Journal of Applied Analysis and Computation
Volume 11, Number 6, December 2021, 2981–3000
Website:http://www.jaac-online.com
DOI:10.11948/20210086

APPROXIMATE CONTROLLABILITY OF SOBOLEV TYPE FRACTIONAL EVOLUTION EQUATIONS OF ORDER $\alpha \in (1, 2)$ VIA RESOLVENT OPERATORS*

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Abstract In this paper, the existence and approximate controllability of mild solutions for $\alpha \in (1, 2)$-order fractional evolution equations of Sobolev type are investigated in abstract spaces. Firstly, we introduce a new concept of mild solution of the concerned problem. Then by using fixed point theorems and the theory of resolvent operator, some existence results are obtained. At last, the approximate controllability of the $\alpha \in (1, 2)$-order fractional evolution equation is proved without assuming the approximate controllability of corresponding linear problem. An example is presented in the last section to illustrate the obtained abstract results.

Keywords Approximate controllability, fractional evolution equations, fractional resolvent family, existence, nonlocal condition.

MSC(2010) 47A10, 93B05.

1. Introduction

Let H be a Hilbert space endowed with the norm $\|\cdot\|$. In the present work, we investigate the approximate controllability of Sobolev type fractional control system with nonlocal conditions of the form

$$\begin{cases} {}^C D_t^\alpha (Ex)(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J := [0, b], \\ Ex(0) = x_0 - g(x), \quad (Ex)'(0) = y_0 - h(x), \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $A : D(A) \subset H \to H$ is a densely defined and closed linear operator in H and $E : D(E) \subset H \to H$ is a closed linear operator, the control u is given in $L^2(J, U)$, U is a Hilbert space, B is a bounded linear operator from U to H, f, g and h are appropriate functions to be specified later.

In recent years, fractional differential equations have been regarded as one of the most powerful tools in modeling many phenomena in various fields, such as chemistry, biology, physics, control theory, etc. See [1] for more details.

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Controllability of fractional systems is becoming more active both in control theory and mathematics. A large number of researches focused on the controllability of $\alpha \in (0,1)$-order fractional evolution equations, see the papers [5, 6, 12, 14]. However, the researches on the fractional evolution equations of order $1 < \alpha < 2$ are seldom. In 2013, Li et al. [9] proved the exact controllability of $\alpha \in (1,2]$-order Caputo fractional differential system

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J, \\ x(0) + g(x) = x_0, \quad x'(0) = y_0, \end{cases} \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous α-order cosine family $\{C_\alpha(t)\}_{t\geq 0}$ in a Banach space X. By utilizing Sadovskii's fixed point theorem and vector-valued operator theory, the authors in [9] proved the exact controllability of control system (1.2). In 2017, Lian et al. [11] studied the existence and approximate controllability of fractional evolution equation of order $\alpha \in (1,2)$

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J, \\ x(0) + g(x) = x_0, \quad x'(0) + h(x) = y_0, \end{cases} \tag{1.3}$$

where A generates a strongly continuous α-order cosine family $\{C_\alpha(t)\}_{t\geq 0}$ in a Hilbert space X. By using Schauder's fixed point theorem and approximate technique, They showed the existence and approximate controllability of fractional control system (1.3).

Sobolev type differential equations are applied to model many physical phenomena, hence they have received great attention in recent years. The fundamental theory of Sobolev type fractional differential equations of order $\alpha \in (0,1)$ has been established. Benchaabane et al. [2] established a set of sufficient conditions for the existence and uniqueness of mild solutions to a class of nonlinear fractional Sobolev type stochastic differential equations in Hilbert spaces. Li et al. [10] studied the existence of mild solutions for fractional integro-differential equations of Sobolev type with nonlocal condition in a separable Banach space. In 2013, Fečkan et al. [7] concerned with the exact controllability of $\alpha \in (0,1)$-order fractional functional evolution equation of Sobolev type in Banach space X

$$\begin{cases} {}^C D_t^\alpha (Ex)(t) + Ax(t) = f(t, x_t) + Bu(t), \quad t \in J, \\ x(t) = \vartheta(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.4}$$

where $\vartheta \in C([-\tau, 0], D(E))$, A and E satisfy the following assumptions:

(A1) A and E are linear operators, and A is closed.

(A2) $D(E) \subset D(A)$ and E is bijective.

(A3) Linear operator E^{-1} is compact.

In this case, the linear operator $-AE^{-1}$ is bounded and generates a uniformly continuous semigroup $\{T(t)\}_{t\geq 0}, T(t) := e^{-AE^{-1}}$. By using Schauder's fixed point theorem, they proved the exact controllability of fractional functional evolution equation (1.4). In 2017, Chang et al. [3] treated the approximate controllability of $\alpha \in (1, 2)$-order fractional differential system of Sobolev type in Banach space X

$$\begin{cases} {}^C D_t^\alpha (Ex)(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J, \\ (Ex)(0) = Ex_0, \quad (Ex)'(0) = Ey_0, \end{cases} \tag{1.5}$$

where the pair (A, E) generates an (α, β)-resolvent family $\{C^E_{\alpha,\beta}(t)\}_{t \geq 0}$ for suitable constants $\alpha, \beta > 0$ (which is first introduced by Ponce in [13]). Later, Chang et al. [4] extended the fractional differential system (1.5) to the stochastic case. By setting up minimizing sequences twice, they established the optimal state-control pair of the limited Lagrange optimal systems governed by the $\alpha \in (1, 2)$-order fractional stochastic differential equation. Very recently, In our work [16], we investigated the nonlocal controllability of fractional control system (1.1) when g and h are Lipschitz continuous. However, to the best of our knowledge that there is no work reported on the approximately controllable of Sobolev type fractional evolution system (1.1) of order $\alpha \in (1, 2)$.

In this paper, with the definition of fractional resolvent family (see Definition 2.1) generated by the pair (A, E)(which is first introduced by Ponce in [13]), we remove the assumptions $(A2), (A3)$ of [7] on operators A and E, and present the concept of mild solution of (1.1) by using Laplace Transform. Secondly, we investigate the existence of mild solutions of the fractional control system (1.1) when the nonlocal function g is Lipschitz continuous, or completely continuous, or continuous. At last, some approximate controllability results are proved without assuming the approximate controllability of corresponding linear system. It is worth to emphasize that throughout this paper we do not assume any compactness on the nonlocal function h. An example is given in the last section to illustrate the application of the abstract results.

2. Preliminaries

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Then the H-valued continuous functions set $C(J, H)$ is a Banach space with the norm $\|x\|_C = \sup_{t \in J} \|x(t)\|$. Let $L^p(J, H)(1 \leq p < +\infty)$ be the H-valued p-order Bochner integrable functions set, which is a Banach space with the norm $\|f\|_{L^p} = (\int_0^b \|f(t)\|^p dt)^{\frac{1}{p}}$. We denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators from X to Y, $\mathcal{B}(X) := \mathcal{B}(X, X)$ for short.

For $\alpha > 0$, let $n := \lceil \alpha \rceil$ denote the smallest integer greater than or equal to α. If $u \in C^n(J, H)$ and the integral $\int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds$ exists in the Bochner sense, then we adopt the definition of Caputo fractional derivative as the following:

$${}^C D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, \quad t > 0, \alpha \in (n-1, n).$$

For more details of the fractional calculus, we refer to [1].

From (1.23) of [1], Laplace Transform of Caputo fractional derivatives is given by

$$\widehat{{}^C D_t^\alpha} u(\lambda) = \lambda^\alpha \widehat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0) \lambda^{\alpha-1-k} \tag{2.1}$$

where $\alpha > 0$ and $n = \lceil \alpha \rceil$.

Define a set $\Phi_E(A)$ by $\Phi_E(A) := \{\lambda \in \mathbb{C} \mid (\lambda E - A) : D(A) \cap D(E) \to H$ is invertible and $(\lambda E - A)^{-1} \in \mathcal{B}(H, D(A) \cap D(E))\}$.

Let $R(\lambda E, A) := (\lambda E - A)^{-1}$. Then $R(\lambda^\alpha E, A) := (\lambda^\alpha E - A)^{-1}$. A strongly continuous family $\{T(t)\}_{t \geq 0} \subset \mathcal{B}(H)$ is said to be exponentially bounded if there are constants $N \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq N e^{\omega t}, \qquad t \geq 0.$$

Definition 2.1. Let A and E be two closed linear operators in H, whose domains are $D(A)$ and $D(E)$ satisfying $D(A) \cap D(E) \neq \{0\}$. For $\alpha, \beta > 0$, if there exist a constant $\omega \geq 0$ and a strongly continuous function $C^E_{\alpha,\beta} : [0, \infty) \to \mathcal{B}(H)$ such that $C^E_{\alpha,\beta}(t)$ is exponentially bounded, $\{\lambda^\alpha : Re\lambda > \omega\} \subset \Phi_E(A)$, and for all $x \in H$,

$$\lambda^{\alpha-\beta} R(\lambda^\alpha E, A)x = \int_0^\infty e^{-\lambda t} C^E_{\alpha,\beta}(t)x dt, \qquad Re\lambda > \omega, \tag{2.2}$$

then we call $\{C^E_{\alpha,\beta}(t)\}_{t \geq 0}$ the (α, β)-resolvent family generated by the pair (A, E).

For $1 < \beta \leq 2$ and $\alpha > 0$, it follows from [4, 13] that

$$C^E_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta - 1)} \int_0^t (t - s)^{\beta-2} C^E_{\alpha,1}(s) ds, \qquad t > 0. \tag{2.3}$$

Definition 2.2. The (α, β)-resolvent family $\{C^E_{\alpha,\beta}(t)\}_{t \geq 0}$ is said to be compact, if $C^E_{\alpha,\beta}(t)$ is a compact operator for $t > 0$.

Now, we derive the appropriate definition of mild solutions of (1.1). Let $\{C^E_{\alpha,1}(t)\}_{t \geq 0}$ be the $(\alpha, 1)$-resolvent family generated by the pair (A, E). Then $\{C^E_{\alpha,1}(t)\}_{t \geq 0}$ is exponentially bounded. Define

$$S^E_{\alpha,1}(t) := \int_0^t C^E_{\alpha,1}(s) ds, \qquad t \geq 0 \tag{2.4}$$

and

$$P^E_{\alpha,1}(t) := \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha-2} C^E_{\alpha,1}(s) ds, \qquad t \geq 0. \tag{2.5}$$

Then by (2.2), we have

$$\lambda^{\alpha-1} R(\lambda^\alpha E, A)x = \int_0^\infty e^{-\lambda t} C^E_{\alpha,1}(t)x dt, \qquad Re\lambda > \omega,\ x \in H. \tag{2.6}$$

In view of (2.3), (2.4) and (2.6), we get

$$\lambda^{\alpha-2} R(\lambda^\alpha E, A)x = \int_0^\infty e^{-\lambda t} S^E_{\alpha,1}(t)x dt, \qquad Re\lambda > \omega,\ x \in H. \tag{2.7}$$

From (2.5), we have

$$\begin{aligned}\int_0^\infty e^{-\lambda t} P^E_{\alpha,1}(t)x dt =& \frac{1}{\Gamma(\alpha - 1)} \int_0^\infty e^{-\lambda t} \int_0^t (t - s)^{\alpha-2} C^E_{\alpha,1}(s)x ds dt \\ =& \frac{1}{\Gamma(\alpha - 1)} \int_0^\infty \int_s^\infty e^{-\lambda t} (t - s)^{\alpha-2} C^E_{\alpha,1}(s)x dt ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha-1)} \int_0^\infty C_{\alpha,1}^E(s) \int_0^\infty e^{-\lambda(\theta+s)} \theta^{\alpha-2} x d\theta ds \\
&= \frac{1}{\lambda^{\alpha-1}} \int_0^\infty e^{-\lambda s} C_{\alpha,1}^E(s) x ds.
\end{aligned}
$$

Combining this fact with (2.6), we obtain that

$$
R(\lambda^\alpha E, A)x = \int_0^\infty e^{-\lambda t} P_{\alpha,1}^E(t) x dt, \qquad Re\lambda > \omega,\ x \in H. \tag{2.8}
$$

Applying Laplace Transform to (1.1), it follows from (2.1) that

$$
\begin{aligned}
\widehat{{}^C D_t^\alpha (Ex)}(\lambda) &= \lambda^\alpha E \widehat{x}(\lambda) - \lambda^{\alpha-1} Ex(0) - \lambda^{\alpha-2}(Ex)'(0) \\
&= A\widehat{x}(\lambda) + \widehat{F}(\lambda) + B\widehat{u}(\lambda).
\end{aligned}
$$

By virtue of (2.6)-(2.8), we deduce that

$$
\begin{aligned}
\widehat{x}(\lambda) =& \lambda^{\alpha-1} R(\lambda^\alpha E, A) Ex(0) + \lambda^{\alpha-2} R(\lambda^\alpha E, A)(Ex)'(0) \\
&+ R(\lambda^\alpha E, A)[\widehat{F}(\lambda) + B\widehat{u}(\lambda)] \\
=& \int_0^\infty e^{-\lambda t} C_{\alpha,1}^E(t) Ex(0) dt + \int_0^\infty e^{-\lambda t} S_{\alpha,1}^E(t)(Ex)'(0) dt \\
&+ \int_0^\infty e^{-\lambda t} P_{\alpha,1}^E(t)[\widehat{F}(\lambda) + B\widehat{u}(\lambda)] dt \\
=& \int_0^\infty e^{-\lambda t} C_{\alpha,1}^E(t) Ex(0) dt + \int_0^\infty e^{-\lambda t} S_{\alpha,1}^E(t)(Ex)'(0) dt \\
&+ \int_0^\infty e^{-\lambda t} P_{\alpha,1}^E(t)\Big[\int_0^\infty e^{-\lambda s} f(s, x(s)) ds\Big] dt + \int_0^\infty e^{-\lambda t} P_{\alpha,1}^E(t) B\widehat{u}(\lambda) dt \\
=& \int_0^\infty e^{-\lambda t} C_{\alpha,1}^E(t) Ex(0) dt + \int_0^\infty e^{-\lambda t} S_{\alpha,1}^E(t)(Ex)'(0) dt \\
&+ \int_0^\infty \int_s^\infty e^{-\lambda \theta} P_{\alpha,1}^E(\theta - s) f(s, x(s)) d\theta ds + \int_0^\infty e^{-\lambda t} P_{\alpha,1}^E(t) B\widehat{u}(\lambda) dt \\
=& \int_0^\infty e^{-\lambda t} C_{\alpha,1}^E(t) Ex(0) dt + \int_0^\infty e^{-\lambda t} S_{\alpha,1}^E(t)(Ex)'(0) dt \\
&+ \int_0^\infty e^{-\lambda t} \int_0^t P_{\alpha,1}^E(t-s) f(s, x(s)) ds dt \\
&+ \int_0^\infty e^{-\lambda t} \int_0^t P_{\alpha,1}^E(t-s) Bu(s) ds dt.
\end{aligned}
$$

Hence, the inverse of Laplace Transform yields

$$
x(t) = C_{\alpha,1}^E(t)[x_0 - g(x)] + S_{\alpha,1}^E(t)[y_0 - h(x)] + \int_0^t P_{\alpha,1}^E(t-s)[f(s, x(s)) + Bu(s)] ds,\ t \in J. \tag{2.9}
$$

Based on the above discussion, we now give the definition of mild solution of (1.1) below.

Definition 2.3. A function $x \in C(J, H)$ is called the mild solution of (1.1) if it satisfies the integral equation (2.9).

In the remaining of this work, we make the following assumption on the pair (A, E).

(H_{AE}) The pair (A, E) generates an $(\alpha, 1)$-resolvent family $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$, which is compact and norm continuous for all $t > 0$. Put

$$M := \sup_{t\in J} \|C^E_{\alpha,1}(t)\| < +\infty.$$

Remark 2.1. By Proposition 16 of [13], let $1 < \alpha < 2$ and $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$ be the $(\alpha, 1)$-resolvent family generated by the pair (A, E). If $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$ is continuous in the uniform operator topology for all $t > 0$, then $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$ is a compact operator for all $t > 0$ if and only if $(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{\frac{1}{\alpha}}$.

Lemma 2.1 (Lemma 1, [16]). *Let the assumption (H_{AE}) hold. Then for any $t \in J$ and $x \in H$, we have*

$$\|S^E_{\alpha,1}(t)x\| \leq Mb\|x\|, \qquad \|P^E_{\alpha,1}(t)x\| \leq \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|x\|.$$

The Lemmas 3.4 and 3.5 of [6] imply the following result.

Lemma 2.2. *Let the assumption (H_{AE}) hold. Then*

(i) $\lim_{\xi\to 0^+} \|C^E_{\alpha,1}(t+\xi) - C^E_{\alpha,1}(\xi)C^E_{\alpha,1}(t)\| = 0, \qquad \forall t > 0;$

(ii) $\lim_{\xi\to 0^+} \|C^E_{\alpha,1}(t) - C^E_{\alpha,1}(\xi)C^E_{\alpha,1}(t-\xi)\| = 0, \qquad \forall t > 0.$

Lemma 2.3. *Let the assumption (H_{AE}) hold. Then $S^E_{\alpha,1}(t)$ and $P^E_{\alpha,1}(t)$ are compact operators for $t \geq 0$.*

Proof. By using Lemma 2.2 and a similar approach employed in the proof of Lemma 2.10 of [11], we can derive the conclusion of this lemma. So we omit the detail here. □

Lemma 2.4 (Lemma 2, [16]). *Let the assumption (H_{AE}) hold. The operators $S^E_{\alpha,1}(t)$ and $P^E_{\alpha,1}(t)$ are equi-continuous for $t \in J$.*

Next, we introduce the definition of approximate controllability which is used in this paper.

Definition 2.4. The control system (1.1) is said to be approximately controllable on J if there exists a control $u \in L^2(J, U)$ such that $\overline{R_b(f)} = H$, where $R_b(f) = \{x(b; u) : x \text{ is the mild solution of (1.1) on } J \text{ for some } u \in L^2(J, U)\}$.

Consider the fractional linear control system corresponding to (1.1) of the form

$$\begin{cases} {}^C D^\alpha_t (Ex)(t) = Ax(t) + Bu(t), \qquad t \in J, \\ Ex(0) = x_0, \quad (Ex)'(0) = y_0, \end{cases} \tag{2.10}$$

where $1 < \alpha < 2$ and $x_0, y_0 \in H$.

For the sake of simplicity, we denote

$$\Psi^b_0 := \int_0^b P^E_{\alpha,1}(b-s)BB^*(P^E_{\alpha,1})^*(b-s)ds,$$

$$R(\delta, \Psi_0^b) := (\delta I + \Psi_0^b)^{-1},$$

where B^* and $(P_{\alpha,1}^E)^*(t)$ are the adjoint operators of B and $P_{\alpha,1}^E(t)$, respectively. For the controllability of linear system (2.10), we give the following lemma.

Lemma 2.5 (Theorem 2.3, [12]). *The following statements are equivalent:*

(i) The linear control system (2.10) *is approximately controllable;*

(ii) The operator Ψ_0^b is positive, i.e. $\langle x^*, \Psi_0^b x^*\rangle > 0$ for all nonzero $x^* \in H^*$;

(iii) For any $x \in H$, $\|\delta R(\delta, \Psi_0^b)x\| \to 0$ as $\delta \to 0^+$ in the strong topology.

Remark 2.2. From *(iii)* of Lemma 2.5, without loss of generality, we suppose that $\|R(\delta, \Psi_0^b)\| \le \frac{1}{\delta}$ for all $\delta > 0$.

Let us define a bounded linear operator $G_b : L^2(J, H) \to H$ by

$$G_b\sigma = \int_0^b P_{\alpha,1}^E(b-s)\sigma(s)ds, \qquad \sigma \in L^2(J, H).$$

We suppose the following:

(H_B) For every $\sigma \in L^2(J, H)$, there is a function $\xi \in \overline{K(B)}$ such that

$$G_b\sigma = G_b\xi,$$

where $K(B)$ means the range of B and $\overline{K(B)}$ is its closure.

(H_C) For each $x \in H$, one has

$$C_{\alpha,1}^E(t)x \in D(A), \qquad S_{\alpha,1}^E(t)x \in D(A), \qquad t > 0.$$

By the definition of $P_{\alpha,1}^E(t)$, we have

$$\frac{d(P_{\alpha,1}^E(t))^2}{dt}x = 2P_{\alpha,1}^E(t)\frac{dP_{\alpha,1}^E(t)}{dt}x, \qquad t > 0.$$

Lemma 2.6. *Let the assumptions (H_B) and (H_C) hold. Then the linear control system* (2.10) *is approximately controllable on J.*

Proof. The idea comes from [8, 15]. Since the approximate controllability of system (2.10) is equivalent to $\overline{R_b(0)} = H$, it is sufficient to prove $D(A) \subset \overline{R_b(0)}$ due to $\overline{D(A)} = H$, i.e., for any $\epsilon > 0$ and $\rho \in D(A)$, there is a control function $u \in L^2(J, U)$ such that

$$\|\rho - C_{\alpha,1}^E(b)x_0 - S_{\alpha,1}^E(b)y_0 - G_bBu\| < \epsilon.$$

For any given $\rho \in D(A)$, we see that there is a function $\sigma \in L^2(J, H)$ such that $G_b\sigma = \rho - C_{\alpha,1}^E(b)x_0 - S_{\alpha,1}^E(b)y_0$, for example,

$$\sigma(t) = \Big[\frac{1}{bP_{\alpha,1}^E(b-t)} - P_{\alpha,1}^E(b-t) + 2t\frac{dP_{\alpha,1}^E(b-t)}{dt}\Big]\Big[\rho - C_{\alpha,1}^E(b)x_0 - S_{\alpha,1}^E(b)y_0\Big], t \in J.$$

In view of (H_B), for this $\sigma \in L^2(J, H)$, there is a function $\xi \in \overline{K(B)}$ such that

$$G_b\sigma = G_b\xi.$$

Since $\xi \in \overline{K(B)}$, for any $\epsilon > 0$, there is a control function $u \in L^2(J, U)$ such that

$$\|\xi - Bu\| < \frac{\Gamma(\alpha)}{Mb^{\alpha-\frac{1}{2}}}\epsilon.$$

Consequently, for any $\epsilon > 0$ and $\rho \in D(A)$, there is a control function $u \in L^2(J, U)$ satisfying

$$\begin{aligned}&\|\rho - C^E_{\alpha,1}(b)x_0 - S^E_{\alpha,1}(b)y_0 - G_b Bu\|\\ =&\|G_b\sigma - G_b Bu\|\\ =&\|G_b\xi - G_b Bu\|\\ =&\|\int_0^b P^E_{\alpha,1}(b-s)[\xi(s) - Bu(s)]ds\|\\ \leq&\frac{Mb^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\|\eta - Bu\|_{L^2}\\ <&\epsilon.\end{aligned}$$

This completes the proof. □

To end of this section, we present a fixed point theorem, on which the proofs of main results are based.

Lemma 2.7 (Sadovskii's Fixed Point Theorem)**.** *Let Q be a condensing operator on Banach space X, i.e. Q is continuous and takes bounded sets into bounded sets, and $\gamma(Q(D)) < \gamma(D)$ for every bounded set D of X with $\gamma(D) > 0$. If $Q(S) \subset S$ for a convex closed and bounded subset S of X, then Q has at least one fixed point in S (where $\gamma(\cdot)$ denotes the Kuratowski measure of non-compactness).*

3. Existence of mild solutions

3.1. The case g is Lipschitz continuous

In order to prove the existence of mild solutions of (1.1), we first make the following assumptions:

(H_1) $f : J \times H \to H$ satisfies the Carath*é*odory condition, i.e., for each $x \in H$, the function $f(\cdot, x) : J \to H$ is strongly measurable; for each $t \in J$, the function $f(t, \cdot) : H \to H$ is continuous.

(H_2) For every $r > 0$, there is a function $\varphi_r \in L^1(J, \mathbb{R}^+)$ satisfying $\lim\limits_{r\to\infty} \frac{\|\varphi_r\|_{L^1}}{r} = \sigma_1 < +\infty$ such that

$$\sup_{\|x\|\leq r} \|f(t, x)\| \leq \varphi_r(t), \qquad t \in J.$$

(H_3) $g : C(J, H) \to H$ and there exists a constant $L_g > 0$ such that

$$\|g(x) - g(y)\| \leq L_g\|x - y\|_C$$

for all $x, y \in C(J, H)$.

(H_4) $h : C(J, H) \to H$ is continuous and there exists a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim\limits_{r\to\infty} \frac{\psi(r)}{r} = \sigma_2 < +\infty$ such that

$$\|h(x)\| \leq \psi(\|x\|_C)$$

for all $x \in C(J,H)$.

(H_5) $B : U \to H$ is a linear bounded operator, and put $M_B := \|B\|$.

For each $r > 0$, set $B_r := \{x \in C(J,H) : \|x(t)\| \leq r, t \in J\}$. Then B_r is clearly a closed, bounded, convex subset of $C(J,H)$. For every $\delta > 0$, $x_b \in H$ and $x \in C(J,H)$, we define the control u_x in the following way:

$$u_x(t) = B^*(P^E_{\alpha,1})^*(b-t)R(\delta, \Psi^b_0)p(x), \tag{3.1}$$

where

$$p(x) = x_b - C^E_{\alpha,1}(b)[x_0 - g(x)] - S^E_{\alpha,1}(b)[y_0 - h(x)] - \int_0^b P^E_{\alpha,1}(b-s)f(s,x(s))ds.$$

Then $x(\cdot) = x(\cdot; u_x(\cdot)) \in C(J,H)$ defined by (2.9) is the mild solution of (1.1) corresponding to the control u_x.

Lemma 3.1. *Let the assumptions (H_{AE}) and $(H_2) - (H_5)$ hold. Then for each $x \in B_r$, we have*

$$\|u_x(t)\| \leq \frac{MM_B b^{\alpha-1}\|R(\delta, \Psi^b_0)\|}{\Gamma(\alpha)}\|p(x)\|, \tag{3.2}$$

where

$$\|p(x)\| \leq \|x_b\| + M(\|x_0\| + L_g r + \|g(0)\|) + Mb(\|y_0\| + \psi(r)) + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}. \tag{3.3}$$

Proof. By applying the assumptions (H_{AE}) and $(H_2) - (H_5)$ and using Lemma 2.1, we can show the inequalities (3.2) and (3.3) directly. So, we omit the detail here. □

Based on our assumptions, we define two operators $Q_1, Q_2 : C(J,H) \to C(J,H)$ by

$$(Q_1x)(t) = C^E_{\alpha,1}(t)[x_0 - g(x)], \quad t \in J \tag{3.4}$$

and

$$(Q_2x)(t) = S^E_{\alpha,1}(t)[y_0 - h(x)] + \int_0^t P^E_{\alpha,1}(t-s)[f(s,x(s)) + Bu_x(s)]ds, \quad t \in J. \tag{3.5}$$

Next, we will utilize Lemma 2.7 to show the existence of fixed points of the operator $Q := Q_1 + Q_2 : C(J,H) \to C(J,H)$ in B_r. To do this, we first prove some lemmas.

Lemma 3.2. *Let the assumptions (H_{AE}) and $(H_2) - (H_5)$ hold. Then there exists a constant $r > 0$ such that $Q(B_r) \subset B_r$ provided that*

$$(1 + \frac{M^2 M_B^2 b^{2\alpha-1}\|R(\delta, \Psi^b_0)\|}{(\Gamma(\alpha))^2})(ML_g + Mb\sigma_2 + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\sigma_1) < 1. \tag{3.6}$$

Proof. If this is not true, then for each $r > 0$, there exists $\overline{x} \in B_r$ such that $\|(Q\overline{x})(t)\| > r$ for all $t \in J$. By means of (H_{AE}) and $(H_2) - (H_5)$ and Lemmas 2.1 and 3.1, we have

$$\begin{aligned} r <& \|(Q\overline{x})(t)\| \\ \leq& M(\|x_0\| + L_g r + \|g(0)\|) + Mb(\|y_0\| + \psi(r)) \end{aligned}$$

$$
\begin{aligned}
&+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\varphi_r(s)ds+\frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2}\|p(x)\|\\
\leq&\frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2}\|x_b\|+(1+\frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2})\\
&\times\big[M(\|x_0\|+L_gr+\|g(0)\|)\\
&+Mb(\|y_0\|+\psi(r))+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}\big].
\end{aligned}
$$

Dividing in both sides by r and taking the lower limit as $r\to\infty$, we get

$$
1\leq(1+\frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2})(ML_g+Mb\sigma_2+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\sigma_1).
$$

Thus, in view of (3.6), there is a constant $r>0$ such that $Q(B_r)\subset B_r$. □

Lemma 3.3. *Let the assumptions* (H_{AE}) *and* $(H_1)-(H_5)$ *hold. Then* $Q_2:B_r\to B_r$ *is a continuous operator.*

Proof. Let $\{x_n\}_{n\geq1}\subset B_r$ with $x_n\to x$ as $n\to\infty$ in B_r. The continuity of f,g and h yield

$$
\begin{aligned}
&f(t,x_n(t))\to f(t,x(t)),\quad t\in J,\\
&g(x_n)\to g(x)
\end{aligned}
$$

and

$$
h(x_n)\to h(x)
$$

as $n\to\infty$. Moreover, by Lebesgue's dominated convergence theorem, we have

$$
p(x_n)\to p(x)
$$

and

$$
u_{x_n}(t)\to u_x(t),\quad t\in J
$$

as $n\to\infty$. By Lebesgue's dominated convergence theorem again, we get

$$
(Q_2x_n)(t)\to(Q_2x)(t),\quad t\in J
$$

as $n\to\infty$. Therefore, $Q_2:B_r\to B_r$ is a continuous operator. □

Lemma 3.4. *Let the assumptions* (H_{AE}) *and* $(H_2)-(H_5)$ *hold. Then* $Q_2(B_r)$ *is equi-continuous in* $C(J,H)$.

Proof. For $0\leq t_1<t_2\leq b$ and $x\in B_r$, we have

$$
\begin{aligned}
&\|(Q_2x)(t_2)-(Q_2x)(t_1)\|\\
\leq&\|S_{\alpha,1}^E(t_2)-S_{\alpha,1}^E(t_1)\|\big(\|y_0\|+\|h(x)\|\big)\\
&+\|\int_0^{t_1}\big(P_{\alpha,1}^E(t_2-s)-P_{\alpha,1}^E(t_1-s)\big)\big(f(s,x(s))+Bu_x(s)\big)ds\|\\
&+\|\int_{t_1}^{t_2}P_{\alpha,1}^E(t_2-s)\big(f(s,x(s))+Bu_x(s)\big)ds\|\\
\leq&\|S_{\alpha,1}^E(t_2)-S_{\alpha,1}^E(t_1)\|\big(\|y_0\|+\psi(r)\big)
\end{aligned}
$$

$$+\int_0^{t_1}\|P^E_{\alpha,1}(t_2-s)-P^E_{\alpha,1}(t_1-s)\|\|f(s,x(s))+Bu_x(s)\|ds$$
$$+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_{t_1}^{t_2}\varphi_r(s)ds+M_B\|u_x\|_{L^2}(t_2-t_1)^{\frac{1}{2}}\Big).$$

The facts of the equi-continuity of $S^E_{\alpha,1}(t)$ and $P^E_{\alpha,1}(t)$ on J imply

$$\|(Q_2x)(t_2)-(Q_2x)(t_1)\|\to 0$$

as $t_2-t_1\to 0^+$. Hence we conclude the equi-continuity of $Q_2(B_r)$ in $C(J,H)$. □

Lemma 3.5. *Let the assumptions* (H_{AE}) *and* $(H_2)-(H_5)$ *hold. Then the set* $\Omega(t):=\{(Q_2x)(t):x\in B_r\}$ *is relatively compact on* J.

Proof. Set

$$\Omega_1(t):=\Big\{S^E_{\alpha,1}(t)[y_0+h(x)]:x\in B_r\Big\},\qquad t\in J$$

and

$$\Omega_2(t):=\Big\{\int_0^t P^E_{\alpha,1}(t-s)[f(s,x(s)+Bu_x(s))]ds:x\in B_r\Big\},\qquad t\in J.$$

Then $\Omega(t)=\Omega_1(t)+\Omega_2(t)$ for each $t\in J$. In order to prove the relative compactness of $\Omega(t)$ for $t\in J$, we show that the sets $\Omega_1(t)$ and $\Omega_2(t)$ are relatively compact for $t\in J$, respectively. For every $x\in B_r$, since $\|y_0+h(x)\|\le\|y_0\|+\psi(r)$, by the compactness of $S^E_{\alpha,1}(t)$ for $t\ge 0$, we easily see that the set $\Omega_1(t)$ is relatively compact for $t\in J$. It remains to prove that the set $\Omega_2(t)$ is relatively compact on J. For $t=0$, the set $\Omega_2(0)$ is obviously relatively compact. For $0<t\le b$, let $0<\varepsilon<t$. Define

$$\Omega_2^\varepsilon(t)=\Big\{\int_0^{t-\varepsilon}P^E_{\alpha,1}(t-s)[f(s,x(s)+Bu_x(s))]ds:x\in B_r\Big\}.$$

Owing to the compactness of $P^E_{\alpha,1}(t)$ for every $t\ge 0$(see the Lemma 2.3), the set

$$\mathcal{Q}_\varepsilon:=\{P^E_{\alpha,1}(t-s)[f(s,x(s)+Bu_x(s))]:x\in B_r,s\in(0,t-\varepsilon)\}$$

is relatively compact for $\varepsilon\in(0,t)$. Then $\overline{conv(\mathcal{Q}_\varepsilon)}$ is a compact set and $\Omega_2^\varepsilon(t)\subset(t-\varepsilon)\overline{conv(\mathcal{Q}_\varepsilon)}$. Thus, the set $\Omega_2^\varepsilon(t)$ is relatively compact in H for every $\varepsilon\in(0,t)$. Moreover, we have

$$\begin{aligned}&\Big\|\int_0^t P^E_{\alpha,1}(t-s)[f(s,x(s)+Bu_x(s))]ds-\int_0^{t-\varepsilon}P^E_{\alpha,1}(t-s)[f(s,x(s)+Bu_x(s))]ds\Big\|\\ \le&\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\Big[\int_{t-\varepsilon}^t\varphi_r(s)ds+M_B\varepsilon^{\frac{1}{2}}\|u_x\|_{L^2}\Big]\\ \to&0\end{aligned}$$

as $\varepsilon\to 0^+$, which implies the relative compactness of the set $\Omega_2(t)$ for $t>0$. Therefore, the set $\Omega(t)$ is relatively compact in H for $t\in J$. □

Theorem 3.1. *Suppose that the assumptions* (H_{AE}) *and* $(H_1)-(H_5)$ *are satisfied. In addition the condition* (3.6) *holds. Then the fractional control system* (1.1) *admits one mild solution on* J.

Proof. For every $\alpha \in (1,2)$ and $x_b \in H$, we define the operator $Q : C(J,H) \to C(J,H)$ by $Q = Q_1 + Q_2$, where Q_1 and Q_2 are given in (3.4) and (3.5), respectively. It is clear that the mild solution of control system (1.1) is equivalent to the fixed point of Q on J. By Lemma 3.2, $Q(B_r) \subset B_r$ for some $r > 0$. For each $x, y \in B_r$, the assumptions (H_{AE}) and (H_3) lead to

$$\begin{aligned}\|Q_1 x - Q_1 y\|_C =& \sup_{t\in J} \|(Q_1 x)(t) - (Q_1 y)(t)\| \\ =& \sup_{t\in J} \|C^E_{\alpha,1}(t)[g(x) - g(y)]\| \\ \le& ML_g \|x - y\|_C.\end{aligned}$$

The inequality (3.6) yields $ML_g < 1$. So, $Q_1 : B_r \to B_r$ is a contractive operator. By means of Lemmas 3.3, 3.4 and 3.5, $Q_2 : B_r \to B_r$ is a completely continuous operator according to Ascoli-Arzela theorem. Thus, in view of properties of the measure of non-compactness, we conclude that $\gamma(Q_1(B_r)) \le ML_g\gamma(B_r) < \gamma(B_r)$ and $\gamma(Q_2(B_r)) = 0$. Therefore, $\gamma(Q(B_r)) = \gamma(Q_1(B_r)) + \gamma(Q_2(B_r)) < \gamma(B_r)$, which means that $Q : B_r \to B_r$ is a condensing operator. By Lemma 2.7, the operator Q has at least one fixed point on J, which is the mild solution of (1.1). □

If the functions f and h satisfy the following growth conditions:

$(H_2)'$ There exist two functions $a_1, a_2 \in L^1(J, \mathbb{R}^+)$ such that

$$f(t,x) \le a_1(t)x + a_2(t), \quad \forall t \in J, \quad x \in H.$$

$(H_4)'$ $h : C(J,H) \to H$ is continuous and there exist two constants $c_1, c_2 > 0$ such that

$$\|h(x)\| \le c_1\|x\|_C + c_2, \quad \forall x \in C(J,H)$$

then (H_2) and (H_4) are satisfied by choosing $\varphi_r(t) = a_1(t)r + a_2(t), t \in J$ and $\psi(\theta) = c_1\theta + c_2, \theta \in \mathbb{R}^+$ with $\sigma_1 = \|a_1\|_{L^1}$ and $\sigma_2 = c_1$, respectively.

Particularly, if the functions f and h are uniformly bounded, that is, the following conditions hold:

$(H_2)''$ There exists a constant $N_1 > 0$ such that

$$\|f(t,x)\| \le N_1, \quad \forall t \in J, \quad x \in H.$$

$(H_4)''$ $h : C(J,H) \to H$ is continuous and there exists a constant $N_2 > 0$ such that

$$\|h(x)\| \le N_2, \quad \forall x \in H$$

then by Theorem 3.1, we obtain the following existence result.

Corollary 3.1. *Let the assumptions $(H_{AE}), (H_1), (H_2)'', (H_3), (H_4)''$ and (H_5) hold. Then the fractional control system (1.1) possesses a mild solution on J provided that*

$$ML_g\big(1 + \frac{M^2 M_B^2 b^{2\alpha-1}\|R(\delta, \Psi_0^b)\|}{(\Gamma(\alpha))^2}\big) < 1.$$

3.2. The case g is completely continuous

Sometimes, the Lipschitz condition is not easy to verify in application. So if we replace the condition (H_3) by

$(H_3)^*$ $g : C(J,H) \to H$ is completely continuous and there exist constants $d_1, d_2 > 0$ such that

$$\|g(x)\| \leq d_1\|x\|_C + d_2, \quad \forall x \in C(J,H)$$

then we can obtain the following existence theorem.

Theorem 3.2. *Let the assumptions* $(H_{AE}), (H_1), (H_2), (H_3)^*, (H_4)$ *and* (H_5) *hold. Then the fractional control system* (1.1) *has at least one mild solution on* J *provided that*

$$(1 + \frac{M^2 M_B^2 b^{2\alpha-1}\|R(\delta, \Psi_0^b)\|}{(\Gamma(\alpha))^2})(Md_1 + Mb\sigma_2 + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\sigma_1) < 1.$$

Proof. By the assumption $(H_3)^*$, $Q_1 : C(J,H) \to C(J,H)$ is clearly continuous, where Q_1 is given in (3.4). By the strong continuity of $\{C_{\alpha,1}^E(t)\}_{t\geq 0}$, $Q_1(B_r)$ is equicontinuous in $C(J,H)$. So we only prove the relative compactness of $Q_1(B_r)(t)$ on J. Indeed, since $g : C(J,H) \to H$ is completely continuous, it follows that $g(B_r)$ is relatively compact. Hence, the exponential boundedness of $C_{\alpha,1}^E(t)$ for $t \geq 0$ yields that $Q_1(B_r)(t)$ is a relatively compact set on J. Thus, $Q_1 : B_r \to B_r$ is completely continuous. By Lemmas 3.3, 3.4 and 3.5, $Q_2 : B_r \to B_r$ is completely continuous. Therefore, $Q = Q_1 + Q_2 : B_r \to B_r$ is completely continuous. By applying Schauder's fixed point theorem, we conclude the existence of fixed point of Q in B_r, which is the mild solution of fractional control system (1.1). □

By Corollary 3.1, we can obtain the following corollary.

Corollary 3.2. *Let the assumptions* $(H_{AE}), (H_1), (H_2)'', (H_4)''$ *and* (H_5) *hold. In addition, the function* g *satisfies the condition*

$(H_3)^{**}$ $g : C(J,H) \to H$ *is compactly continuous and there exists a constant* $N_3 > 0$ *such that*

$$\|g(x)\| \leq N_3, \quad \forall x \in H.$$

Then the fractional control system (1.1) *has at least one mild solution on* J.

3.3. The case g is continuous

In this section, we assume that the nonlocal function g satisfies the condition

$(H_3)^{***}$ $g : C(J,X) \to X$ is continuous and there exists a constant $N_4 > 0$ such that

$$\|g(x)\| \leq N_4, \quad \forall x \in C(J,X).$$

Moreover, there exists a constant $\varrho \in (0,b)$ such that $g(x) = g(y)$ for any $x, y \in C(J,X)$ with $x(s) = y(s)$ for $s \in [\varrho, b]$.

In order to prove the existence of mild solutions of control system (1.1), for $n \geq 1$, we first consider the following approximate problem

$$\begin{cases} {}^C D_t^\alpha (Ex)(t) = Ax(t) + f(t,x(t)) + Bu_x(t), & t \in J, \\ Ex(0) = x_0 - C_{\alpha,1}^E(\frac{1}{n})g(x), \quad (Ex)'(0) = y_0 - h(x), \end{cases} \tag{3.7}$$

where u_x is defined by (3.1) and

$$p(x) = x_b - C_{\alpha,1}^E(b)[x_0 - C_{\alpha,1}^E(\frac{1}{n})g(x)] - S_{\alpha,1}^E(b)[y_0 - h(x)] - \int_0^b P_{\alpha,1}^E(b-s)f(s,x(s))ds. \tag{3.8}$$

Lemma 3.6. *Let the assumptions* $(H_{AE}), (H_1), (H_2), (H_3)^{***}, (H_4)$ *and* (H_5) *hold. Then the fractional approximate problem* (3.7) *possesses at least one mild solution on* J *provided that*

$$(1+\frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2})(Mb\sigma_2+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\sigma_1)<1. \tag{3.9}$$

Proof. For $n \geq 1$, we define an operator $\Phi_n : C(J,X) \to C(J,X)$ by

$$\begin{aligned}(\Phi_n x)(t) =& C_{\alpha,1}^E(t)[x_0 - C_{\alpha,1}^E(\frac{1}{n})g(x)] + S_{\alpha,1}^E(t)[y_0 - h(x)]\\ &+ \int_0^t P_{\alpha,1}^E(t-s)[f(s,x(s)) + Bu_x(s)]ds, \quad t \in J.\end{aligned}$$

By Definition 2.3, the mild solution of fractional approximate problem (3.7) is equivalent to the fixed point of Φ_n. We will apply Schauder's fixed point theorem to show that the operator Φ_n has at least one fixed point in $C(J,X)$. For this purpose, we divide the proof into three steps.

Step I, there is a constant $r > 0$ such that $\Phi_n(B_r) \subset B_r$ is continuous, where $B_r = \{x \in C(J,X) : \|x(t)\| \leq r, t \in J\}$. If this is not true, for $\forall r > 0$, there exists $\widetilde{x} \in B_r$ such that $\|(\Phi_n\widetilde{x})(t)\| > r$ for all $t \in J$. By means of the assumptions listed above and (3.8), we have

$$\|p(x)\| \leq \|x_b\| + M(\|x_0\| + MN_4) + Mb(\|y_0\| + \psi(r)) + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}$$

and

$$\begin{aligned}r < \|(\Phi_n\widetilde{x})(t)\| \leq& \frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2}\|x_b\|\\ &+ (1+\frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2})\big[M(\|x_0\| + MN_4)\\ &+ Mb(\|y_0\| + \psi(r)) + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}\big].\end{aligned}$$

Dividing r in both sides of the above inequality and taking lower limit as $r \to \infty$, we obtain that

$$1 \leq (1+\frac{M^2M_B^2b^{2\alpha-1}\|R(\delta,\Psi_0^b)\|}{(\Gamma(\alpha))^2})(Mb\sigma_2+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\sigma_1).$$

This contradicts to (3.9). Hence, there is a constant $r > 0$ such that $\Phi_n(B_r) \subset B_r$. Similar to the proof of Lemma 3.3, we can easily show that $\Phi_n : B_r \to B_r$ is continuous.

Step II, the set $\Lambda_n(\cdot) := \{C_{\alpha,1}^E(\cdot)[x_0 - C_{\alpha,1}^E(\frac{1}{n})g(x)] : x \in B_r\}$ is relatively compact in $C(J,X)$. For any $0 \leq t_1 < t_2 \leq b$ and $x \in B_r$, Since

$$\|\Lambda_n(t_2) - \Lambda_n(t_1)\| = \|\big[C_{\alpha,1}^E(t_2) - C_{\alpha,1}^E(t_1)\big]\big[x_0 - C_{\alpha,1}^E(\frac{1}{n})g(x)\big]\| \to 0$$

as $t_2 - t_1 \to 0$ due to the strong continuity of $C_{\alpha,1}^E(t)$ for $t \geq 0$, it follows that the set $\Lambda_n(\cdot)$ is equi-continuous in $C(J,X)$. On the other hand, owing to the uniform boundedness of g and compactness of $C^E_{\alpha,1}(\frac{1}{n})$, the set $\Lambda_n(0)$ is relatively compact. For $0 < t \leq b$, Since $C^E_{\alpha,1}(t)$ is compact for $t > 0$ and $\|x_0 - C^E_{\alpha,1}(\frac{1}{n})g(x)\| \leq \|x_0\| + MN_4$ for all $x \in B_r$, we obtain that the set $\Lambda_n(t)$ is relatively compact for $0 < t \leq b$. Therefore, the set $\Lambda_n(\cdot)$ is relatively compact in $C(J, X)$ by Ascoli-Arzela theroem.

Step III, the operator $\Phi_n : B_r \to B_r$ is completely continuous. Since the set $\Lambda_n(\cdot)$ is relatively compact in $C(J, X)$, Combining this fact with the conclusions of Lemmas 3.4 and 3.5, we deduce that $\Phi_n : B_r \to B_r$ is relatively compact in $C(J, X)$. Thus, $\Phi_n : B_r \to B_r$ is completely continuous. Consequently, the operator Φ_n admits a fixed point in B_r for $n \geq 1$ by using Schauder's fixed point theorem, and it is the mild solution of fractional approximate problem (3.7). □

Let

$$S(u) := \{x_n \in C(J, X) : x_n = \Phi_n x_n, n \geq 1\}.$$

Then $S(u)$ is called the solution set of fractional approximate problem (3.7). Define

$$\widehat{x_n}(t) = \begin{cases} x_n(t), & t \in [\varrho, b], \\ x_n(\varrho), & t \in [0, \varrho], \end{cases}$$

where ϱ comes from the assumption $(H_3)^{***}$. Then $g(\widehat{x_n}) = g(x_n)$ owing to $(H_3)^{***}$. According to Lemma 3.6 and Theorem 3.7 of [11], we can obtain the following lemma.

Lemma 3.7. *Let the assumptions* $(H_{AE}), (H_1), (H_2), (H_3)^{***}, (H_4)$ *and* (H_5) *hold. Then the solution set* $S(u)$ *is relatively compact in* $C(J, X)$.

Theorem 3.3. *Let the assumptions* $(H_{AE}), (H_1), (H_2), (H_3)^{***}, (H_4)$ *and* (H_5) *hold. In addition, the inequality* (3.9) *is satisfied. Then the fractional control system* (1.1) *has a mild solution on* J.

Proof. By Lemma 3.7, since the solution set $S(u)$ is relatively compact in $C(J, X)$, there is a subsequence $\{x_n : n \geq 1\} \subset S(u)$ converging to some x_* in $C(J, X)$. By the definitions of $S(u)$ and Φ_n, we have

$$\begin{aligned} x_n(t) =& C^E_{\alpha,1}(t)[x_0 - C^E_{\alpha,1}(\frac{1}{n})g(x_n)] + S^E_{\alpha,1}(t)[y_0 - h(x_n)] \\ &+ \int_0^t P^E_{\alpha,1}(t-s)[f(s, x_n(s)) + Bu_{x_n}(s)]ds, \quad t \in J, \end{aligned}$$

where

$$u_{x_n}(t) = B^*(P^E_{\alpha,1})^*(b-t)R(\delta, \Psi^b_0)p(x_n)$$

and

$$\begin{aligned} p(x_n) =& x_b - C^E_{\alpha,1}(b)[x_0 - C^E_{\alpha,1}(\frac{1}{n})g(x_n)] - S^E_{\alpha,1}(b)[y_0 - h(x_n)] \\ &- \int_0^b P^E_{\alpha,1}(b-s)f(s, x_n(s))ds. \end{aligned}$$

Taking the limit as $n \to \infty$ in both sides of above identities, by the continuity of functions g, h and f and Lebesgue's dominated convergence theorem, we obtain that

$$x_*(t) = C^E_{\alpha,1}(t)[x_0 - g(x_*)] + S^E_{\alpha,1}(t)[y_0 - h(x_*)]$$

$$+\int_0^t P_{\alpha,1}^E(t-s)[f(s,x_*(s))+Bu_{x_*}(s)]ds, \qquad t\in J,$$

where

$$u_{x_*}(t)=B^*(P_{\alpha,1}^E)^*(b-t)R(\delta,\Psi_0^b)p(x_*)$$

and

$$p(x_*)=x_b-C_{\alpha,1}^E(b)[x_0-g(x_*)]-S_{\alpha,1}^E(b)[y_0-h(x_*)]-\int_0^b P_{\alpha,1}^E(b-s)f(s,x_*(s))ds.$$

Therefore, $x_*\in C(J,X)$ is the mild solution of fractional control system (1.1). □

By Theorem 3.3, we can obtain the following corollary.

Corollary 3.3. *Let the assumptions* $(H_{AE}),(H_1),(H_2)'',(H_3)^{***},(H_4)''$ *and* (H_5) *hold. Then the fractional control system* (1.1) *has a mild solution on* J*.*

4. Approximate controllability

Now, we are in the position to state and prove our main results of approximate controllability.

Theorem 4.1. *Suppose that the assumptions* (H_{AE}), (H_B), (H_C), (H_1), $(H_2)''$, (H_3), $(H_4)''$ *and* (H_5) *are satisfied. If the function* g *is uniformly bounded and* $ML_g<1$*, then the fractional control system* (1.1) *is approximately controllable on* J*.*

Proof. For every $\delta>0$ and $x_b\in H$, we choose the control u_δ by

$$u_\delta(t)=B^*(P_{\alpha,1}^E)^*(b-t)R(\delta,\Psi_0^b)p(x_\delta),$$

where

$$p(x_\delta)=x_b-C_{\alpha,1}^E(b)[x_0-g(x_\delta)]-S_{\alpha,1}^E(b)[y_0-h(x_\delta)]-\int_0^b P_{\alpha,1}^E(b-s)f(s,x_\delta(s))ds.$$

By Corollary 3.1, the fractional control system (1.1) has a mild solution corresponding to u_δ expressed by

$$\begin{aligned}x_\delta(t;u_\delta):=&x_\delta(t)=C_{\alpha,1}^E(t)[x_0-g(x_\delta)]+S_{\alpha,1}^E(t)[y_0-h(x_\delta)]\\&+\int_0^t P_{\alpha,1}^E(t-s)[f(s,x_\delta(s))+Bu_\delta(s)]ds, \qquad t\in J.\end{aligned}$$

Owing to $\delta R(\delta,\Psi_0^b)=I-\Psi_0^bR(\delta,\Psi_0^b)$, we have

$$x_\delta(b;u_\delta)=x_b-\delta R(\delta,\Psi_0^b)p(x_\delta). \tag{4.1}$$

In view of (H_{AE}) and Lemma 2.3, $\{C_{\alpha,1}^E(t)\}_{t>0}$ and $\{S_{\alpha,1}^E(t)\}_{t\geq 0}$ are compact. Then the sets $\{C_{\alpha,1}^E(b)[x_0-g(x_\delta)]:\delta>0\}$ and $\{S_{\alpha,1}^E(b)[y_0-h(x_\delta)]:\delta>0\}$ are relatively compact due to the uniform boundedness of g and h. Therefore, they have subsequences, still denoted by themselves, tend to some x' and y' in H as $\delta\to 0^+$, respectively. Moreover, the condition $(H_2)''$ yields

$$\int_0^b \|f(s,x_\delta(s;u_\delta))\|^2ds\leq N_1^2b.$$

Combining this fact with the reflexive property of $L^2(J,H)$, there exists $\phi \in L^2(J,H)$ such that a subsequence of $\{f(\cdot, x_\delta(\cdot; u_\delta)) : \delta > 0\}$, still denoted by itself, weakly converges to $\phi(\cdot)$ in $L^2(J,H)$ as $\delta \to 0^+$. Then the compactness of $\{P^E_{\alpha,1}(t)\}_{t\geq 0}$ implies

$$\int_0^b P^E_{\alpha,1}(b-s)\big[f(s, x_\delta(s; u_\delta)) - \phi(s)\big]ds \to 0$$

as $\delta \to 0^+$. Denote by

$$\mu := x_b - x' - y' - \int_0^b P^E_{\alpha,1}(b-s)\phi(s)ds.$$

Then $\mu \in H$ and

$$\|p(x_\delta) - \mu\| \to 0 \tag{4.2}$$

as $\delta \to 0^+$. By virtue of (4.1), (4.2) and Lemmas 2.5 and 2.6, we get

$$\begin{aligned}\|x_\delta(b; u_\delta) - x_b\| =&\|\delta R(\delta, \Psi_0^b)p(x_\delta)\|\\ \leq&\|\delta R(\delta, \Psi_0^b)\|\|p(x_\delta) - \mu\| + \|\delta R(\delta, \Psi_0^b)\mu\| \to 0\end{aligned}$$

as $\delta \to 0^+$. Therefore, the fractional control system (1.1) is approximately controllable on J. □

By Corollaries 3.2 and 3.3, we can obtain the following approximate controllability theorems.

Theorem 4.2. *Suppose that* $(H_{AE}), (H_B), (H_C), (H_1), (H_2)'', (H_3)^{**}, (H_4)''$ *and* (H_5) *are satisfied. Then the fractional control system* (1.1) *is approximately controllable on* J.

Proof. The proof is similar to the one of Theorem 4.1, we omit it here. □

Theorem 4.3. *Suppose that* $(H_{AE}), (H_B), (H_C), (H_1), (H_2)'', (H_3)^{***}, (H_4)''$ *and* (H_5) *are satisfied. Then the fractional control system* (1.1) *is approximately controllable on* J.

Proof. The proof is similar to the one of Theorem 4.1, we also omit the detail here. □

Remark 4.1. In Theorem 4.3, if we choose $E \equiv I$, where I is the identity operator, then it is a nature improvement of Theorem 3.8 of [11].

Remark 4.2. In [3, 11, 12], when the authors considered the approximate controllability of the nonlinear evolution equations, they always supposed the approximate controllability of the corresponding linear system. But in our Theorems 4.1, 4.2 and 4.3, this assumption is removed by adding (H_B) and (H_C).

5. Applications

Consider the fractional differential control system of Sobolev type in the following

$$\begin{cases} {}^C D_t^{\alpha}\left(x(t,\eta) - \dfrac{\partial^2 x(t,\eta)}{\partial \eta^2}\right) = \dfrac{\partial^2 x(t,\eta)}{\partial \eta^2} + \dfrac{1}{e^{3t}\big(20 + x^2(t,\eta)\big)} + \tau u(t,\eta), \\ \qquad\qquad\qquad\qquad t \in [0,1], \eta \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,1], \\ [(I - \dfrac{\partial^2}{\partial \eta^2})x](0,\eta) = [x_0 - g(x)](\eta), \quad \eta \in [0,\pi], \\ [(I - \dfrac{\partial^2}{\partial \eta^2})x]'_t(0,\eta) = [y_0(\eta) - 1 - \sin(2 + x^3(\cdot,\eta))], \quad \eta \in [0,\pi], \end{cases} \tag{5.1}$$

where $\alpha \in (1,2), \rho > 0$ is a constant.

Let $H = U = L^2[0,\pi]$. We define

$$D(A) = D(E) = \{x \in H : x \in W^{2,2}[0,\pi], x(0) = x(\pi) = 0\},$$
$$Ax = \frac{\partial^2 x}{\partial \eta^2}, \qquad Ex = (I - \frac{\partial^2}{\partial \eta^2})x.$$

Noting that A has eigenvalues $-n^2, n \in \mathbb{N}$ with corresponding eigenvectors $e_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns)$, and $\{e_n : n \in \mathbb{N}\}$ is a complete orthonormal system in H, by Example 6.3 of [3], the pair (A, E) generates an $(\alpha, 1)$-resolvent family $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$ given by

$$C^E_{\alpha,1}(t)x = \sum_{n=1}^{\infty} \mathcal{E}_{\alpha,n}(t)\langle x, e_n\rangle e_n, \qquad \forall x \in H,$$

where $\mathcal{E}_{\alpha,n}(t) = \sum\limits_{k=0}^{\infty} \frac{(-1)^k n^{2k} t^{\alpha k}}{(1+n^2)^k \Gamma(\alpha k + 1)}$. Then $C^E_{\alpha,1}(t)$ is norm continuous for $t > 0$ and $\|C^E_{\alpha,1}(t)\| \leq 2$. Moreover, the operator $(\lambda^\alpha E - A)^{-1}$ is compact for $\lambda > 0$. By Remark 2.1, $C^E_{\alpha,1}(t)$ is a compact operator for all $t > 0$. Thus, the assumption (H_{AE}) holds.

For $t \in [0,1]$, let

$$x(t)(\eta) = x(t,\eta), \qquad Bu(t)(\eta) = \tau u(t,\eta),$$
$$f(t, x(t))(\eta) = \frac{1}{e^{3t}\big(20 + x^2(t,\eta)\big)}$$

and

$$h(x)(\eta) = 1 + \sin(2 + x^3(\cdot,\eta)).$$

Then

$$\|f(t,x(t))\| \leq \frac{1}{20}, \qquad \|h(x)\| \leq 2.$$

Hence the assumptions $(H_2)''$ and $(H_4)''$ are satisfied with $N_1 = \frac{1}{20}$ and $N_2 = 2$, respectively.

Consequently, if one of the following conditions are satisfied:

(i) $g(x)(\eta)$ is Lipschitz continuous with Lipschitz constant $L_g \in (0, \frac{1}{2})$ and uniformly bounded;

(ii) $g(x)(\eta)$ is completely continuous and uniformly bounded;

(iii) $g(x)(\eta)$ satisfies the condition $(H_3)^{***}$.

then the control system (5.1) is approximately controllable provided that the conditions (H_B) and (H_C) are satisfied.

Remark 5.1. The technique used in this paper can be applied to investigate the existence and approximate controllability of mild solutions of the following Riemann-Liouville fractional control system of Sobolev type in the Hilbert space H

$$\begin{cases} {}^L D_t^\alpha (Ex)(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J, \\ E(g_{2-\alpha} * x)(0) = x_0 - g(x), \\ [E(g_{2-\alpha} * x)]'(0) = y_0 - h(x), \end{cases} \tag{5.2}$$

where ${}^L D_t^\alpha$ denotes the Riemann-Liouville fractional derivative operator of order $\alpha \in (1, 2)$.

The mild solution of the control system (5.2) is defined by

$$\begin{aligned} x(t) =& C^E_{\alpha,\alpha-1}(t)[x_0 - g(x)] + C^E_{\alpha,\alpha}(t)[y_0 - h(x)] \\ &+ \int_0^t C^E_{\alpha,\alpha}(t-s)[f(s, x(s)) + Bu(s)]ds, \quad t \in J, \end{aligned}$$

where $C^E_{\alpha,\alpha-1}(t)$ is the $(\alpha, \alpha - 1)$ resolvent family generated by the pair (A, E), which satisfies

$$\lambda R(\lambda^\alpha E, A)x = \int_0^\infty e^{-\lambda t} C^E_{\alpha,\alpha-1}(t)x dt, \quad Re\lambda > \omega, \quad x \in H,$$

$C^E_{\alpha,\alpha}(t)$ is given by $C^E_{\alpha,\alpha}(t) = \int_0^t C^E_{\alpha,\alpha-1}(s)ds$, which satisfies

$$R(\lambda^\alpha E, A)x = \int_0^\infty e^{-\lambda t} C^E_{\alpha,\alpha}(t)x dt, \quad Re\lambda > \omega, \quad x \in H.$$

6. Conclusion

In this paper, the existence and approximate controllability of the $\alpha \in (1, 2)$ ordered Sobolev type fractional evolution system (1.1) are investigated in the Hilbert space H. For any $\alpha, \beta > 0$, with the help of the resolvent family $\{C^E_{\alpha,\beta}(t)\}_{t\geq 0}$ generated by (A, E), the definition of the mild solution of (1.1) is given by utilizing Laplace Transform. Then some sufficient conditions for the existence of mild solutions of fractional evolution system (1.1) is established by using fixed point theorems. The approximate controllability of fractional evolution system (1.1) is also discussed without assuming the approximate controllability of corresponding linear system. In our discussion, we assume that the nonlocal function g is Lipschitz continuous, or completely continuous, or continuous, and the nonlocal function h is continuous without any compactness conditions. Particularly, the existence and compactness of E^{-1} are not needed in our work, hence the results obtained in this work improve and extend some existing results.

Acknowledgements.

The author thanks the editors and the anonymous reviewers for their helpful and valuable suggestions on improving this paper.

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