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MULTIPLE SOLUTIONS FOR SOME NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH THREE-POINT BOUNDARY CONDITIONS VIA VARIATIONAL APPROACH*

Yongfang Wei[1] and Zhanbing Bai[1,†]

Abstract This paper investigates a class of nonlinear impulsive differential equations with three-point boundary conditions. By using the critical point theory and the variational method, gives the results of multiple solutions. We study the non-local boundary value problem by variational method, and give its variational structure. Finally, two examples are given to prove the results.

Keywords Variational methods, impulsive differential equations, critical point theory, non-local boundary conditions.

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1. Introduction

In recent years, impulsive differential equations have attracted much attention. Because many phenomena in life will change suddenly due to the influence of external factors, so it is more suitable to describe this mutation by impulse differential equations. The characteristic of impulsive differential systems is to describe the impact of sudden phenomena on the system state. Therefore, impulsive differential equation has been widely used in recent years, especially in the field of biological mathematics. For example, since oral and injected drugs often enter the human body in the form of pulse, it is more reasonable to use impulse differential equation to describe the changes of drug concentration in the human body. In addition, impulse differential equations have been fully applied in pest control, environmental governance and other aspects, and many outstanding achievements have been achieved.

Because of the wide range of applications, some scholars studied the existence of solutions for some impulsive differential equations with local boundary conditions. And through fixed point theorems, upper and lower solution theorems, etc., they obtained some excellent results [1–6, 9–11, 13, 14, 16–19, 21–29, 31, 32]. We can see some of the results for impulsive differential equations as the followings. In [16], Nieto and O'Regan studied the nonlinear impulsive boundary-value problem in

†The corresponding author. Email address: zhanbingbai@163.com (Z. Bai)
1College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, China
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space $H_0^1([0,T])$:

$$\begin{cases} -y''(t) - \lambda y(t) = f(t, y(t)), \ a.e. \ t \in [0,T], \\ -\Delta y'(t_j) = I_j(y(t_j)), \quad j = 1,2,...,p, \\ y(0) = y(T) = 0, \end{cases}$$

where $\Delta y'(t_j) = y'(t_j^+)) - y'(t_j^-)$ as $y'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} y'(t)$, $j = 1,2,...,p$, they gave a new approach via variational methods and critical point theory, to obtain the existence of solutions to the impulsive problems. In [23], Tian and Ge studied the nonlinear impulsive Sturm-Liouville boundary value problem in space $H^1(0,T)$:

$$\begin{cases} -y''(t) - \lambda y(t) = f(t, y(t)), \ t \neq t_j, \ a.e. \ t \in [0,T], \\ -\Delta y'(t_j) = I_j(y(t_j)), \quad j = 1,2,...,l, \\ \alpha y'(0) - \beta y(0) = 0, \quad r y'(T) + \sigma y(T) = 0, \end{cases}$$

with the functional

$$\begin{aligned} \psi(y) =& \frac{1}{2}\int_0^T [(y'(t))^2 - \lambda(y(t))^2]dt + \frac{\beta}{2\alpha}y^2(0) + \frac{\sigma}{2\gamma}y^2(T) \\ & - \int_0^T [F(t,y(t)) - f(t,0)y(t)]dt - \sum_{j=1}^{l}\left[\int_0^{y(t_j)} I_j(s)ds - I_j(0)y(t_j)\right], \end{aligned}$$

by applying a new approach via critical point theory and variational methods, they obtained the existence results of positive solutions. In [28], Xiao, Nieto and Luo studied the following nonlinear second order impulsive boundary value problems:

$$\begin{cases} -(p(t)y'(t))' + r(t)y'(t) + q(t)y(t) = f(t, y(t)), \ a.e. \ t \in Q, \\ -\Delta(p(t_j)y'(t_j)) = I_j(y(t_j)), \quad j = 1,2,...,n, \\ y(0) = 0, \quad a_1 y(1) + y'(1) = 0, \end{cases}$$

they established the conditions for the existence of multiple solutions, and dealt with a class of problems which was not considered before.

In the previous articles, almost no one considered the multi-point boundary value problem via variational method. Almost of them, gave the results of local boundary value problems. Recently, in [12] bai et.al studied the three-point boundary value problem:

$$\begin{cases} (P(t)y'(t))' + f(t, y(t)) = 0, \ a.e. \ t \in [0,1], \\ y(0) = 0, \quad y(1) = \beta y(\eta), \end{cases}$$

they got the existence and multiplicity of solutions for the second-order differential equation with three-point boundary condition via Mountain Pass Lemma, gave the results for a non-local boundary value problem by variational method.

As far as we know, the impulsive differential system with non-local boundary conditions, has not been considered by the variational method and critical point theory. Therefore, this paper considers the impulsive differential system with three-point boundary conditions as following :

$$\begin{cases} -(J(t)y'(t))' = \nabla H(t, y(t)), \ t \neq t_k, \ a.e. \ t \in [0,1], \\ -\Delta(J(t_k)y'(t_k)) = I_k(y(t_k)), \quad k = 1, 2, ..., q, \\ y(0) = 0, \quad y(1) = \gamma y(\eta), \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < ... < t_{q_1} < t_{q_1+1} = \eta < t_{q_1+2} < ... < t_q < t_{q+1} = 1$, $\gamma > 0$, $0 < \eta < 1$, and $\Delta(J(t_k)y'(t_k)) = J(t_k^+)y'(t_k^+) - J(t_k^-)y'(t_k^-)$ for $y'(t_k^{\pm}) = \lim_{t \to t_k^{\pm}} y'(t)$, $k = 1, 2, ..., q$.

The rest of the paper is organized as follows. Section 2 will give the preliminary theorems, and prove that the critical point of functional ψ is the classical solution of BVPs (1.1). Section 3 will present the results of the classical solutions respectively, under some suitable conditions. Section 4 will give two examples to verify the results.

2. Preliminaries

Let $Z = \{y \in H^1([0,1], \mathbb{R}^n) | y(0) = 0, \ y(1) = \gamma y(\eta)\}$ with the norm

$$||y||_Z = \left[\int_0^1 |y(t)|^2 dt + \int_0^1 |y'(t)|^2 dt\right]^{\frac{1}{2}}, \ \forall y \in Z,$$

then Z is a closed subspace of reflexive Banach space $H^1([0,1], \mathbb{R}^n)$. Because $y(0) = 0$ and $\int_0^1 |y(t)|^2 dt \leq \frac{4}{\pi^2} \int_0^1 |y'(t)|^2 dt$, we can take the norm in Z as

$$||y|| = \left(\int_0^1 |y'(t)|^2 dt\right)^{\frac{1}{2}}, \forall y \in Z,$$

we can see detail in [8]. Let $r > 0$, $B_r = \{y \in Z| \ ||y|| < r\}$, and B_r is a subset of Z. Defined the functional $\psi : Z \to \mathbb{R}$ by

$$\psi(y) = \int_0^1 \left[\left(\frac{1}{2}J(t)y'(t), y'(t)\right) - H(t, y(t))\right] dt - \sum_{k=1}^{q} \int_0^{y(t_k)} I_k(t) dt, \forall y, \upsilon \in Z, \tag{2.1}$$

then the derivative is

$$\langle \psi'(y), \upsilon \rangle = \int_0^1 \left[(J(t)y'(t), \upsilon'(t)) - (\nabla H(t, y(t)), \upsilon(t))\right] dt - \sum_{k=1}^{q} I_k(y(t_k))\upsilon(t_k). \tag{2.2}$$

Lemma 2.1 ([12]). *For each $y \in Z$, it holds that*

$$\|y\|_\infty \leq \|y\|. \tag{2.3}$$

Lemma 2.2 ([12]). $Z \hookrightarrow\hookrightarrow C([0,1], \mathbb{R}^n)$.

Lemma 2.3 ([15]). *If ψ is weakly lower semi-continuous on a reflexive Banach space Z and has a bounded minimizing sequence, then ψ has a minimum on Z.*

Definition 2.1. Let Z be a real Banach space, $\psi \in C^1(Z,\mathbb{R})$. If for any sequence $\{y_i\} \subset Z$ with

$$\psi'(y_i) \text{ being bounded and } \lim_{i\to\infty} \psi'(y_i) \to \theta$$

contains a convergent subsequence, then the functional ψ is called satisfying the Palais-Smale $(PS)_c$ condition.

Lemma 2.4 ([7]). *Let Z be a real Banach space and $\psi \in C^1(Z,\mathbb{R})$ be a lower bounded functional which satisfies the $(PS)_c$ condition, then ψ have the minimum value in Z, i.e. there exists $y_0 \in Z$ such that*

$$\psi(y_0) = \inf_{y\in Z} \psi(y),$$

then y_0 is a critical point of ψ.

Lemma 2.5 ([7]). *Let Z be a real Banach space, assume $\psi \in C^1(Z,\mathbb{R})$ satisfy the $(PS)_c$ condition. Assume there are $y_0, y_1 \in Z$ and a bounded open neighborhood Ω of y_0 such that $y_1 \in Z\backslash\Omega$ and*

$$\inf_{\upsilon\in\partial\Omega} \psi(\upsilon) > \max\{\psi(y_0), \psi(y_1)\}.$$

Let

$$\Gamma = \{\sigma|\sigma \in C([0,1], Z)|\sigma(0) = y_0, \sigma(1) = y_1\},$$

and

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \psi(\sigma(s)).$$

Then there exists a critical point y^* of ψ such that $\psi'(y^*) = \theta$ and $\psi(y^*) = c$ and

$$c = \psi(y^*) > \max\{\psi(y_0), \psi(y_1)\}.$$

Lemma 2.6 ([20]). *Let E be a real Banach space, $\psi \in C^1(E,\mathbb{R})$ be even, and ψ satisfy the $(PS)_c$ condition. If $E = X \oplus Z$, where X is finite dimensional, then ψ satisfies the following:*
(i) $\psi(\theta) = 0$;
(ii) there exist $\delta > 0$ and $\tau > 0$ such that $\psi|_{\partial B_\tau \cap Z} \geq \delta$;
(iii) if $E_1 \subset E$, where E_1 is a finite dimensional subspace, there exists $r = r(E_1)$ such that $\psi(y) \leq 0$ for $\forall y \in E_1$ with $\|y\| > r$.
Then ψ possesses an unbounded sequence of critical values.

Lemma 2.7. *The critical point of functional ψ is the classical solution of BVPs (1.1).*

Proof. By the properties of H, J and I_k, we can get $\psi \in C^1(Z,\mathbb{R})$. Then assume $y_0 \in Z$ is a critical point of ψ, and by (2.2), there is

$$\langle \psi'(y_0), \upsilon\rangle = 0,\ \forall\ \upsilon\ \in\ Z.$$

Let $T_1 = (t_0, t_{q_1+1})$, $T_2 = (t_{q_1+1}, t_{q+1})$, where $t_0 = 0$, $t_{q_1+1} = \eta$, $t_{q+1} = 1$, and

$$Y_i = \{y \in C^\infty([0,1];\mathbb{R}^n)|\overline{supp\ y} \subset T_i\} \subset Z,\ i = 1,2.$$

According to (2.2), we can get

$$\langle \psi'(y_0), \upsilon\rangle = 0,\ \forall \upsilon \in Y_i,\ i = 1,2,$$

that is

$$\int_0^1 [(J(t)y_0'(t), \upsilon'(t)) - (\nabla H(t, y_0(t)), \upsilon(t))]\, dt - \sum_{k=1}^{q} I_k(y_0(t_k))\upsilon(t_k) = 0. \tag{2.4}$$

Consider $t \in T_1$, there is

$$\begin{aligned}
&\int_0^\eta (J(t)y_0'(t), \upsilon'(t))\, dt \\
=& \int_0^\eta ((J(s)y_0'(s))', \upsilon(\eta) - \upsilon(s))\, ds \\
=& \sum_{k=0}^{q_1} \int_{t_k^+}^{t_{k+1}^-} ((J(s)y_0'(s))', \upsilon(t_{k+1}) - \upsilon(s))\, ds \\
=& \sum_{k=0}^{q_1} \upsilon(t_{k+1}) \int_{t_k^+}^{t_{k+1}^-} (J(s)y_0'(s))' ds - \int_0^\eta ((J(s)y_0'(s))', \upsilon(s)) ds \\
=& \sum_{k=1}^{q_1} [J(t_{k+1}^-)y_0'(t_{k+1}^-)\upsilon(t_k) - J(t_k^+)y_0'(t_k^+)\upsilon(t_k)] - \int_0^\eta ((J(s)y_0'(s))', \upsilon(s)) ds \\
=& \sum_{k=1}^{q_1} \Delta\left(J(t_k)y_0'(t_k)\right)\upsilon(t_k) - \int_0^\eta ((J(t)y_0'(t))', \upsilon(t))\, dt,
\end{aligned}$$

then

$$\begin{aligned}
&- \int_0^\eta ((J(t)y_0'(t))', \upsilon(t))\, dt \\
=& - \sum_{k=1}^{q_1} \Delta\left(J(t_k)y_0'(t_k)\right)\upsilon(t_k) + \int_0^\eta (J(t)y_0'(t), \upsilon'(t))\, dt \\
=& - \sum_{k=1}^{q_1} I_k(y_0(t_k))\upsilon(t_k) + \int_0^\eta (J(t)y_0'(t), \upsilon'(t))\, dt,\ \forall \upsilon \in Y_1.
\end{aligned}$$

Similarly we can get

$$\begin{aligned}
&- \int_{T_i} ((J(t)y_0'(t))', \upsilon(t))\, dt \\
=& - \sum_{k=1}^{q_1} \Delta\left(J(t_k)y_0'(t_k)\right)\upsilon(t_k) + \int_{T_i} (J(t)y_0'(t), \upsilon'(t))\, dt \\
=& - \sum_{k=1}^{q_1} I_k(y_0(t_k))\upsilon(t_k) + \int_{T_i} (J(t)y_0'(t), \upsilon'(t))\, dt,\ \forall \upsilon \in Y_i,\ i = 1, 2,
\end{aligned}$$

and by (2.4), there is

$$\begin{aligned}
&- \int_{T_i} ((J(t)y_0'(t))', \upsilon(t))\, dt \\
=& - \sum_{k=1}^{q} I_k(y_0(t_k))\upsilon(t_k) + \int_{T_i} (J(t)y_0'(t), \upsilon'(t))\, dt
\end{aligned}$$

$$= \int_{T_i} (\nabla H(t, y_0(t)), \upsilon(t))\, dt,\ \forall \upsilon \in Y_i,\ i = 1, 2. \tag{2.5}$$

So by $\{\upsilon|_{T_i} \mid \upsilon \in Y_i\} = C_0^\infty(T_i; \mathbb{R}^n)$ and (2.5), one has

$$-(J(t)y_0'(t))' = \nabla H(t, y_0(t)),\ a.e.\ t \in T_i, i = 1, 2,$$

then there is

$$-(J(t)y_0'(t))' = \nabla H(t, y_0(t)),\ a.e.\ t \in [0, 1],$$

thus $y_0(t)$ satisfies the first differential equation in (1.1).

Next to show that $y_0(t)$ satisfies the impulsive condition in (1.1), we have

$$\sum_{k=1}^{q} [\Delta(J(t_k)y_0'(t_k)) + I_k(y_0(t_k))]\, \upsilon(t_k) = 0,\ \forall \upsilon \in Z,\ k = 1, 2, ..., q. \tag{2.6}$$

Without loss of generality, let $k_0 \in \{1, 2, .., q\}$ satisfy

$$\Delta(J(t_{k_0})y_0'(t_{k_0})) + I_k(y_0(t_{k_0})) \neq 0, \tag{2.7}$$

assume $\upsilon(t) = \prod_{j=0,j\neq k_0}^{q+1}(t - t_j)$, there is

$$\begin{aligned}
&\sum_{m=1}^{q} [\Delta(J(t_m)y_0'(t_m)) + I_m(y_0(t_m))]\, \upsilon(t_m) \\
&= \sum_{m=1}^{q} [\Delta(J(t_m)y_0'(t_m)) + I_m(y_0(t_m))] \prod_{j=0,j\neq k_0}^{q+1} (t_m - t_j) \\
&= [\Delta(J(t_{k_0})y_0'(t_{k_0})) + I_{k_0}(y_0(t_{k_0}))] \prod_{j=0,j\neq k_0}^{q+1} (t_{k_0} - t_j) \neq 0,
\end{aligned}$$

obviously it contradicts (2.6). Thus $y_0(t)$ makes the impulsive conditions in (1.1) held. So we can get the critical point of $\psi(y)$ is the classical solution of BVPs (1.1). □

3. Main Results

We assume the following conditions in the rest of the paper:
$(A1)$ $J : [0, 1] \to \mathbb{R}^{n\times n}$ is a continuous symmetric matrix, $\{\lambda_j(t)\}$ is the eigenvalue of $J(t)$ and

$$0 < p_1 \leq \min_{0\leq t\leq 1} \min_{1\leq j\leq n} \lambda_j(t) \leq \max_{0\leq t\leq 1} \max_{1\leq j\leq n} \lambda_j(t) \leq p_2;$$

$(A2)$ there are constants $\alpha \geq \beta > \frac{2p_2}{p_1}$, for $(t, y) \in [0, 1] \times \mathbb{R}^n$ such that

$$0 < \beta H(t, y) \leq y\ \nabla H(t, y),\ 0 < \alpha \int_0^y I_k(s)ds \leq yI_k(y),\ k = 1, 2, ..., q,$$

where $H(t, y) = \int_0^y \nabla H(t, s)ds$, and $\nabla H(t, y)$ is a C^1 function.;
$(A3)$ for $t \in [0, 1], y \in \mathbb{R}^n$ and $k = 1, 2, .., q$, there is

$$\overline{\lim}_{y\to\theta}\ \frac{H(t, y)}{|y|^2} = 0,\ \text{and}\ \ \overline{\lim}_{y\to\theta}\ \frac{\int_0^y I_k(s)ds}{|y|^2} = 0.$$

Theorem 3.1. *If assumptions $(A1)$ and $(A3)$ satisfied, then BVPs (1.1) has at least one classical solution.*

Proof. Let $\{y_i\}$ be a weakly convergent sequence, and $y_i \rightharpoonup y$ in Z, then $||y|| \leq \underline{\lim}_{i\to\infty}||y_i||$, and

$$\begin{aligned}\lim_{i\to\infty}\psi(y_i) &= \lim_{i\to\infty}\left[\int_0^1\left(\frac{1}{2}J(t)|y_i'(t)|^2 - H(t,y_i(t))\right)dt - \sum_{k=1}^{q}\int_0^{y_i(t_k)} I_k(t)dt\right] \\ &\geq \int_0^1\left[\frac{1}{2}J(t)|y(t)|^2 - H(t,y(t))\right]dt - \sum_{k=1}^{q}\int_0^{y(t_k)} I_k(t)dt = \psi(y),\end{aligned}$$

that is $\psi(y) \leq \underline{\lim}_{i\to\infty}\psi(y_i)$, thus ψ is weakly lower semi-continuous.

By the assumption $(A3)$, we can get

$$H(t,y) \leq l|y|^2,\ \int_0^y I_k(s)ds \leq l_k|y|^2,\ t\in[0,1],\ y\in\mathbb{R}^n,\ k=1,2,..,q,$$

where positive constants l and l_k satisfy $l+\sum_{k=1}^{q} l_k < \frac{p_1}{2}$. Then by the assumption $(A1)$, Lemma 2.1 and (2.1), there is

$$\begin{aligned}\psi(y) &= \int_0^1 \frac{1}{2}J(t)|y'(t)|^2dt - \int_0^1 H(t,y(t))dt - \sum_{k=1}^{q}\int_0^{y(t_k)} I_k(s)ds \\ &\geq \frac{p_1}{2}||y||^2 - l\int_0^1|y|^2dt - \sum_{k=1}^{q} l_k|y|^2 \geq \frac{p_1}{2}||y||^2 - (l+\sum_{k=1}^{q} l_k)||y||_\infty^2 \\ &\geq \left[\frac{p_1}{2} - (l+\sum_{k=1}^{q} l_k)\right]||y||^2 > 0,\end{aligned}$$

then $\psi(y) \to +\infty$ if $||y|| \to +\infty$, thus ψ has a bounded minimizing sequence. Then by Lemma 2.3, ψ has a minimum on Z, so there exists a $y_1 \in Z$ such that $\psi(y_1) = \inf_{y\in Z}\psi(y)$, thus y_1 is a classical solution of BVPs (1.1) by Lemma 2.4. $\square$

Lemma 3.1. *If assumptions $(A1)$ and $(A2)$ hold, then ψ satisfies the $(PS)_c$ condition.*

Proof. Let $\{y_i\}$ be a sequence in Z, such that $\{\psi(y_i)\}$ is bounded, and $\lim_{i\to\infty}\psi'(y_i)=0$, then there exists nonnegative constants D_1 such that $|\psi(y_i)| \leq D_1$. Then by (2.2), there is

$$\begin{aligned}\langle\psi'(y_i), y_i\rangle = \int_0^1 &\left[(J(t)y_i'(t), y_i'(t)) - (\nabla H(t,y_i(t)), y_i(t))\right]dt \\ &- \sum_{k=1}^{q} I_k(y_i(t_k))y_i(t_k),\end{aligned}$$

then according to the assumption $(A1)$, we can get

$$\int_0^1 \nabla H(t,y_i(t))y_i(t)dt + \sum_{k=1}^{q} I_k(y_i(t_k))y_i(t_k)$$

$$= \int_0^1 J(t)|y_i'(t)|^2 dt - \langle \psi'(y_i), y_i \rangle \le p_2 ||y_i||^2 - \langle \psi'(y_i), y_i \rangle.$$

By the assumption $(A2)$ and (2.1), there is

$$\begin{aligned}
D_1 &\ge \psi(y_i) \\
&= \int_0^1 \left[\left(\frac{1}{2} J(t) y_i'(t), y_i'(t) \right) - H(t, y_i(t)) \right] dt - \sum_{k=1}^q \int_0^{y_i(t_k)} I_k(t) dt \\
&\ge \frac{p_1}{2} ||y_i||^2 - \frac{1}{\beta} \int_0^1 \nabla H(t, y_i(t)) y_i(t) dt - \frac{1}{\alpha} \sum_{k=1}^q I_k(y_i(t_k)) y_i(t_k) \\
&\ge \frac{p_1}{2} ||y_i||^2 - \frac{1}{\beta} \left[\int_0^1 \nabla H(t, y_i(t)) y_i(t) dt + \sum_{k=1}^q I_k(y_i(t_k)) y_i(t_k) \right] \\
&\ge (\frac{p_1}{2} - \frac{p_2}{\beta}) ||y_i||^2 + \frac{1}{\beta} \langle \psi'(y_i), y_i \rangle,
\end{aligned}$$

because $\frac{p_1}{2} - \frac{p_2}{\beta} > 0$, and $\langle \psi'(y_i), y_i \rangle \to 0$ as $i \to +\infty$, thus $\{y_i\}$ is bounded in Z. Then exist a subsequence $\{y_j\}$ of sequence $\{y_i\}$ such that $y_j \rightharpoonup y$ in Z, by Lemma 2.2, there is $y_j \to y$ in $C[0,1]$, so we can get

$$\begin{aligned}
&\langle \psi'(y_j) - \psi'(y), y_j - y \rangle \\
= &\int_0^1 \left(J(t)(y_j'(t) - y'(t)), y_j'(t) - y'(t) \right) dt \\
&- \int_0^1 (\nabla H(t, y_j(t)) - \nabla H(t, y(t)), y_j(t) - y(t)) dt \\
&- \sum_{k=1}^q [I_k(y_j(t_k)) - I_k(y(t_k))] [y_j(t_k) - y(t_k)],
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^q [I_k(y_j(t_k)) - I_k(y(t_k))] [y_j(t_k) - y(t_k)] \to 0, \\
&\int_0^1 (\nabla H(t, y_j(t)) - \nabla H(t, y(t)), y_j(t) - y(t)) dt \to 0,
\end{aligned}$$

then $\langle \psi'(y_j) - \psi'(y), y_j - y \rangle \to 0$ in Z, thus $||y_j - y|| \to 0$ as $j \to +\infty$. Therefore $y_j \to y$ in Z, that is sequence $\{y_i\} \subset Z$ has a convergent subsequence. Then ψ satisfies the $(PS)_c$ condition. □

Theorem 3.2. *If assumptions $(A1)$, $(A2)$ and $(A3)$ hold, then BVPs (1.1) has at least two classical solutions.*

Proof. By the assumption $(A2)$ and (2.1), we have $\psi(\theta) = 0$. And ψ satisfies $(PS)_c$ condition by Lemma 3.1.
Step 1, according to the assumption $(A3)$, given $+\sum_{k=1}^q l_k = \frac{p_1}{4} > 0$, there exists $M > 0$ such that for $||y|| \le M$

$$H(t, y) \le l|y|^2, \quad \int_0^y I_k(s) ds \le l_k |y|^2, \ k = 1, 2, .., q,$$

for $\forall y \in \partial B_M$, $||y|| = M$, then

$$\begin{aligned}\psi(y) &= \int_0^1 \frac{1}{2}J(t)|y'(t)|^2 dt - \int_0^1 H(t,y(t))dt - \sum_{k=1}^{q}\int_0^{y(t_k)} I_k(s)ds \\ &\geq \frac{p_1}{2}||y||^2 - l\int_0^1 |y|^2 dt - \sum_{k=1}^{q} l_k|y|^2 \geq \frac{p_1}{2}||y||^2 - l||y||^2 - \sum_{k=1}^{q} l_k||y||^2 \\ &\geq \frac{p_1}{2}||y||^2 - (l + \sum_{k=1}^{q} l_k)||y||^2 = \frac{p_1}{4}M^2 > 0,\end{aligned}$$

there is a constant $a = \frac{p_1}{4}M^2 > 0$ such that $\psi(y)|_{y\in\partial B_M} > a$. And $\psi(\hat{y}) \leq \psi(\theta) = 0 < \psi(y)$, as $y \in \partial B_M$, $\theta \in B_M$, ψ is a lower bounded functional which satisfies the $(PS)_c$ condition, thus $\hat{y}$ is a critical point of ψ by Lemma 2.4.
Step 2, by assumption $(A2)$, for any $y \in Z\backslash B_M$, that is for $||y|| \geq M$, there is

$$H(t,y) \geq h|y|^{\beta} + d, \quad \int_0^y I_k(s)ds \geq h_k|y|^{\alpha} + d_k,\ k = 1,2,..,q,$$

where h, h_k, d, d_k are positive constants. Then for $r \in \mathbb{R}\backslash\{0\}$ one has

$$\begin{aligned}\psi(rx) &= \int_0^1 \left[\frac{1}{2}J(t)(ry'(t), ry'(t)) - H(t, ry(t))\right] dt - \sum_{k=1}^{q}\int_0^{ry(t_k)} I_k(s)ds \\ &\leq \frac{p_2 r^2}{2}||y||^2 - \int_0^1 (h|ry|^{\beta} + d)dt - \sum_{k=1}^{q}(h_k|ry|^{\alpha} + d_k) \\ &\leq \frac{p_2 r^2}{2}||y||^2 - r^{\beta}h\int_0^1 |y|^{\beta}dt - r^{\alpha}\left(\sum_{k=1}^{q} h_k|y|^{\alpha}\right) - d - \sum_{k=1}^{q} d_k,\end{aligned}$$

because $\alpha \geq \beta > 2$, there is $\psi(ry) \to -\infty$ as $r \to +\infty$. Then choose y_1 with $y_1 \in Z\backslash B_M$ sufficiently large such that $\psi(y_1) < 0$, thus there is $\max\{\psi(\hat{y}), \psi(y_1)\} < \inf\{\psi(y), y \in \partial B_M\}$. Then by Lemma 2.5, there is a critical point y^* and $\psi(y^*) > \max\{\psi(\hat{y}), \psi(y_1)\}$. Therefore $\hat{y}$ and y^* are two different critical points of ψ, by Lemma 2.7, BVPs (1.1) has at least two classical solutions. □

Proposition 3.1. *Under the conditions of Theorem 3.2, and if $H(t,y)$ and $I_k(y)$ are odd functions with respect to y, then BVPs* (1.1) *has infinitely many classical solutions.*

Proof. By the (2.1), we can get ψ is a even functional, let $Z_1 \subset Z$ be a finite dimensional subspace. By Theorem 3.2, it's easy to prove the (i) and (ii) in Lemma 2.6, and there is $\psi(\theta) = 0$ and ψ satisfies $(PS)_c$ condition.
For any $r \in \mathbb{R}\backslash\{0\}$ and $y \in Z_1$, there is

$$\begin{aligned}\psi(ry) &= \int_0^1 \left[\frac{1}{2}J(t)(ry'(t), ry'(t)) - H(t, ry(t))\right] dt - \sum_{k=1}^{q}\int_0^{ry(t_k)} I_k(s)ds \\ &\leq \frac{br^2}{2}||y||^2 - \int_0^1 (h|ry(t)|^{\beta} + d)dt - \sum_{k=1}^{q}(h_k|ry(t)|^{\alpha} + d_k) \\ &\leq \frac{br^2}{2}||y||^2 - r^{\beta}h\int_0^1 |y(t)|^{\beta}dt - r^{\alpha}\left(\sum_{k=1}^{q} h_k|y(t_k)|^{\alpha}\right) - d - \sum_{k=1}^{q} d_k,\end{aligned}$$

because $\alpha \geq \beta > 2$, there exists $y_r > 0$ such that $\psi(y_r) < 0$ and $||y_r|| > M$, then we get (iii) of the Lemma 2.6. Therefore BVPs (1.1) possesses infinitely many classical solutions. $\square$

Remark 3.1. Under different conditions, this paper gives different results of the classical solutions for the problem. In particular, $H(t,y) = 0$ and $\int_0^y I_k(t)dt = 0$ can be true respectively, the results are also correct.

4. Example

Example 4.1. Let $y = (y_1, y_2) \in \mathbb{R}^2 \setminus \{\theta\}$. Consider the following boundary value problem:

$$\begin{cases} -(3y_i'(t))' = y_i^{\frac{3}{2}}(t), \ t \neq t_k \ a.e. \ t \in [0,1], \\ -\Delta(3y_i'(t_k)) = 4y_i^3(t_k), \quad k = 1,2,...,q, \\ y_i(0) = 0, \quad y_i(1) = \gamma y_i(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $i = 1, 2$. Compare with (1.1), we choose $\gamma = 2$, $\nabla H(t,y) = (y_1^{\frac{3}{2}}, y_2^{\frac{3}{2}})$, $I(y) = 4(y_1^3, y_2^3)$, in this case we have

$$H(t,y) = \frac{2}{5}\left(y_1^{\frac{5}{2}}, y_2^{\frac{5}{2}}\right), \int_0^y I(s)ds = (y_1^4, y_2^4).$$

Then there is

$$\begin{aligned} \overline{\lim}_{y\to\theta} \frac{H(t,y)}{|y|^2} &= \overline{\lim}_{y\to\theta} \frac{\frac{2}{5}\left(y_1^{\frac{5}{2}}, y_2^{\frac{5}{2}}\right)}{y_1^2+y_2^2} = \overline{\lim}_{y\to\theta}\left(\frac{y_1^{\frac{3}{2}}}{2y_1+2y_2}, \frac{y_2^{\frac{3}{2}}}{2y_2+2y_1}\right) \\ &= \overline{\lim}_{y\to\theta} \frac{3}{8}\left(y_1^{\frac{1}{2}}, y_2^{\frac{1}{2}}\right) = \theta, \end{aligned}$$

and

$$\begin{aligned} \overline{\lim}_{y\to\theta} \frac{\int_0^y I_k(s)ds}{|y|^2} &= \overline{\lim}_{y\to\theta} \frac{(y_1^4, y_2^4)}{y_1^2+y_2^2} = \overline{\lim}_{y\to\theta}\left(\frac{4y_1^3}{2y_1+2y_2}, \frac{4y_2^3}{2y_2+2y_1}\right) \\ &= \overline{\lim}_{y\to\theta} 3(y_1^2, y_2^2) = \theta \end{aligned}$$

where $k = 1, 2, .., q$, thus conditions $(A3)$ are satisfied. By Theorem 3.1 the problem (4.1) has at least one classical solution.

Example 4.2. Let $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ be a positive function, and $\beta = 2.5, \alpha = 3$. Consider the following boundary value problem:

$$\begin{cases} -(2y_i'(t))' = y_i^2(t), \ t \neq t_k \ a.e. \ t \in [0,1], \\ -\Delta(2y_i'(t_k)) = \frac{1}{2}y_i^3(t_k), \quad k = 1,2,...,q, \\ y_i(0) = 0, \quad y_i(1) = \gamma y_i(\frac{1}{2}), \end{cases} \tag{4.2}$$

where $i = 1, 2, 3$. Compare with (1.1), we choose $\gamma = 3$, $\nabla H(t,y) = (y_1^2, y_2^2, y_3^2)$, $I(y) = \frac{1}{3}(y_1^3, y_2^3, y_3^3)$, in this case, we have

$$H(t,y) = \frac{1}{3}\left(y_1^3, y_2^3, y_3^3\right), \int_0^y I(s)ds = \frac{1}{12}(y_1^4, y_2^4, y_3^4),$$

$H(t,y)$ and $I_k(y)$ are both odd functions, condition $(A4)$ are satisfied. And there is

$$0 < \beta H(t,y) = \frac{5}{6}\left(y_1^3, y_2^3, y_3^3\right) \le y\nabla H(t,y) = \left(y_1^3, y_2^3, y_3^3\right),$$
$$0 < \alpha \int_0^y I_k(s)ds = \frac{1}{4}(y_1^4, y_2^4, y_3^4) \le yI_k(y) = \frac{1}{3}(y_1^4, y_2^4, y_3^4),$$

then condition $(A2)$ are satisfied, and

$$\overline{\lim}_{y\to\theta}\ \frac{H(t,y)}{|y|^2} = \overline{\lim}_{y\to\theta}\ \frac{\frac{5}{6}\left(y_1^3, y_2^3, y_3^3\right)}{y_1^2+y_2^2+y_3^2} = \overline{\lim}_{y\to\theta}\ \frac{5}{6}(y_1, y_2, y_3) = 0,$$
$$\overline{\lim}_{y\to\theta}\ \frac{\int_0^y I_k(t)dt}{|y|^2} = \overline{\lim}_{y\to\theta}\ \frac{\frac{1}{12}(y_1^4, y_2^4, y_3^4)}{y_1^2+y_2^2+y_3^2} = \overline{\lim}_{y\to\theta}\ \frac{1}{6}(y_1^2, y_2^2, y_3^2) = 0,$$

so condition $(A3)$ are satisfied. Therefore problem (4.2) possesses infinitely many classical solutions by Proposition 3.1.

5. Conclusion

In this paper, the interesting points mainly include the followings:
(i) impulsive equation with three-point boundary condition has not been studied by variational method and critical point theory before;
(ii) chooses an appropriate space instead of the functional to contain the boundary conditions, is a novel and practical theoretical idea;
(iii) the nonlinear term $\nabla H(t,y)$ and the impulse term $I_k(y)$ are both super-linear growth.

In addition, $H(t,y) = 0$ and $\int_0^y I_k(t)dt = 0$ can be true respectively.

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