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# A SHADOWING LEMMA FOR RANDOM DYNAMICAL SYSTEMS

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**Abstract** This paper proves a shadowing lemma for the random dynamical systems generated by a class of random parabolic equations. We propose random versions of Newton's method and solution-tracing theory to obtain our main theorem. This result applies to $C^1$ random dynamical systems on Banach space without assuming the corresponding map to be a diffeomorphism. We also provide sufficient conditions to assure the measurability of the resulting solution. This measurability can be verified as long as a proper subsequence of the initial iteration sequence is measurable.

**Keywords** Shadowing lemma, random dynamical system, hyperbolic set.

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## 1. Introduction

As well known, the shadowing lemma has been widely used as an efficient tool to solve various problems in theoretical analysis and numerical computation. For example, through this lemma, Lin [24] substantiated the validity of the expansions for a singularity perturbed boundary value problem; Dellnitz and Melbourne [10] proved the density of periodic points with given symmetry; Chow and Van Vleck [8] estimated the global error of the solution for initial value ordinary differential equations. Another vital application of shadowing lemma is chaotic behavior detection. Due to the mysterious nature of chaotic orbits, the rigorous proof of chaotic behavior had been a rather complicated problem. However, the shadowing lemma vastly simplifies this proof procedure and provides new perspectives to investigate the orbital properties of chaotic dynamical systems.

The celebrated Birkhoff-Smale theorem has proposed that homoclinic points may cause quite complex behavior. By the construction of Smale horseshoe map [29], the essence of chaotic dynamical systems was gradually revealed and explicitly expressed as some iterates of diffeomorphism, which are topologically conjugate to a Bernoulli shift with finite symbols. After being put forward and proved (Anosov [1], Bowen [4]) and then successively generalized (Franke and Selgrade [15], Robinson [28], Guckenheimer etc [17]), the shadowing lemma was first used by Palmer [25] to construct the conjugation relationship for proving Birkhoff-Smale Theorem. It bridges the existence of pseudo-orbit to the fact that the discretization of the initial dynamical system admits a Bernoulli shift as a subsystem. After that, the shadowing lemma earned more and more attention and interest in many aspects.

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In deterministic cases, Chow etc [6] proved a shadowing lemma for $C^1$ maps in infinite-dimensional space, which may not be diffeomorphisms. Coomes etc [9] proposed a shadowing lemma for flows. Some other important results can be seen in Palmer [26], Pilyugin [27], Gan [16], Delshams etc [11], and recently Han and Wen [19], Delshams etc [12], Li [22]. In addition to the results mentioned before, the shadowing lemma is also of great significance in random dynamical systems theory. Chow and Van Vleck [7] proved a shadowing lemma for random diffeomorphisms. To take one step further, Gundlach [18] introduced stable and unstable manifolds for stationary orbits of the finite-dimensional random dynamical system corresponding to diffeomorphisms $\varphi(n, \omega)$, and then proved that the random hyperbolic set contains an invariant subset on which the dynamics are conjugate to random shifts. In [20], He etc first defined a type of hyperbolicity on full measure invariant sets and proved the shadowing property of diffeomorphisms by the assumptions of equicontinuity and uniformity of the Oseledec splitting. Though there are some results in random cases, most of them only took into account of the random mappings which are diffeomorphisms. In this paper, we weaken this assumption and consider the mappings that are not necessarily diffeomorphisms.

We intend to prove a shadowing lemma for a $C^1$ random dynamical system, which is generated by the solution operator of a random parabolic evolution equation. The time map of this system is not required to be a diffeomorphism. The process to achieve this goal involves the theory of invariant manifolds (see Deng and Xiao [13], Bento and Vilarinho [3]) and random exponential dichotomy (see Lian and Lu [23], Zhou etc [30]). Moreover, by following the ingenious method in [6], we present a random version of Newton's method, which plays an essential part in our proof. This attempt brings up a set of brand new problems. The most tricky one of them is the measurability of the tracing solution. To solve these problems, we propose a strategy centering around the strong measurability of specific operators. It is worth mentioning that, rather than the whole initial iteration sequence, the measurability of a proper subsequence will suffice to guarantee that the entire resulting solution is measurable. The detailed derivation and proof will be elucidated progressively in the following few sections.

The outline of this paper is as follows. In Section 2, we introduce some basic definitions and specify the system we will study. In particular, we generalize the idea of hyperbolic set to this random case and present their critical properties, which contribute to the main result as stated in Theorem 2.1. Section 3 investigates a set of random linear difference equations which serve as a discretization of the system we are interested in. As a preparation to prove the main result, a shadowing lemma for this linearized system is presented and proved in Section 4. Finally, Section 5 gives the proof of the main result.

## 2. Random Parabolic Equations

In this section, we first review some basic concepts about random dynamical systems, which are referenced from Arnold [2]. Then we describe the system we are concerned with and generalize some definitions for it. At the end of this section, we state the main theorem.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let time $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ and endow it with Borel $\sigma$-algebra $\mathcal{B}(\mathbb{T})$. $X$ is a separable Banach space with norm $|\cdot|$ and Borel $\sigma$-algebra $\mathcal{B}$.

**Definition 2.1.** A family of mappings $(\theta^t)_{t\in\mathbb{T}}$ from $\Omega$ into itself is called a **metric dynamical system** if

$(i)$ $(\omega,t)\to\theta^t\omega$ is $\mathcal{F}\otimes\mathcal{B}(\mathbb{T})$-measurable;

$(ii)$ $\theta^0=Id_\Omega$, $\theta^{t+s}=\theta^t\circ\theta^s$ for all $t,s\in\mathbb{T}$;

$(iii)$ $\theta^t$ preserves the probability measure $\mathbb{P}$.

For the dynamical systems with discrete time, part $(i)$ of the definition above can be reduced to $(i)'$ $\omega\to\theta^t\omega$ is $\mathcal{F}$ measurable for each $t\in\mathbb{T}$.

**Definition 2.2.** A **random dynamical system** (or RDS) on $X$ over a metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\theta^t)$ with time $\mathbb{T}$ is a mapping

$$\phi:\mathbb{T}\times\Omega\times X\to X,\qquad (t,\omega,x)\to\phi(t,\omega,x),$$

which satisfies

$(i)$ $\phi$ is $\mathcal{B}(\mathbb{T})\otimes\mathcal{F}\otimes\mathcal{B}$-measurable;

$(ii)$ The mappings $\phi(t,\omega):=\phi(t,\omega,\cdot)$ form a cocycle over $\theta^t$:

$$\begin{aligned}&\phi(0,\omega)=Id_X\qquad \text{for all } \omega\in\Omega,\\ &\phi(t+s,\omega)=\phi(t,\theta^s\omega)\circ\phi(s,\omega)\qquad \text{for all } s,t\in\mathbb{T},\omega\in\Omega.\end{aligned}$$

If in addition $\phi$ is differentiable with respect to $x$, and both $\phi$ and its derivative $D\phi$ are continuous with respect to $(t,x)$, then $\phi$ is called a $\boldsymbol{C^1}$ **random dynamical system**.

Consider the random parabolic evolution equation

$$\dot{x}+Ax=f(\theta^t\omega,x),\tag{2.1}$$

where $A$ is a sectorial operator on $X$ with $\operatorname{Re}\sigma(A)>0$. By Theorem 1.3.4 on Henry [21, p20], $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-At}\}_{t\geqslant 0}$. And also, for $0\leqslant\alpha\leqslant 1$, the fractional power operator $A^\alpha:\mathscr{D}(A^\alpha)\to X$ is well-defined, and the space $X^\alpha=\mathscr{D}(A^\alpha)$ is Banach with graph norm $|x|_\alpha=|A^\alpha x|$. Both of these two norms can induce corresponding measurable norms $|\cdot|_.$ and $|\cdot|_{(\cdot,\alpha)}$ on $\Omega\times X$ and $\Omega\times X^\alpha$ respectively, i.e., the mappings $(\omega,x)\to|x|_\omega$ and $(\omega,x)\to|x|_{(\omega,\alpha)}$ are $\mathcal{F}\otimes\mathcal{B}$-measurable. The corresponding norm of the operators from $\{\theta^m\omega\}\times X$ to $\{\theta^n\omega\}\times X$ (resp. from $\{\theta^m\omega\}\times X^\alpha$ to $\{\theta^n\omega\}\times X^\alpha$) is denoted by $|\cdot|_{(m,n,\omega)}$ (resp. $|\cdot|_{(m,n,\omega,\alpha)}$). For the sake of conciseness, if a point is obviously on a certain fiber, we will omit the part of the subscript which indicates its location on base space. Assume $f(\cdot,x)$ is $(\mathcal{F},\mathcal{B})$-measurable for each $x\in X$, and $f(\omega,\cdot)$ is Lipschitz continuous for each $\omega\in\Omega$ with Lipschitz constant $\mathcal{L}(\omega)$ satisfying

$$\int_a^b\mathcal{L}(\theta^s\omega)ds<\infty,\qquad \text{for } -\infty<a<b<\infty.$$

Then according to Theorem 3.5 in Caraballo etc [5], given initial condition $x(t_0,\omega)=x\in X^\alpha$, the random differential equation (2.1) has a unique mild solution $x(t,\omega)$ on any interval $t\in[t_0,t_1]$ for any $\omega\in\Omega$ and it generates a continuous random dynamical system, which is given by

$$\phi(t,\theta^{t_0}\omega,x)=e^{-At}x+\int_0^t e^{-A(t-s)}f(\theta^{s+t_0}\omega,\phi(s,\theta^{t_0}\omega,x))ds.\tag{2.2}$$

Note that (2.2) implies $\phi(0,\theta^{t_0}\omega,x)=x$ for all $\omega$. Assume in addition that $x\to f(\omega,x)$ is $C^1$ and Lipschitz continuous, $t\to f(\theta^t\omega,x)$ is locally Hölder continuous, and also in the sense of operator norm, $x\to Df(\omega,x)$ is Lipschitz continuous, $t\to Df(\theta^t\omega,x)$ is Hölder continuous. Then $\phi$ is a $C^1$ random dynamical system.

In order to present a random version of Shadowing Lemma for this system precisely, we need to introduce some notations and definitions. First, denote the metric on $X$ measuring the distance between a point $x$ and a set $U\subset X$ as

$$d(x,U)=\inf_{y\in U}|x-y|.$$

Then a mapping $\Lambda:\Omega\to 2^X$ is called a random set if $\omega\to d(x,\Lambda(\omega))$ is measurable for any $x\in X$. In addition, if each $\Lambda(\omega)$ is closed (compact), $\Lambda$ is said to be a random closed (compact) set. If $\Lambda^c$ is a random closed set, $\Lambda$ is called a random open set. A set-valued mapping of this kind can also define a subbundle $\{(\omega,x)\in\Omega\times X|x\in\Lambda(\omega)\}$, which is also denoted by $\Lambda$. To avoid causing confusion, we will explicitly specify which one is mentioned each time they appear. We say that a subbundle $\Lambda$ is forward invariant under random dynamical system $\phi$, if $\phi(t,\theta^{t_0}\omega,\Lambda(\theta^{t_0}\omega))\subset\Lambda(\theta^{t+t_0}\omega)$ for all $t\geqslant 0$.

**Definition 2.3.** A random bounded linear operator $T:\Omega\to\mathscr{L}(X)$ is called **strongly measurable** on $X$, if the mapping $\omega\to T(\omega)x$ is $(\mathcal{F},\mathcal{B})$-measurable for each $x\in X$. If for every $\omega\in\Omega$, $T(\omega)$ is invertible, then $T$ is called **invertible**.

Now we are ready to define the hyperbolicity of subbundle in the sense of random dynamical system.

**Definition 2.4.** A forward invariant subbundle $\Lambda$ under RDS $\phi$ is called **hyperbolic**, if $D\phi$ admits **exponential dichotomy** on $\Lambda$, which means the following conditions are satisfied:

(i) For each $(\omega,x)\in\Lambda$, there is a splitting

$$X^\alpha=E^s(\omega,x)\oplus E^u(\omega,x),$$

which is invariant under $D\phi$ in the following sense,

$$D\phi(t,\theta^{t_0}\omega,x)E^s(\theta^{t_0}\omega,x)\subset E^s(\theta^{t+t_0}\omega,\phi(t,\theta^{t_0}\omega,x)),$$
$$D\phi(t,\theta^{t_0}\omega,x)E^u(\theta^{t_0}\omega,x)\subset E^u(\theta^{t+t_0}\omega,\phi(t,\theta^{t_0}\omega,x)),$$

for all $(\theta^{t_0}\omega,x)\in\Lambda$ and $t>0$. And this splitting is also measurable and continuous, which means the splitting projection $\mathbf{P}(\omega,x)$ with range $E^s(\omega,x)$ and nullspace $E^u(\omega,x)$ is strongly measurable in $(\omega,x)$ and uniformly continuous in the operator norm.

(ii) $D\phi(t,\theta^{t_0}\omega,x):E^u(\theta^{t_0}\omega,x)\to E^u(\theta^{t+t_0}\omega,\phi(t,\theta^{t_0}\omega,x))$ is an isomorphism with bounded inverse

$$D\phi(t,\theta^{t_0}\omega,x)^{-1}:E^u(\theta^{t+t_0}\omega,\phi(t,\theta^{t_0}\omega,x))\to E^u(\theta^{t_0}\omega,x)$$

satisfying $D\phi(t,\theta^{t_0}\omega,x)^{-1}(I-\mathbf{P}(\theta^{t_0}\omega,x))$ is strongly measurable in $(\omega,x)$. And there exist constants $K\geqslant 1$, $\beta>0$ such that for all $(\theta^{t_0}\omega,x)\in\Lambda$, $t>0$,

$$|D\phi(t,\theta^{t_0}\omega,x)\mathbf{P}(\theta^{t_0}\omega,x)|\leqslant Ke^{-\beta t},$$
$$|D\phi(t,\theta^{t_0}\omega,x)^{-1}(I-\mathbf{P}(\theta^{t+t_0}\omega,\phi(t,\theta^{t_0}\omega,x)))|\leqslant Ke^{-\beta t}.$$

Let $\bigcup_{n\in\mathbb{Z}}[\tau_{n-1},\tau_n]=\mathbb{R}$ be a partition of $\mathbb{R}$ with $0<\tau\leqslant\tau_n-\tau_{n-1}\leqslant 2\tau$ for all $n$, where $\tau>0$ is a constant.

**Definition 2.5.** Let $\delta:\Omega\to\mathbb{R}^+$ be a $\theta^t$ invariant random variable, i.e., $\delta(\theta^t\omega)=\delta(\omega)$ for all $t\geqslant 0$. A sequence $\{x_n(t,\omega)\}_{n\in\mathbb{Z}}$, $t\in[\tau_{n-1},\tau_n]$ is called an $\boldsymbol{(\omega,\delta)}$ **pseudo-solution** for (2.1) if for all $n$,

$$|x_n(\tau_n,\omega)-x_{n+1}(\tau_n,\omega)|_\alpha\leqslant\delta(\omega),$$

and

$$\sup\{|h_n(\theta^t\omega)|:\tau_{n-1}\leqslant t\leqslant\tau_n\}\leqslant\delta(\omega),$$

where

$$h_n(\theta^t\omega):=\dot{x}_n(t,\omega)+Ax_n(t,\omega)-f(\theta^t\omega,x_n(t,\omega)).$$

**Definition 2.6.** Let $\varepsilon:\Omega\to\mathbb{R}^+$ be a $\theta^t$ invariant random variable. An $\omega$-solution $x(t,\omega)$ of (2.1) is said to $\boldsymbol{(\omega,\varepsilon)}$**-shadows** the $(\omega,\delta)$ pseudo-solution $\{x_n(t,\omega)\}$ if $x(t,\omega)$ is defined for all $t\in\mathbb{R}$ and

$$|x(t,\omega)-x_n(t,\omega)|\leqslant\varepsilon(\omega)\qquad\text{for }\tau_{n-1}\leqslant t\leqslant\tau_n,n\in\mathbb{Z}.$$

Our main result then can be summarized as follows.

**Theorem 2.1.** *Suppose $\Lambda$ is a forward invariant hyperbolic subbundle of random dynamical system (2.2). There is a positive $\theta^t$ invariant random variable $\Delta(\omega)$, such that for each $\omega\in\Omega$, both $f(\omega,x)$ and $Df(\omega,x)$ are bounded in the $\Delta$-neighborhood $O$ of $\Lambda$, which is $\omega$-wise defined as $O(\omega)=\{x\in X|d(x,\Lambda(\omega))\leqslant\Delta(\omega)\}$. Let $\{x_n(t,\omega)\}_{n\in\mathbb{Z}}$, $\tau_{n-1}\leqslant t\leqslant\tau_n$ be a $(\omega,\delta)$ pseudo-solution of (2.1), where $x_n(t,\omega)$ is in a $\delta(\omega)$-neighborhood of $\Lambda(\theta^t\omega)$. Then there exists a positive $\theta^t$ invariant random variable $\varepsilon_0$, such that if $0<\varepsilon(\omega)\leqslant\varepsilon_0(\omega)$, then there is an $\varepsilon$-dependent positive random variable $\delta(\omega,\varepsilon)$, such that if $\delta(\omega)\leqslant\delta(\omega,\varepsilon)$, there exists a unique $\omega$-solution $x(t,\omega)$ of (2.1) which $(\omega,\varepsilon)$-shadows $\{x_n(t,\omega)\}$. Furthermore, if there exist positive integers $k$ and $l$ satisfying $16K^3\lambda^k\leqslant 1$, $k\leqslant l$, such that $\{x_n(\tau_{n-1},\omega)\}$ possesses a measurable subsequence $\{x_{n_i}(\tau_{n_i-1},\omega)\}_{i\in\mathbb{Z}}$ with $k\leqslant n_{i+1}-n_i\leqslant l$, then $x(t,\omega)$ is measurable for all $t$.*

# 3. Random Difference Equations and its Linearization

In this section, we change the angle of view to consider a nonautonomous random difference equation with discrete time and its linearization. This procedure will show a clearer picture of the behavior of the system we care about, since the dynamics of a system on hyperbolic invariant set can be reflected by the exponential dichotomy of its linearized system.

Let $\varphi:\Omega\times X\to X$ be a measurable mapping and $\varphi(\omega,\cdot)$ is $C^1$ for each $\omega\in\Omega$. Consider the system generated by $\{\varphi(\theta^n\omega,\cdot)\}_{n\in\mathbb{Z}}$. The role of $\varphi$ in this system is analogous to the time-one map in a classic random dynamical system, but it is not necessarily a diffeomorphism, and the invertibility about it will be specified at once.

**Definition 3.1.** For fixed $\omega\in\Omega$, an $\boldsymbol{\omega}$**-orbit** of $\{\varphi(\theta^n\omega,\cdot)\}_{n\in\mathbb{Z}}$ is a sequence of points $\{x_n\}_{n\in\mathbb{Z}}$ with $x_n\in\{\theta^n\omega\}\times X$ for all $n$, which satisfies

$$x_{n+1}=\varphi(\theta^n\omega,x_n)\qquad\text{for all }n\in\mathbb{Z}.$$

A sequence of mappings $x_n : \Omega \to X$ is called an **orbit** of $\varphi$, if $\{x_n(\omega)\}_{n\in\mathbb{Z}}$ is an $\omega$-orbit of $\{\varphi(\theta^n\omega, \cdot)\}_{n\in\mathbb{Z}}$ for every $\omega \in \Omega$.

Such $\{x_n\}_{n\in\mathbb{Z}}$ can also be called a solution of measurable difference equation

$$u_{n+1} = \varphi(\theta^n\omega, u_n).$$

Since $\varphi$ is measurable, a unique measurable forward solution will be generated if initial value $x_m(\omega) = \xi$ is preassigned for each $\omega \in \Omega$. Here $m$ represents the initial time and will be set 0. Assume $\Lambda : \Omega \to 2^X$ is a random set, and the subbundle $\Lambda$ it defines is invariant under $\varphi$, i.e., $\varphi(\theta^n\omega, \Lambda(\theta^n\omega)) \subset \Lambda(\theta^{n+1}\omega)$ for all $n$. There is a $\theta^n$ invariant positive function $\Delta(\omega)$, such that for all $n \in \mathbb{Z}$, $\omega \in \Omega$, $\varphi(\theta^n\omega, \cdot)$ and $D\varphi(\theta^n\omega, \cdot)$ are uniformly bounded and continuous in a closed $\Delta(\omega)$-neighborhood $O(\omega)$ of $\Lambda(\omega)$. Let the bound of $|D\varphi(\theta^n\omega, \cdot)|$ be $M$.

**Definition 3.2.** Let $\Lambda$ be an invariant subbundle under $\varphi$. It is called **hyperbolic**, if there exist constants $K \geqslant 1$, $0 \leqslant \lambda < 1$ and for all $(\omega, x) \in \Lambda$ a splitting

$$X = E^s(\omega, x) \oplus E^u(\omega, x),$$

which is invariant under $D\varphi(\theta^n\omega, x)$ in the following sense,

$$\begin{aligned} D\varphi(\theta^n\omega, x)E^s(\theta^n\omega, x) \subset E^s(\theta^{n+1}\omega, \varphi(\theta^n\omega, x)), \\ D\varphi(\theta^n\omega, x)E^u(\theta^n\omega, x) \subset E^u(\theta^{n+1}\omega, \varphi(\theta^n\omega, x)), \end{aligned}$$

and depends measurably on $(\omega, x) \in \Lambda$, for each fixed $\omega \in \Omega$ continuously on $x \in \Lambda(\omega)$, i.e., there is a strongly measurable projection $\mathbf{P}(\omega, x)$ with range $E^s(\omega, x)$ and nullspace $E^u(\omega, x)$, which satisfies $|\mathbf{P}(\omega, x)| \leqslant K, |I - \mathbf{P}(\omega, x)| \leqslant K$ and is uniformly continuous in operator norm with respect to $x \in \Lambda(\omega)$, such that $D\varphi(\theta^n\omega, x) : E^u(\theta^n\omega, x) \to E^u(\theta^{n+1}\omega, \varphi(\theta^n\omega, x))$ is an isomorphism with strongly measurable inverse $D\varphi^{-1}(\theta^n\omega, x) : E^u(\theta^{n+1}\omega, \varphi(\theta^n\omega, x)) \to E^u(\theta^n\omega, x)$. Furthermore, for any pair of integers $m < n$, the finite sequence $x_m(\omega)$, $x_{m+1}(\omega) = \varphi(\theta^m\omega, x_m(\omega))$, $x_{m+2}(\omega) = \varphi(\theta^{m+1}\omega, x_{m+1}(\omega))$, $\cdots$, $x_n(\omega) = \varphi(\theta^{n-1}\omega, x_{n-1}(\omega))$ satisfies

$$\begin{aligned} &|D\varphi(\theta^n\omega, x_n(\omega)) \cdots D\varphi(\theta^m\omega, x_m(\omega))\mathbf{P}(\theta^m\omega, x_m(\omega))| \leqslant K\lambda^{n-m+1}, \\ &|D\varphi^{-1}(\theta^m\omega, x_m(\omega)) \cdots D\varphi^{-1}(\theta^n\omega, x_n(\omega))(I - \mathbf{P}(\theta^{n+1}\omega, x_{n+1}(\omega)))| \leqslant K\lambda^{n-m+1}. \end{aligned}$$

**Definition 3.3.** Let $\delta : \Omega \to \mathbb{R}^+$ be a $\theta^n$ invariant function. For any $\omega \in \Omega$, a sequence $\{y_n(\omega)\}_{n\in\mathbb{Z}}$, where $y_n(\omega) \in \{\theta^n\omega\} \times X$, is called an $\boldsymbol{(\omega, \delta)}$ **pseudo-orbit** of $\{\varphi(\theta^n\omega, \cdot)\}_{n\in\mathbb{Z}}$, if

$$|y_{n+1}(\omega) - \varphi(\theta^n\omega, y_n(\omega))| \leqslant \delta(\omega) \qquad \text{for all } n \in \mathbb{Z}.$$

**Definition 3.4.** Let $\varepsilon : \Omega \to \mathbb{R}^+$ be a $\theta^n$ invariant function. An $\omega$-orbit $\{x_n(\omega)\}_{n\in\mathbb{Z}}$, where $x_n(\omega) \in \{\theta^n\omega\} \times X$, is said to $\boldsymbol{(\omega, \varepsilon)}$**-shadow** the $(\omega, \delta)$ pseudo-orbit $\{y_n(\omega)\}$, if

$$|x_n(\omega) - y_n(\omega)| \leqslant \varepsilon(\omega) \qquad \text{for all } n \in \mathbb{Z}.$$

To conclude the assumptions above and provide an auxiliary result for our main theorem, we present the following theorem and prove it in the next few sections.

**Theorem 3.1.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\theta^n)_{n\in\mathbb{Z}}$ be a metric dynamical system defined on $\Omega$. $X$ is a separable Banach space equipped with Borel $\sigma$-algebra $\mathcal{B}$. $\Lambda$ is a hyperbolic invariant subbundle for $C^1$ measurable mappings $\{\varphi(\theta^n\omega, \cdot)\}_{n\in\mathbb{Z}}$ with constants $K \geqslant 1$, $0 \leqslant \lambda < 1$. Then there exists a positive $\theta^n$ invariant function $\varepsilon_0(\omega)$, such that for each $\theta^n$ invariant function $\varepsilon(\omega)$ satisfying $0 < \varepsilon(\omega) \leqslant \varepsilon_0(\omega)$, there is a $\delta(\omega) > 0$ such that if $\{y_n(\omega)\}_{n\in\mathbb{Z}}$, $y_n(\omega) \in \Lambda(\theta^n\omega)$, is an $(\omega, \delta)$ pseudo-orbit for $\{\varphi(\theta^n\omega, \cdot)\}_{n\in\mathbb{Z}}$, then there is a unique orbit $\{x_n(\omega)\}_{n\in\mathbb{Z}}$ which $(\omega, \varepsilon)$-shadows $\{y_n(\omega)\}_{n\in\mathbb{Z}}$. Moreover, if there exist positive integers $k$ and $l$ satisfying $16K^3\lambda^k \leqslant 1$, $0 < k \leqslant l < \infty$ such that $\{y_n(\omega)\}_{n\in\mathbb{Z}}$ possesses a measurable subsequence $\{y_{n_i}(\omega)\}_{i\in\mathbb{Z}}$ with $k \leqslant n_{i+1} - n_i \leqslant l$ for any $i \in \mathbb{Z}$, then the orbit $\{x_n(\omega)\}_{n\in\mathbb{Z}}$ is measurable for all $n \in \mathbb{Z}$.*

Let $A_n : \Omega \to \mathscr{L}(X)$, $n \in \mathbb{Z}$, be a sequence of strongly measurable operators, then equations

$$x_{n+1} = A_n(\omega)x_n, \qquad n \in \mathbb{Z}, \tag{3.1}$$

constitute a measurable linear difference equation set, and the corresponding transition matrices of it are as follows

$$\Phi(n, m, \omega) = \begin{cases} A_{n-1}(\omega) \circ A_{n-2}(\omega) \circ \cdots \circ A_m(\omega) & \text{for } n > m, \\ I & \text{for } n = m. \end{cases}$$

**Definition 3.5.** We say the measurable linear difference equation set (3.1) has **exponential dichotomy** if there exist constants $K \geqslant 1$, $0 \leqslant \lambda < 1$ and strongly measurable projections $\mathbf{P}_n$ on $\{\theta^n\omega\} \times X$, such that for all $\omega \in \Omega$ and any arbitrary pair of integers $n \geqslant m$, we have

$$\Phi(n, m, \omega)\mathbf{P}_m(\omega) = \mathbf{P}_n(\omega)\Phi(n, m, \omega),$$
$$|\Phi(n, m, \omega)\mathbf{P}_m(\omega)| \leqslant K\lambda^{n-m}.$$

In addition, $\Phi(n, m, \omega) : \mathcal{N}(\mathbf{P}_m(\omega)) \to \mathcal{N}(\mathbf{P}_n(\omega))$ is an isomorphism with bounded strongly measurable inverse $\Phi(n, m, \omega)^{-1} = \Phi(m, n, \omega) : \mathcal{N}(\mathbf{P}_n(\omega)) \to \mathcal{N}(\mathbf{P}_m(\omega))$ satisfying

$$|\Phi(m, n, \omega)(I - \mathbf{P}_n(\omega))| \leqslant K\lambda^{n-m}.$$

Now we introduce a product space $\Pi X = \cdots \times X \times X \times X \times \cdots$ of bounded sequences $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}}$, $x_n \in X$. It is a Banach space if we endow it with norm

$$\|\mathbf{x}\| = \sup_{n\in\mathbb{Z}} |x_n|.$$

This norm also induces a measurable norm

$$\|\mathbf{x}\|_\omega = \sup_{n\in\mathbb{Z}} |x_n|_\omega$$

on $\Omega \times \Pi X$, since the norm on $\Omega \times X$ is measurable. The corresponding operator norm from $\{\theta^m\omega\} \times \Pi X$ to $\{\theta^n\omega\} \times \Pi X$ is denoted by $\|\cdot\|_{(m,n,\omega)}$. Unless it is necessary, the subscripts will be omitted. Define $\pi_n : \Pi X \to X$ as $\pi_n(\mathbf{x}) = x_n$ to be the coordinate map. Recall that $\mathcal{B}$ is the Borel $\sigma$-algebra on $X$, then the product Borel $\sigma$-algebra $\bigotimes \mathcal{B}$ on $\Pi X$ is generated by $\{\pi_n^{-1}(V_n) | V_n \in \mathcal{B} \text{ for } n \in \mathbb{Z}\}$, where $\pi_n^{-1}(V_n) = \Pi_{m\in\mathbb{Z}} V_m$ and $V_m = X$ for $m \neq n$. By proposition 1.3 in Folland [14, p23], $\bigotimes \mathcal{B}$ is also generated by $\{\Pi_{n\in\mathbb{Z}} V_n | V_n \in \mathcal{B}\}$. So we have

**Lemma 3.1.** *If $\mathbf{x}(\omega) = \{x_n(\omega)\}_{n\in\mathbb{Z}}$, then $\mathbf{x}(\omega) : \Omega \to \Pi X$ is measurable if and only if $x_n(\omega) : \Omega \to X$ is measurable for all $n \in \mathbb{Z}$.*

This lemma leads us directly to a fact about strongly measurable operators.

**Corollary 3.1.** *Let $T_n : \Omega \to \mathscr{L}(X)$, $n \in \mathbb{Z}$, be a sequence of strongly measurable operators on $X$, then the operator $T : \Omega \to \mathscr{L}(\Pi X)$ defined as*

$$(T(\omega)\mathbf{x})_n = T_n(\omega)x_n$$

*is strongly measurable on $\Pi X$.*

We also need another simple lemma about strongly measurable operators.

**Lemma 3.2.** *Let $T_1$, $T_2 : \Omega \to \mathscr{L}(X)$ be two strongly measurable operators, then $-T_1$, $-T_2$, $T_1 + T_2$, and $T_1 \circ T_2$ are all strongly measurable on $X$.*

With the assistance of the results above, we are safe to draw the following conclusion.

**Proposition 3.1.** *Assume $A_i : \Omega \to \mathscr{L}(X)$, $i \in \mathbb{Z}$ are strongly measurable satisfying*

$$\sup_{n\in\mathbb{Z}} |A_i(\omega)| < \infty \qquad \text{for all } \omega \in \Omega.$$

*$Q_i : \Omega \to \mathscr{L}(X)$ are strongly measurable projections with properties that there exist positive constants $K$, $M_1$, $M_2$ satisfying $K \geqslant 1$, $8KM_1 \leqslant 1$, $8M_2 \leqslant 1$, such that for all $i \in \mathbb{Z}$, $\omega \in \Omega$,*

$$\begin{aligned}
&|Q_i(\omega)| \leqslant K, |I - Q_i(\omega)| \leqslant K, |A_i(\omega)Q_i(\omega)| \leqslant M_1,\\
&|Q_{i+1}(\omega)A_i(\omega)(I - Q_i(\omega))| \leqslant M_2,\\
&|(I - Q_{i+1}(\omega))A_i(\omega)Q_i(\omega)| \leqslant M_2.
\end{aligned}$$

*Suppose $(I - Q_{i+1}(\omega))A_i(\omega) : \mathcal{N}(Q_i(\omega)) \to \mathcal{N}(Q_{i+1}(\omega))$ has an inverse $B_i(\omega)$ such that*

$$|B_i(\omega)(I - Q_{i+1}(\omega))| \leqslant M_1,$$

*and $B_i(\omega)(I - Q_i(\omega))$ is strongly measurable. Then the random operator $L : \Omega \to \mathscr{L}(\Pi X)$ defined as $(L(\omega)\mathbf{x})_i = x_i - A_{i-1}(\omega)x_{i-1}$ is strongly measurable and invertible with a strongly measurable inverse $L^{-1}$ which satisfies $\left\|L^{-1}(\omega)\right\| \leqslant 2K + 1$ for all $\omega$. And if $\mathbf{x} = \mathbf{x}(\cdot) = \{x_i(\cdot)\}_{i\in\mathbb{Z}}$, where $x_i : \Omega \to X$ is measurable for all $i$, then $\{(L^{-1}(\omega)\mathbf{x}(\omega))_i\}$ is a measurable sequence.*

**Proof.** By Lemma 3.2 in [6], for arbitrarily fixed $\omega$, $L(\omega)$ is invertible and its inverse satisfies $\|L(\omega)^{-1}\| \leqslant 2K + 1$. Then as a random operator, $L$ is invertible and its inverse possesses the same norm estimation. To elaborate further, if we define operator $S(\omega)$ on $\Pi X$ as $(S(\omega)\mathbf{h})_i = Q_i(\omega)h_i - B_i(\omega)(I - Q_{i+1}(\omega))h_{i+1}$, we have $\|I - L(\omega)S(\omega)\| \leqslant \frac{1}{2}$, then $L(\omega)S(\omega)$ has an inverse $T(\omega)$ with an expansion expression

$$T(\omega) = (L(\omega)S(\omega))^{-1} = (I - (I - L(\omega)S(\omega)))^{-1} = \sum_{j=0}^{\infty}(I - L(\omega)S(\omega))^j,$$

which is convergent in the operator norm on $\mathscr{L}(\Pi X)$, since $\|T(\omega)\| \leqslant (1 - \|I - L(\omega)S(\omega)\|)^{-1} \leqslant 2$. So the inverse of $L(\omega)$ can be written as

$$L(\omega)^{-1} = S(\omega)T(\omega) = \sum_{j=0}^{\infty} S(\omega)(I - L(\omega)S(\omega))^j. \tag{3.2}$$

By the measurability assumption of $A_i$, $B_i$, $Q_i$, Lemma 3.2 and Corollary 3.1, $S$ and $L$ are both strongly measurable. Then the expanded formula (3.2) implies $L^{-1}$ is strongly measurable. So for each $\mathbf{x} \in \Pi X$, $\omega \to L^{-1}(\omega)\mathbf{x}$ is measurable. Note that $\mathbf{x} \to L^{-1}(\omega)\mathbf{x}$ is continuous for each $\omega \in \Omega$, then $(\omega, \mathbf{x}) \to L^{-1}(\omega)\mathbf{x}$ is measurable. For any $\mathbf{x} = \mathbf{x}(\omega) = \{x_i(\omega)\}$, where $x_i(\omega)$ is measurable for all $i$, the mapping $\omega \to (\omega, \mathbf{x}(\omega))$ is measurable from $\Omega$ to $\Omega \times \Pi X$ by Lemma 3.1. By compositing the two mappings above, we have $\omega \to L^{-1}(\omega)\mathbf{x}(\omega)$ is measurable. Again by Lemma 3.1, $\{(L^{-1}(\omega)\mathbf{x}(\omega))\}$ is a measurable sequence, which completes the proof. □

**Remark 3.1.** The property of operator $L^{-1}$ in this proposition that it maps a measurable sequence to another could apply to all strongly measurable operators on separable product Banach space.

## 4. Proof of Theorem 3.1

In this section, we first prove a random Newton's method, and especially we prove that if the initial function of iteration is measurable, the unique resulting solution is measurable. Then we follow the procedure in [6] to give proof of Theorem 3.1.

**Proposition 4.1** (Newton's Method)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta^n)_{n\in\mathbb{Z}})$ be a metric dynamical system, $Y$ be a separable Banach space with Borel $\sigma$-algebra $\mathcal{B}(Y)$, $U(\omega) \subset Y$ be a random open subset and the subbundle defined by it is denoted by $U$. Suppose $\mathscr{F} : U \to Y$ is a mapping satisfying that $\omega \to \mathscr{F}(\omega, x)$ is measurable for each $x$, $(\omega, x) \to \mathscr{F}(\omega, x)$ is measurable, and $x \to \mathscr{F}(\omega, x)$ is $C^1$ for each $\omega$. Let $y : \Omega \to Y$ be a measurable function such that $y(\omega) \in U(\omega)$ for all $\omega \in \Omega$, and $D\mathscr{F}(\omega, y(\omega))^{-1}$ exists and is strongly measurable. Let $\varepsilon_0$ be a positive random variable such that for each $\omega \in \Omega$, if $\|x - y(\omega)\| \leqslant \varepsilon_0(\omega)$, we have*

$$\|D\mathscr{F}(\omega, x) - D\mathscr{F}(\omega, y(\omega))\| \leqslant (2\|D\mathscr{F}(\omega, y(\omega))^{-1}\|)^{-1}.$$

*Then if $0 < \varepsilon(\omega) \leqslant \varepsilon_0(\omega)$ and*

$$\|\mathscr{F}(\omega, y(\omega))\| \leqslant \varepsilon(\omega)(2\|D\mathscr{F}(\omega, y(\omega))^{-1}\|)^{-1},$$

*the random equation*

$$\mathscr{F}(\omega, x) = 0 \tag{4.1}$$

*has a unique measurable solution $x : \Omega \to Y$ such that $\|x(\omega) - y(\omega)\| \leqslant \varepsilon(\omega)$ for all $\omega \in \Omega$.*

**Proof.** For each $\omega \in \Omega$, $\mathscr{F}(\omega, x)$ can be expanded with respect to $x$ near $y(\omega)$ as following

$$\mathscr{F}(\omega, x) = \mathscr{F}(\omega, y(\omega)) + D\mathscr{F}(\omega, y(\omega))(x - y(\omega)) + \eta(\omega, x),$$

where $\eta(\omega, x) = o(x)$. Let $\varepsilon(\omega)$ be a random variable satisfying $0 < \varepsilon(\omega) \leqslant \varepsilon_0(\omega)$ for any $\omega$. Define operator $\mathcal{T}(\omega, \cdot)$ on $B_\varepsilon(\omega) = \{x \in Y : \|x - y(\omega)\| \leqslant \varepsilon(\omega)\}$ as

$$\mathcal{T}(\omega, x) := y(\omega) - D\mathscr{F}(\omega, y(\omega))^{-1}(\mathscr{F}(\omega, y(\omega)) + \eta(\omega, x)). \tag{4.2}$$

Then $\mathcal{T}(\omega, \cdot)$ is a contraction mapping on $B_\varepsilon(\omega)$ by Proposition 4.1 in [6]. By Banach contraction mapping principle, for each $\omega \in \Omega$, $\mathcal{T}(\omega, \cdot)$ has a unique fixed point $x(\omega)$, which is the unique $\omega$-solution of equation (4.1). After taking all values for $\omega$, the map $x : \Omega \to Y$ is the unique solution of (4.1).

Now we turn to the measurability of this unique solution. Set $x_0(\omega) = y(\omega)$. It can be seen from the process above that $x(\omega)$ is the limit of iteration sequence $x_n(\omega) = \mathcal{T}(\omega, x_{n-1}(\omega))$ for $n = 1, 2, \cdots$. The mapping $\omega \to \mathscr{F}(\omega, y(\omega))$ is measurable since it is the composition of measurable maps $\omega \to (\omega, y(\omega))$ and $(\omega, x) \to \mathscr{F}(\omega, x)$. Similarly, $\eta(\omega, y(\omega))$ is measurable. The operator $D\mathscr{F}(\omega, y(\omega))^{-1}$ is strongly measurable by assumption. Then every approximate solution we get from each step of iteration formula (4.2) is measurable. So the limit $x(\omega)$ is also measurable. $\square$

**Lemma 4.1.** *Let $X$ be a Banach space equipped with Borel $\sigma$-algebra $\mathcal{B}$. $(\Omega, \mathcal{F})$ is a measurable space. Let $P$, $Q : \Omega \to \mathscr{L}(X)$ be strongly measurable projections such that $|P(\omega)| \leqslant K$, $|I - P(\omega)| \leqslant K$ for all $\omega \in \Omega$. Then if $|P(\omega) - Q(\omega)| < \frac{1}{2K}$, the operator $J(\omega) = P(\omega)Q(\omega) + (I - P(\omega))(I - Q(\omega))$ is invertible with a strongly measurable inverse $J^{-1}(\omega)$ satisfying $|J^{-1}(\omega)| \leqslant (1 - 2K|P(\omega) - Q(\omega)|)^{-1}$. And $J(\omega)\mathcal{R}(Q(\omega)) = \mathcal{R}(P(\omega)), J(\omega)\mathcal{N}(Q(\omega)) = \mathcal{N}(P(\omega))$.*

**Proof.** For each fixed $\omega \in \Omega$, we know that $|I - J(\omega)| < 1$ through Lemma 5.1 in [6]. So $J(\omega)$ is invertible with inverse

$$J^{-1}(\omega) = \sum_{j=0}^{\infty}(I - J(\omega))^j = I + (I - J(\omega)) + (I - J(\omega))^2 + \cdots, \tag{4.3}$$

thus,

$$\begin{aligned}\|J^{-1}(\omega)\| &\leqslant \|I\| + \|I - J(\omega)\| + \|I - J(\omega)\|^2 + \cdots \\ &= (1 - \|I - J(\omega)\|)^{-1} \leqslant (1 - 2K\|P(\omega) - Q(\omega)\|)^{-1}.\end{aligned}$$

$J(\omega)\mathcal{R}(Q(\omega)) \subset \mathcal{R}(P(\omega))$, $J(\omega)\mathcal{N}(Q(\omega)) \subset \mathcal{N}(P(\omega))$ and the equalities follow from the invertibility of $J(\omega)$. Since $\omega$ is arbitrary, $J$ is invertible as a random operator. In addition, the strong measurability of $P$ and $Q$, the definition of $J$, the expansion formula (4.3) and Lemma 3.2 imply that $J$ and $J^{-1}$ are both strongly measurable. $\square$

Recall that $M$ is the uniform bound of $|D\varphi(\theta^n\omega, \cdot)|$ on $O(\omega)$, the $\Delta(\omega)$ neighborhood of $\Lambda(\omega)$.

**Lemma 4.2.** *Let $l < \infty$ be a positive integer. If $\{y_n(\omega)\}_{n\in\mathbb{Z}}$ is an $(\omega, \delta)$ pseudo-orbit of $\{\varphi(\theta^n\omega, \cdot)\}_{n\in\mathbb{Z}}$ with $y_n(\omega) \in \Lambda(\theta^n\omega)$ for all $n$. Then its subsequence $\{y_{n_i}(\omega)\}_{i\in\mathbb{Z}}$ satisfying $n_{i+1} - n_i \leqslant l$ is an $(\omega, \sum_{j=0}^{l-1} M^j\delta)$ pseudo-orbit of mappings $\{\varphi(\theta^{n_{i+1}-1}\omega, \cdot) \circ \cdots \circ \varphi(\theta^{n_i+1}\omega, \cdot) \circ \varphi(\theta^{n_i}\omega, \cdot)\}_{i\in\mathbb{Z}}$.*

**Proof.** For any arbitrary pair of $n_i$ and $n_{i+1}$, we have $n_{i+1} - n_i \in \{1, 2, \cdots, l\}$. We intend to prove this lemma by examining all these $l$ possibilities inductively.

If $n_{i+1}-n_i=1$, we have

$$|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_i}\omega,y_{n_i}(\omega))|\leqslant\delta(\omega)=\sum_{j=0}^{n_{i+1}-n_i-1}M^j\delta(\omega)\leqslant\sum_{j=0}^{l-1}M^j\delta(\omega),$$

since $\{y_n(\omega)\}_{n\in\mathbb{Z}}$ is an $(\omega,\delta)$ pseudo-orbit of $\{\varphi(\theta^n\omega,\cdot)\}_{n\in\mathbb{Z}}$.

Suppose

$$|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_{i+1}-1}\omega,\cdots\varphi(\theta^{n_i+1}\omega,\varphi(\theta^{n_i}\omega,y_{n_i}(\omega)))\cdots)|\leqslant\sum_{j=0}^{n_{i+1}-n_i-1}M^j\delta(\omega)$$

for the case of $n_{i+1}-n_i=a$, where $a\in\{1,\cdots,l-1\}$. Then for $n_{i+1}-n_i=a+1$,

$$\begin{aligned}
&|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_{i+1}-1}\omega,\cdots\varphi(\theta^{n_i+1}\omega,\varphi(\theta^{n_i}\omega,y_{n_i}(\omega)))\cdots)|\\
\leqslant&|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_{i+1}-1}\omega,y_{n_{i+1}-1}(\omega))|\\
&+|\varphi(\theta^{n_{i+1}-1}\omega,y_{n_{i+1}-1}(\omega))-\varphi(\theta^{n_{i+1}-1}\omega,\cdots\varphi(\theta^{n_i+1}\omega,\varphi(\theta^{n_i}\omega,y_{n_i}(\omega)))\cdots)|\\
\leqslant&\delta(\omega)+M\cdot|y_{n_{i+1}-1}(\omega)-\varphi(\theta^{n_{i+1}-2}\omega,\cdots\varphi(\theta^{n_i+1}\omega,\varphi(\theta^{n_i}\omega,y_{n_i}(\omega)))\cdots)|\\
\leqslant&\sum_{j=0}^{a}M^j\delta(\omega)\leqslant\sum_{j=0}^{l-1}M^j\delta(\omega).
\end{aligned}$$

The conclusion is drawn from the arbitrariness of pair $n_i$ and $n_{i+1}$. □

**Lemma 4.3.** *Let $\{y_n(\omega)\}_{n\in\mathbb{Z}}$ be an $(\omega,\delta)$ pseudo-orbit for $\{\varphi(\theta^n\omega,\cdot)\}_{n\in\mathbb{Z}}$, and $\{x_n(\omega)\}_{n\in\mathbb{Z}}$ be an $\omega$-orbit of $\{\varphi(\theta^n\omega,\cdot)\}_{n\in\mathbb{Z}}$ such that its subsequence $\{x_{n_i}(\omega)\}_{i\in\mathbb{Z}}$ satisfying $n_{i+1}-n_i\leqslant l$ can $(\omega,\varepsilon)$-shadow $\{y_{n_i}(\omega)\}_{i\in\mathbb{Z}}$. Set $\epsilon(\omega)=\max\{\varepsilon(\omega),\delta(\omega)\}$. Then if $(\sum_{j=0}^{l}M^j)\epsilon(\omega)\leqslant\Delta(\omega)$, orbit $\{x_n(\omega)\}_{n\in\mathbb{Z}}$ will $(\omega,\sum_{j=0}^{l}M^j\epsilon)$-shadow $\{y_n(\omega)\}_{n\in\mathbb{Z}}$.*

**Proof.** For any pair of $n_i$ and $n_{i+1}$, we have $n_{i+1}-n_i\in\{1,\cdots,l\}$. If $n_{i+1}-n_i=1$, then

$$\begin{aligned}
&|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_i}\omega,x_{n_i}(\omega))|\\
\leqslant&|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_i}\omega,y_{n_i}(\omega))|+|\varphi(\theta^{n_i}\omega,y_{n_i}(\omega))-\varphi(\theta^{n_i}\omega,x_{n_i}(\omega))|\\
\leqslant&\delta(\omega)+M\varepsilon(\omega)\leqslant(1+M)\epsilon(\omega).
\end{aligned}$$

Suppose for the case of $n_{i+1}-n_i=a$, where $a\in\{1,\cdots,l-1\}$,

$$|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_{i+1}-1}\omega,\cdots\varphi(\theta^{n_i+1}\omega,\varphi(\theta^{n_i}\omega,x_{n_i}(\omega)))\cdots)|\leqslant\sum_{j=0}^{a}M^j\epsilon(\omega),$$

then if $n_{i+1}-n_i=a+1$,

$$\begin{aligned}
&|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_{i+1}-1}\omega,\cdots\varphi(\theta^{n_i+1}\omega,\varphi(\theta^{n_i}\omega,x_{n_i}(\omega)))\cdots)|\\
\leqslant&|y_{n_{i+1}}(\omega)-\varphi(\theta^{n_{i+1}-1}\omega,y_{n_{i+1}-1}(\omega))|\\
&+|\varphi(\theta^{n_{i+1}-1}\omega,y_{n_{i+1}-1}(\omega))-\varphi(\theta^{n_{i+1}-1}\omega,\cdots\varphi(\theta^{n_i+1}\omega,\varphi(\theta^{n_i}\omega,x_{n_i}(\omega)))\cdots)|\\
\leqslant&\delta(\omega)+M\sum_{j=0}^{a}M^j\varepsilon(\omega)\leqslant\sum_{j=0}^{a+1}M^j\epsilon(\omega)\leqslant\sum_{j=0}^{l}M^j\epsilon(\omega).
\end{aligned}$$

Since $n_i$ and $n_{i+1}$ are arbitrary, this conclusion does follow. $\square$

Before proceeding to the next step, we need to introduce and reiterate some parameters and symbols. Let $k$, $l$ be two positive integers such that $0 < k \leqslant l < \infty$ and $16K^3\lambda^k \leqslant 1$. Pick an arbitrary subsequence $\{n_i\}_{i\in\mathbb{Z}}$ of $\{\cdots,-1,0,1,\cdots\}$, which satisfies $k \leqslant n_{i+1}-n_i \leqslant l$ for all $i \in \mathbb{Z}$. Let $F(i,\omega,\cdot)=\varphi(\theta^{n_{i+1}-1}\omega,\cdot)\circ\cdots\circ\varphi(\theta^{n_i}\omega,\cdot)$, then subbundle $\Lambda$ is invariant under $F$ as $F(i,\omega,\Lambda(\theta^{n_i}\omega))\subset\Lambda(\theta^{n_{i+1}}\omega)$. Define new random variables as

$$\mu(\omega,\eta)=\sup_i\{|DF(i,\omega,x)-DF(i,\omega,y)|: x\in\{\theta^{n_i}\omega\}\times X, y\in\Lambda(\theta^{n_i}\omega), |x-y|\leqslant\eta\},$$

$$\nu(\omega,\eta)=\sup\{|\mathbf{P}(\omega,x)-\mathbf{P}(\omega,y)|: x\in\{\omega\}\times X, y\in\Lambda(\omega), |x-y|\leqslant\eta\}.$$

Because of the uniform continuity of $DF(i,\omega,\cdot)$ and $\mathbf{P}(\omega,\cdot)$, we have $\mu(\omega,\eta)\to 0$, $\nu(\omega,\eta)\to 0$ for all $\omega\in\Omega$ if $\eta\to 0$. A positive random variable $\varepsilon_0$ can be chosen $\omega$-wise such that $\varepsilon_0(\omega)<\Delta(\omega)$ and $\mu(\omega,\varepsilon_0(\omega))<\frac{1}{4K+2}$. For given $0<\bar{\varepsilon}(\omega)\leqslant\varepsilon_0(\omega)$, let $\bar{\delta}(\omega)$ be a positive $\theta^n$ invariant random variable such that $\bar{\delta}(\omega)\leqslant\Delta(\omega)$, $(4K+2)\bar{\delta}(\omega)\leqslant\bar{\varepsilon}(\omega)$, $8M^lK\nu(\omega,\bar{\delta}(\omega))\leqslant 1$, $4K\nu(\omega,\bar{\delta}(\omega))\leqslant 1$ for all $\omega\in\Omega$.

We first prove Shadowing Lemma for random mapping sequence $\{F(i,\omega,\cdot)\}_{i\in\mathbb{Z}}$. Let $\bar{\mathbf{y}}(\omega)=\{\bar{y}_i(\omega)\}_{i\in\mathbb{Z}}$ be an $(\omega,\bar{\delta})$ pseudo-orbit of $\{F(i,\omega,\cdot)\}$, where $\bar{y}_i(\omega)\in\Lambda(\theta^{n_i}\omega)$ are all measurable. Define $\mathscr{F}: U\to\Pi X$ by

$$(\mathscr{F}(\omega,\mathbf{x}))_i := x_i - F(i-1,\omega,x_{i-1}),$$

where

$$U=\{(\omega,\mathbf{x})\in\Omega\times\Pi X \,|\, \sup|x_i-\bar{y}_i(\omega)|<\Delta(\omega)\}.$$

$\mathscr{F}(\omega,\mathbf{x})$ is $C^1$ with derivative $(D\mathscr{F}(\omega,\mathbf{x})\mathbf{h})_i = h_i - DF(i-1,\omega,x_{i-1})h_{i-1}$. Define $L(\omega):\Pi X\to\Pi X$ by

$$(L(\omega)\mathbf{h})_i := (D\mathscr{F}(\omega,\bar{\mathbf{y}}(\omega))\mathbf{h})_i = h_i - DF(i-1,\omega,\bar{y}_{i-1}(\omega))h_{i-1}.$$

Let $A_i(\omega)=DF(i,\omega,\bar{y}_i(\omega))$, $Q_i(\omega)=\mathbf{P}(\theta^{n_i}\omega,\bar{y}_i(\omega))$. By the assumptions of hyperbolicity of $\Lambda$, for all $i\in\mathbb{Z}$, $\omega\in\Omega$, we have $|A_i(\omega)|\leqslant M^l$, $|Q_i(\omega)|\leqslant K$, $|I-Q_i(\omega)|\leqslant K$, $|A_i(\omega)Q_i(\omega)|\leqslant K\lambda^k$ and $A_i(\omega)(I-Q_i(\omega)):\mathcal{N}(Q_i(\omega))\to\mathcal{N}(\mathbf{P}(\theta^{n_{i+1}}\omega,F_i(\omega,\bar{y}_i(\omega))))$ is invertible with strongly measurable inverse having norm bounded by $K\lambda^k$. In addition, we can calculate

$$\begin{aligned}&|Q_{i+1}(\omega)A_i(\omega)(I-Q_i(\omega))|\\ =&|(\mathbf{P}(\theta^{n_{i+1}}\omega,\bar{y}_{i+1}(\omega))-\mathbf{P}(\theta^{n_{i+1}}\omega,F_i(\omega,\bar{y}_i(\omega))))A_i(\omega)(I-Q_i(\omega))|\\ \leqslant&\nu(\theta^{n_{i+1}}\omega,\delta_1(\omega))\cdot M^lK\leqslant\frac{1}{8}.\end{aligned}$$

And similarly we have $|(I-Q_{i+1}(\omega))A_i(\omega)Q_i(\omega)|\leqslant\frac{1}{8}$ for all $\omega\in\Omega$. Let $J_i(\omega)=Q_{i+1}(\omega)P(\theta^{n_{i+1}}\omega,F_i(\omega,\bar{y}_i(\omega)))+(I-Q_{i+1}(\omega))(I-P(\theta^{n_{i+1}}\omega,F_i(\omega,\bar{y}_i(\omega))))$. Since

$$\begin{aligned}&|Q_{i+1}(\omega)-P(\theta^{n_{i+1}}\omega,F(i,\omega,\bar{y}_i(\omega)))|\\ =&|P(\theta^{n_{i+1}}\omega,\bar{y}_{i+1}(\omega))-P(\theta^{n_{i+1}}\omega,F(i,\omega,\bar{y}_i(\omega)))|\\ \leqslant&\nu(\theta^{n_{i+1}}\omega,\delta_1(\omega))<\frac{1}{2K},\end{aligned}$$

by Lemma 4.1, $J_i(\omega)$ is invertible with strongly measurable inverse operator satisfying $|J_i^{-1}(\omega)|\leqslant(1-2K\nu(\omega,\delta_1(\omega)))^{-1}\leqslant 2$, and $J_i(\omega)(\mathcal{N}(P(\theta^{n_{i+1}}\omega,F_i(\omega,\bar{y}_i(\omega)))))=\mathcal{N}(Q_{i+1}(\omega))$. Then

$$(I-Q_{i+1}(\omega))A_i(\omega)(I-Q_i(\omega))=J_i(\omega)A_i(\omega)(I-Q_i(\omega)):\mathcal{N}(Q_i(\omega))\to\mathcal{N}(Q_{i+1}(\omega))$$

is invertible with strongly measurable inverse, which is denoted by $B_i(\omega)$, and $|B_i(\omega)(I-Q_{i+1}(\omega))| \leqslant 2K^2\lambda^k$. Let $M_1 = 2K^2\lambda^k$, $M_2 = \frac{1}{8}$. Then by Proposition 3.1, $L(\omega)$ is invertible with $\|L(\omega)^{-1}\| \leqslant 2K+1$. Since $D\mathscr{F}(\omega,\cdot) : \Pi X \to \mathscr{L}(\Pi X)$ is continuous, we can choose $\varepsilon_0(\omega)$ small enough so that

$$\|D\mathscr{F}(\omega,\mathbf{x}) - D\mathscr{F}(\omega,\bar{\mathbf{y}}(\omega))\| \leqslant (\|2D\mathscr{F}(\omega,\bar{\mathbf{y}}(\omega))^{-1}\|)^{-1}$$

for $\|\mathbf{x}-\bar{\mathbf{y}}(\omega)\| \leqslant \varepsilon_0(\omega)$. Furthermore,

$$\begin{aligned}\|\mathscr{F}(\omega,\bar{\mathbf{y}}(\omega))\| &= \sup|\bar{y}_i(\omega) - F(i-1,\omega,\bar{y}_{i-1}(\omega))| \leqslant \delta_1(\omega) \leqslant \frac{\bar{\varepsilon}(\omega)}{4K+2} \leqslant \frac{\bar{\varepsilon}(\omega)}{2\|L(\omega)^{-1}\|}\\ &= \bar{\varepsilon}(\omega)(2\|D\mathscr{F}(\omega,\bar{\mathbf{y}}(\omega))^{-1}\|)^{-1}.\end{aligned}$$

Consider $\Pi X$ to be space $Y$ in Proposition 4.1, then there exists a unique measurable solution $\bar{\mathbf{x}}(\omega) = \{\bar{x}_i(\omega)\}$ of $\{F(i,\omega,\cdot)\}$ such that $\|\bar{\mathbf{x}}(\omega) - \bar{\mathbf{y}}(\omega)\| \leqslant \bar{\varepsilon}(\omega)$.

For $0 < \varepsilon(\omega) \leqslant \varepsilon_0(\omega)$, let $\bar{\varepsilon}(\omega) = \varepsilon(\omega)/\sum_{j=0}^{l} M^j$, $\delta(\omega) = \sum_{j=0}^{l} M^j\bar{\delta}(\omega)$. If $\{y_n(\omega)\}_{n\in\mathbb{Z}}$ is an $(\omega,\delta)$ pseudo-orbit of $\{\varphi(\theta^n\omega,\cdot)\}$, which possesses a measurable subsequence $\{y_{n_i}(\omega)\}$ satisfying $k \leqslant n_{i+1} - n_i \leqslant l$, then by lemma 4.2, $\{\bar{y}_i(\omega)\}_{n\in\mathbb{Z}}$ with $\bar{y}_i(\omega) = y_{n_i}(\omega)$ is a $(\omega,\bar{\delta})$ pseudo-orbit for $\{F(i,\omega,\cdot)\}$. So there exists a unique measurable orbit $\{\bar{x}_i(\omega)\}$ which $(\omega,\bar{\varepsilon})$-shadows $\{\bar{y}_i(\omega)\}$. Define $x_{n_i}(\omega) = \bar{x}_i(\omega)$ and refill the points between each two elements in this subsequence by natural way, so $\{x_n(\omega)\}$ is a measurable orbit of $\{f(n,\omega,\cdot)\}$. It follows from Lemma 4.3 and $\max\{\bar{\varepsilon}(\omega),\delta(\omega)\} = \bar{\varepsilon}(\omega)$ that $\{x_n(\omega)\}$ can $(\omega,\varepsilon)$-shadow $\{y_n(\omega)\}$. And the uniqueness of $\{x_n(\omega)\}$ follows from the uniqueness of $\{x_{n_i}(\omega)\}$.

## 5. Proof of Theorem 2.1

**Lemma 5.1.** *Let the hypotheses of Theorem 2.1 hold and $M$ be the uniform bound for $Df(\omega,x)$ in $O$. Let $x(t,\omega)$ be a solution of (2.1) with initial condition $x(t_0,\omega) = x_0$ for all $\omega$, such that $x(t,\omega) \in \Lambda(\theta^t\omega)$. Let $y(t,\omega)$ be a solution of the initial value problem*

$$\dot{y}(t,\omega) + Ay(t,\omega) = f(\theta^t\omega, y) + h(\theta^t\omega), \qquad y(t_0,\omega) = y_0$$

*for $t \in [t_0, t_0+2\tau]$, where $y(t,\omega) \in O(\theta^t\omega)$ for all $t \in [t_0,t_0+2\tau]$ and $\omega \in \Omega$, $|y_0 - x_0|_\alpha \leqslant \delta(\omega)$, mapping $t \to h(\theta^t\omega)$ is continuous and $\sup|h(\theta^t\omega)| \leqslant \delta(\omega)$. Then there exists a constant $C \geqslant 1$ such that $|y(t,\omega) - x(t,\omega)|_\alpha \leqslant C\delta(\omega)$ for $t \in [t_0,t_0+2\tau]$.*

**Proof.** According to [21] Theorem 1.3.4 and Theorem 1.4.3, there exist positive constants $C_1$, $C_2$ and $c$ such that for $t \geqslant 0$,

$$|e^{-At}|_{\mathscr{L}(X^\alpha,X^\alpha)} \leqslant C_1 e^{-ct}, \qquad |e^{-At}|_{\mathscr{L}(X,X^\alpha)} \leqslant C_2 t^{-\alpha} e^{-ct}.$$

Let $z(t,\omega) = y(t,\omega) - x(t,\omega)$, then

$$\begin{aligned}|z(t,\omega)|_\alpha =& |e^{-A(t-t_0)}(y_0-x_0) + \int_{t_0}^{t} e^{-A(t-s)}\{f(\theta^s\omega,y(s,\omega)) - f(\theta^s\omega,x(s,\omega))\}ds\\ &+ \int_{t_0}^{t} e^{-A(t-s)}h(\theta^s\omega)ds|_\alpha\\ \leqslant& C_1 e^{-c(t-t_0)}\delta(\omega) + \int_{t_0}^{t} MC_2(t-s)^{-\alpha}e^{-c(t-s)}|z(s,\omega)|_\alpha ds\end{aligned}$$

$$+\int_{t_0}^{t} C_2(t-s)^{-\alpha} e^{-c(t-s)}\delta(\omega)ds.$$

So by the inequality above and Lemma 7.1.1 in [21],

$$\begin{aligned}
e^{ct}|z(t,\omega)|_\alpha \leqslant & C_1 e^{ct_0}\delta(\omega) + \int_{t_0}^{t} MC_2(t-s)^{-\alpha} e^{cs}|z(s,\omega)|_\alpha ds \\
& + \int_{t_0}^{t} C_2(t-s)^{-\alpha} e^{cs}\delta(\omega)ds \\
\leqslant & C_1 e^{ct_0}\delta(\omega) + \int_{t_0}^{t} C_2(t-s)^{-\alpha} e^{cs}\delta(\omega)ds \\
& + \int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(MC_2\Gamma(1-\alpha))^n(t-s)^{n-n\alpha-1}}{\Gamma(n-n\alpha)} \\
& \cdot (C_1 e^{ct_0}\delta(\omega) + \int_{t_0}^{s} C_2(s-\tau)^{-\alpha} e^{ct}\delta(\omega)d\tau)ds.
\end{aligned}$$

Since the series in the inequality above converges, there indeed exists a constant $C \geqslant 1$ such that $|z(t,\omega)|_\alpha \leqslant C\delta(\omega)$ for $t \in [t_0, t_0 + 2\tau]$. ☐

By our assumptions in this theorem, for each $\omega \in \Omega$ and every $t \in [t_0, t_0 + 2\tau]$, there exists a closed $\Delta_1(\omega)$-neighborhood $O_1(\omega)$ of $\Lambda(\omega)$, where $\Delta_1(\omega)$ is $\theta^t$ invariant, such that for given initial point $x(t_0,\omega) = x_0 \in O_1(\theta^{t_0}\omega)$, the random dynamical system (2.2) is well defined and both $\phi(t,\theta^{t_0}\omega,x)$ and $D\phi(t,\theta^{t_0}\omega,x)$ are uniformly bounded and continuous.

First, we investigate a special case in which $h_n(\theta^t\omega) = 0$ and $x_n(t,\omega) \in \Lambda(\theta^t\omega)$ for all $t \in [\tau_{n-1},\tau_n]$, $n \in \mathbb{Z}$. In order to take advantage of the result we have obtained in the past few sections, consider $X^\alpha$ as $X$, $\phi(\tau_n - \tau_{n-1}, \theta^{\tau_{n-1}}\omega, x)$ as $\varphi(\theta^n\omega, x)$, $\Lambda(\theta^{\tau_{n-1}}\omega)$ as $\Lambda(\theta^n\omega)$, and $x_n(\tau_{n-1},\omega)$ as $y_n(\omega)$. Since

$$|\phi(\tau_n-\tau_{n-1},\theta^{\tau_{n-1}}\omega,x_n(\tau_{n-1},\omega))-x_{n+1}(\tau_n,\omega)|_\alpha = |x_n(\tau_n,\omega)-x_{n+1}(\tau_n,\omega)|_\alpha \leqslant \delta(\omega),$$

by Theorem 3.1, there exists $\varepsilon_1(\omega) > 0$ such that for each $\varepsilon(\omega)$ satisfying $0 < \varepsilon(\omega) \leqslant \varepsilon_1(\omega)$, there is an $\varepsilon$-dependent random variable $\delta_1(\omega,\varepsilon)$, if $\delta(\omega) \leqslant \delta_1(\omega,\varepsilon)$, there is a unique orbit $\{x(\tau_{n-1},\omega)\}_{n\in\mathbb{Z}}$ such that $|x(\tau_{n-1},\omega) - x_n(\tau_{n-1},\omega)|_\alpha \leqslant \varepsilon(\omega)$ for all $n$. Furthermore, if the sequence $\{x_n(\tau_{n-1},\omega)\}_{n\in\mathbb{Z}}$ possesses a proper measurable subsequence, $\{x(\tau_{n-1},\omega)\}_{n\in\mathbb{Z}}$ is measurable. Then the resulting orbit $x(\omega,t) = \phi(t-\tau_{n-1},\theta^{\tau_{n-1}}\omega, x(\tau_{n-1},\omega))$ on $[\tau_{n-1},\tau_n]$ is measurable, as a composition of two measurable mappings.

Now we set $0 \leqslant \varepsilon(\omega) \leqslant \varepsilon_0(\omega) = \frac{1}{2}\min\{\Delta(\omega),\varepsilon_1(\omega)\}$, $\delta(\omega) \leqslant \delta(\omega,\varepsilon) = \min\{(C+2)^{-1}\delta_1(\omega,\frac{\varepsilon}{2C}), \frac{\varepsilon(\omega)}{2C}\}$. Based on the special case above, it is easier to analyze the general situation where $h_n(\theta^t\omega) \not\equiv 0$, $x_n(t,\omega)$ is in a $\delta(\omega)$-neighborhood of $\Lambda(\theta^t\omega)$, $\{x_n(\tau_{n-1},\omega)\}$ possesses $\{x_{n_i}(\tau_{n_i-1},\omega)\}$ as a measurable subsequence and $k \leqslant n_{i+1} - n_i \leqslant l$ for all $i$. We choose $y_n(\omega) \in \Lambda(\theta^{\tau_{n-1}}\omega)$ for each $\omega$ and $n$ such that $|y_n(\omega) - x_n(\tau_{n-1},\omega)|_\alpha \leqslant \delta(\omega)$ and especially $y_{n_i}(\omega)$ can be chosen to be measurable for each $i$. Let $\bar{x}_n(t,\omega) = \varphi(t-\tau_{n-1},\theta^{\tau_{n-1}}\omega,y_n(\omega))$ for $t \in [\tau_{n-1},\tau_n]$. Then by Lemma 5.1, there exists a constant $C > 1$ such that

$$|\bar{x}_n(t,\omega) - x_n(t,\omega)|_\alpha \leqslant C\delta(\omega) \qquad \text{for } t \in [\tau_{n-1},\tau_n], n \in \mathbb{Z}.$$

Then

$$|\bar{x}_{n+1}(\tau_n,\omega) - \bar{x}_n(\tau_n,\omega)|_\alpha$$

$$\begin{aligned}&\leqslant |\bar{x}_{n+1}(\tau_n,\omega)-x_{n+1}(\tau_n,\omega)|_\alpha+|x_{n+1}(\tau_n,\omega)-x_n(\tau_n,\omega)|_\alpha+|x_n(\tau_n,\omega)-\bar{x}_n(\tau_n,\omega)|_\alpha\\ &\leqslant (C+2)\delta(\omega)\leqslant \delta_1(\omega,\frac{\varepsilon}{2C}).\end{aligned}$$

So $\{\bar{x}_n(t,\omega)\}$ is a $(\omega,\delta_1(\omega,\frac{\varepsilon}{2C}))$ pseudo-solution of (2.1) with $h_n(\theta^t\omega)\equiv 0$ and $\bar{x}_n(t,\omega)\in\Lambda(\theta^t\omega)$ for all $t\in[\tau_{n-1},\tau_n]$, $n\in\mathbb{Z}$. This coincides with the situation we have considered. So by the first part of this proof, there exists a unique measurable solution $x(t,\omega)$ of (2.1) such that

$$|x(\tau_{n-1},\omega)-\bar{x}_n(\tau_{n-1},\omega)|_\alpha\leqslant\frac{\varepsilon(\omega)}{2C}\qquad\text{for all } n\in\mathbb{Z}.$$

Then

$$\begin{aligned}&|x(\tau_{n-1},\omega)-x_n(\tau_{n-1},\omega)|_\alpha\\ \leqslant&|x(\tau_{n-1},\omega)-\bar{x}_n(\tau_{n-1},\omega)|_\alpha+|\bar{x}_n(\tau_{n-1},\omega)-x_n(\tau_{n-1},\omega)|_\alpha\\ \leqslant&\frac{\varepsilon(\omega)}{2C}+\delta(\omega).\end{aligned}$$

Again by Lemma 5.1,

$$|x(t,\omega)-x_n(t,\omega)|_\alpha\leqslant C(\frac{\varepsilon(\omega)}{2C}+\delta(\omega))\leqslant\varepsilon(\omega).$$

This means that we have at least one measurable solution which $(\omega,\varepsilon)$-shadows $\{x_n(t,\omega)\}$.

Suppose that $\tilde{x}(t,\omega)$ is another solution which $(\omega,\varepsilon)$-shadows $\{x_n(t,\omega)\}$. Then

$$\begin{aligned}&|\tilde{x}(\tau_{n-1},\omega)-\bar{x}_n(\tau_{n-1},\omega)|_\alpha\\ \leqslant&|\tilde{x}(\tau_{n-1},\omega)-x_n(\tau_{n-1},\omega)|_\alpha+|x_n(\tau_{n-1},\omega)-\bar{x}_n(\tau_{n-1},\omega)|_\alpha\\ \leqslant&\varepsilon(\omega)+\delta(\omega)\leqslant\varepsilon_1(\omega).\end{aligned}$$

Since

$$|\bar{x}_{n+1}(\tau_n,\omega)-\bar{x}_n(\tau_n,\omega)|_\alpha\leqslant\delta_1(\omega,\frac{\varepsilon}{2C})\leqslant\delta_1(\omega,\varepsilon_1),$$

$\{\tilde{x}(\tau_{n-1},\omega)\}$ should be the unique orbit that $(\omega,\varepsilon_1)$-shadows the $\delta_1(\omega,\varepsilon_1)$ pseudo-orbit $\{\bar{x}_n(\tau_{n-1},\omega)\}$. By the uniqueness of such orbit, $\tilde{x}(\tau_{n-1},\omega)=x(\tau_{n-1},\omega)$, and this completes the proof.

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