E$. That is, for any $\epsilon > 0$ and every desired final state $u_b \in E$, there exists a control $y \in L^2(J, U)$ such that u satisfies $\|u(b) - u_b\| < \epsilon$.

In order to state the problem, we introduce the following two operators defined on Banach space E by

$$\Gamma_0^b = \int_0^b T_q(b-s)BB^*T_q^*(b-s)ds, \tag{2.13}$$

is the controllability Grammian and

$$R(\mu, \Gamma_0^b) = (\mu I + \Gamma_0^b)^{-1}, \quad \mu > 0, \tag{2.14}$$

where $B^*, T_q^*(t)$ denote the adjoint operators of $B, T_q(t)$ respectively.

In the following, it will be showed that the system (1.1) is approximately controllable if for all $\mu > 0$ and $u_b \in E$, there exists a continuous function $u \in C_{\beta,2-\alpha}(J,E)$ such that

$$\begin{aligned} u(t) =& g_{2q-1+\beta(2-\alpha)}(t)u_0 + (g_{2q-1+\beta(2-\alpha)} * AT_q)(t)u_0 + (g_{\beta(2-\alpha)} * T_q)(t)u_1 \\ &+ \int_0^t T_q(t-s)f(s,u(s))ds + \int_0^t T_q(t-s)By(s)ds, \end{aligned}$$

where the function y_μ is the control function defined by

$$\begin{aligned} y_\mu(t) &= B^*T_q(b-t)R(\mu,\Gamma_0^b)p(u(\cdot)), \\ p(u(\cdot)) =& u_b - g_{2q-1+\beta(2-\alpha)}(b)u_0 - (g_{2q-1+\beta(2-\alpha)} * AT_q)(b)u_0 \\ &- (g_{\beta(2-\alpha)} * T_q)(b)u_1 - \int_0^b T_q(b-s)f(s,u(s))ds. \end{aligned}$$

3. Existence of mild solutions

In this section, by means of the measure of noncopmactness and Ascoli-Arzela Theorem, we will state some sufficient conditions on the existence of mild solution. First of all, we introduce the following assumptions:

(H1) The cosine family $C(t)$ is continuous in the uniform operator topology for every $t > 0$, and $\{C(t) : t \geq 0\}$ is exponentially bounded, i.e., there exist $M \geq 0$ and $\omega \geq 0$, such that $\|C(t)\|_{\mathcal{L}(E)} \leq Me^{\omega t}$ for $t \geq 0$.

(H2) For each $t \in J'$, the function $f(t,\cdot) : E \to E$ is continuous and for each $u \in E$, the function $f(\cdot,u) : J' \to E$ is strongly measure. Moreover, there exists a function $m \in L^1(J',\mathbb{R}^+)$ such that

$$I_{0+}^\alpha m \in C(J',\mathbb{R}^+), \quad \lim_{t\to 0+} t^{(1-\beta)(2-\alpha)} I_{0+}^\alpha m(t) = 0$$

and

$$\|f(t,u)\| \leq m(t) \quad \text{for all} \;\; u \in E \;\; \text{and almost all } t \in J.$$

(H3) There exists a function $\psi \in L^1(J',\mathbb{R}^+)$ such that $\|By(t)\| \leq \psi(t)$ for all $y \in L^2(J,U)$ and $t \in J'$.

For any $u \in C_{\beta,2-\alpha}(J,E)$, we can define operator $\mathcal{P}$ as follows

$$(\mathcal{P}u)(t) = g_{2q-1+\beta(2-\alpha)}(t)u_0 + (g_{2q-1+\beta(2-\alpha)} * AT_q)(t)u_0 + (g_{\beta(2-\alpha)} * T_q)(t)u_1$$

$$+\int_0^t T_q(t-s)By(s)ds+\int_0^t T_q(t-s)f(s,u(s))ds. \tag{3.1}$$

It is easy to see that

$$\lim_{t\to 0^+} t^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(t)=\frac{u_0}{\Gamma(2q-1+\beta(2-\alpha))}.$$

For any $v\in C(J,E)$, set $u(t)=t^{(\beta-1)(2-\alpha)}v(t), t\in J'$. Then, $u\in C_{\beta,2-\alpha}(J,E)$.

Define the operator $\mathscr{P}$ as follows

$$(\mathscr{P}v)(t)=\begin{cases} t^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(t), & \text{for } \ t\in(0,b],\\ \frac{u_0}{\Gamma(2q-1+\beta(2-\alpha))}, & \text{for } \ t=0. \end{cases} \tag{3.2}$$

It is verity that there exists a fixed point u satisfying operator equation $u=\mathcal{P}u$. Further, from the definition above, u is a mild solution of the problem (1.1) in $C_{\beta,2-\alpha}(J,E)$ if and only if v satisfies the operator equation $v=\mathscr{P}v$ in $C(J,E)$, where $u(t)=t^{(\beta-1)(2-\alpha)}v(t)$ for $t\in J'$.

Theorem 3.1. *Assume that the operator A generates a strongly continuous cosine family $\{C(t):t\geq 0\}$. If the assumptions (H1)-(H3) are satisfied, then the problem (1.1) exists at least one mild solution in $B_r^\alpha(J')$.*

Proof. We consider the operator $\mathscr{P}:C(J,E)\to C(J,E)$ defined by (3.2). By direct calculation, we know that the operator $\mathscr{P}$ is well defined. From Definition 2.6, it is easy to verify that the mild solution of problem (1.1) is equivalent to the fixed point the operator $\mathscr{P}$ defined by (3.2).

Step 1. We show that the operator $\mathscr{P}$ maps $B_r(J)$ into $B_r(J)$. For any $v\in B_r(J)$, let $u(t)=t^{(\beta-1)(2-\alpha)}v(t)$ for $t\in J'$. Then $u\in B_r^\alpha(J')$. For $t\in J$, in view of Lemma 2.7, 2.8, we get that

$$\begin{aligned}
&\|(\mathscr{P}v)(t)\|\\
=&\|t^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(t)\|\leq\|t^{(1-\beta)(2-\alpha)}g_{2q-1+\beta(2-\alpha)}(t)u_0\|\\
&+\|t^{(1-\beta)(2-\alpha)}(g_{2q-1+\beta(2-\alpha)}*AT_q)(t)u_0\|+\|t^{(1-\beta)(2-\alpha)}(g_{\beta(2-\alpha)}*T_q)(t)u_1\|\\
&+\Big\|\int_0^t t^{(1-\beta)(2-\alpha)}T_q(t-s)f(s,u(s))ds\Big\|+\Big\|\int_0^t t^{(1-\beta)(2-\alpha)}T_q(t-s)By(s)ds\Big\|\\
\leq& Mt^\alpha E^2_{q,\beta(2-\alpha)+4q-1}(\omega t^q)\|u_0\|_A+MtE^2_{q,\beta(2-\alpha)+2q}(\omega t^q)\|u_1\|\\
&+Mt^{1-\beta(2-\alpha)}E^2_{q,2q}(\omega t^q)\int_0^t[m(s)+\psi(s)]ds\\
\leq& Mb^\alpha E^2_{q,\beta(2-\alpha)+4q-1}(\omega b^q)\|u_0\|_A+MbE^2_{q,\beta(2-\alpha)+2q}(\omega b^q)\|u_1\|_C\\
&+Mb^{1-\beta(2-\alpha)}E^2_{q,2q}(\omega b^q)\Big(\|m\|_{L^1(J,\mathbb{R}^+)}+\|\psi\|_{L^1(J,\mathbb{R}^+)}\Big)\leq r.
\end{aligned} \tag{3.3}$$

Thus, $\|\mathscr{P}v\|_C\leq r$, for any $v\in B_r(J)$.

Step 2. We will prove that $\{\mathscr{P}v:v\in B_r(J)\}$ is equicontinuous. For $v\in B_r(J)$, let $u(t)=t^{(\beta-1)(2-\alpha)}v(t)$, $t\in(0,b]$, then $u\in B_r^\alpha(J')$. For $t_1=0,\ 0<t_2\leq b$, by (3.1), (3.2) and the assumption (H2), we get that

$$\|(\mathscr{P}v)(t_2)-(\mathscr{P}v)(0)\|$$

$$
\begin{aligned}
&=\Big\|t_2^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(t_2)-t_1^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(0)\Big\|\\
&=\Big\|t_2^{(1-\beta)(2-\alpha)}(g_{2q-1+\beta(2-\alpha)}*AT_q)(t_2)u_0+t_2^{(1-\beta)(2-\alpha)}(g_{\beta(2-\alpha)}*T_q)(t_2)u_1\\
&\quad+t_2^{(1-\beta)(2-\alpha)}\int_0^{t_2}T_q(t_2-s)[By(s)+f(s,u(s))]ds\Big\|\\
&\le Mt_2^{\alpha}E^2_{q,\beta(2-\alpha)+4q-1}(\omega t_2^q)\|u_0\|_A+Mt_2E^2_{q,\beta(2-\alpha)+2q}(\omega t_2^q)\|u_1\|\\
&\quad+Mt_2^{1-\beta(2-\alpha)}E^2_{q,2q}(\omega t_2^q)\int_0^{t_2}[m(s)+\psi(s)]ds\to 0,\qquad\text{as } t_2\to 0.
\end{aligned}
$$

Let $\epsilon>0$ and $0<\epsilon<t_1<t_2\le b$ be given, in view of the definitions of operators $\mathscr{P}$ and $\mathcal{P}$, we have

$$
\begin{aligned}
&\|(\mathscr{P}v)(t_2)-(\mathscr{P}v)(t_1)\|\\
=&\Big\|t_2^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(t_2)-t_1^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(t_1)\Big\|\\
\le&\Big\|t_2^{(1-\beta)(2-\alpha)}g_{2q-1+\beta(2-\alpha)}(t_2)u_0-t_1^{(1-\beta)(2-\alpha)}g_{2q-1+\beta(2-\alpha)}(t_1)u_0\Big\|\\
&+\Big\|t_2^{(1-\beta)(2-\alpha)}(g_{2q-1+\beta(2-\alpha)}*AT_q)(t_2)u_0\\
&-t_1^{(1-\beta)(2-\alpha)}(g_{2q-1+\beta(2-\alpha)}*AT_q)(t_1)u_0\Big\|\\
&+\Big\|t_2^{(1-\beta)(2-\alpha)}(g_{\beta(2-\alpha)}*T_q)(t_2)u_1-t_1^{(1-\beta)(2-\alpha)}(g_{\beta(2-\alpha)}*T_q)(t_1)u_1\Big\|\\
&+\Big\|\int_0^{t_2}t_2^{(1-\beta)(2-\alpha)}T_q(t_2-s)[By(s)+f(s,u(s))]ds\\
&-\int_0^{t_1}t_1^{(1-\beta)(2-\beta)}T_q(t_1-s)[By(s)+f(s,u(s))]ds\Big\|\\
\le&\Big(t_2^{(1-\beta)(2-\alpha)}-t_1^{(1-\beta)(2-\alpha)}\Big)\\
&\times\int_0^{t_1}\Big\|\Big(g_{2q-1+\beta(2-\alpha)}(t_2-s)-g_{2q-1+\beta(2-\alpha)}(t_1-s)\Big)AT_q(s)u_0\Big\|ds\\
&+t_2^{(1-\beta)(2-\alpha)}\int_{t_1}^{t_2}\|g_{2q-1+\beta(2-\alpha)}(t_2-s)AT_q(s)u_0\|ds+\Big(t_2^{(1-\beta)(2-\alpha)}-t_1^{(1-\beta)(2-\alpha)}\Big)\\
&\times\int_0^{t_1}\Big\|\Big(g_{\beta(2-\alpha)}(t_2-s)-g_{\beta(2-\alpha)}(t_1-s)\Big)T_q(s)u_1\Big\|ds+t_2^{(1-\beta)(2-\alpha)}\\
&\times\int_{t_1}^{t_2}\|g_{2q-1+\beta(2-\alpha)}(t_2-s)T_q(s)u_1\|ds\\
&+t_2^{(1-\beta)(2-\alpha)}\int_0^{t_1-\epsilon}\|T_q(t_2-s)-T_q(t_1-s)\|[By(s)+f(s,u(s))]ds\\
&+\Big(t_2^{(1-\beta)(2-\alpha)}-t_1^{(1-\beta)(2-\alpha)}\Big)\int_0^{t_1-\epsilon}\|T_q(t_1-s)\|[By(s)+f(s,u(s))]ds\\
&+t_2^{(1-\beta)(2-\alpha)}\int_{t_1-\epsilon}^{t_1}\|T_q(t_2-s)-T_q(t_1-s)\|[By(s)+f(s,u(s))]ds\\
&+t_2^{(1-\beta)(2-\alpha)}\int_{t_1}^{t_2}\|T_q(t_2-s)\|[By(s)+f(s,u(s))]ds
\end{aligned}
$$

$$
\begin{aligned}
\leq & \Big(t_2^{(1-\beta)(2-\alpha)} - t_1^{(1-\beta)(2-\alpha)}\Big) \\
& \times \int_0^{t_1} \Big\| \Big(g_{2q-1+\beta(2-\alpha)}(t_2 - s) - g_{2q-1+\beta(2-\alpha)}(t_1 - s)\Big) AT_q(s)u_0 \Big\| ds \\
& + t_2^{(1-\beta)(2-\alpha)} \int_{t_1}^{t_2} \| g_{2q-1+\beta(2-\alpha)}(t_2 - s) AT_q(s)u_0 \| ds \\
& + \Big(t_2^{(1-\beta)(2-\alpha)} - t_1^{(1-\beta)(2-\alpha)}\Big) \int_0^{t_1} \Big\| \Big(g_{\beta(2-\alpha)}(t_2 - s) - g_{\beta(2-\alpha)}(t_1 - s)\Big) T_q(s)u_1 \Big\| ds \\
& + t_2^{(1-\beta)(2-\alpha)} \int_{t_1}^{t_2} \| g_{2q-1+\beta(2-\alpha)}(t_2 - s) T_q(s)u_1 \| ds \\
& + t_2^{(1-\beta)(2-\alpha)} \sup_{s\in[0,t_1-\epsilon]} \| T_q(t_2 - s) - T_q(t_1 - s) \| \int_0^{t_1-\epsilon} [m(s) + \psi(s)] ds \\
& + M t_1^{2q-1} E_{q,2q}^2(\omega t_1^q) \Big(t_2^{(1-\beta)(2-\alpha)} - t_1^{(1-\beta)(2-\alpha)}\Big) \int_0^{t_1-\epsilon} [m(s) + \psi(s)] ds \\
& + M t_2^{1-\beta(2-\alpha)} E_{q,2q}^2(\omega t_2^q) \int_{t_1}^{t_2} [m(s) + \psi(s)] ds \\
& + t_2^{(1-\beta)(2-\alpha)} \sup_{s\in[t_1-\epsilon,t_1]} \| T_q(t_2 - s) - T_q(t_1 - s) \| \int_{t_1-\epsilon}^{t_1} [m(s) + \psi(s)] ds
\end{aligned} \tag{3.4}
$$

Hence, in view of Lemma 2.5, the inequality (3.4) tends to zero as $t_2 - t_1 \to 0$ and $\epsilon \to 0$. Therefore, we have

$$\Big\| (\mathscr{P}v)(t_2) - (\mathscr{P}v)(t_1) \Big\| \to 0$$

independently of $v \in B_r(J)$ as $t_2 \to t_1$, which means that $\{\mathscr{P}v : v \in B_r(J)\}$ is equicontinuous.

Step 3. Now we show that $\mathscr{P}$ is continuous in $B_r(J)$. To show this, for any $v_n, v \in B_r, n = 1, 2, \ldots$, with $\lim_{n\to\infty} v_n = v$, we get

$$\lim_{n\to\infty} v_n(t) = v(t), \quad \text{and} \quad \lim_{n\to\infty} t^{(\beta-1)(2-\alpha)} v_n(t) = t^{(\beta-1)(2-\alpha)} v(t), \quad \text{for } t \in J'.$$

Thus, by (H2), we get that

$$\lim_{n\to\infty} f(t, u_n(t)) = \lim_{n\to\infty} f(t, t^{(\beta-1)(2-\alpha)} v_n(s)) = f(t, t^{(\beta-1)(2-\alpha)} v(s)) = f(t, u(t)).$$

On the one hand, by the assumption (H2), we get for each $t \in J'$,

$$(t-s)^{2q-1} \| f(s, u_n) - f(s, u) \| \leq 2(t-s)^{2q-1} m(s), \quad \text{a.e. in } [0, t).$$

On the other hand, the function $s \to 2(t-s)^{2q-1} m(s)$ is integrable for $s \in [0, t)$ and $t \in J'$. By Lebesgue dominated convergence theorem, we have

$$\int_0^t (t-s)^{2q-1} \| f(s, u_n(s)) - f(s, u(s)) \| ds \to 0, \quad \text{as } n \to \infty.$$

For $t \in J, u_n, u \in B_r$, we have

$$\| (\mathscr{P}v_n)(t) - (\mathscr{P}v)(t) \|$$

$$
\begin{aligned}
&= \|t^{(1-\beta)(2-\alpha)}(\mathcal{P}u_n)(t) - t^{(1-\beta)(2-\alpha)}(\mathcal{P}u)(t)\| \\
&\leq \int_0^t t^{(1-\beta)(2-\alpha)} T_q(t-s)\|f(s,u_n(s)) - f(s,u(s))\|ds \\
&\leq Mt^{(1-\beta)(2-\alpha)} E^2_{2q,q}(\omega t^q) \int_0^t (t-s)^{2q-1}\|f(s,u_n(s)) - f(s,u(s))\|ds \\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned}
$$

which implies that $\mathscr{P}v_n \to \mathscr{P}v$ pointwise on J as $n \to \infty$. From Step 2, one has that $\mathscr{P}v_n \to \mathscr{P}v$ uniformly on J as $n \to \infty$ and so $\mathscr{P}$ is a continuous operator.

Step 4. We will prove that $\mathcal{H}(t) = \{(\mathscr{P}v)(t), v \in B_r(J)\}$ is relatively compact in E for any $t \in J$. Obviously, $\mathcal{H}(0)$ is relatively compact in E. Let $t \in (0, b]$ be fixed. For any $\epsilon \in (0,t)$, we define an operator $\mathscr{P}_\epsilon$ on $B_r(J)$ by

$$
\begin{aligned}
(\mathscr{P}_\epsilon v)(t) =& \frac{u_0}{\Gamma(2q-1+\beta(2-\alpha))} + t^{(1-\beta)(2-\alpha)} \int_0^t g_{2q-1+\beta(2-\alpha)}(t-s) AT_q(s)u_0 ds \\
&+ t^{(1-\beta)(2-\alpha)} \int_0^t g_{\beta(2-\alpha)}(t-s)T_q(s)u_1 ds \\
&+ t^{(1-\beta)(2-\alpha)} \int_0^{t-\varepsilon} T_q(t-s-\epsilon)[By(s) + f(s, s^{(\beta-1)(2-\alpha)}v(s))]ds.
\end{aligned}
$$

Thus, in view of Lemma 2.7 and 2.8, we get that the set

$$
\Big\{T_q(t-s-\epsilon)[By(s) + f(s, s^{(\beta-1)(2-\alpha)}v(s))], \quad v \in B_r(J)\Big\}
$$

is relatively compact for each $\epsilon \in (0, t-s)$. Thus, for $v \in B_r(J)$, we obtain the set $\mathcal{H}(t) := \{(\mathscr{P}_\varepsilon v)(t),\ v \in B_r(J)\}$ is relatively compact in E. On the other hand, for any $v \in B_r(J)$, we have

$$
\begin{aligned}
&\|(\mathscr{P}_\epsilon v)(t) - (\mathscr{P}v)(t)\| \\
\leq& t^{(1-\beta)(2-\alpha)} \int_{t-\varepsilon}^t T_q(t-s)By(s)ds \\
&+ t^{(1-\beta)(2-\alpha)} \int_{t-\epsilon}^t T_q(t-s)f(s, s^{(\beta-1)(2-\alpha)}v(s))ds \\
\leq& Mt^{1+\beta(2-\alpha)} E^2_{q,2q}(\omega t^q) \int_{t-\epsilon}^t [By(s) + f(s, s^{(\beta-1)(2-\alpha)}v(s))]ds \\
\leq& Mt^{1+\beta(2-\alpha)} E^2_{q,2q}(\omega t^q) \int_{t-\epsilon}^t [By(s) + f(s,u(s))]ds \\
\leq& Mt^{1+\beta(2-\alpha)} E^2_{q,2q}(\omega t^q) \Big[\int_{t-\epsilon}^t m(s)ds + \int_{t-\epsilon}^t \psi(s)ds\Big] \\
\to& 0, \quad \epsilon \to 0.
\end{aligned}
$$

As a result, there are relatively compact sets arbitrarily close to the set $\mathcal{H}$ for every $t \in J'$. This implies that the set $\mathcal{H}$ is relatively compact in E for every J'. Moreover, $\mathcal{H}$ is relatively compact at $t = 0$. Hence $\mathcal{H}$ is relatively compact in E for every J. We can deduce that $\{\mathscr{P}v, v \in B_r(J)\}$ is relatively compact by Arzelà-Ascoli theorem and thus $\mathscr{P}$ is a completely continuous operator on $C(J, E)$. Therefore, by Schauder fixed point theorem, we obtain that $\mathscr{P}$ has a fixed point $v^* \in C(J, E)$. Let $u^*(t) = t^{(\beta-1)(2-\alpha)}v^*(t)$. Then, u^* is a mild solution of the problem (1.1). This completes the proof Theorem 3.1. $\square$

Remark 3.1. The assumption (H2) is replaced by the following assumption:

(H2') there exists a constant $\mu_1 \in (0, \mu)$ and $m \in L^{\frac{1}{\mu_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t,u)\| \leq m(t), \quad \text{for all} \;\; u \in E \;\; \text{and almost} \;\; t \in J.$$

Corollary 3.1. *Assume that (H1), (H2') and (H3) hold, then the problem (1.1) has at least one mild solution in $B_r^\alpha(J')$.*

4. Approximate controllability

In this section, we establish the sufficient conditions for the controllability of system (1.1).

Theorem 4.1. *Assume that the operator A generates a strongly continuous cosine family $\{C(t) : t \geq 0\}$. If the assumptions (H1), (H3), and the following assumptions condition are satisfied:*

(H4) For each $t \in J'$, the function $f(t, \cdot) : E \to E$ is continuous and for each $u \in E$, the function $f(\cdot, u) : J' \to E$ is strongly measure. Moreover, there exists a function $m \in L^2(J', \mathbb{R}^+)$ such that

$$I_{0+}^\alpha m \in C(J', \mathbb{R}^+), \quad \lim_{t\to 0+} t^{(1-\beta)(2-\alpha)} I_{0+}^\alpha m(t) = 0$$

and

$$\|f(t,u)\| \leq m(t) \quad \textit{for all} \;\; u \in E \;\; \textit{and almost all} \; t \in J.$$

(H5) $\mu R(\mu, \Gamma_0^a) \to 0$ as $\mu \to 0^+$ in the strong operator topology.

Then the semilinear control system (1.1) is approximately controllable on J.

Proof. It is easily to know that the assumption (H2) $\Rightarrow$ (H4). For $\mu > 0$, define the operator $\mathcal{P}_\mu$ on $C(J, E)$ as follows,

$$\begin{aligned}(\mathcal{P}_\mu u)(t) =& g_{2q-1+\beta(2-\alpha)}(t)u_0 + (g_{2q-1+\beta(2-\alpha)} * AT_q)(t)u_0 \\ &+ (g_{\beta(2-\alpha)} * T_q)(t)u_1 + \int_0^t T_q(t-s)By_\mu(s)ds \\ &+ \int_0^t T_q(t-s)f(s,u(s))ds, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} y_\mu(t) =& B^*T_q(b-t)R(\mu, \Gamma_0^b)p(u(\cdot)), \\ p(u(\cdot)) =& u_b - g_{2q-1+\beta(2-\alpha)}(b)u_0 - (g_{2q-1+\beta(2-\alpha)} * AT_q)(b)u_0 \\ &- (g_{\beta(2-\alpha)} * T_q)(b)u_1 - \int_0^b T_q(b-s)f(s,u(s))ds. \end{aligned}$$

Let $v_\mu(t) = t^{(1-\beta)(2-\alpha)}u_\mu(t)$ be the fixed point of $\mathscr{P}_\mu$ in $B_{r_0}(J)$. By Theorem 3.1, any fixed point of $\mathscr{P}_\mu = t^{(1-\beta)(2-\alpha)}\mathcal{P}_\mu$ is a mild solution of the control system (1.1) under the control

$$y_\mu(t) = B^*T_q(b-t)R(\mu,\Gamma_0^b)p(v(\cdot)),$$

where

$$\begin{aligned}p(v(\cdot)) =& t^{(1-\beta)(2-\alpha)}u_b\\ &- t^{(1-\beta)(2-\alpha)}g_{2q-1+\beta(2-\alpha)}(b)u_0 - t^{(1-\beta)(2-\alpha)}(g_{2q-1+\beta(2-\alpha)} * AT_q)(b)u_0\\ &- t^{(1-\beta)(2-\alpha)}(g_{\beta(2-\alpha)} * T_q)(b)u_1 - t^{(1-\beta)(2-\alpha)}\int_0^b T_q(b-s)f(s,u(s))ds\end{aligned}$$

and satisfies

$$\begin{aligned}v_\mu(b) =& b^{(1-\beta)(2-\alpha)}u_\mu(b) = t^{(1-\beta)(2-\alpha)}g_{2q-1+\beta(2-\alpha)}(b)u_0\\ &+ b^{(1-\beta)(2-\alpha)}(g_{2q-1+\beta(2-\alpha)} * AT_q)(b)u_0\\ &+ b^{(1-\beta)(2-\alpha)}(g_{\beta(2-\alpha)} * T_q)(b)u_1 + b^{(1-\beta)(2-\alpha)}\int_0^b T_q(t-s)By_\mu(s)ds\\ &+ b^{(1-\beta)(2-\alpha)}\int_0^b T_q(b-s)f(s,u_\mu(s))ds\\ =& b^{(1-\beta)(2-\alpha)}u_b - p(v_\mu(\cdot)) + \int_0^b T_q(b-s)BB^*T_q^*(b-s)R(\mu,\Gamma_0^b)p(v_\mu(\cdot))ds\\ =& b^{(1-\beta)(2-\alpha)}u_b - p(v_\mu(\cdot)) + \Gamma_0^b R(\mu,\Gamma_0^b)p(v_\mu(\cdot))\\ =& b^{(1-\beta)(2-\alpha)}u_b - (\mu I + \Gamma_0^b)R(\mu,\Gamma_0^b)p(v_\mu(\cdot)) + \Gamma_0^b R(\mu,\Gamma_0^b)p(v_\mu(\cdot))\\ =& b^{(1-\beta)(2-\alpha)}u_b - \mu R(\mu,\Gamma_0^b)p(v_\mu(\cdot))\\ =& v_b - \mu R(\mu,\Gamma_0^b)p(v_\mu(\cdot)).\end{aligned}$$

Now, from the assumption (H4), we have

$$\Big(\int_0^b \|f(s,u_\mu(s))\|^2 ds\Big)^{\frac{1}{2}} \le \Big(\int_0^b m^2(s)ds\Big)^{\frac{1}{2}} < \infty,$$

which implies that the sequence $\{f(\cdot,u_\mu(\cdot))|\mu>0\}$ is bounded in $L^2(J,U)$. Thus, there exists a subsequence of $\{f(\cdot,u_\mu(\cdot))|\mu>0\}$, still denoted by $\{f(\cdot,u_\mu(\cdot))|\mu>0\}$, which weakly converges to some point $\mathcal{F}(\cdot) \in L^2(J,U)$. Let

$$\begin{aligned}\omega :=& u_b - g_{2q-1+\beta(2-\alpha)}(b)u_0 - (g_{2q-1+\beta(2-\alpha)} * AT_q)(b)u_0\\ &- (g_{\beta(2-\alpha)} * T_q)(b)u_1 - \int_0^b T_q(b-s)\mathcal{F}(s)ds.\end{aligned}$$

Thus, we have

$$\Big\|p(u_\mu(\cdot)) - \omega\Big\| \le \Big\|\int_0^b T_q(b-s)[f(s,u_\mu(s)) - \mathcal{F}(s)]ds\Big\|. \tag{4.2}$$

From the fact that $T_q(t)$ is compact operator for $t>0$. And similarly to the proof of the compactness of the operator $\mathscr{P}$ in Theorem 3.1, one can easily verify that the mapping

$$g(t) \to \int_0^t T_q(t-s)g(s)ds$$

is compact for $t \in J$. Therefore, we have

$$\int_0^b T_q(b-s)[f(s,u_\mu(s)) - \mathcal{F}(s)]ds \to 0 \quad \text{as} \quad \mu \to 0^+. \tag{4.3}$$

Thus, from (4.2) and (4.3), we get

$$\Big\|p(u_\mu(\cdot)) - \omega\Big\| \to 0 \quad \text{as} \quad \mu \to 0^+. \tag{4.4}$$

And by (4.4) and the assumption (H5), we have

$$\begin{aligned}
\|v_\mu(b) - v_b\| &= \|t^{(1-\beta)(2-\alpha)}(u_\mu(b) - u_b)\| \\
&\leq \|t^{(1-\beta)(2-\alpha)}\mu R(\mu, \Gamma_0^b)p(u_\mu(\cdot))\| \\
&\leq \|t^{(1-\beta)(2-\alpha)}\mu R(\mu, \Gamma_0^b)\omega\| \\
&\quad + \|t^{(1-\beta)(2-\alpha)}\mu R(\mu, \Gamma_0^b)\| \cdot \Big\|p(u_\mu(\cdot)) - \omega\Big\| \\
&\to 0 \quad \text{as} \quad \mu \to 0^+.
\end{aligned} \tag{4.5}$$

This implies that

$$\|u_\mu(b) - u_b)\| \to 0 \quad \text{as} \quad \mu \to 0^+.$$

This concludes that the control system (1.1) is approximately controllable on J. This completes the proof of Theorem 4.1. □

Remark 4.1. In Theorem 4.1, the assumption (H4) is replaced by the following assumption:

(H6) The linear fractional evolution system is approximately controllable on $(0, b]$.

Observe that linear fractional evolution control problem

$$\begin{cases} D_{0+}^{\alpha,\beta}u(t) = Au(t) + By(t), \quad t \in (0,b], \\ (g_{(1-\beta)(2-\alpha)} * u)(0) = u_0, \quad (g_{(1-\beta)(2-\alpha)} * u)'(0) = u_1, \end{cases} \tag{4.6}$$

corresponding to (1.1) is approximately controllable on J' if and only if the operator $\mu R(\mu, \Gamma_0^a) \to 0$ as $\mu \to 0^+$ in the strong operator topology. For more details see [52].

5. Applications

In this section, we present two examples, which illustrate the applicability of our main results.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the sufficiently smooth boundary $\partial\Omega$.

Example 5.1. We consider the following Hilfer fractional wave equations:

$$\begin{cases} D_{0+}^{\alpha,\beta}u(t,x) = A(x,D)u(t,x) + +\frac{t^2\sin(2\pi t)}{1+|u(t,x)|} + \kappa y(t,x), \quad x\in\Omega,\ t\in(0,b], \\ u(t,x)=0, \quad x\in\partial\Omega, \quad t\in(0,b], \\ (g_{(1-\beta)(2-\alpha)} * u)(0,x) = u_0(x), \quad \partial_t(g_{(1-\beta)(2-\alpha)} * u)(0,x) = u_1(x), \quad x\in\Omega, \end{cases} \tag{5.1}$$

where $D_{0+}^{\alpha,\beta}$is the Hilfer fractional derivative, $0\le\beta\le1,\ 1<\alpha<2,\ f: J\times\Omega\to H$ is continuous. $A(x,D)$ is the second order linear elliptic operator:

$$A(x,D)u(t,x) = \sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{N}\frac{\partial}{\partial x_i}(b_i(x)u) + c(x)u,\ x\in\Omega,$$

and satisfies the uniformly elliptic condition:

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2, \quad \xi\in\mathbb{R}^N,\ x\in\overline{\Omega},$$

where the coefficient functions $a_{ij}=a_{ji}\in C^1(\overline{\Omega}), b_i, c\in C(\overline{\Omega}), i,j=1,2,\ldots,N$ and $\nu>0$ is a constant.

Let $H=L^2(\Omega)$ be a Hilbert space with the L^2-norm $\|\cdot\|_2$, we define

$$\mathcal{D}(A) = H_0^1(\Omega)\cap H^2(\Omega), \quad Au = A(x,D)u, \quad u\in\mathcal{D}(A).$$

It is well known that the operator A generates a strongly continuous, exponentially bounded cosine family on H, i.e., the assumption condition (H1) is satisfied.

Let

$$u(t)(x) = u(t,x), \quad f(t,u(t))(x) = f(t,u(t,x)) = \frac{t^2\sin(2\pi t)}{1+|u(t,x)|}.$$

We define the bounded linear operator $B: U := E\to E$ by $By(t)=\kappa y(t,x)$. Hence, we can write the problem (5.1) into the abstract fractional evolution equations (1.1) in the Hilbert space H.

To study this problem, we assume the following conditions:

(i) There exists a essential bounded function $h_r(t)$ such that for any $t\in[0,b],\ x\in\Omega$ and $u\in L^2(\Omega)$ satisfying $(\int_\Omega|u(x)^2|dx)^{\frac12}\le r$ for some $r>0$

$$\Big(\int_\Omega |f(t,u(t,x))|^2dx\Big)^{\frac12} \le h_r(t).$$

Theorem 5.1. *If the assumptions (i) is satisfied, then the problem (5.1) has at least one mild solution $u\in C(J\times[0,\pi])$ and it is approximately controllable on J.*

Proof. By the assumptions (i) one can easily verify that conditions (H3)-(H5) are satisfied with $\psi(t)=\kappa y(t)$. Therefore, our conclusion follows Theorem 3.1 and Theorem 4.1. This completes the proof of Theorem 5.1. □

Example 5.2. We consider the following Hilfer fractional evolution equations:

$$\begin{cases} D_{0+}^{\alpha,\beta}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + By(t,x) + f(t,u(t,x)), \quad x \in [0,\pi],\ t \in J, \\ u(t,0) = u(t,\pi) = 0, \quad x \in [0,\pi],\ t \in J, \\ \left(I_{0+}^{(1-\beta)(2-\alpha)}u(0,x)\right) = u_0(x), \quad \left(I_{0+}^{(1-\beta)(2-\alpha)}u(0,x)\right)' = u_1(x), \quad x \in [0,\pi], \end{cases} \tag{5.2}$$

where $D_{0+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $0 \le \beta \le 1, 1 < \alpha < 2$, κ is a constants, $J = [0,b], y \in L^2(J, L^2(0,\pi;\mathbb{R}))$.

Let $E = L^2(0,\pi)$. If $e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$, then $\{e_n : n = 1,2,\ldots\}$ is an orthonormal base for E. The operator $A : E \to E$ is defined by

$$Au = \frac{\partial^2 u}{\partial x^2} = u''(x),$$

where $\mathcal{D}(A) = \{u \in E :\ u'' \in E,\ u(0) = u(\pi) = 0\}$. Here, clearly, the operator A is the infinitesimal generator of a strongly continuous, exponentially bounded cosine family of operators on E. The operator A has infinite series representation:

$$Au = \sum_{n=1}^{\infty} -n^2(u,e_n)e_n, \quad u \in \mathcal{D}(A).$$

Moreover, the operator A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$ on E which is given by

$$C(t)u = \sum_{n=1}^{\infty} \cos nt(u,e_n)e_n, \quad u \in E,$$

and the associated sine family $S(t), t \in \mathbb{R}$ on E which is given by

$$S(t)u = \sum_{n=1}^{\infty} \frac{1}{n}\sin nt(u,e_n)e_n, \quad u \in E.$$

Define an infinite dimensional space U by

$$U = \left\{ y = \sum_{n=2}^{\infty} y_n e_n(x) \Big| \sum_{n=2}^{\infty} y_n^2 < \infty \right\} \subset L^2([0,\pi]).$$

Then norm in U is defined by $\|y\| = \sqrt{\sum_{n=2}^{\infty} y_n^2}$.

Define a linear operator $B : U \to E$ by

$$(By)(x) = 2y_2 e_1(x) + \sum_{n=2}^{\infty} y_n e_n(x), \quad \text{for} \ \ y = \sum_{n=2}^{\infty} y_n e_n(x) \in U.$$

By simple calculation, we can get $\|B\| \le 2$. Therefore B is a bounded linear operator.

Let

$$u(t)(x) = u(t,x), \quad f(t,u(t))(x) = f(t,u(t,x)), \quad t \in J,\ x \in [0,\pi],$$

then we can transform the problem (5.2) into the abstract form of (1.1).

Theorem 5.2. *If the following assumption*

(i) There exists a Lebesgue measure function $h_r(t)$ such that for any $t \in [0,b], x \in [0,\pi]$ and $u \in L^2(0,\pi)$ satisfying $\|u\| \leq r$ for some $r > 0$

$$\|f(t, u(t,x))\| \leq h_r(t);$$

is satisfied, then the problem (5.2) has at least one mild solution $u \in C(J \times [0,\pi])$ and it is approximate controllable on J.

Proof. Since the conditions (H1), (H3)-(H5) are satisfied. Therefore, our conclusion follows Theorem 3.1 and Theorem 4.1. This completes the proof of Theorem 5.1. □

6. Conclusions

In this paper, we deal with a class of nonlinear fractional evolution equations in Banach spaces by using Hilfer fractional derivative, which generalized the famous Riemann-Liouville fractional derivative. The definition of mild solutions for studied problem was given based on a cosine family generated by the operator A and probability density function. Combining the techniques of fractional calculus with Schauder's fixed-point theorem, we establish the existence of mild solutions as well as approximate controllability for the desired problem. Lastly, we presented theoretical and practical applications to support the validity of the study.

The results obtained improve and extend some related conclusions on this topic. When $\beta = 1$, the fractional equation (1.1) simplifies to the classical Caputo fractional differential equations; When $\beta = 0$, the fractional equation (1.1) simplifies to the classical Riemman-Liouville fractional differential equations. When $0 \leq \beta \leq 1$, $0 < \alpha < 1$, we assume that A generate a strongly continuous semigroup $\{C(t)\}_{t\geq 0}$ of bounded linear operator on E, the fractional equation (1.1) simplifies to evolution equation with Hilfer fractional derivative which has been studied by Gu et al. [25].

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