Journal of Applied Analysis and Computation
Volume 11, Number 6, December 2021, 2947–2958
Website:http://www.jaac-online.com
DOI:10.11948/20210070

# INTERESTING DETERMINANTS AND INVERSES OF SKEW LOEPLITZ AND FOEPLITZ MATRICES*

Qingyan Meng[1], Xiaoyu Jiang[2,†] and Zhaolin Jiang[1,†]

**Abstract** In this paper, we show that there is an intimate relationship between Toeplitz matrix, tridiagonal Toeplitz matrix, the Fibonacci number, the Lucas number, and the Golden Ratio. We introduce skew Loeplitz and skew Foeplitz matrices and derive their determinants and inverses by construction. Specifically, the determiant of $n \times n$ skew Loeplitz matrix can be expressed by the $(n+1)$st Fibonacci number. The inverse of skew Loeplitz matrix is sparse and can be expressed by the $n$th and $(n+1)$st Fibonacci numbers. Similarly, the determinant of $n \times n$ skew Foeplitz matrix also can be expressed by the $(n+1)$st Lucas number. The inverse of skew Foeplitz matrix can be expressed by only seven elements with each element being the explicit expression of the Lucas or Fibonacci numbers. We also calculate the determinants and inverses of skew Lankel and skew Fankel matrices.

**Keywords** determinant, inverse, Fibonacci number, Lucas number, skew Foeplitz matrix, skew Loeplitz matrix.

**MSC(2010)** 15A09, 15A15, 65F50.

## 1. Introduction

As is well-known, Toeplitz matrix families have important applications in various disciplines including fractional differential equation [6, 13, 14, 17, 20, 21], integral equations [19].

The main research objects of this paper are the explicit determinants and inverses of two special matrices, which are called skew Loeplitz and Foeplitz matrix, respectively, and defined as follows.

A skew Loeplitz matrix is a Toeplitz matrix of the form

$$\mathbf{T}_{L,n,-1} = \left(t_{i,j}\right)_{n\times n}, \tag{1.1}$$

where

$$t_{i,j} = \begin{cases} L_{j-i+1}, \ 1 \le i \le j \le n, \\ -L_{-(i-j+1)}, \ 1 \le j < i \le n, \end{cases}$$

†The corresponding author.
Email: jxy19890422@sina.com(X. Y. Jiang); jzh1208@sina.com(Z. L. Jiang)
[1]School of Mathematics and Statistics, Linyi University, Linyi, 276000, China
[2]School of Information Science and Technology, Linyi University, Linyi, 276000, China

*The authors were supported by Natural Science Foundation of Shandong Province (Nos. ZR2020MA051, ZR2020QA035) and the PhD Research Foundation of Linyi University (No.LYDX2018BS052).

and $L_1, L_{\pm 2}, \ldots, L_{\pm n}$ are Lucas numbers.

A skew Foeplitz matrix is a Toeplitz matrix of the form

$$\mathbf{T}_{F,n,-1} = (f_{i,j})_{n\times n}, \tag{1.2}$$

where

$$f_{i,j} = \begin{cases} F_{j-i+1}, \ 1 \le i \le j \le n, \\ -F_{-(i-j+1)}, \ 1 \le j < i \le n, \end{cases}$$

and $F_1, F_{\pm 2}, \ldots, F_{\pm n}$ are Fibonacci numbers.

Recently, many scholars showed the explicit determinants and inverses of the special matrices involving famous numbers. More specifically, Shen et al. [15] proposed circulant matrices involving Fibonacci and Lucas numbers and compute their explicit determinants and inverses. Moreover, Jiang et al. [9] considered circulant type matrices with the $k$-Fibonacci & $k$-Lucas numbers and presented the explicit determinants and inverses by construction. Zheng and Shon [24] proposed the invertibility criterion of the generalized Lucas skew circulant type matrices and provided their determinants and inverses. Besides, Bozkurt and Tam [2] provided determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers. Following year, they [8] evaluated the determinants and inverses for Tribonacci skew circulant type matrices. Shen et al. [16] considered generalized Tribonacci circulant type matrices, including the circulant and left circulant. Determinants and inverses of Ppoeplitz and Ppankel matrices have been obtained in [22]. Explicit expression of determinants and inverse matrices for Foeplitz and Loeplitz matrices were represented in [11]. Determinant and inverse of a Gaussion Fibonacci skew-Hermitian Toeplitz matrix was studied by Jiang and Sun in [10]. Determinant and inverse of skew Peoeplitz matrix was considered by Han and Jiang in [7]. Determinants and inverses of symmetric Poeplitz and Qoeplitz matrix were investigated in [3]. Determinants and inverses of skew symmetric generalized Loeplitz matrices and Foeplitz matrices were proposed in [4] and [5], respectively. Akbulak and Bozkurt [1] gave upper and lower bounds for the spectral norms of the Fibonacci and Lucas Toeplitz matrix.

The Fibonacci numbers $\{F_n\}$ and Lucas numbers $\{L_n\}$ are respectively defined by the following recurrence relations [18]:

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1, n \ge 2; \\ L_n &= L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1, n \ge 2. \end{aligned}$$

The rule can be used to extend the sequence backwards. Hence

$$F_{-n} = (-1)^{n+1} F_n, \quad L_{-n} = (-1)^n L_n.$$

The following identities are easy to verify

$$(i) \sum_{i=1}^{n} F_{k+i} a^i = \frac{aF_{k+1} + a^2 F_k - a^{n+1}(F_{k+n+1} + aF_{k+n})}{1 - a - a^2}, a \neq \frac{-1 \pm \sqrt{5}}{2}, \tag{1.3}$$

$$(ii) \sum_{i=1}^{n} F_{k-i} a^i = \frac{aF_{k-1} + a^2 F_k - a^{n+1}(F_{k-n-1} + aF_{k-n})}{1 + a - a^2}, a \neq \frac{1 \pm \sqrt{5}}{2}, \tag{1.4}$$

$$(iii)\sum_{i=1}^{n} L_{k+i}a^i = \frac{aL_{k+1} + a^2L_k - a^{n+1}(L_{k+n+1} + aL_{k+n})}{1 - a - a^2},\ a \neq \frac{-1 \pm \sqrt{5}}{2}, \quad (1.5)$$

$$(iv)\sum_{i=1}^{n} L_{k-i}a^i = \frac{aL_{k-1} + a^2L_k - a^{n+1}(L_{k-n-1} + aL_{k-n})}{1 + a - a^2},\ a \neq \frac{1 \pm \sqrt{5}}{2}, \quad (1.6)$$

where $a$ is a complex number and $k$ is an integer.

This paper is organized as follows. In Section 2, the determinant and inverse of skew Loeplitz matrix are provided. Section 3 is devoted to calculating the determinant and inverse of skew Foeplitz matrix. In Section 4, the determinants and inverses of skew Lankel and skew Fankel matrices are given. Finally, we present an algorithm at Section 5.

## 2. Determinant and inverse of skew Loeplitz matrix

In this section, we compute the determinant and the inverse of the matrix $\mathbf{T}_{L,n,-1}$ in the following Theorem 2.1 and 2.2 below, respectively.

**Theorem 2.1.** *Let $\mathbf{T}_{L,n,-1}$ be the $n \times n$ skew Loeplitz matrix given in (1.1). Then we have*

$$\det \mathbf{T}_{L,n,-1} = 5^{n-1}F_{n+1}, \quad (2.1)$$

*where $F_{n+1}$ is the $(n+1)$st Fibonacci number.*

**Proof.** Obviously, $\det \mathbf{T}_{L,1,-1} = 1$ and $\det \mathbf{T}_{L,2,-1} = 10$ both satisfy the equation (2.1). In the case $n > 2$, let $A_n = (a_{ij})_{n\times n}$ and $B_n = (b_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1,\ j = n+1-i,\ 1 \leq i \leq n, \\ -1,\ n+2-i \leq j \leq n+3-i,\ 3 \leq i \leq n, \\ 0,\ \text{else}, \end{cases} \quad (2.2)$$

and

$$b_{ij} = \begin{cases} 1,\ j = i = 1, \\ 1,\ j = n+2-i,\ 2 \leq i \leq n, \\ 0,\ \text{else}. \end{cases} \quad (2.3)$$

Apparently, $A_n$ and $B_n$ are invertible, and

$$\det A_n \det B_n = (-1)^{n-1}. \quad (2.4)$$

Multiplying $\mathbf{T}_{L,n,-1}$ by $A_n$ from the left yields

$$A_n\mathbf{T}_{L,n,-1} = \left(\begin{array}{c:ccccc:c} -L_{-n} & -L_{-(n-1)} & -L_{-(n-2)} & \cdots & -L_{-3} & -L_{-2} & L_1 \\ -L_{-(n-1)} & -L_{-(n-2)} & -L_{-(n-3)} & \cdots & -L_{-2} & L_1 & L_2 \\ \hdashline 0 & & & 5\hat{\mathbf{I}}_{n-2} & & & 0 \end{array}\right)_{n\times n},$$

where $\hat{\mathbf{I}}_n$ is the "reverse unit matrix", having ones along the secondary diagonal and zeros elsewhere.

And multiplying the above matrix by $B_n$ from the right, we obtain

$$A_n\mathbf{T}_{L,n,-1}B_n = \left(\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right) =$$

$$\left(\begin{array}{cc|cccccc} -L_{-n} & L_1 & -L_{-2} & -L_{-3} & -L_{-4} & \cdots & -L_{-(n-2)} & -L_{-(n-1)} \\ -L_{-(n-1)} & L_2 & L_1 & -L_{-2} & -L_{-3} & \cdots & -L_{-(n-3)} & -L_{-(n-2)} \\ \hline 0 & 0 & 5 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 5 \end{array}\right)_{n\times n}. \tag{2.5}$$

Taking the determinant for both sides of (2.5) and by the identity $-3L_{-n}+L_{-(n-1)} = 5F_{n+1}$, we have

$$\det(A_n\mathbf{T}_{L,n,-1}B_n) = \det E \det H = (-1)^{n-1}5^{n-1}F_{n+1}. \tag{2.6}$$

Using the formula $\det A_n \det B_n = (-1)^{n-1}$ and (2.6), we obtain $\det \mathbf{T}_{L,n,-1}$ as (2.1). $\square$

**Remark 2.1.** Theorem 2.1 gives the relationship between the skew Loeplitz matrix and the Fibonacci number. From the perspective of number theory, the $(n+1)$st Fibonacci number can be represented by the determinant of the $n\times n$ skew Loeplitz matrix.

**Theorem 2.2.** *Let $\mathbf{T}_{L,n,-1}$ be the $n\times n$ skew Loeplitz matrix given in (1.1) for a positive integer $n>2$. Then*

$$\mathbf{T}^{-1}_{L,n,-1} = \frac{1}{5}\begin{pmatrix} \frac{F_n}{F_{n+1}} & -1 & 0 & 0 & \cdots & \cdots & 0 & \frac{(-1)^{n+1}}{F_{n+1}} \\ 1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & -1 & -1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 & -1 & -1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -1 & -1 \\ \frac{1}{F_{n+1}} & 0 & \cdots & \cdots & \cdots & 0 & 1 & \frac{F_n}{F_{n+1}} \end{pmatrix}_{n\times n}, \tag{2.7}$$

*where $F_i$ is the ith Fibonacci number, $i=n,\ n+1$.*

**Proof.** For $n>2$, in order to obtain the inverse of $T_{L,n,-1}$, we write

$$\mathbf{T}^{-1}_{L,n,-1} = B_n(A_n\mathbf{T}_{L,n,-1}B_n)^{-1}A_n, \tag{2.8}$$

where $A_n, B_n$ are the same as Theorem 1.

According to the Theorem in [23, p.19], the equation (2.5), $L_{n-1}L_{n-k}-L_nL_{n-k-1}=(-1)^{n-k}L_k$ and $3L_{n-k}+L_{n-k-1}=5F_{n-k-1}$ where $k$ is an integer, we have

$$(A_n\mathbf{T}_{L,n,-1}B_n)^{-1}$$
$$=\begin{pmatrix} \frac{(-1)^{n+1}3}{5F_{n+1}} & \frac{(-1)^{n+2}}{5F_{n+1}} & \frac{(-1)^{n+5}F_3}{5F_{n+1}} & \frac{(-1)^{n+6}F_4}{5F_{n+1}} & \cdots & \frac{(-1)^{2n+2}F_n}{5F_{n+1}} \\ \frac{L_{n-1}}{5F_{n+1}} & \frac{L_n}{5F_{n+1}} & \frac{L_{n-2}}{25F_{n+1}} & \frac{L_{n-3}}{25F_{n+1}} & \cdots & \frac{1}{25F_{n+1}} \\ 0 & 0 & \frac{1}{5} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} \end{pmatrix}_{n\times n}. \tag{2.9}$$

According to formulas (2.2), (2.8), (2.9) and the relation between the Fibonacci number and the Lucas number, we can obtain the inverse of $\mathbf{T}_{L,n,-1}$ as (2.7), which completes the proof. □

**Remark 2.2.** Equation (2.7) can be appreciated in many different ways, and it is easy to see that top-left and bottom-right corner entries of $5\mathbf{T}_{L,n,-1}^{-1}$ get closer and closer to the Golden Ratio. In fact, skew Loeplitz matrix, tridiagonal Toeplitz matrix with perturbed corner entries, the Fibonacci number, the Lucas number, and the Golden Ratio are all connected by Equation (2.7).

## 3. Determinant and inverse of skew Foeplitz matrix

In this section, we compute the determinant and the inverse of the matrix $\mathbf{T}_{F,n,-1}$ in the following Theorem 3.1 and 3.2 below, respectively.

**Theorem 3.1.** *Let $\mathbf{T}_{F,n,-1}$ be the $n\times n$ skew Foeplitz matrix given in (1.2) and $n\geq 3$. We have*

$$\det \mathbf{T}_{F,n,-1}=\frac{2^n+(-1)^{n+1}L_{n+1}}{5}. \tag{3.1}$$

**Proof.** Obviously, $\det \mathbf{T}_{F,3,-1}=3$ satisfying the equation (3.1). We now compute $\det \mathbf{T}_{F,n,-1}$ for $n\geq 3$. Multiply $\mathbf{T}_{F,n,-1}$ by $A_n$ and $\hat{\mathbf{I}}_n$ from left and right, respectively, we obtain

$$A_n\mathbf{T}_{F,n,-1}\hat{\mathbf{I}}_n = \begin{pmatrix} 1 & \mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \cdots & \mu_3 & \mu_2 & \mu_1 \\ 1 & \eta_{n-1} & \eta_{n-2} & \eta_{n-3} & \cdots & \eta_3 & \eta_2 & \eta_1 \\ 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 2 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}_{n\times n}, \tag{3.2}$$

where $\hat{\mathbf{I}}_n$ is the reverse unit matrix, $A_n$ is given by (2.2),

$$\mu_i = (-1)^{n+1-i}F_{n+1-i}, i = 1, 2, \ldots, n-1,$$
$$\eta_i = (-1)^{n-i}F_{n-i}, i = 1, 2, \ldots, n-2,\ \eta_{n-1} = F_1.$$

By using the Laplace expansion of the determinant of the matrix $A_n\mathbf{T}_{F,n,-1}\hat{\mathbf{I}}_n$ along the first column and Lemma 2 in [12], we get

$$\begin{aligned}\det(A_n\mathbf{T}_{F,n,-1}\hat{\mathbf{I}}_n) &= \det D_{n-1}(\ [\eta_i]_{i=1}^{n-1},\ 2,\ \ -1) - \det D_{n-1}(\ [\mu_i]_{i=1}^{n-1},\ 2,\ \ -1) \\ &= \sum_{i=1}^{n-1}(\eta_i-\mu_i)2^{i-1} = \sum_{i=1}^{n-2}(-1)^{n-i}F_{n+2-i}2^{i-1}, \end{aligned} \tag{3.3}$$

where $D_{n-1}(\ [\eta_i]_{i=1}^{n-1},\ 2,\ \ -1)$ and $D_{n-1}(\ [\mu_i]_{i=1}^{n-1},\ 2,\ \ -1)$ are in the following form

$$D_{n-1}(\ [g_i]_{i=1}^{n-1},\ 2,\ \ -1) = \begin{pmatrix} g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_2 & g_1 \\ -1 & 2 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}_{n-1\times n-1}.$$

From the equation (1.4), (3.3) and $F_n + F_{n+2} = L_{n+1}$, we get

$$\det(A_n\mathbf{T}_{F,n,-1}\hat{\mathbf{I}}_n) = \frac{2^n + (-1)^{n+1}L_{n+1}}{5}. \tag{3.4}$$

According to (3.4) and $\det A_n = \det \hat{\mathbf{I}}_n = (-1)^{\frac{n(n-1)}{2}}$, we obtain $\det \mathbf{T}_{F,n,-1}$ as (3.1). $\square$

**Remark 3.1.** Theorem 3.1 gives the relationship between the skew Foeplitz matrix and the Lucas number. From the perspective of number theory, the $(n+1)$st Lucas number can be represented by the determinant of the $n \times n$ skew Foeplitz matrix.

**Theorem 3.2.** *Let $\mathbf{T}_{F,n,-1}$ be the $n \times n$ invertible skew Foeplitz matrix given in (1.2) and $n \geq 5$. Then $\mathbf{T}_{F,n,-1}^{-1}$ is of the form*

$$\mathbf{T}_{F,n,-1}^{-1} = \begin{pmatrix} p_3 & p_2 & 2^{n-3}p_1 & \cdots & 2^2p_1 & 2p_1 & p_1 \\ p_4 & p_5 & p_2 & \ddots & \ddots & \ddots & 2p_1 \\ 2p_4 & p_6 & p_5 & \ddots & \ddots & \ddots & 2^2p_1 \\ 2^2p_4 & 2p_6 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & p_5 & p_2 & 2^{n-3}p_1 \\ 2^{n-3}p_4 & 2^{n-4}p_6 & \cdots & 2p_6 & p_6 & p_5 & p_2 \\ p_7 & 2^{n-3}p_4 & \cdots & 2^2p_4 & 2p_4 & p_4 & p_3 \end{pmatrix}_{n\times n}, \tag{3.5}$$

*where*

$$p_1 = -\frac{1}{\det \mathbf{T}_{F,n,-1}}, \tag{3.6}$$

$$p_2 = -\frac{2^{n+2} + (-1)^n L_{n+1}}{2^{n+2} - (-1)^n 16 L_{n+1}}, \tag{3.7}$$

$$p_3 = -\frac{2^{n+1} + (-1)^n L_{n-2}}{20} p_1, \tag{3.8}$$

$$p_4 = (-1)^{n+1} p_1 F_{n+1}, \tag{3.9}$$

$$p_5 = \frac{2^{n+1} + (-1)^{n+1} 7 L_{n+1}}{2^{n+1} + (-1)^{n+1} 4 L_{n+1}}, \tag{3.10}$$

$$p_6 = (-1)^n p_1 L_{n+1}, \tag{3.11}$$

$$p_7 = \frac{(-1)^{n+1}(1 + 2L_{n+1})}{5 \det \mathbf{T}_{F,n,-1} - 3 \cdot 2^{n-2}}, \tag{3.12}$$

$$\det \mathbf{T}_{F,n,-1} = \frac{2^n + (-1)^{n+1} L_{n+1}}{5}. \tag{3.13}$$

**Proof.** To obtain the inverse of $\mathbf{T}_{F,n,-1}$, we split the inverse $\mathbf{T}_{F,n,-1}^{-1}$ as the following form

$$\mathbf{T}_{F,n,-1}^{-1} = \hat{\mathbf{I}}_n (A_n \mathbf{T}_{F,n,-1} \hat{\mathbf{I}}_n)^{-1} A_n. \tag{3.14}$$

The matrix $A_n \mathbf{T}_{F,n,-1} \hat{\mathbf{I}}_n$ in (3.2) is partitioned as follows

$$A_n \mathbf{T}_{F,n,-1} \hat{\mathbf{I}}_n = \left(\begin{array}{c:c} M & N \\ \hdashline Q & Y \end{array}\right)$$

$$
= \left(\begin{array}{cc:cccccc}
F_1 & -F_{-2} & -F_{-3} & -F_{-4} & \cdots & -F_{-(n-2)} & -F_{-(n-1)} & -F_{-n} \\
F_2 & F_1 & -F_{-2} & -F_{-3} & \cdots & -F_{-(n-3)} & -F_{-(n-2)} & -F_{-(n-1)} \\
\hdashline
0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & -1 & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & 0 & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \ddots & 2 & 0 \\
0 & 0 & 0 & \cdots & \cdots & 0 & -1 & 2
\end{array}\right)_{n\times n}. \tag{3.15}
$$

By Lemma 1 in [11], we have

$$
Y^{-1} = (x_{i,j})_{(n-2)\times(n-2)}, \tag{3.16}
$$

where

$$
x_{i,j} = \begin{cases} x_{i-j+1}, \ 1 \le j \le i \le n-2, \\ 0, \ \text{else}, \end{cases}
$$

in which

$$
x_i = \frac{1}{2^i}, i = 1, 2, \ldots, n-2.
$$

Using the equation (1.4) and $F_{n-k-1} + F_{n-k+1} = L_{n-k}$ where $k$ is an integer, we have

$$
(M - NY^{-1}Q)^{-1} = \begin{pmatrix} zw & -zu \\ -z & z \end{pmatrix}, \tag{3.17}
$$

where

$$
u = \frac{2 + (-\frac{1}{2})^{n-2}L_n}{5}, \ w = \frac{6 - (-\frac{1}{2})^{n-2}L_{n-1}}{5}, \ z = \frac{5}{4 - (-\frac{1}{2})^{n-2}L_{n+1}}.
$$

From (3.15), Theorem in [23, p.19], (3.16), (3.17), (7-10) and the relation between the Fibonacci number and Lucas number, we obtain

$$
(A_n \mathbf{T}_{F,n,-1}\hat{\mathbf{I}}_n)^{-1} = (y_{i,j})_{n\times n}, \tag{3.18}
$$

where

$$
\begin{aligned}
y_{1,1} &= \frac{6 - (-\frac{1}{2})^{n-2}L_{n-1}}{4 - (-\frac{1}{2})^{n-2}L_{n+1}}, \\
y_{1,2} &= -\frac{2 + (-\frac{1}{2})^{n-2}L_n}{4 - (-\frac{1}{2})^{n-2}L_{n+1}}, \\
y_{1,j} &= \frac{(-1)^{j+1}[2F_j + (-\frac{1}{2})^{n-j}F_{n+1} + (\frac{1}{2})^{n-2}F_{j-1-n}]}{4 - (-\frac{1}{2})^{n-2}L_{n+1}}, \qquad j = 3, 4, \ldots, n,
\end{aligned}
$$

$$y_{2,1} = -\frac{5}{4-(-\frac{1}{2})^{n-2}L_{n+1}},$$
$$y_{2,2} = \frac{5}{4-(-\frac{1}{2})^{n-2}L_{n+1}},$$
$$y_{2,j} = \frac{(-1)^j[L_j-(-\frac{1}{2})^{n+1-j}L_{n+1}]}{4-(-\frac{1}{2})^{n-2}L_{n+1}}, \qquad j=3,4,\ldots,n,$$
$$y_{i,1} = -\frac{5}{2^i+(-1)^{n-1}(\frac{1}{2})^{n-i}L_{n+1}}, \qquad i=3,4,\ldots,n,$$
$$y_{i,2} = \frac{5}{2^i+(-1)^{n-1}(\frac{1}{2})^{n-i}L_{n+1}}, \qquad i=3,4,\ldots,n,$$
$$y_{k,j} = \begin{cases} \frac{2^k+(-1)^j 2^{k-j+1}L_j}{2^{2k-j+1}+(-1)^{n-1}(\frac{1}{2})^{n-2k+j-1}L_{n+1}}, & k \geq j, k, j=3,4,\ldots,n, \\ \\ \frac{(-1)^j[L_j-(-\frac{1}{2})^{n+1-j}L_{n+1}]}{2^k+(-1)^{n-1}(\frac{1}{2})^{n-k}L_{n+1}}, & k<j, k, j=3,4,\ldots,n. \end{cases}$$

According to formulas (2.2), (3.14), (3.18) and the relation between the Fibonacci number and Lucas number, we can obtain the inverse of $\mathbf{T}_{F,n,-1}$ as (3.5). □

**Remark 3.2.** We note that $\mathbf{T}_{F,n,-1}^{-1}$ is a symmetric matrix with respect to secondary diagonal, i.e., a sub-symmetric matrix. In this situation, we only need to work out 7 entries and it is easy to compute the inverse of $\mathbf{T}_{F,n,-1}$ by (3.5).

## 4. Determinants and inverses of skew Lankel and skew Fankel matrices

In this section, based on the relation between skew Loeplitz and skew Lankel matrices, we calculate the determinant and inverse of skew Lankel matrix. Also, we get the corresponding results of skew Fankel matrix.

Now, we show the definitions of the matrices $\mathbf{H}_{L,n,-1}$ and $\mathbf{H}_{F,n,-1}$ given as follows.

A skew Lankel matrix is a Hankel matrix of the form

$$\mathbf{H}_{L,n,-1} = (h_{i,j})_{n\times n}, \tag{4.1}$$

where

$$h_{i,j} = \begin{cases} L_{n-i-j+2}, & 2 \leq i+j \leq n+1, \\ -L_{-(i+j-n)}, & n+1 < i+j \leq 2n, \end{cases}$$

and $L_1, L_{\pm 2}, \ldots, L_{\pm n}$ are Lucas numbers.

A skew Fankel matrix is a Hankel matrix of the form

$$\mathbf{H}_{F,n,-1} = (g_{i,j})_{n\times n}, \tag{4.2}$$

where

$$g_{i,j} = \begin{cases} F_{n-i-j+2}, & 2 \leq i+j \leq n+1, \\ -F_{-(i+j-n)}, & n+1 < i+j \leq 2n, \end{cases}$$

and $F_1, F_{\pm 2}, \ldots, F_{\pm n}$ are Fibonacci numbers.

It is easy to check that

$$\mathbf{H}_{L,n,-1} = \mathbf{T}_{L,n,-1}\hat{\mathbf{I}}_n, \tag{4.3}$$

$$\mathbf{H}_{F,n,-1} = \mathbf{T}_{F,n,-1}\hat{\mathbf{I}}_n, \tag{4.4}$$

where $\hat{\mathbf{I}}_n$ is the reverse unit matrix.

**Theorem 4.1.** *Let $\mathbf{H}_{L,n,-1}$ be the $n \times n$ invertible skew Lankel matrix given in (4.1). Then we have*

$$\det \mathbf{H}_{L,n,-1} = (-1)^{\frac{n(n-1)}{2}} 5^{n-1} F_{n+1}. \tag{4.5}$$

**Proof.** *From* (4.3), *it follows that* $\det \mathbf{H}_{L,n,-1} = \det \mathbf{T}_{L,n,-1} \det \hat{\mathbf{I}}_n$. *Then we can obtain this result by using Theorem 2.1 and* $\det \hat{\mathbf{I}}_n = (-1)^{\frac{n(n-1)}{2}}$. □

**Theorem 4.2.** *Let $\mathbf{H}_{L,n,-1}$ be the $n \times n$ invertible skew Lankel matrix given in (4.1) for a positive integer $n > 2$. Then $\mathbf{H}_{L,n,-1}^{-1}$ is of the form*

$$\mathbf{H}_{L,n,-1}^{-1} = \hat{\mathbf{I}}_n \mathbf{T}_{L,n,-1}^{-1}, \tag{4.6}$$

*where $\mathbf{T}_{L,n,-1}^{-1}$ is the same as Theorem 2.2.*

**Proof.** We can obtain this conclusion by using (4.3) and Theorem 2.2. □

**Theorem 4.3.** *Let $\mathbf{H}_{F,n,-1}$ be the $n \times n$ skew Fankel matrix given in (4.2) and $n \geq 3$. We have*

$$\det \mathbf{H}_{F,n,-1} =(-1)^{\frac{n(n+1)}{2}} \frac{[(-2)^n - L_{n+1}]}{5}. \tag{4.7}$$

**Proof.** It follows from (4.4) that $\det \mathbf{H}_{F,n,-1} = \det \mathbf{T}_{F,n,-1} \det \hat{\mathbf{I}}_n$. Then we obtain the desired result by using Theorem 3.1 and $\det \hat{\mathbf{I}}_n = (-1)^{\frac{n(n-1)}{2}}$. □

**Theorem 4.4.** *Let $\mathbf{H}_{F,n,-1}$ be the $n \times n$ invertible skew Fankel matrix given in (4). Then $\mathbf{H}_{F,n,-1}^{-1}$ is of the form*

$$\mathbf{H}_{F,n,-1}^{-1} = \hat{\mathbf{I}}_n \mathbf{T}_{F,n,-1}^{-1}, \tag{4.8}$$

*where $\mathbf{T}_{F,n,-1}^{-1}$ is the same as in Theorem 3.2.*

**Proof.** We can obtain this conclusion by using (4.4) and Theorem 3.2. □

# 5. An algorithm for the inverses of the matrices $\mathbf{T}_{F,n,-1}$ and $\mathbf{H}_{F,n,-1}$

In this section, we demonstrate an algorithm for finding the inverses of $\mathbf{T}_{F,n,-1}$ and $\mathbf{H}_{F,n,-1}$.

An algorithm for finding $\mathbf{T}_{F,n,-1}^{-1}$ and $\mathbf{H}_{F,n,-1}^{-1}$ is as follows:

**Algorithm:** By Theorem 3.2 and Theorem 4.4, we proceed with

**Step 1**: Compute $p_i$ $(i = 1,\ 2,\ \ldots,\ 7)$ via the formulas (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12), respectively;

**Step 2**: By the formula (3.5), we obtain $\mathbf{T}_{F,n,-1}^{-1}$;

**Step 3**: By the formula (4.8), we obtain $\mathbf{H}_{F,n,-1}^{-1}$.

## References


[1] M. Akbulak and D. Bozkurt, *On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers*, Hacet. J. Math. Stat., 2008, 37(2), 89–95.

[2] D. Bozkurt and T. Y. Tam, *Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas Numbers*, Appl. Math. Comput., 2012, 219, 544–551.

[3] J. Chen, *Determinants and inverses of symmetric Poeplitz and Qoeplitz matrix*, J. Adv. Math. Comput. Sci., 2017, 24(5), 1–20.

[4] X. Chen, *Exact determinants and inverses of skew symmetric generalized Loeplitz matrices*, J. Adv. Math. Comput. Sci., 2019, 33(6), 1–11.

[5] X. Chen, *Determinants and inverses of skew symmetric generalized Foeplitz matrices*, J. Adv. Math. Comput. Sci., 2019, 33(4), 1–12.

[6] Q. Feng and F. Meng, *Explicit solutions for space-time fractional partial differential equations in mathematical physics by a new generalized fractional Jacobi elliptic equation-based sub-equation method*, Optik., 2016, 127, 7450–7458.

[7] M. Han and Z. Jiang, *Determinant and inverse of skew Peoeplitz matrix*, J. Adv. Math. Comput. Sci., 2018, 28(4), 1–21.

[8] X. Jiang and K. Hong, *Explicit inverse matrices of Tribonacci skew circulant type matrices*, Appl. Math. Comput., 2015, 268, 93–102.

[9] Z. Jiang, Y. Gong and Y. Gao, *Invertibility and explicit inverses of circulant-type matrices with k-Fibonacci and k-Lucas number*, Abstr. Appl. Anal., 2014, 238953.

[10] Z. Jiang and J. Sun, *Determinant and inverse of a Gaussion Fibonacci skew-Hermitian Toeplitz matrix*, J. Nonlinear Sci. Appl., 2017, 10, 3694–3707.

[11] Z. Jiang, W. Wang, Y. Zheng, B. Zuo and B. Niu, *Interesting explicit expression of determinants and inverse matrices for Foeplitz and Loeplitz matrices*, Mathematics, 2019, 7(10), 939.

[12] L. Liu and Z. Jiang, *Explicit form of the inverse matrices of Tribonacci circulant type matrices*, Abstr. Appl. Anal., 2015, 169726.

[13] R. Malti and M. Thomassin, *Differentiation similarities in fractional pseudo-state space representations and the subspace-based methods*, Fract. Calc. Appl. Anal., 2013, 16(1), 273–287.

[14] J. Shao, Z. Zheng and F. Meng, *Oscillation criteria for fractional differential equations with mixed nonlinearities*, Adv. Differ. Equ., 2013, 323.

[15] S. Shen, J. Cen and Y. Hao, *On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers*, Appl. Math. Comput., 2015, 217, 9790–9797.

[16] S. Shen, W. Liu and L. Feng, *Explicit inverses of generalized Tribonacci circulant type matrices*, Hacet. J. Math. Stat., 2019, 48(3), 689–699.

[17] Y. Sun and F. Meng, *Interval criteria for oscillation of second-order differential equations with mixed nonlinearities*, Appl. Math. Comput., 2008, 198, 375–381.

[18] K. Thomas, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, 2001.

[19] N. X. Thao, V. K. Tuan and N. Hong, *Generalized convolution transforms and Toeplitz plus Hankel integral equation*, Fract. Calc. Appl. Anal., 2008, 11(2), 153–174.

[20] J. Wang and F. Meng, *Interval oscillation criteria for second order partial differential systems with delays*, J. Comput. Appl. Math., 2008, 212, 397–405.

[21] R. Xu and F. Meng, *Some new weakly singular integral inequalities and their applications to fractional differential equations*, Journal of Inequalities and Applications, 2016, 1, 1–16.

[22] B. Zuo, Z. Jiang and D. Fu, *Determinants and inverses of Ppoeplitz and Ppankel matrices*, Special Matrices, 2018, 6, 201–215.

[23] F. Zhang, *The Schur Complement and Its Applications*, Springer Science & Business Media, 2006.

[24] Y. Zheng and S. Shon, *Exact determinants and inverses of generalized Lucas skew circulant type matrices*, Appl. Math. Comput., 2015, 270, 105–113.