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HOPF BIFURCATION AT A DEGENERATE SINGULAR POINT IN 3-DIMENSIONAL VECTOR FIELD*

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Abstract The work of this paper focuses on investigating limit cycle bifurcation for a degenerate singular point in 3-Dimensional vector fields. By making two appropriate transformations and making use of singular values methods to compute focal values carefully, we give the expressions of the first five Lyapunov constants at the origin that is a degenerate singular point. Moreover, we obtain the considered system can bifurcate 5 limit cycles near the origin. In terms of results on limit cycle bifurcation from degenerate singular point in 3-Dimensional vector field, it is less seen in published references..

Keywords 3-Dimensional vector field, degenerate singular point, limit cycle bifurcation, focal values.

MSC(2010) 34C07, 34C23.

1. Introduction

For the limit cycle bifurcation of polynomial differential system, many good published results focus on 2-dimensional systems because of attracting effect from the "Hilbert 16-th problem", up to now, it is a hot topic, for example some recent works (see [1, 6, 13, 17–19, 21, 23–26, 29, 32] etc). Recently we also carried out some investigations about the Hopf bifurcation for 2-dimensional polynomial differential systems and obtained some good results (see [9, 12, 12]. In terms of the limit cycle bifurcation from non-degenerate singular point of 3-dimensional polynomial systems, a lot of published references showed this is also an attractive topic, although there is relatively little literatures on this subject. For example: [2] and [20] studied two classes of 3-dimensional polynomial systems and considered their limit cycle bifurcation behavior by using the averaging method, [12] offered a kind of method (singular values method) to investigate the limit cycle bifurcation in 3-Dimensional vector field and showed a class of 3-Dimensional quadratic systems could bifurcate 8 limit cycle, [32] considered the limit cycle bifurcation for a class of 3-d quadratic system with quadratic perturbation and showed it could bifurcate ten limit cycles, [28]

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introduced computational algebra approach to study Hopf bifurcations in R^3, A. Buica et.al studied the multiple Hopf bifurcation in R^3 in Ref. [4] and introduced some methods and results in [3], Ref. [12] made use of singular values method given by [12] and computational algebra approach to investigate the Hopf bifurcation for a class of cubic Kolmogorov model in 3-dimensional vector field and obtained it could yield 5 small limit cycles. Romanovski and his coauthors studied the center problem or integrability for several classes of 3-D systems and obtained some good results which were shown in [15, 16, 27]. There are else many articles about the limit cycle bifurcation of polynomial differential system, we will not list one by one. It can be seen that most of these conclusions are concerned with the bifurcation behavior of non-degenerate singular point. Of course, there are a few articles about the bifurcation behavior of degenerate singular point in 2-dimensional vector field, [5, 15, 30] offered some cases about this class of degenerate singular points. For the limit cycle bifurcation of degenerate singular point in 3-dimensional vector field, it is less to see this kind of cases. At present, [22] considered the limit cycle bifurcation for a degenerate singular point of the following 3-Dimensional systems:

$$\begin{cases} \frac{dx}{d\tau} = -H_y(x,y) + P_{2n}(x,y,z) + \varepsilon P_{2n-1}(x,y), \\ \frac{dy}{d\tau} = H_x(x,y) + Q_{2n}(x,y,z) + \varepsilon Q_{2n-1}(x,y), \\ \frac{du}{d\tau} = R_{2n}(x,y,z) + \varepsilon c z^{2n-1}, \end{cases} \tag{1.1}$$

in which

$$\begin{aligned} H &= \frac{1}{2n}(x^{2l}+y^{2l})^m,\ n = lm, \\ =P_{2n-1} &= x(p_1x^{2n-2} + p_2x^{2n-3}y + \cdots + p_{2n-1}y^{2n-2}), \\ =Q_{2n-1} &= y(p_1x^{2n-2} + p_2x^{2n-3}y + \cdots + p_{2n-1}y^{2n-2}), \end{aligned}$$

and proved using the averaging theory of first order that, moving the parameter ε from $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small, from the origin it could bifurcate $2n-1$ limit cycles, and that using the averaging theory of second order from the origin it could bifurcate $3n-1$ limit cycles when $l = 1$.

We also try to carry out some researches about bifurcation behavior of degenerate singular points in 3-D vector field in order to supplement some cases about Hopf bifurcation of 3-d system, hence, in this paper, we will study the limit cycle bifurcation for a degenerate singular point in 3-Dimensional vector field by making use of our method (singular values method). Considered system is as follows:

$$\begin{cases} \dfrac{dx}{dt} =(-y+\delta x)(x^2+y^2)^{\frac{4}{3}} + u(A_1u + A_2y)(x^2+y^2) \\ \qquad + 2ux(A_1ux + B_2uy + A_2xy + B_3xy) + 2B_1x^3y, \\ \dfrac{dy}{dt} =(x+\delta y)(x^2+y^2)^{\frac{4}{3}} + (B_2u^2 + B_3ux + B_1x^2)(x^2+y^2) \\ \qquad + 2yu(A_1ux + B_2uy + A_2xy + B_3xy) + 2B_1x^2y^2, \\ \dfrac{du}{dt} =-u(x^2+y^2)^{\frac{4}{3}} + Cxy(x^2+y^2) \\ \qquad + 2u^2(A_1ux + B_2uy + A_2xy + B_3xy) + 2B_1x^2yu, \end{cases} \tag{1.2}$$

in which A_1, A_2, B_1, B_2, B_3, C, are six real parameters and they are not zero. Obviously, the origin of system (1.2) is a degenerate singular point.

We will make use of singular values method offered by article [12] to study the limit cycle bifurcation of the origin. In order to investigate the bifurcation behavior at the origin of system (1.2), we at first make two appropriate transformations of system (1.2) which let the origin of system (1.2) be changed into the origin of system (3.3). Next, we carry out investigation on limit cycle bifurcation at the origin of system (3.3) according to the method offered by article [12]. Via using computer Algebra system Mathematica 7.0 to compute carefully, we obtain the expressions of the first five focal values at the origin of system (3.3), and we show that the origin of system (3.3) can become a fine focus of fifth order. Moreover, we give the condition that from this point can bifurcate 5 limit cycles. From the relation between the origin of system (1.2) and system (3.3), we give the bifurcation behavior of the origin of system (1.2), namely system (1.2) can bifurcate 5 limit cycles from the origin. Of course perhaps our result can be improved, but we think it is a class of new interesting problem for investigating the Hopf bifurcation problem of a degenerate singular point in 3-D vector fields which will attract more attentions.

The remainder of this paper is organized as follows. In section 2, we introduce a method to study Hopf bifurcation of the elementary focus point for 3-Dimensional polynomial differential systems which will show the relation between focal values of real system and singular point values of the corresponding complex system at the origin. This kind of method is introduced in [12]. In section 3, we make two appropriate transformations which let research of the origin of system (1.2) be reduced to investigate the origin of system (3.3). For system (3.3), we obtain the condition that the origin of system (3.3) can be a fine focus of fifth order and bifurcate 5 small limit cycles. Moreover, we give the result that the degenerate singular point (the origin) of system (1.2) can bifurcate 5 limit cycles by comparing the relation between the degenerate singular point (the origin) of system (1.2) and the elementary focus point of system (3.3).

2. Our preliminary method to study the 3-Dimensional Hopf bifurcations

For 3-Dimensional Hopf bifurcations system, the singular values method is used to study the Hopf bifurcations problems of the elementary focus point in [12], it is a valid method for investigating Hopf bifurcation of the elementary focus point in 3-Dimensional vector field. Consider the following 3-Dimensional Hopf bifurcations systems:

$$\begin{cases} \dfrac{dx}{dt} = -y + \delta x + \displaystyle\sum_{k+j+l=2}^{\infty} A_{jkl} x^k y^j u^l = X(x,y,u), \\ \dfrac{dy}{dt} = x + \delta y + \displaystyle\sum_{k+j+l=2}^{\infty} B_{jkl} x^k y^j u^l = Y(x,y,u), \\ \dfrac{du}{dt} = -gu + \displaystyle\sum_{k+j+l=2}^{\infty} d_{jkl} x^k y^j u^l = U(x,y,u), \end{cases} \tag{2.1}$$

in which $x, y, u, g, t, A_{jkl}, B_{jkl}, d_{jkl}, \delta \in \mathbf{R}\,(k,j,l \in \mathbf{N})$. This kind of method focuses on obtaining the expressions of focal values in real system and those of singular values in corresponding complex system by comparing the relation between them.

For system (2.1), there exists a center manifold $u = u(x, y)$, which can be expressed as the polynomial series about x and y as follows:

$$u = x^2 + y^2 + h.o.t., \tag{2.2}$$

in which $h.o.t.$ stands for higher-order term about x and y. It is clear that u can be expanded only from the beginning of a square item. However, here we will consider the implicit function formal series about u, x and y. By means of the following complex transformation

$$z = x + iy, w = x - iy, u = u, T = it, i = \sqrt{-1}, \tag{2.3}$$

system $(2.1)|_{\delta=0}$ is changed into the following complex system

$$\begin{cases} \dfrac{dz}{dT} = z + \displaystyle\sum_{k+j+l=2}^{\infty} a_{jkl} z^k w^j u^l = Z(z, w, u), \\ \dfrac{dw}{dT} = -w - \displaystyle\sum_{k+j+l=2}^{\infty} b_{jkl} z^k w^j u^l = -W(z, w, u), \\ \dfrac{du}{dT} = igu + \displaystyle\sum_{k+j+l=2}^{\infty} \tilde{d}_{jkl} z^k w^j u^l = \tilde{U}(z, w, u), \end{cases} \tag{2.4}$$

where $z, w, T, a_{jkl}, b_{jkl}, \tilde{d}_{jkl} \in \mathbf{C}\,(k, j, l \in \mathbf{N})$. It is clear that the coefficients of (2.4) a_{jkl} and b_{jkl} are a pair of conjugate complex numbers, i.e. $\overline{a_{jkl}} = b_{jkl}, j \geq 0, k \geq 0, l \geq 0, j + k + l \geq 2$, Hence system (2.1) and (2.4) can be called as concomitant systems each other. For convenience, we may as well write $\tilde{d}_{jkl}, \tilde{U}$ as d_{jkl}, U in the following investigation.

Ref. [12] obtain the computing method of the singular values at the origin of system (2.4) , namely the following Lemma.

Lemma 2.1 ([12]). *For system* (2.4)*, Let* $c_{110} = 1$, $c_{101} = c_{011} = c_{200} = c_{020} = 0$, $c_{kk0} = 0, k = 2, 3, \cdots$*, then the terms of the following formal series can be derived successively and uniquely:*

$$F(z, w, u) = zw + \sum_{\alpha+\beta+\gamma=3}^{\infty} c_{\alpha\beta\gamma} z^\alpha w^\beta u^\gamma \tag{2.5}$$

satisfying

$$\frac{dF}{dT} = \frac{\partial F}{\partial z} Z - \frac{\partial F}{\partial w} W + \frac{\partial F}{\partial u} U = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}, \tag{2.6}$$

and we have let $c_{\alpha,\beta,\gamma} = 0$ *if* $\alpha < 0$ *or* $\beta < 0$ *or* $\gamma < 0$ *or* $\gamma = 0, \alpha = \beta$*, and* $c_{\alpha\beta\gamma}$ *of* (2.5) *is determined by the following recursive formula if* $\alpha \neq \beta$ *or* $\alpha = \beta, \gamma \neq 0$*:*

$$\begin{aligned} c_{\alpha\beta\gamma} =& \frac{1}{\beta - \alpha - id\gamma} \times \sum_{k+j+l=3}^{\alpha+\beta+\gamma+2} [(\alpha - k + 1) a_{k,j-1,l} - (\beta - j + 1) b_{j,k-1,l} \\ &+ (\gamma - l) d_{k-1,j-1,l+1}] \times c_{\alpha-k+1,\beta-j+1,\gamma-l}, \end{aligned} \tag{2.7}$$

and for any positive integer m*,* μ_m *of* (2.6) *is determined by the following recursive formula:*

$$\mu_m = \sum_{k+j=3}^{2m+2} [(m - k + 1) a_{k,j-1,0} - (m - j + 1) b_{j,k-1,0}] c_{m-k+1,m-j+1,0}. \tag{2.8}$$

Obtained μ_m from (2.8) is called as mth singular values at the origin of (2.4) . Ref. [12] gave the relation between mth singular values and mth Lyapunov constants (or called focal values), namely the following Lemma.

Lemma 2.2 ([12]). *For the mth Lyapunov constants of focal values of system* $(2.1)|_{\delta=0}$ *and the mth singular values of system* (2.4) *and any positive integer* m, *the following assertion holds:*

$$v_{2m+1}(2\pi) = i\pi(\mu_m + \sum_{k=1}^{m-1} \xi_m^{(k)} \mu_k), \tag{2.9}$$

in which $v_{2k+1}(2\pi), (k = 1, 2, \cdots, m-1)$ *are the kth Lyapunov constants (or focal values) at the origin of* (2.1), $\mu_k (k = 1, 2, \cdots, m-1)$ *are the kth singular values at the origin of system* (2.4) , $\xi_m^{(k)}, (k = 1, 2, \cdots, m-1)$ *are polynomial functions on coefficients of system* (2.4) .

Definition 2.1. For system $(2.1)|_{\delta=0}$, if $v_1(2\pi) \neq 1$, then the origin is called a rough focus(strong focus); if $v_1(2\pi) = 1$, and $v_2(2\pi) = v_3(2\pi) = \cdots = v_{2k}(2\pi) = 0, v_{2k+1}(2\pi) \neq 0$, then the origin is called a fine focus (weak focus) of order k, in which the quantity of $v_{2k+1}(2\pi), k = 1, 2, \cdots$ is the kth focal value (or Liapunov Constants) at the origin of system (2.1); if $v_1(2\pi) = 1$, and for any positive integer k, $v_{2k+1}(2\pi) = 0$, then the origin is called a center.

According to Lemma 2.1 and Lemma 2.2, the following lemma holds

Lemma 2.3 ([12]). *For the mth Lyapunov constants of focal values of system* $(2.1)|_{\delta=0}$ *and the mth singular values of system* (2.4) , *the following relation holds:*

$$v_{2m+1}(2\pi) = i\pi\mu_m, \tag{2.10}$$

if $\mu_k = 0, k = 1, 2, \cdots, m-1$.

Further, Ref. [12] gave the following two results to study the Hopf bifurcation behavior.

Lemma 2.4 ([12]). *For Hopf bifurcation system* (2.1), *the following conclusions hold:*

(i) System (2.1) *can bifurcate* m *limit cycles at most in a small enough neighborhood at the origin of* (2.1), *if expressions of focal values can be expressed in the following form under a suitable coefficients' perturbation:*

$$v_1(2\pi, \epsilon) - 1 = \lambda_0 \epsilon^{l_0+N} + o(\epsilon^{l_0+N+1}), \tag{2.11}$$

$$v_{2k+1}(2\pi, \epsilon) = \lambda_k \epsilon^{l_k+N} + o(\epsilon^{l_k+N+1}), k = 1, 2, \cdots, 0 < |\epsilon| \ll 1, \tag{2.12}$$

in which $l_0, l_1, \cdots, l_m, m, N$ *are positive integers and* $l_m = 0, \lambda_m \neq 0$.

(ii) If conditions (2.11) *and* (2.12) *hold, and* $\lambda_k \lambda_{k-1} < 0\, (k = 1, 2, \cdots, m)$, $l_{k-1} - l_k > l_k - l_{k+1}, (k = 1, 2, \cdots, m-1)$, *then equation* $\sum_{k=0}^{m} \lambda_k \epsilon^{l_k} h^{2k} = 0$ *has* m *positive solutions, i.e.,*

$$h_k(\epsilon) = \sqrt{(-\frac{\lambda_{k-1}}{\lambda_k})\epsilon^{l_{k-1}-l_k}} + o(\epsilon^{\frac{l_{k-1}-l_k}{2}}). \tag{2.13}$$

Accordingly, system (2.1) *can bifurcate* m *limit cycles which are close to circles* $x^2 + y^2 = (-\frac{\lambda_{k-1}}{\lambda_k})\epsilon^{l_{k-1}-l_k}$.

Lemma 2.5 ([12]). *If the origin of unperturbed system* (2.1) *is a fine focus of* $n-th$ *order, then the origin of disturbed system* (2.1) *can bifurcate* n *limit cycles under a suitable perturbation.*

Obviously, for 3-Dimensional Hopf bifurcations system (2.1), we can obtain the first m-th singular values by using recursive formula offered by Lemma 2.1, moreover we can judge whether the origin of (2.1) will be a m-th fine focus. Next, according to Lemma 2.5, we can obtain the origin of (2.1) can bifurcate m limit cycles if the origin of (2.1) is a m-th fine focus.

3. Limit cycle bifurcation of system (1.2)

Lemma 2.1–Lemma 2.5 offered a kind of method to investigate the Hopf bifurcation of elementary focus point in 3-Dimensional vector field, while we will focus on the limit cycle bifurcation of the degenerate singular point (the origin) of system (1.2). In order to study the limit cycle bifurcation at the origin of system (1.2), we may as well make some appropriate transformations so as to carry out our investigation.

3.1. The reduction of system (1.2)

By means of the following homeomorphous transformation

$$x = x_1(x_1^2 + y_1^2),\ y = y_1(x_1^2 + y_1^2),\ u = u_1(x_1^2 + y_1^2), \tag{3.1}$$

and time transformation

$$d\tau = (x_1^2 + y_1^2)^4 dt, \tag{3.2}$$

system (1.2) is turned into the following form

$$\begin{cases} \dfrac{dx_1}{d\tau} = -y_1 + \delta x_1 + A_1 u^2 + A_2 y_1 u_1, \\ \dfrac{dy_1}{d\tau} = x_1 + \delta y_1 + B_1 x_1^2 + B_2 u_1^2 + B_3 x_1 u_1, \\ \dfrac{du_1}{d\tau} = -u_1 + C x_1 y_1. \end{cases} \tag{3.3}$$

Clearly, the origin of system (1.2) becomes the origin of (3.3) correspondingly under transformations (3.1) and (3.2). From transformation (3.1), it is easy to obtain the conclusion that the origin of system (1.2) can also bifurcate k limit cycles if the origin of system (3.3) can bifurcate k small limit cycles. On the other hand, the computation of focal values plays pivotal role to study the limit cycle bifurcation of system (3.3) according to method offered by [12]. In this sense, the focal values at the origin of system (3.3) can be called general focal values at origin of system (1.2). Next it is necessary to compute the focal values or general focal values under center manifold $u_1 = x_1^2 + y_1^2 + h.o.t.$.

3.2. The general focal values of system (1.2)

It can be seen that system (3.3) belongs to the class of system (2.1), hence we can investigate the limit cycle bifurcation behavior by using the method by [12].

Let

$$z = x_1 + iy_1,\ w = x_1 - iy_1,\ u = u_1,\ T = i\tau,\ i = \sqrt{-1}, \tag{3.4}$$

system (3.3) $|_\delta = 0$ becomes

$$\begin{cases} \dfrac{dz}{dT} = z + \dfrac{1}{4}B_1(z^2 + w^2) + (B_2 - iA_1)u^2 + \dfrac{1}{2}(A_2 + B_3)uw \\ \qquad - \dfrac{1}{2}(A_2 u - B_3 u - B_1 w)z, \\ \dfrac{dw}{dT} = -w - \dfrac{1}{4}B_1(z^2 + w^2) - (B_2 + iA_1)u^2 + \dfrac{1}{2}(A_2 - B_3)uw \\ \qquad - \dfrac{1}{2}(A_2 u + B_3 u + B_1 w)z, \\ \dfrac{du}{dT} = iu + \dfrac{1}{4}C(w^2 - z^2). \end{cases} \tag{3.5}$$

According to Lemma 2.1, we can obtain the following recursive formula of the singular values of the origin of system (3.5) by using computer Algebra system Mathematica 7.0.

Theorem 3.1. *Let $c_{k,j,l} = 0$ if $k < 0$ or $j < 0$ or $l < 0$ or $l = 0, k = j$, and if $k \neq j$ or $k = j, l \neq 0$, c_{kjl} is determined by the following recursive formula:*

$$\begin{aligned} c_{k,j,l} = & -\frac{1}{4(j-k-il)}(Cc_{-2+k,j,1+l} + Clc_{-2+k,j,1+l} + B_1c_{-2+k,1+j,l} + B_1jc_{-2+k,1+j,l} \\ & + B_1c_{-1+k,j,l} + 2B_1jc_{-1+k,j,l} - B_1kc_{-1+k,j,l} + 2A_2c_{-1+k,1+j,-1+l} \\ & + 2B_3c_{-1+k,1+j,-1+l} + 2A_2jc_{-1+k,1+j,-1+l} + 2B_3jc_{-1+k,1+j,-1+l} - Cc_{k,-2+j,1+l} \\ & - Clc_{k,-2+j,1+l} - B_1c_{k,-1+j,l} + B_1jc_{k,-1+j,l} - 2B_1kc_{k,-1+j,l} - 2A_2jc_{k,j,-1+l} \\ & + 2B_3jc_{k,j,-1+l} + 2A_2kc_{k,j,-1+l} - 2B_3kc_{k,j,-1+l} + 4iA_1c_{k,1+j,-2+l} \\ & + 4B_2c_{k,1+j,-2+l} + 4iA_1jc_{k,1+j,-2+l} + 4B_2jc_{k,1+j,-2+l} - B_1c_{1+k,-2+j,l} \\ & - B_1kc_{1+k,-2+j,l} - 2A_2c_{1+k,-1+j,-1+l} - 2B_3c_{1+k,-1+j,-1+l} - 2A_2kc_{1+k,-1+j,-1+l} \\ & - 2B_3kc_{1+k,-1+j,-1+l} + 4iA_1c_{1+k,j,-2+l} - 4B_2c_{1+k,j,-2+l} + 4iA_1kc_{1+k,j,-2+l} \\ & - 4B_2kc_{1+k,j,-2+l}), \end{aligned}$$

in addition, for any positive integer j, μ_j is determined by the following recursive formula:

$$\begin{aligned} \mu_{j-1} = & \frac{1}{4}(-Cc_{-2+j,j,1} - B_1c_{-2+j,1+j,0} - B_1jc_{-2+j,1+j,0} - B_1c_{-1+j,j,0} - B_1jc_{-1+j,j,0} \\ & + Cc_{j,-2+j,1} + B_1c_{j,-1+j,0} + B_1jc_{j,-1+j,0} + B_1c_{1+j,-2+j,0} + B_1jc_{1+j,-2+j,0}). \end{aligned}$$

By making use of the recursive formula of Theorem 3.1, we can obtain the singular values at the origin of system (3.5) with help of computer algebra system Mathematica 7.0, namely the following theorem.

Theorem 3.2. *The simplified expressions of the first five singular values at the origin of system* (3.5) *are as follows:*

$$\begin{aligned} \mu_1 = & -\frac{i}{20}(A_2 + B_3)C, \\ \mu_2 = & \frac{iCB_1}{1200}(25A_2B_1 - 70B_1B_3 + 54A_1C + 22B_2C), \\ \mu_3 = & \frac{iC^2}{736440000}[(3347619A_1^2 - 9964150A_1B_2 - 3434431B_2^2)C^2 - 475B_1^3(86099A_1 + 3942B_2)], \\ \mu_4 = & -\frac{iC^6}{45475170000000B_1(86099A_1 + 3942B_2)^2}m_4, \\ \mu_5 = & -\frac{iC^8}{127172859060780000000000B_1^2(86099A_1 + 3942B_2)^3}m_5, \end{aligned}$$

where

$$\begin{aligned} m_4 = & 1372328508460045614558A_1^5 - 1316781479020086046443A_1^4B_2 \\ & - 2738536719956146718841A_1^3B_2^2 - 4935337386291807670874A_1^2B_2^3 \\ & + 947409725662684456014A_1B_2^4 + 975829407980036385 6B_2^5, \\ m_5 = & 1297403394924790703138396915618379A_1^7 \\ & + 2204998752178358559079867087363 68A_1^6B_2 \\ & - 8204926501612658110679519467756935A_1^5B_2^2 \\ & - 2477303818460171440677202943596104 0A_1^4B_2^3 \\ & - 1389096423936903428937494675272847 5A_1^3B_2^4 \\ & - 3239163577332233019751853146115648A_1^2B_2^5 \\ & - 3917117093326889506951642114872 01A_1B_2^6 \end{aligned}$$

$$- 180511119235343083860296074479 60B_2^7.$$

In the above expressions of μ_n $n \in \{2,3,4,5\}$, *we have let* $\mu_1 = \cdots \mu_{n-1} = 0$.

Proof. The course of proof on Theorem 3.2 can be realized by computer with help of computer algebra system Mathematica 7.0. According to the recursive formula of Theorem 3.1, we have

$$\mu_1 = -\frac{i}{20}(A_2 + B_3)C,$$

because $C \neq 0$, then we let $A_2 + B_3 = 0$ which deduce that

$$B_3 = -A_2, \tag{3.6}$$

moreover we have

$$\mu_2 = \frac{iCB_1}{1200}(25A_2B_1 - 70B_1B_3 + 54A_1C + 22B_2C).$$

From $CB_1 \neq 0$, we let $25A_2B_1 - 70B_1B_3 + 54A_1C + 22B_2C = 0$, then

$$A_2 = -\frac{2(27A_1C + 11B_2C)}{95B_1}, \tag{3.7}$$

at this time we have

$$\begin{aligned}\mu_3 =& \frac{iC^2}{736440000}[(3347619A_1^2 - 9964150A_1B_2 - 3434431B_2^2)C^2 \\ & - 475B_1^3(86099A_1 + 3942B_2)].\end{aligned}$$

According to $\mu_3 = 0$ and $C \neq 0$, we have

$$B_1^3 = \frac{(3347619A_1^2 - 9964150A_1B_2 - 3434431B_2^2)C^2}{475(86099A_1 + 3942B_2)}, \tag{3.8}$$

under conditions (3.6) (3.7) (3.8), we obtain

$$\mu_4 = -\frac{iC^6}{45475170000000B_1(86099A_1 + 3942B_2)^2}m_4,$$

and

$$\mu_5 = -\frac{iC^8}{127172859060780000000000B_1^2(86099A_1 + 3942B_2)^3}m_3.$$

While the remainder from dividing the polynomial m_3 in A_1 by m_4 is m_5, hence

$$\mu_5 = -\frac{iC^8}{127172859060780000000000B_1^2(86099A_1 + 3942B_2)^3}m_5.$$

From the above course of computation, it can be seen that the expression of μ_n, $n \in \{2,3,4,5\}$ is obtained after letting $\mu_1 = \cdots \mu_{n-1} = 0$. □

According to the relation between the singular values and the focal values, we have the following theorem.

Theorem 3.3. *The first five focal values at the origin of system* (3.3) *(or the first five general focal values at the origin of system* (1.2)*) are expressed as follows:*

$$v_3 = \frac{\pi}{20}(A_2 + B_3)C,$$

$$v_5 = -\frac{\pi C B_1}{1200}(25A_2B_1 - 70B_1B_3 + 54A_1C + 22B_2C),$$
$$v_7 = -\frac{\pi C^2}{736440000}[(3347619A_1^2 - 9964150A_1B_2 - 3434431B_2^2)C^2 - 475B_1^3(86099A_1 + 3942B_2)],$$
$$v_9 = \frac{\pi C^6}{45475170000000B_1(86099A_1 + 3942B_2)^2}m_4,$$
$$v_{11} = \frac{\pi C^8}{12717285906078000000000B_1^2(86099A_1 + 3942B_2)^3}m_5,$$

in which the expressions of m_4, m_5 *are the same as those of Theorem 3.2.*

3.3. Limit cycle bifurcation of system (1.2)

At first, we may as well consider the limit cycles bifurcation at the origin of system (3.3). From Lemma 2.5 and Theorem 3.3, we only need to consider the number of order that origin of system (3.3) become a fine focus. We can obtain the following theorem.

Theorem 3.4. *The origin of system* (3.3) *can become a fine focus of 5th order if and only if the following condition holds:*

$$\begin{aligned} &B_3 = -A_2,\ A_2 = -\tfrac{2(27A_1C+11B_2C)}{95B_1},\\ &B_1^3 = \tfrac{(3347619A_1^2-9964150A_1B_2-3434431B_2^2)C^2}{475(86099A_1+3942B_2)},\\ &m_4 = 0,\ m_5 \neq 0. \end{aligned} \tag{3.9}$$

Proof. According to the proof course of Theorem 3.2, the above conclusion is clear. □

In fact, under condition (3.9), the real number solutions about A_1, A_2, B_1, B_2, B_3, C such that $v_3 = v_5 = v_7 = v_9 = 0$ and $v_{11} \neq 0$. If conditions (3.6) (3.7) (3.8) hold, then $v_3 = v_5 = v_7 = 0$. We can obtain the solution of $m_4 = m_5 = 0$ is $A_1 = 0, B_2 = 0$, while $A_1 \neq 0, B_2 \neq 0$, hence equations group $m_4 = m_5 = 0$ don't hold, which deduce $v_9 = v_{11} = 0$ don't hold at the same time. Next we try to find the relation between A_1 and B_2 if the origin of system (3.3) is a fine focus of 5th order. Let $m_4 = 0$, we have $B_2 = \lambda A_1$, in which λ is the real number root of the following equation

$$137232850846045614558 - 1316781479020086046443\lambda - 2738536719956146718841\lambda^2 - 493533738629180767087\lambda^3 + 94740972566268445601\lambda^4 + 9758294079800363856\lambda^5 = 0. \tag{3.10}$$

Equation (3.10) have 5 real number roots, namely

$$\lambda \approx -12.0509,\ \lambda \approx -3.32558,\ \lambda \approx -0.641669,\ \lambda \approx 0.0878996,\ \lambda \approx 6.22153. \tag{3.11}$$

Hence $B_2 = \lambda A_1$, λ satisfies (3.11) and $A_1 \neq 0$.

From the above analysis, the real number roots about A_1, A_2, B_1, B_2, B_3, C such that $v_3 = v_5 = v_7 = v_9 = 0$ and $v_{11} \neq 0$ have an infinite number of groups of real numbers.

According to Theorem 3.3 and Theorem 3.4, we will deduce the following theorem.

Theorem 3.5. *Suppose that the origin of system* (3.3) *is a fine focus of 5th order, then under a certain parameters' perturbed condition, the origin of system* (3.3) *can bifurcate 5 small limit cycles in which 3 limit cycles can be 3 stable cycles, and system* (1.2) *can bifurcate 5 limit cycles, in which 3 limit cycles can be 3 stable cycles.*

Proof. If condition (3.11) holds, then the origin of unperturbed system (3.3) is a fine focus of 5th order, and the Jacobian of the functions group (v_3, v_5, v_7, v_9) with respect to the variables group (B_3, A_2, B_1, A_1) is as follows:

$$J = \begin{vmatrix} \frac{\partial v_3}{\partial B_3} & \frac{\partial v_3}{\partial A_2} & \frac{\partial v_3}{\partial B_1} & \frac{\partial v_3}{\partial A_1} \\ \frac{\partial v_5}{\partial B_3} & \frac{\partial v_5}{\partial A_2} & \frac{\partial v_5}{\partial B_1} & \frac{\partial v_5}{\partial A_1} \\ \frac{\partial v_7}{\partial B_3} & \frac{\partial v_7}{\partial A_2} & \frac{\partial v_7}{\partial B_1} & \frac{\partial v_7}{\partial A_1} \\ \frac{\partial v_9}{\partial B_3} & \frac{\partial v_9}{\partial A_2} & \frac{\partial v_9}{\partial B_1} & \frac{\partial v_9}{\partial A_1} \end{vmatrix} = -\frac{B_1^3 C^{10}\pi^4}{5937238195200000000000(86099A1 + 3942B2)^2} m_1, \quad (3.12)$$

in which

$$\begin{aligned} m_1 =& 35446833674981044103487726A_1^5 - 22404227763412521796245353 4A_1^4 B_2 \\ & - 256548283412692993125804483A_1^3 B_2^2 - 323859352502013910970136 66A_1^2 B_2^3 \\ & - 120481229923356080655144 07A_1 B_2^4 - 130688981009723284271634 6B_2^5. \end{aligned}$$

By computing, we can obtain the resultant of m_1 and m_4 on A_1 is as follows:

$$r = Resultant[m_1, m_4, A_1] = 8721246660991825 \cdots 8750000000B_2^{25}$$

in which $B_2 \neq 0$, so $m_4 = 0$ deduces that $m_1 \neq 0$. In addition, $B_1 \neq 0$, $C \neq 0$, $B_2 \neq 0$, hence $J \neq 0$ under condition (3.9).

Obviously, equations group $v_3 = v_5 = v_7 = v_9 = 0$ have many real number solutions on B_3, A_2, B_1, A_1 such that $v_{11} \neq 0$. We may as well let $B_3 = b_3$, $A_2 = a_2$, $B_1 = b_1$, $A_1 = a_1$ are a group of solutions satisfying $v_3 = v_5 = v_7 = v_9 = 0, v_{11} \neq 0$.

Give a suitable perturbation about these parameters, we may as well let

$$v_3(2\pi, \varepsilon) = \varepsilon_1,\ v_5(2\pi, \varepsilon) = \varepsilon_2,\ v_7(2\pi, \varepsilon) = \varepsilon_3,\ v_9(2\pi, \varepsilon) = \varepsilon_4, \quad (3.13)$$

in which ε_1, ε_2, ε_3, ε_4 are a group of arbitrary real numbers. Because $J \neq 0$, then according to existence theorem of implicit function, equation (3.13) has a group of solutions as follows:

$$\begin{aligned} B_3 &= b_3 + f_1(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \\ A_2 &= a_2 + f_2(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \\ B_1 &= b_1 + f_3(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \\ A_1 &= a_1 + f_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4). \end{aligned} \quad (3.14)$$

Obviously, given perturbations by (3.14) will let (3.13) hold.

Here we can let

$$\begin{aligned} &\delta = \tfrac{1}{2}c_0\epsilon^{10} + o(\epsilon^{10}),\ \epsilon_1 = c_1\pi\epsilon^8 + o(\epsilon^9),\ \epsilon_2 = c_2\pi\epsilon^6 + o(\epsilon^7), \\ &\epsilon_3 = c_3\pi\epsilon^4 + o(\epsilon^5),\ \epsilon_4 = c_4\pi\epsilon^2 + o(\epsilon^3), \end{aligned} \quad (3.15)$$

in which

$$\begin{aligned} &c_0 = -14400j_0,\ c_1 = 21076j_0,\ c_2 = -7645j_0,\ c_3 = 1023j_0, \\ &c_4 = -55j_0,\ j_0 = v_{11}|_{\epsilon=0, M=0, A=a, B=b, C=c} \neq 0. \end{aligned} \quad (3.16)$$

At this time, the focal values at the origin of perturbed system (3.3) (or the general focal values at the origin of perturbed system (1.2)) are follows:

$$\begin{aligned}
&v_1(2\pi,\epsilon)-1=e^{2\pi\delta}-1=c_0\pi\epsilon^{10}+o(\epsilon^{11}),\\
&v_3(2\pi,\epsilon)=c_1\pi\epsilon^{8}+o(\epsilon^{9}),\\
&v_5(2\pi,\epsilon)=c_2\pi\epsilon^{6}+o(\epsilon^{7}),\\
&v_7(2\pi,\epsilon)=c_3\pi\epsilon^{4}+o(\epsilon^{5}),\\
&v_9(2\pi,\epsilon)=c_4\pi\epsilon^{2}+o(\epsilon^{3}),\\
&v_{11}(2\pi,\epsilon)=v_9|_{\epsilon=0}+o(\epsilon).
\end{aligned} \tag{3.17}$$

At this time, Poincaré succession function for the origin of system (3.3) is changed into the following form:

$$\begin{aligned}
d(\epsilon h)&=r(2\pi,\epsilon h)-\epsilon h\\
&=(v_1(2\pi,\epsilon)-1)\epsilon h+v_2(2\pi,\epsilon)(\epsilon h)^2+v_3(2\pi,\epsilon)(\epsilon h)^3+\ldots+v_{11}(2\pi,\epsilon)(\epsilon h)^{11}+\cdots\\
&=\pi\epsilon^{11}h[g(h)+\epsilon hG(h,\epsilon)],
\end{aligned} \tag{3.18}$$

in which

$$\begin{aligned}
g(h)&=c_0+c_1h^2+c_2h^4+c_3h^6+c_4h^8+j_0h^{10}\\
&=-14400j_0+21076j_0h^2-7645j_0h^4+1023j_0h^6-55j_0h^8+j_0h^{10}\\
&=j_0(h^2-1)(h^2-4)(h^2-9)(h^2-16)(h^2-25),
\end{aligned} \tag{3.19}$$

and $G(h,\epsilon)$ is an analytic function about h and ϵ at $(0,0)$.

Clearly, $g(h)=0$ has 5 simple positive real number solutions namely $1,2,3,4,5$. From implicit function theorem, the number of positive zero points of equation $d(\epsilon h)=0$ is equal to one of $g(h)=0$, and these positive zero points are close to $1,2,3,4,5$ if $0<|\epsilon|\ll 1$. The above analysis shows there exists 5 small limit cycles in a small enough neighborhood at the origin of system (3.3), which are close to cycles $x_1^2+y_1^2=k^2\epsilon^2, k=1,2,3,4,5$. At the same time, when $v_{11}<0$, there exists 3 stable limit cycles which are close to cycles $x_1^2+y_1^2=k^2\epsilon^2$, $k=1,3,5$. Correspondingly, system (1.2) has 5 limit cycles which are close to cycles $x^2+y^2=k^6\epsilon^6$, $(k\in\{1,2,3,4,5\})$. At the same time, when $v_{11}<0$, there exists 3 stable limit cycles which are close to spheres $x^2+y^2=k^6\epsilon^6$, $k=1,3,5$. □

4. Conclusion

The work of this paper focuses on the limit cycle bifurcation problem at infinity for a class of polynomial systems in 3-Dimensional vector fields. By making two appropriate transformations and making use of singular values methods on center manifolds to compute and simply the general focal values carefully, we give the expressions of the first five focal values of the infinity and prove the conditions of the fifth fine focuses. Moreover, we obtain the infinity can bifurcate 5 large limit cycles and the relative positions and stability of these limit cycles are given. Similar published results are hardly seen, and the result of the number of large limit cycles is new.

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