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TRAVELING WAVE SOLUTIONS OF THE GENERALIZED (2+1)-DIMENSIONAL KUNDU-MUKHERJEE-NASKAR EQUATION*

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Abstract In this paper, we consider two types of traveling wave systems of the generalized Kundu-Mukherjee-Naskar equation. Firstly, due to the integrity, we obtain their energy functions. Then, the dynamical system method is applied to study bifurcation behaviours of the two types of traveling wave systems to obtain corresponding global phase portraits in different parameter bifurcation sets. According to them, every bounded and unbounded orbits can be identified clearly and investigated carefully which correspond to various traveling wave solutions of the generalized Kundu-Mukherjee-Naskar equation exactly. Finally, by integrating along these orbits and calculating some complicated elliptic integral, we obtain all type I and type II traveling wave solutions of the generalized Kundu-Mukherjee-Naskar equation without loss.

Keywords GKMN equation, traveling wave solution, bifurcation, dynamical system.

MSC(2010) 35C07, 35Q55, 34C23.

1. Introduction

In this paper, we consider the following generalized (2+1)-dimensional Kundu-Mukherjee-Naskar (GKMN) equation [7]

$$iq_t + aq_{xy} + ibq(qq_x^* - q^*q_x) = 0, \quad i = \sqrt{-1}, \tag{1.1}$$

where complex function $q = q(x, y, t)$ represents the profile of soliton, q^* is the complex conjugation of $q(x, y, t)$, x, y and t are the spatial and temporal variables respectively. Real parameters a and b are the dispersion coefficient and nonlinear coefficient respectively. This equation can be used to describe optical wave propagation through coherently excited resonant waveguides, especially in the phenomenon of the bending of light beams [3].

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When the coefficients $a = 1$ and $b = 2$, the GKMN equation degenerates to the classical KMN equation

$$iq_t + q_{xy} + 2iq(qq_x^* - q^* q_x) = 0, \tag{1.2}$$

which was first proposed by Kundu, Mukherjee and Naskar in 2014 [8–10]. Besides description of the dynamics of optical soliton propagation in optical fibers, Eq. (1.2) also can be used extensively to address the problems of oceanic rogue waves, hole waves and an ion acoustic wave in a magnetized plasma [10, 12, 14].

Since the KMN equation is completely integrable and possesses the dynamic characteristics similar to the standard nonlinear Schrödinger equation [10,12], it can be regarded as an integrable generalization of the well-known nonlinear Schrödinger equation

$$q_t + iq_{xx} + 2i|q|^2 q = 0. \tag{1.3}$$

Unlike the conventional Kerr type nonlinearity in the Eq. (1.3), the nonlinear terms of Eq. (1.2) can be considered as the current nonlinearity caused by chirality [1]. Hence, Eq. (1.3) only allows bright (dark) soliton for focusing (defocusing) nonlinearity or with anomalous (normal) dispersion, whereas Eq. (1.2) admits both bright and dark soliton solutions regardless of its positive spatial dispersive term [17].

Traveling wave solutions of the KMN equation have been always focused on by people. In 2015, Mukherjee showed the connection between Eq. (1.2) and Kadomtsev Petviashvili equation and obtained its one-soliton solution, two-soliton solution and static lump solution by using Hirota bilinear method [12]. In 2017, Wen [20] pointed out that the solutions, satisfying the focusing nonlinear Schrödinger equation

$$q_y + iq_{xx} + 2i|q|^2 q = 0, \tag{1.4}$$

and the complex modified Korteweg-de Vries equation

$$q_t + q_{xxx} + 6|q|^2 q_x = 0, \tag{1.5}$$

must satisfy Eq. (1.2). So, by generalizing the n-fold Darboux transformation of Eqs. (1.4) and (1.5) to the perturbation $(n; M)$-fold Darboux transformation [20], he obtained the higher order rogue wave solutions of Eq. (1.2). Recently, as a generalization model of Eq. (1.2), the GKMN equation has aroused people's more extensive interests and attentions. In 2018, Peng applied ansatz method to obtain the bright soliton, dark soliton and power series solutions of Eq. (1.1) and constructed the complexitons through the tanh method [13]. In 2019, Yıldırım got bright, dark, singular, combo bright-dark, combo singular and singular periodic solitons of Eq. (1.1) by using the modified simple equation method, Riccati function method and so on [21–25]. In the same year, Ekici explored the plane wave solutions of Eq. (1.1) via the extended trial function method [3]. Later, Kudryashov used the Jacobi elliptic functions [7] to construct the general solution of the Eq. (1.1). With the aid of extended rational sinh-Gordon equation expansion method [18], Sulaiman obtained the trigonometric functions solutions of Eq. (1.1). In addition, Jhangeer applied the direct extended algebraic approach to Eq. (1.1) to derive the complex waves [6]. In 2020, Rizvi [15] got dark, bright, periodic U-shaped and singular solitons through the generalized Kudryashov method. Subsequently, Kumar [11] discussed singular, dark, combined dark-singular solitons and other hyperbolic solutions by using the csch method, extended tanh-coth method and extended rational sinh-cosh method.

Meanwhile, Talarposhti [19] and Ghanbari [4] derived some new solitary solutions by using the Exp-function method. More recently, Rezazadeh constructed the analytical solutions of Eq. (1.1) by utilizing the functional variable method [16].

Although various concise and efficient methods have been put forward to obtain so many profound results about traveling wave solutions of the Eq. (1.1), there still exist some problems unsolved. Firstly, due to the limitations caused by both the ansatz equations and the assumption about solutions in these direct methods, some solutions of Eq. (1.1) could be lost. In addition, we note that unbounded traveling wave solutions of Eq. (1.1) have not been reported in previous work. So, in this paper, we will try to apply the bifurcation method of dynamical system [2, 5] to solve these problems. This method allows detailed analysis of phase space geometry of the traveling wave system of Eq. (1.1). Through exploring various orbits of traveling wave system of Eq. (1.1), we construct its traveling wave solutions uniformly, including the bounded traveling wave solutions and unbounded ones.

Our paper is organized as follows: In section 2, we derive two types of traveling wave systems of Eq. (1.1), including a singular traveling wave system. Then, by studying bifurcation of traveling wave solutions, we try to give global phase portraits of the two types of traveling wave systems. In section 3, by calculating some complicated elliptic integrals along various orbits, we construct bounded and unbounded traveling wave solutions of Eq. (1.1) uniformly.

2. Traveling wave systems and bifurcation analysis

In this section, inspired by previous work [7, 22], we firstly derive two types of traveling wave systems of Eq. (1.1). Then, we study bifurcation of the two traveling wave systems (2.3) and (2.8) by dynamical system method.

2.1. Two types of traveling wave systems of the GKMN equation

Firstly, we assume that the type I traveling wave solution has the form

$$q(x,y,t) = p(\xi)\exp(\psi(x,y,t)i), \tag{2.1}$$

where real function $p(\xi)$ is the portion of the amplitude with $\xi = x + my - ct$ and real function $\psi(x,y,t) = \kappa x + \omega y - rt + \theta$ is the phase component. Parameters m, c, r and θ are reals and represent the inverse width of the soliton in the direction y, wave velocity, wave number and phase constant respectively. Real parameters κ and ω denote the frequencies along the x and y directions respectively.

Substituting the solution (2.1) into Eq. (1.1), we obtain

$$\begin{cases} \text{real part: } amp'' + (r - a\kappa\omega)p + 2\kappa bp^3 = 0, \\ \text{imaginary part: } (-c + am\kappa + a\omega)p' = 0, \end{cases} \tag{2.2}$$

where $'$ denotes $d/d\xi$. From the imaginary part in (2.2), we have

$$c = am\kappa + a\omega.$$

It means that system (2.2) has the equivalent form

$$\begin{cases} p^{'} = y, \\ y^{'} = \dfrac{a\kappa\omega - r}{am}p - \dfrac{2\kappa b}{am}p^3, \end{cases} \tag{2.3}$$

with constraint $c = am\kappa + a\omega$. Obviously, system (2.3) is a Hamiltonian system with the energy function

$$H(p, y) = \frac{1}{2}y^2 - \frac{a\kappa\omega - r}{2am}p^2 + \frac{\kappa b}{2am}p^4. \tag{2.4}$$

Then, we consider the type II traveling wave solution

$$q(x, y, t) = \phi(\xi)\exp((\varphi(\xi) - \mu t)i), \quad \xi = x + my - ct, \tag{2.5}$$

where $\phi(\xi)$ is the amplitude function, $\varphi(\xi)$ is the phase function and real parameters m, c and μ represent the inverse width of the soliton in the direction y, wave velocity and wave frequency respectively. Here, we suppose that the amplitude function $\phi(\xi) \neq 0$, otherwise the solution $q(x, y, t)$ of Eq. (1.1) degenerates to the trivial solution. Substituting (2.5) into (1.1) and separating the real and imaginary parts, we have

$$\begin{cases} \text{real part: } am\phi^{''} + n\phi\varphi^{'} + c\phi - am\phi(\varphi^{'})^2 + 2b\phi^3\varphi^{'} = 0, \\ \text{imaginary part: } -c\phi^{'} + 2am\phi^{'}\varphi^{'} + am\phi\varphi^{''} = 0, \end{cases} \tag{2.6}$$

where $'$ denotes $d/d\xi$. From the second equation in (2.6), we can solve

$$\varphi^{'} = \frac{e}{am\phi^2} + \frac{c}{2am}, \tag{2.7}$$

where e is the integral constant. Plugging (2.7) into the real part in (2.6), we obtain

$$\begin{cases} \phi^{'} = y, \\ y^{'} = \dfrac{\alpha_1\phi^6 + \alpha_2\phi^4 + \alpha_3}{\phi^3}, \end{cases} \tag{2.8}$$

where $\alpha_1 = -\dfrac{cb}{a^2m^2}$, $\alpha_2 = -\dfrac{\mu}{am} - \dfrac{c^2 + 8be}{4}$ and $\alpha_3 = \dfrac{e^2}{a^2m^2} \neq 0$. If $\alpha_3 = 0$, system (2.8) degenerates to system (2.3). System (2.8) is a singular traveling wave system. With the transformation $d\xi = \phi^3 d\zeta$, it can be converted to the associated regular system

$$\begin{cases} \phi^{'} = \phi^3 y, \\ y^{'} = \alpha_1\phi^6 + \alpha_2\phi^4 + \alpha_3, \end{cases} \tag{2.9}$$

where $'$ stands for $d/d\zeta$, which has the energy function

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{\alpha_1}{4}\phi^4 - \frac{\alpha_2}{2}\phi^2 + \frac{\alpha_3}{2}\frac{1}{\phi^2}. \tag{2.10}$$

When $\phi \neq 0$, systems (2.8) has the same vector field as system (2.9). So, the function (2.10) is also the energy function of system (2.8).

2.2. Bifurcation analysis

Firstly, we study the distribution and properties of equilibria of systems (2.3) and (2.9).

Theorem 2.1. *When $\kappa b(a\kappa\omega - r) > 0$, system (2.3) has three equilibria $P_1(-\sqrt{\frac{a\kappa\omega - r}{2\kappa b}}, 0)$, $P_2(0,0)$ and $P_3(\sqrt{\frac{a\kappa\omega - r}{2\kappa b}}, 0)$. If $\frac{\kappa b}{am} < 0 (> 0)$, P_1 and P_3 are saddles(centers), while P_2 is a center(saddle). When $\kappa b(a\kappa\omega - r) < 0$, system (2.3) has only one simple equilibrium $P_2(0,0)$. If $\frac{\kappa b}{am} < 0 (> 0)$, P_2 is a saddle(center). When $a\kappa\omega - r = 0$, system (2.3) has a unique degenerate equilibrium $P_2(0,0)$. If $\frac{\kappa b}{am} < 0 (> 0)$, P_2 is still a saddle(center).*

Proof. When $\kappa b(a\kappa\omega - r) \neq 0$, by the theory of dynamical system and the properties of Hamiltonian system [2, 5, 26], it is easy to verify the corresponding results above.

Especially, when $a\kappa\omega - r = 0$, we see that system (2.3) has a unique equilibrium $P_2(0,0)$ with the degenerate Jacobian matrix

$$M(P_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In this case, system (2.3) has the associated normal form

$$\begin{cases} p' = y, \\ y' = a_k p^k [1 + f_1(p)] + b_n p^n y [1 + f_2(p)] + y^2 f_3(p,y) = -\dfrac{2\kappa b}{am} p^3, \end{cases}$$

where $k = 3$, $a_k = -\dfrac{2\kappa b}{am}$, $b_n = 0$, $f_1(p) = 0$, $f_2(p) = 0$ and $f_3(p,y) = 0$. From the fact that k is an odd number, the degenerate equilibrium P_2 is a saddle when $a_k > 0$, whereas P_2 is a center when $a_k < 0$ and $b_n = 0$ according to the qualitative theory of differential equation [26, Theorem 7.2, Chapter 2]. □

Furthermore, we note that when $\kappa b(a\kappa\omega - r) > 0$, the energy of three equilibria has the following relationship

$$\begin{aligned} h(P_2) &= 0, \\ h(P_1) \equiv h(P_3) &= -\frac{(a\kappa\omega - r)^2}{8ab\kappa m}, \end{aligned}$$

which means that the energy of saddles P_1 and P_3 is always equivalent. So, according to the properities of Hamilton system [2], we have the following results:

Case 1. When $\kappa b(a\kappa\omega - r) > 0$ and $\dfrac{\kappa b}{am} < 0$, there exist two heteroclinic orbits Γ^0 and Γ_0 connecting the saddles P_1 and P_3. Center P_2 is surrounded by a family of periodic orbits

$$\gamma(h) = \{H(p,y) = h, h \in \big(0, -\frac{(a\kappa\omega - r)^2}{8ab\kappa m}\big)\}.$$

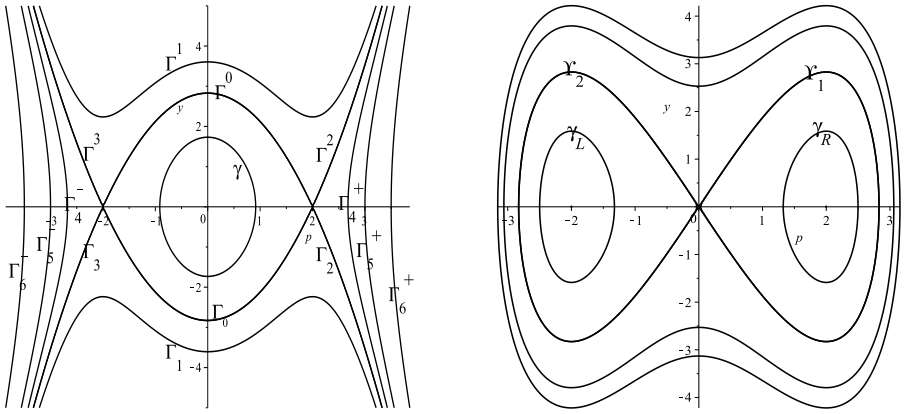
$\gamma(h)$ tends to P_2 as h$\to$0, and tends to Γ^0 and Γ_0 as h$\to-\dfrac{(a\kappa\omega-r)^2}{8ab\kappa m}$. Except the periodic orbit and the heteroclinic orbits, other orbits of system (2.3) are unbounded. Please see Fig. 1(a).

Case 2. When $\kappa b(a\kappa\omega-r)>0$ and $\dfrac{\kappa b}{am}>0$, all orbits are bounded. There exist two homoclinic orbits Υ_1 and Υ_2 connecting the saddle P_2. Centers P_1 and P_3 are surrounded by two families of periodic orbits

$$\gamma_{_R}(h)=\{H(p,y)=h, h\in\big(-\frac{(a\kappa\omega-r)^2}{8ab\kappa m},0\big)\},$$
$$\gamma_{_L}(h)=\{H(p,y)=h, h\in\big(-\frac{(a\kappa\omega-r)^2}{8ab\kappa m},0\big)\}.$$

$\gamma_{_R}(h)$ and $\gamma_{_L}(h)$ tend to P_1 and P_3 respectively as h$\to-\dfrac{(a\kappa\omega-r)^2}{8ab\kappa m}$, and tend to Υ_1 and Υ_2 respectively as h$\to$0 shown in Fig. 1(a).

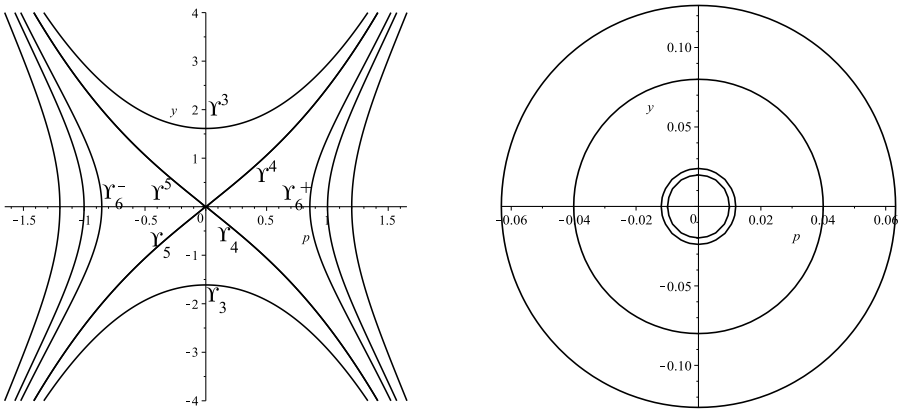
Case 3. When $\kappa b(a\kappa\omega-r)<0$ or $a\kappa\omega-r=0$, system (2.3) has only one equilibrium. If $\dfrac{\kappa b}{am}<0(>0)$, all orbits of system (2.3) are unbounded(bounded) shown in Figs. 2 and 3.



(a) $\dfrac{\kappa b}{am}=-\dfrac{1}{2}$, $\dfrac{a\kappa\omega-r}{am}=-4$. (b) $\dfrac{\kappa b}{am}=\dfrac{1}{2}$, $\dfrac{a\kappa\omega-r}{am}=4$.

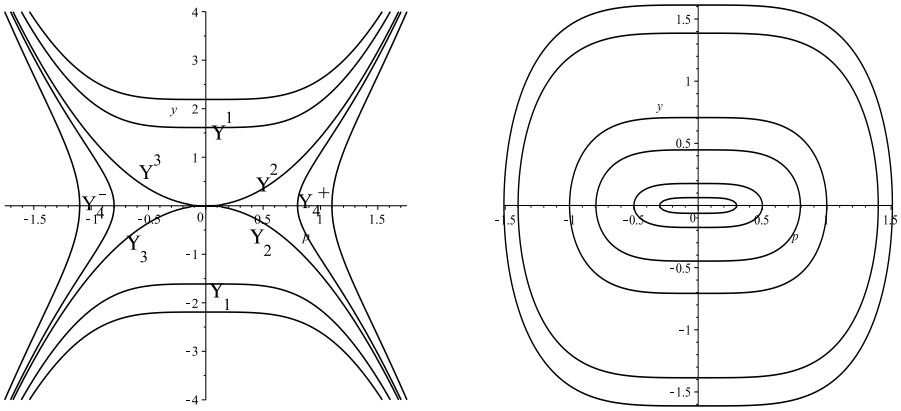
Figure 1. Phase portraits of system (2.3) for $\kappa b(a\kappa\omega-r)>0$.

Theorem 2.2. *When $\alpha_1>0$, $\alpha_2<0$ and $0<\alpha_3<-\dfrac{4\alpha_2^3}{27\alpha_1^2}$, system (2.9) has two pairs of equilibria $\widetilde{P}_{1,2}(\pm u_1,0)$ and $\widetilde{P}_{3,4}(\pm u_2,0)$, where $\widetilde{P}_{1,2}$ are centers and $\widetilde{P}_{3,4}$ are saddles. When $\alpha_1>0$, $\alpha_2<0$ and $\alpha_3=-\dfrac{4\alpha_2^3}{27\alpha_1^2}$, system (2.9) has a pair of cusps $\widetilde{P}_{5,6}(\pm u_3,0)$. When either $\alpha_1>0$, $\alpha_2\geq 0$ or $\alpha_1>0$, $\alpha_2<0$, $\alpha_3>-\dfrac{4\alpha_2^3}{27\alpha_1^2}$, system (2.9) has no equilibrium. When $\alpha_1<0$, system (2.9) has a pair of centers $\widetilde{P}_{7,8}(\pm u_4,0)$.*



(a) $\frac{\kappa b}{am} = -2,\ \frac{a\kappa\omega - r}{am} = 4.$ (b) $\frac{\kappa b}{am} = 2,\ \frac{a\kappa\omega - r}{am} = -4.$

Figure 2. Phase portraits of system (2.3) for $\kappa b(a\kappa\omega - r) < 0$.



(a) $\frac{\kappa b}{am} = -\frac{1}{2},\ a\kappa\omega - r = 0.$ (b) $\frac{\kappa b}{am} = \frac{1}{2},\ a\kappa\omega - r = 0.$

Figure 3. Phase portraits of system (2.3) for $a\kappa\omega - r = 0$.

Proof. Similar to the proof of Theorem 2.1, we only give the detail proof for degenerate equilibria for simplicity. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_3 = -\frac{4\alpha_2^3}{27\alpha_1^2}$, a direct computation shows that system (2.9) has a pair of degenerate equilibria $\widetilde{P}_{5,6}(\pm u_3, 0)$ with the degenerate Jacobian matrix

$$M(\widetilde{P}_{5,6}) = \begin{bmatrix} 0 & \pm u_3^3 \\ 0 & 0 \end{bmatrix}.$$

In order to judge the type of the equilibrium $\widetilde{P}_5(u_3, 0)$, we make the homeomor-

phic transformation

$$u = \phi - u_3, v = y + \frac{1}{u_3^3}(3u_3^2\alpha y + 3u_3\alpha^2 y + \alpha^3 y), \tag{2.11}$$

which transforms system (2.9) to the normal form

$$\begin{cases} u' = v, \\ v' = a_k u^k[1 + g_1(u)] + b_n u^n v[1 + g_2(u)] + v^2 g_3(u, v), \end{cases}$$

where $'$ stands for $d/d\varsigma$ and $d\varsigma = u_3^3 d\zeta$, $k = 2$, $M = \dfrac{\alpha_2 + 3u_3^2\alpha_1}{\alpha_1}$, $a_k = \dfrac{4\alpha_1 M}{u_3}$, $g_1(u) = \dfrac{1}{4u_3^5 M}\big((8u_3^5 M + 16u_3^4 M)u + (12u_3^5 + 24u_3^4 + 25u_3^3 M)u^2 + (42u_3^4 + 24u_3^3 + 19u_3^2 M)u^3 + (54u_3^3 + 9u_3^2 + 7u_3 M)u^4 + (33u_3^2 + M)u^5 + 9u_3 u^6 + u^7\big)$, $b_n = 0$ and $g_3(u, v) = \dfrac{3u_3^2 + 6u_3 u + 3u^2}{u_3^3 + 3u_3^2 u + 3u_3 u^2 + u^3}$. From the fact that $k = 2$ is an even number and $b_n = 0$, we come to the conclusion that equilibrium $\widetilde{P}_5(u_3, 0)$ is a cusp according to the qualitative theory of differential equation [26, Theorem 7.3, Chapter 2].

Similarly, applying another homeomorphic transformation

$$u = \phi + u_3, v = y - \frac{1}{u_3^3}(3u_3^2\alpha y - 3u_3\alpha^2 y + \alpha^3 y), \tag{2.12}$$

to system (2.9), one can check equilibrium $\widetilde{P}_6(-u_3, 0)$ is also a cusp. □

Based on the properties of the system (2.9), the bifurcation results of system (2.8) are given as follows.

Case I. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $0 < \alpha_3 < -\dfrac{4\alpha_2^3}{27\alpha_1^2}$, there exist two homoclinic orbits Π_0^+ and Π_0^- connecting saddles $\widetilde{P}_3$ and $\widetilde{P}_4$ respectively. Centers $\widetilde{P}_1$ and $\widetilde{P}_2$ are surrounded by two families of periodic orbits, respectively

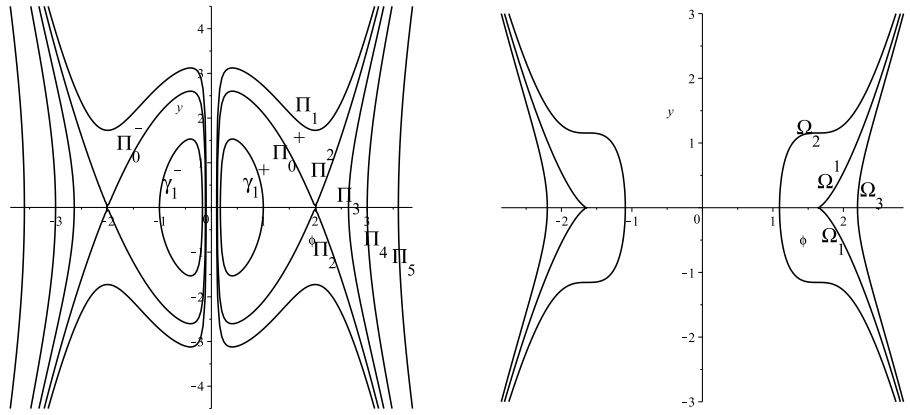
$$\gamma_1^+(h) = \{H(\phi, y) = h, h \in \big(h(u_1, 0), h(u_2, 0)\big)\},$$
$$\gamma_1^-(h) = \{H(\phi, y) = h, h \in \big(h(-u_1, 0), h(-u_2, 0)\big)\}.$$

$\gamma_1^+(h)$ and $\gamma_1^-(h)$ respectively tend to $\widetilde{P}_{1,2}(\pm u_1, 0)$ as h$\to h(\pm u_1, 0)$, and tend to Π_0^+ and Π_0^- as h$\to h(\pm u_2, 0)$. All orbits are unbounded except for the periodic orbits and the homoclinic orbits shown in Fig. 4(a).

Case II. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_3 = -\dfrac{4\alpha_2^3}{27\alpha_1^2}$, all orbits of system (2.8) are unbounded. In the positive ϕ-axis, the orbit Ω^1 is different from orbit Ω_1. More precisely, the ω-limit set of Ω^1 and the α-limit set of Ω_1 correspond to the same equilibrium $\widetilde{P}_5(u_3, 0)$ shown in Fig. 4(b).

Case III. When $\alpha_1 > 0$, $\alpha_2 \geq 0$ or $\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_3 > -\dfrac{4\alpha_2^3}{27\alpha_1^2}$, there only exist unbounded orbits of system (2.8) shown in Fig. 4(c).

Case IV. When $\alpha_1 < 0$, there exist two families of periodic orbits shown in Fig. 4(d).



(a) $\alpha_1 = 1, \alpha_2 = -4, \alpha_3 = 0.1$. (b) $\alpha_1 = 1, \alpha_2 = -4, \alpha_3 = \frac{256}{27}$.

(c) $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 0.1$. (d) $\alpha_1 = -1, \alpha_2 = 0, \alpha_3 = 0.1$.

Figure 4. Phase portraits of system (2.8).

3. Exact solutions of systems (2.3) and (2.8)

In this section, we seek the explicit expressions of bounded and unbounded solutions of systems (2.3) and (2.8).

3.1. Bounded solutions of system (2.3)

According to the bifurcation results in Theorem 2.1, to seek bounded solutions of system (2.3), there are three cases need to be discussed.

1. When $\kappa b(a\kappa\omega - r) > 0$ and $\dfrac{\kappa b}{am} < 0$, we consider two subcases as follows.
(1) Consider the periodic orbits shown in Fig. 1(a), whose energy is lower than energy of the saddle P_1, but higher than energy of center P_2. Any one of them can be expressed by

$$y = \pm\sqrt{-\frac{\kappa b}{am}}\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)},$$

where p_1, p_2, p_3 and p_4 satisfy the constraint condition $p_1 < -\sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_2 < p < p_3 < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_4$. Assuming that the period is $2T_0$ and choosing initial value $p(0) = p_2$, we have

$$\int_{p_2}^{p} \sqrt{-\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)}} = \int_0^{\xi} d\xi, \quad 0 < \xi < T_0,$$

$$-\int_{p}^{p_2} \sqrt{-\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)}} = \int_{\xi}^{0} d\xi, \quad -T_0 < \xi < 0,$$

which can be rewritten as

$$\int_{p_2}^{p} \sqrt{-\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)}} = \mid \xi \mid, \quad -T_0 < \xi < T_0.$$

By calculating the elliptic integral

$$\int_{p_2}^{p} \frac{dp}{\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)}} = g \cdot sn^{-1}\left(\sqrt{\frac{(p_3-p_1)(p-p_2)}{(p_3-p_2)(p-p_1)}}, k\right),$$

where $g = \dfrac{2}{\sqrt{(p_3-p_1)(p_4-p_2)}}$, $k^2 = \dfrac{(p_3-p_2)(p_4-p_1)}{(p_4-p_2)(p_3-p_1)}$, we get the first type of periodic solution of system (2.3)

$$p_{b_1}(\xi) = p_1 + \frac{(p_2-p_1)(p_3-p_1)}{(p_3-p_1) - (p_3-p_2)sn^2\left(\sqrt{\frac{-\kappa b(p_4-p_2)(p_3-p_1)}{4am}}\xi\right)}, \quad -T_0 < \xi < T_0.$$

(2) Consider the heteroclinic orbit Γ^0 shown in Fig. 1(a), whose energy is equal to energy of the saddle P_1. It can be expressed by

$$y = \sqrt{-\frac{\kappa b}{am}}\sqrt{(p-p_5)^2(p_6-p)^2},$$

where p_5 and p_6 satisfy the constraint condition $-\sqrt{\dfrac{a\kappa\omega - r}{2\kappa b}} = p_5 < p < p_6 = \sqrt{\dfrac{a\kappa\omega - r}{2\kappa b}}$. Choosing initial value $p(0) = \dfrac{p_5 + p_6}{2} = 0$, we have

$$\int_0^{p} \sqrt{-\frac{am}{\kappa b}} \frac{dp}{(p-p_5)(p_6-p)} = \int_0^{\xi} d\xi, \quad -\infty < \xi < +\infty.$$

Noting that

$$\int_0^{p} \frac{dp}{(p-p_5)(p_6-p)} = \frac{2}{p_6-p_5} \tanh^{-1} \frac{2p - (p_5+p_6)}{p_6-p_5},$$

we obtain the expression of kink wave solution of system (2.3)

$$p_{b_2}(\xi) = \frac{p_6-p_5}{2} \tanh\left(\frac{p_6-p_5}{2}\sqrt{-\frac{\kappa b}{am}}\xi\right), \quad -\infty < \xi < +\infty.$$

Applying similar calculation to another heteroclinic orbit Γ_0 shown in Fig. 1(a), we can get the corresponding kink wave solution of system (2.3) as follows

$$p_{b_{2'}}(\xi) = \frac{p_6 - p_5}{2} \tanh(\frac{p_5 - p_6}{2}\sqrt{-\frac{\kappa b}{am}}\xi), \quad -\infty < \xi < +\infty.$$

2. When $\kappa b(a\kappa\omega - r) > 0$ and $\dfrac{\kappa b}{am} > 0$, we need to consider three subcases in this case.

(1) Consider the family of periodic orbits inside the homoclinic orbit Υ_2 shown in Fig. 1(b), whose energy is lower than energy of the saddle P_2, but higher than energy of center P_1. Any one of them can be expressed by

$$y = \pm\sqrt{\frac{\kappa b}{am}}\sqrt{(p - p_7)(p_8 - p)(p_9 - p)(p_{10} - p)},$$

where p_7, p_8, p_9 and p_{10} satisfy the constraint condition $p_7 < p < p_8 < 0 < p_9 < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_{10}$. Assuming that the period is $2T_1$, similar to the calculation of solution $p_{b_1}(\xi)$, we get the second type of periodic solution of system (2.3)

$$p_{b_3}(\xi) = p_{10} - \frac{(p_{10} - p_8)(p_{10} - p_7)}{(p_{10} - p_8) + (p_8 - p_7)sn^2(\sqrt{\frac{\kappa b}{am}}\frac{\sqrt{(p_{10} - p_8)(p_9 - p_7)}}{2}\xi)}, \quad -T_1 < \xi < T_1.$$

Similarly, we can give the periodic solutions of system (2.3) corresponding to the family of periodic orbits inside the homoclinic orbit Υ_1 shown in Fig. 1(b)

$$p_{b_{3'}}(\xi) = p_8 + \frac{(p_9 - p_8)(p_{10} - p_8)}{(p_{10} - p_8) - (p_{10} - p_9)sn^2(\sqrt{\frac{\kappa b}{am}}\frac{\sqrt{(p_{10} - p_8)(p_9 - p_7)}}{2}\xi)}, \quad -T_1 < \xi < T_1,$$

where $p_7 < -\sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_8 < 0 < p_9 < p < p_{10}$.

(2) Consider the homoclinic orbits shown in Fig. 1(b), whose energy is equal to energy of the saddle P_2. The homoclinic orbit Υ_2 can be expressed by

$$y = \pm\sqrt{\frac{\kappa b}{am}}\sqrt{(p + p_{11})p^2(p_{11} - p)},$$

where p_{11} and $-p_{11}$ satisfy the constraint condition $-p_{11} < p < 0 < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_{11}$. Choosing initial value $p(0) = -p_{11}$, we have

$$\int_{-p_{11}}^{p} \sqrt{\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p + p_{11})p^2(p_{11} - p)}} = \int_0^{\xi} d\xi, \quad \xi > 0,$$

$$-\int_{p}^{-p_{11}} \sqrt{\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p + p_{11})p^2(p_{11} - p)}} = \int_{\xi}^{0} d\xi, \quad \xi < 0,$$

which can be rewritten as

$$\int_{-p_{11}}^{p} \sqrt{\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p + p_{11})p^2(p_{11} - p)}} = \mid \xi \mid .$$

Noting that

$$\int_{-p_{11}}^{p} \frac{dp}{p\sqrt{(p+p_{11})(p_{11}-p)}} = \frac{\ln(\frac{-p_{11}-\sqrt{{p_{11}}^2-p^2}}{p})}{-p_{11}},$$

we obtain the expression of solitary wave solution of system (2.3)

$$p_{b_4}(\xi) = \frac{-2p_{11}\exp(\sqrt{\frac{\kappa b}{am}}p_{11} \mid \xi \mid)}{\exp(2\sqrt{\frac{\kappa b}{am}}p_{11} \mid \xi \mid)+1}, \quad -\infty < \xi < +\infty.$$

Similarly, we can get another solitary wave solution of system (2.3) corresponding to the homoclinic orbit Υ_1

$$p_{b_{4'}}(\xi) = \frac{2p_{11}\exp(\sqrt{\frac{\kappa b}{am}}p_{11} \mid \xi \mid)}{\exp(2\sqrt{\frac{\kappa b}{am}}p_{11} \mid \xi \mid)+1}, \quad -\infty < \xi < +\infty,$$

where $-p_{11} < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < 0 < p < p_{11}$.

(3) Consider the family of large amplitude periodic orbits shown in Fig. 1(b), whose energy is higher than energy of the saddle P_2. Any one of them can be expressed by

$$y = \pm\sqrt{\frac{\kappa b}{am}}\sqrt{(p+p_{12})(p_{12}-p)(p^2+p_{12}^2-\frac{a\kappa\omega-r}{\kappa b})},$$

where p_{12} and $-p_{12}$ satisfy the constraint condition $p_{12} > 0$, $p_{12}^2 - \frac{a\kappa\omega-r}{\kappa b} > 0$ and $-p_{12} < p < p_{12}$. Assuming that the period is $2T_2$ and choosing initial value $p(0) = \frac{-p_{12}+p_{12}}{2} = 0$, we have

$$\int_0^p \sqrt{\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p+p_{12})(p_{12}-p)(p^2+p_{12}^2-\frac{a\kappa\omega-r}{\kappa b})}} = \int_0^\xi d\xi, \quad 0 < \xi < T_2,$$

$$-\int_p^0 \sqrt{\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p+p_{12})(p_{12}-p)(p^2+p_{12}^2-\frac{a\kappa\omega-r}{\kappa b})}} = \int_\xi^0 d\xi, \quad -T_2 < \xi < 0,$$

which can be rewritten as

$$\int_0^p \sqrt{\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p+p_{12})(p_{12}-p)(p^2+p_{12}^2-\frac{a\kappa\omega-r}{\kappa b})}} = \mid \xi \mid, \quad -T_2 < \xi < T_2.$$

Noting that

$$\int_0^p \frac{dp}{\sqrt{(p+p_{12})(p_{12}-p)(p^2+p_{12}^2-\frac{a\kappa\omega-r}{\kappa b})}}$$
$$= g \cdot sn^{-1}\left(\sqrt{\frac{p^2(2\kappa b p_{12}^2 - a\kappa\omega + r)}{p_{12}^2(\kappa b p^2 + \kappa b p_{12}^2 - a\kappa\omega + r)}}, k\right),$$

where $g=\sqrt{\dfrac{\kappa b}{2\kappa b p_{12}^2-a\kappa\omega+r}}$, $k^2=\dfrac{\kappa b p_{12}^2}{2\kappa b p_{12}^2-a\kappa\omega+r}$, we obtain the third type of periodic solution of system (2.3)

$$p_{b_5}(\xi)=\sqrt{\frac{(\kappa b p_{12}^2-a\kappa\omega+r)p_{12}^2 sn^2(\sqrt{\frac{2\kappa b p_{12}^2-a\kappa\omega+r}{am}}\xi)}{2\kappa b p_{12}^2-a\kappa\omega+r-\kappa b p_{12}^2 sn^2(\sqrt{\frac{2\kappa b p_{12}^2-a\kappa\omega+r}{am}}\xi)}},\quad -T_2<\xi<T_2.$$

3. When $\kappa b(a\kappa\omega-r)\le 0$ and $\dfrac{\kappa b}{am}>0$, one can check that the periodic orbits shown in Figs. 2(b) and 3(b) have the same form as solution $p_{b_5}(\xi)$. We ignore them here for simplicity.

3.2. Unbounded solutions of system (2.3)

In this subsection, we need to consider two cases to get unbounded solutions of system (2.3).

1. When $\kappa b(a\kappa\omega-r)>0$ and $\dfrac{\kappa b}{am}<0$, we have five subcases to discuss.

(1) Consider the first type of unbounded orbits, for example Γ^1 and Γ_1, shown in Fig. 1(a), whose energy h_0 is higher than energy of saddles P_1 and P_3. They can be expressed respectively by

$$y=\pm\sqrt{-\frac{\kappa b}{am}p^4+\frac{a\kappa\omega-r}{am}p^2+2h_0},$$

where $0<p<+\infty$. For the sake of simplicity, we take Γ_1 for example to calculate its corresponding solution. Choosing initial value $p(0)=+\infty$, we have

$$-\int_{+\infty}^{p}\sqrt{-\frac{am}{\kappa b}}\frac{dp}{\sqrt{p^4+\dfrac{r-a\kappa\omega}{\kappa b}p^2-\dfrac{2amh_0}{\kappa b}}}=\int_0^{\xi}d\xi,\quad \xi>0.$$

Noting that

$$\int_p^{+\infty}\frac{dp}{\sqrt{p^4+\dfrac{r-a\kappa\omega}{\kappa b}p^2-\dfrac{2amh_0}{\kappa b}}}=g\cdot cn^{-1}\Big(\frac{p^2-\sqrt{-\frac{2amh_0}{\kappa b}}}{p^2+\sqrt{-\frac{2amh_0}{\kappa b}}},k\Big),$$

where $g=\dfrac{1}{2\sqrt[4]{-\frac{2amh_0}{\kappa b}}}$, $k^2=\dfrac{2\sqrt{-2am\kappa bh_0}+a\kappa\omega-r}{4\sqrt{-2am\kappa bh_0}}$, we obtain the first type of unbounded solution of system (2.3)

$$p_{u_0}(\xi)=\sqrt[4]{-\frac{2amh_0}{\kappa b}}\cdot\sqrt{-1+\frac{2}{1-cn(2\sqrt[4]{-\frac{2\kappa bh_0}{am}}\xi)}},\quad 0<\xi<\xi_0,$$

where $\xi_0=2\sqrt[4]{-\dfrac{am}{2\kappa bh_0}}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{2\sqrt{-2am\kappa bh_0}+a\kappa\omega-r}{4\sqrt{-2am\kappa bh_0}}\cdot\sin^2\theta}}$.

It is not difficult to check that the corresponding unbounded solution of Γ^1 has the same form as $p_{u_0}(\xi)$.

(2) Consider the second type of unbounded orbits Γ^2, Γ_2, Γ^3 and Γ_3 shown in Fig. 1(a), whose energy is equal to energy of saddles P_1 and P_3. The orbits Γ^2 and Γ_2 can be expressed respectively by

$$y = \pm\sqrt{-\frac{\kappa b}{am}}\sqrt{(p-\sqrt{\frac{a\kappa\omega-r}{2\kappa b}})^2(p+\sqrt{\frac{a\kappa\omega-r}{2\kappa b}})^2},$$

where $0 < \sqrt{\dfrac{a\kappa\omega-r}{2\kappa b}} < p < +\infty$. Similar to the discussion above, we only need to discuss the orbit Γ_2. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p}\sqrt{-\frac{am}{\kappa b}}\frac{dp}{(p-\sqrt{\frac{a\kappa\omega-r}{2\kappa b}})(p+\sqrt{\frac{a\kappa\omega-r}{2\kappa b}})} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_p^{+\infty}\frac{dp}{(p-\sqrt{\frac{a\kappa\omega-r}{2\kappa b}})(p+\sqrt{\frac{a\kappa\omega-r}{2\kappa b}})} = \sqrt{\frac{\kappa b}{2(a\kappa\omega-r)}}\ln\left(\frac{p+\sqrt{\frac{a\kappa\omega-r}{2\kappa b}}}{p-\sqrt{\frac{a\kappa\omega-r}{2\kappa b}}}\right),$$

we obtain the second type of unbounded solution of system (2.3)

$$p_{u_1}(\xi) = \sqrt{\frac{a\kappa\omega-r}{2\kappa b}}\cdot\left(1+\frac{2}{\exp(\sqrt{\frac{2(r-a\kappa\omega)}{am}}\xi)-1}\right), \quad \xi > 0.$$

One can check that the corresponding unbounded solution of Γ^2 has the same form as $p_{u_1}(\xi)$. If choosing initial value $p(0) = -\infty$ and applying similar calculation to the orbit Γ^3, we can get the corresponding unbounded solution of system (2.3)

$$p_{u_{1'}}(\xi) = -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}}\cdot\left(1+\frac{2}{\exp(\sqrt{\frac{2(r-a\kappa\omega)}{am}}\xi)-1}\right), \quad \xi > 0,$$

where $-\infty < p < -\sqrt{\dfrac{a\kappa\omega-r}{2\kappa b}} < 0$. And it is not difficult to conclude that the corresponding unbounded solution of the orbit Γ_3 has the same form as $p_{u_{1'}}(\xi)$.

(3) Consider the third type of unbounded orbits, for example Γ_4^+, shown in Fig. 1(a), whose energy is lower than energy of the saddle P_1, but higher than energy of center P_2. It can be expressed by

$$y = \pm\sqrt{-\frac{\kappa b}{am}}\sqrt{(p-p_{1'})(p-p_{2'})(p-p_{3'})(p-p_{4'})},$$

where $p_{1'}$, $p_{2'}$, $p_{3'}$ and $p_{4'}$ satisfy the constraint condition $p_{1'} < -\sqrt{\dfrac{a\kappa\omega-r}{2\kappa b}} < p_{2'} < 0 < p_{3'} < \sqrt{\dfrac{a\kappa\omega-r}{2\kappa b}} < p_{4'} < p < +\infty$. Similarly, we only need to consider

the lower branch of Γ_4^+. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p} \sqrt{-\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p-p_{1'})(p-p_{2'})(p-p_{3'})(p-p_{4'})}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_p^{+\infty} \frac{dp}{\sqrt{(p-p_{1'})(p-p_{2'})(p-p_{3'})(p-p_{4'})}} = g \cdot sn^{-1}(\frac{p_{4'}}{p}, k),$$

where $g = \dfrac{1}{p_{4'}}$, $k^2 = \dfrac{p_{3'}^2}{p_{4'}^2}$, we get the third type of unbounded solution of system (2.3)

$$p_{u_2}(\xi) = \frac{p_{4'}}{sn(p_{4'}\sqrt{-\frac{\kappa b}{am}}\xi)}, \quad 0 < \xi < \xi_1,$$

where $\xi_1 = \dfrac{4}{p_{4'}}\sqrt{-\dfrac{am}{\kappa b}} \cdot \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{p_{3'}^2}{p_{4'}^2}\cdot \sin^2\theta}}$.

If choosing initial value $p(0) = -\infty$ and adopting the similar calculation to the upper branch of unbounded orbit Γ_4^-, we can get the corresponding unbounded solution of system (2.3)

$$p_{u_{2'}}(\xi) = \frac{p_{1'}}{sn(p_{1'}\sqrt{-\frac{\kappa b}{am}}\xi)}, \quad 0 < \xi < \xi_{1'},$$

where $-\infty < p < p_{1'} < -\sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_{2'} < 0 < p_{3'} < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_{4'}$, $\xi_{1'} = -\dfrac{4}{p_{1'}}\sqrt{-\dfrac{am}{\kappa b}} \cdot \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{p_{2'}^2}{p_{1'}^2}\cdot \sin^2\theta}}$.

(4) Consider the fourth type of unbounded orbits, for example Γ_5^+ and Γ_5^-, shown in Fig. 1(a), whose energy is equal to energy of the center P_2. The unbounded orbit Γ_5^+ can be expressed by

$$y = \pm\sqrt{-\frac{\kappa b}{am}}\sqrt{(p+p_{5'})p^2(p-p_{5'})},$$

where $p_{5'}$ satisfies the constraint condition $0 < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_{5'} < p < +\infty$. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p} \sqrt{-\frac{am}{\kappa b}} \frac{dp}{\sqrt{(p+p_{5'})p^2(p-p_{5'})}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_p^{+\infty} \frac{dp}{p\sqrt{(p+p_{5'})(p-p_{5'})}} = \frac{1}{p_{5'}} \arcsin\frac{p_{5'}}{p},$$

we obtain the fourth type of unbounded solution of system (2.3)

$$p_{u_3}(\xi) = p_{5'} \csc(p_{5'}\sqrt{-\frac{\kappa b}{am}}\xi), \quad 0 < \xi < \xi_2,$$

where $\xi_2 = \dfrac{2\pi}{p_{5'}}\sqrt{-\dfrac{am}{\kappa b}}$.

If choosing initial value $p(0) = -\infty$, we can get another unbounded solution of system (2.3) corresponding to the upper branch of unbounded orbit Γ_5^-

$$p_{u_{3'}}(\xi) = p_{5'}\csc(-p_{5'}\sqrt{-\frac{\kappa b}{am}}\xi), \quad 0 < \xi < \xi_{2'},$$

where $-\infty < p < p_{5'} < -\sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < 0$, $\xi_{2'} = -\dfrac{2\pi}{p_{5'}}\sqrt{-\dfrac{am}{\kappa b}}$.

(5) Consider the fifth type of unbounded orbit, for example Γ_6^+, shown in Fig. 1(a), whose energy is lower than energy of the center P_2. It can be expressed by

$$y = \pm\sqrt{-\frac{\kappa b}{am}}\sqrt{(p + p_{6'})(p - p_{6'})(p^2 + p_{6'}^2 - \frac{a\kappa\omega - r}{\kappa b})},$$

where $p_{6'}$ satisfies the constraint condition $0 < \sqrt{\dfrac{a\kappa\omega - r}{\kappa b}} < p_{6'} < p < +\infty$. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p}\sqrt{-\frac{am}{\kappa b}}\frac{dp}{\sqrt{(p + p_{6'})(p - p_{6'})(p^2 + p_{6'}^2 - \dfrac{a\kappa\omega - r}{\kappa b})}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\begin{aligned}&\int_p^{+\infty}\frac{dp}{\sqrt{(p + p_{6'})(p - p_{6'})(p^2 + p_{6'}^2 - \dfrac{a\kappa\omega - r}{\kappa b})}}\\ =&g \cdot sn^{-1}(\sqrt{\frac{2\kappa b p_{6'}^2 - a\kappa\omega + r}{\kappa b p^2 + \kappa b p_{6'}^2 - a\kappa\omega + r}}, k),\end{aligned}$$

where $g = \sqrt{\dfrac{\kappa b}{2\kappa b p_{6'}^2 - a\kappa\omega + r}}$, $k^2 = \dfrac{\kappa b p_{6'}^2 - a\kappa\omega + r}{2\kappa b p_{6'}^2 - a\kappa\omega + r}$, we obtain the fifth type of unbounded solution of system (2.3)

$$p_{u_4}(\xi) = \sqrt{-\frac{\kappa b p_{6'}^2 - a\kappa\omega + r}{\kappa b} + \frac{2\kappa b p_{6'}^2 - a\kappa\omega + r}{\kappa b s n^2(\sqrt{-\frac{2\kappa b p_{6'}^2 - a\kappa\omega + r}{am}}\xi)}}, \quad 0 < \xi < \xi_3,$$

where $\xi_3 = \sqrt{-\dfrac{4am}{2\kappa b p_{6'}^2 - a\kappa\omega + r}} \cdot \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{\kappa b p_{6'}^2 - a\kappa\omega + r}{2\kappa b p_{6'}^2 - a\kappa\omega + r} \cdot \sin^2\theta}}$.

If choosing initial value $p(0) = -\infty$ and applying similar calculation to the upper branch of unbounded orbit Γ_6^-, it is obvious to conclude that the corresponding unbounded solution has the same form as unbounded solution $p_{u_4}(\xi)$.

2. When $\kappa b(a\kappa\omega - r) \le 0$ and $\dfrac{\kappa b}{am} < 0$, we need to consider three subcases in this case.

(1) Consider the first type of unbounded orbits, for example Υ^3, Υ_3, Y^1 and Y_1, shown in Figs. 2(a) and 3(a), whose energy $h_{0'}$ is higher than energy of the saddle

P_2. One can check that the corresponding unbounded solution of Υ_3 has the same form as solution $p_{u_0}(\xi)$. We ignore it here for simplicity.

In particular, when $\kappa b(a\kappa\omega - r) = 0$, the unbounded orbits Y^1 and Y_1 shown in Fig. 3(a) can be expressed respectively by

$$y = \pm\sqrt{-\frac{\kappa b}{am}p^4 + 2h_{0'}},$$

where $0 < p < +\infty$. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p}\sqrt{-\frac{am}{\kappa b}}\frac{dp}{\sqrt{p^4 - \dfrac{2amh_{0'}}{\kappa b}}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_p^{+\infty}\frac{dp}{\sqrt{p^4 - \dfrac{2amh_{0'}}{\kappa b}}} = \frac{1}{2}\sqrt[4]{-\frac{\kappa b}{2amh_{0'}}}\cdot cn^{-1}(\frac{p^2-1}{p^2+1}, k),$$

where $k^2 = \dfrac{1}{2}$, we obtain the sixth type of unbounded solution of system (2.3)

$$p_{u_{5'}}(\xi) = \sqrt{\frac{2}{1 - cn(8h_{0'}\xi)} - 1}, \quad 0 < \xi < \xi_{4'},$$

where $\xi_{4'} = \dfrac{1}{2h_{0'}}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1 - \frac{1}{2}\cdot\sin^2\theta}}$.

(2) Consider the second type of unbounded orbits shown in Figs. 2(a) and 3(a), whose energy is equal to energy of the saddle P_2. The unbounded orbits Υ^4 and Υ_4 can be expressed by

$$y = \pm\sqrt{-\frac{\kappa b}{am}}\sqrt{p^4 + \frac{r - a\kappa\omega}{\kappa b}p^2},$$

where $0 < p < +\infty$. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p}\sqrt{-\frac{am}{\kappa b}}\frac{dp}{p\sqrt{p^2 + \dfrac{r - a\kappa\omega}{\kappa b}}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_p^{+\infty}\frac{dp}{p\sqrt{p^2 + \dfrac{r - a\kappa\omega}{\kappa b}}} = \sqrt{\frac{\kappa b}{r - a\kappa\omega}}\cdot\ln\left(\frac{\sqrt{p^2 + \frac{r-a\kappa\omega}{\kappa b}} + \sqrt{\frac{r-a\kappa\omega}{\kappa b}}}{p}\right),$$

we obtain the seventh type of unbounded solution of system (2.3)

$$p_{u_6}(\xi) = 2\sqrt{\frac{r - a\kappa\omega}{\kappa b}}\cdot\frac{\exp(\sqrt{\frac{a\kappa\omega - r}{am}}\xi)}{\exp(2\sqrt{\frac{a\kappa\omega - r}{am}}\xi) - 1}, \quad \xi > 0.$$

Similar calculation can be applied to the unbounded orbit Υ^5 shown in Fig. 2(a). If choosing $p(0) = -\infty$ and $-\infty < p < 0$, we obtain another unbounded solution of system (2.3) as follows

$$p_{u_{6'}}(\xi) = 2\sqrt{\frac{r - a\kappa\omega}{\kappa b}} \cdot \frac{\exp(\sqrt{\frac{a\kappa\omega - r}{am}}\xi)}{1 - \exp(2\sqrt{\frac{a\kappa\omega - r}{am}}\xi)}, \quad \xi > 0.$$

In particular, when $\kappa b(a\kappa\omega - r) = 0$, the unbounded orbits Y^2 and Y_2 shown in Fig. 3(a) can be expressed respectively by

$$y = \pm\sqrt{-\frac{\kappa b}{am}}p^2,$$

where $0 < p < +\infty$. By a direct calculation, we obtain the corresponding unbounded solution of system (2.3)

$$p_{u_7}(\xi) = \sqrt{-\frac{am}{\kappa b}} \cdot \frac{1}{\xi}, \quad \xi > 0.$$

If choosing $p(0) = -\infty$ and adopting the similar calculation to the unbounded orbit Y^3 shown in Fig. 3(a), we have

$$p_{u_{7'}}(\xi) = -\sqrt{-\frac{am}{\kappa b}} \cdot \frac{1}{\xi}, \quad \xi > 0,$$

where $-\infty < p < 0$.

(3) Consider the third type of unbounded orbits, for example Υ_6^+, Υ_6^-, Y_4^+ and Y_4^-, shown in Figs. 2(a) and 3(a), whose energy is lower than energy of the saddle P_2. It is not difficult to conclude that the corresponding unbounded solutions have the same form as solution $p_{u_4}(\xi)$. Due to the the tedious expressions of the solutions, we ignore them here for simplicity.

3.3. Bounded solutions of system (2.8)

From the energy function $H(\phi, y) = \frac{1}{2}y^2 - \frac{\alpha_1}{4}\phi^4 - \frac{\alpha_2}{2}\phi^2 + \frac{\alpha_3}{2}\frac{1}{\phi^2} = h$ and the first equation of system (2.8), we can obtain the following expression

$$\begin{aligned} \xi &= \int_{\phi_0}^{\phi} \frac{\phi d\phi}{\sqrt{\frac{\alpha_1}{2}\phi^6 + \alpha_2\phi^4 + 2h\phi^2 - \alpha_3}} \\ &= \int_{\psi_0}^{\psi} \frac{d\psi}{2\sqrt{\frac{\alpha_1}{2}\psi^3 + \alpha_2\psi^2 + 2h\psi - \alpha_3}}, \end{aligned} \tag{3.1}$$

where $\phi > 0$ and $\psi = \phi^2$. Due to the symmetry of system (2.8), it is easy to check that the bounded solutions of system (2.8) where $\phi < 0$ have the same expressions of bounded solutions of system (2.8) where $\phi > 0$ except for the difference of signs.

I. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $0 < \alpha_3 < -\frac{4\alpha_2^3}{27\alpha_1^2}$, we have two subcases in this case.

(i) Consider the periodic orbits shown in Fig. 4(a), whose energy is lower than energy of the saddle $\widetilde{P}_3$, but higher than energy of center $\widetilde{P}_1$. Assuming that r_1, r_2 and r_3 satisfy the constraint condition $0 < r_1 < \psi < r_2 < r_3$, the period is $2T_{0'}$ and choosing $\psi_0 = \psi(0) = r_1$, we have

$$\int_{r_1}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}} = \int_0^{\xi} d\xi, \quad 0 < \xi < T_{0'},$$
$$-\int_{r_1}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}} = \int_{\xi}^{0} d\xi, \quad -T_{0'} < \xi < 0,$$

which can be rewritten as

$$\int_{r_1}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}} = \mid \xi \mid, \quad -T_{0'} < \xi < T_{0'}.$$

Noting that

$$\int_{r_1}^{\psi} \frac{d\psi}{\sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}} = g \cdot sn^{-1}(\sqrt{\frac{\psi - r_1}{r_2 - r_1}}, k),$$

where $g = \dfrac{2}{\sqrt{r_3 - r_1}}$, $k^2 = \dfrac{r_2 - r_1}{r_3 - r_1}$, we get the first type of periodic solution of system (2.8)

$$\phi_{b_1}(\xi) = \sqrt{r_1 + (r_2 - r_1)sn^2(\sqrt{\frac{\alpha_1(r_3 - r_1)}{2}}\xi)}, \quad -T_{0'} < \xi < T_{0'}.$$

(ii) Consider the homoclinic orbit Π_0^+ shown in Fig. 4(a), whose energy is equal to energy of the saddle $\widetilde{P}_3$. Assuming that r_4 and r_5 satisfy the constraint condition $0 < r_4 < \psi < r_5$ and choosing $\psi_0 = \psi(0) = r_4$, we have

$$\int_{r_4}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_4)(r_5 - \psi)^2}} = \int_0^{\xi} d\xi, \quad \xi > 0,$$
$$-\int_{\psi}^{r_4} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_4)(r_5 - \psi)^2}} = \int_{\xi}^{0} d\xi, \quad \xi < 0,$$

which can be rewritten as

$$\int_{r_4}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}(r_5 - \psi)\sqrt{\psi - r_4}} = \mid \xi \mid, \quad -\infty < \xi < +\infty.$$

Noting that

$$\int_{r_4}^{\psi} \frac{d\psi}{(r_5 - \psi)\sqrt{\psi - r_4}} = -\frac{1}{\sqrt{r_5 - r_4}} \ln \frac{\sqrt{r_5 - r_4} - \sqrt{\psi - r_4}}{\sqrt{r_5 - r_4} + \sqrt{\psi - r_4}},$$

we obtain the expression of solitary wave solution of system (2.8)

$$\phi_{b_2}(\xi) = \sqrt{r_4 + \frac{(r_5 - r_4)(1 - \exp(\sqrt{2\alpha_1(r_5 - r_4)}\xi))^2}{(1 + \exp(\sqrt{2\alpha_1(r_5 - r_4)}\xi))^2}}, \quad -\infty < \xi < +\infty.$$

II. When $\alpha_1 < 0$, there only exist two families of periodic orbits shown in Fig. 4(d). Assuming that r_6 and r_7 satisfy the constraint conditions $0 < r_6 < \psi < r_7$, $r_6 + r_7 + \dfrac{2\alpha_2}{\alpha_1} > 0$, the period is $2T_{1'}$ and choosing $\psi_0 = \psi(0) = r_6$, we have

$$\int_{r_6}^{\psi} \frac{d\psi}{\sqrt{-2\alpha_1}\sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2\alpha_2}{\alpha_1})}} = \int_0^{\xi} d\xi, \quad 0 < \xi < T_{1'},$$

$$-\int_{\psi}^{r_6} \frac{d\psi}{\sqrt{-2\alpha_1}\sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2\alpha_2}{\alpha_1})}} = \int_{\xi}^{0} d\xi, \quad -T_{1'} < \xi < 0,$$

which can be rewritten as

$$\int_{r_6}^{\psi} \frac{d\psi}{\sqrt{-2\alpha_1}\sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2\alpha_2}{\alpha_1})}} = \mid \xi \mid, \quad -T_{1'} < \xi < T_{1'}.$$

Noting that

$$\begin{aligned}&\int_{r_6}^{\psi} \frac{d\psi}{\sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2\alpha_2}{\alpha_1})}}\\ =&g \cdot sn^{-1}\Big(\sqrt{\frac{(2r_7 + r_6 + \frac{2\alpha_2}{\alpha_1})(\psi - r_6)}{(r_7 - r_6)(\psi + r_6 + r_7 + \frac{2\alpha_2}{\alpha_1})}}, k\Big),\end{aligned}$$

where $g = \dfrac{2}{\sqrt{2r_7 + r_6 + \frac{2\alpha_2}{\alpha_1}}}$, $k^2 = \dfrac{r_7 - r_6}{2r_7 + r_6 + \frac{2\alpha_2}{\alpha_1}}$, we obtain the second type of periodic solution of system (2.8)

$$\phi_{b_3}(\xi) = \sqrt{r_6 + \frac{(2\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)(r_7 - r_6) sn^2(\sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}}\xi)}{\alpha_1(r_7 - r_6) sn^2(\sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}}\xi) - (2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2)}},$$
$$-T_{1'} < \xi < T_{1'}.$$

3.4. Unbounded solutions of system (2.8)

In this subsection, to seek the unbounded solutions of system (2.8) where $\psi > 0$ and $\phi = \sqrt{\psi} > 0$, we need to discuss three cases. If applying similar calculation to the unbounded orbits of system (2.8) where $\phi < 0$, one can check the corresponding unbounded solutions have the same expressions as unbounded solutions where $\phi > 0$ except for the difference of signs.

I. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $0 < \alpha_3 < -\dfrac{4\alpha_2^3}{27\alpha_1^2}$, we have five subcases in this case.

(i) Consider the first type of unbounded orbits, for example Π_1, shown in Fig. 4(a), whose energy is higher than energy of the saddle $\widetilde{P}_3$. We consider to discuss the lower branch of orbit Π_1 for simplicity. Assuming that $r_{1'}$ satisfies the constraint

condition $0 < r_{1'} < \psi < +\infty$, $h_1 \in (h(\widetilde{P}_3), +\infty)$ and choosing $\psi_0 = \psi(0) = +\infty$, we have

$$-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_{1'})[\psi^2 + (\frac{2\alpha_2}{\alpha_1} + r_{1'})\psi + r_{1'} + \frac{4h_1}{\alpha_1}]}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{\psi}^{+\infty} \frac{d\psi}{\sqrt{(\psi - r_{1'})[\psi^2 + (\frac{2\alpha_2}{\alpha_1} + r_{1'})\psi + r_{1'} + \frac{4h_1}{\alpha_1}]}} = g \cdot cn^{-1}(\frac{\psi - r_{1'} - A}{\psi - r_{1'} + A}, k),$$

where $A^2 = \dfrac{2\alpha_1 r_{1'}^2 + (\alpha_1 + 2\alpha_2) r_{1'} + 4h_1}{\alpha_1}$, $g = \dfrac{1}{\sqrt{A}} = \sqrt{\dfrac{\alpha_1}{2\alpha_1 r_{1'}^2 + (\alpha_1 + 2\alpha_2) r_{1'} + 4h_1}}$ and $k^2 = \dfrac{\sqrt{8\alpha_1^2 r_{1'}^2 + 4\alpha_1(\alpha_1 + 2\alpha_2) r_{1'} + 16\alpha_1 h_1} - (2\alpha_2 + 3\alpha_1 r_{1'})}{\sqrt{32\alpha_1^2 r_{1'}^2 + 16\alpha_1(\alpha_1 + \alpha_2) r_{1'} + 64\alpha_1 h_1}}$, we get the first type of unbounded solution of system (2.8)

$$\phi_{u_1}(\xi) = \sqrt{r_{1'} - \sqrt[4]{\frac{2\alpha_1 r_{1'}^2 + (\alpha_1 + 2\alpha_2) r_{1'} + 4h_1}{\alpha_1}} + \frac{2\sqrt[4]{\frac{2\alpha_1 r_{1'}^2 + (\alpha_1 + 2\alpha_2) r_{1'} + 4h_1}{\alpha_1}}}{1 - cn(\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{1'}^2 + (\alpha_1 + 2\alpha_2) r_{1'} + 4h_1}{\alpha_1}}\xi)}},$$

where $0 < \xi < \xi_{0''}$ and

$$\xi_{0''} = \frac{4}{\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{1'}^2 + (\alpha_1 + 2\alpha_2) r_{1'} + 4h_1}{\alpha_1}}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{\sqrt{8\alpha_1^2 r_{1'}^2 + 4\alpha_1(\alpha_1 + 2\alpha_2) r_{1'} + 16\alpha_1 h_1} - (2\alpha_2 + 3\alpha_1 r_{1'})}{\sqrt{32\alpha_1^2 r_{1'}^2 + 16\alpha_1(\alpha_1 + \alpha_2) r_{1'} + 64\alpha_1 h_1}} \cdot \sin^2\theta}}.$$

Similar calculation can be applied to the upper branch of the orbit Π_1. And one can check the corresponding solution has the same form as solution $\phi_{u_1}(\xi)$.

(ii) Consider the second type of unbounded orbit Π_2 shown in Fig. 4(a), whose energy is equal to energy of the saddle $\widetilde{P}_3$. Similar to the discussion above, we only need to discuss the lower branch of orbit Π_2. Assuming that $r_{2'}$ and $r_{3'}$ satisfy the constraint condition $0 < r_{2'} < r_{3'} < \psi < +\infty$ and choosing $\psi_0 = \psi(0) = +\infty$, we have

$$-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_{2'})(\psi - r_{3'})^2}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{\psi}^{+\infty} \frac{d\psi}{(\psi - r_{3'})\sqrt{\psi - r_{2'}}} = -\frac{1}{\sqrt{r_{3'} - r_{2'}}} \ln \frac{\sqrt{\psi - r_{2'}} - \sqrt{r_{3'} - r_{2'}}}{\sqrt{\psi - r_{2'}} + \sqrt{r_{3'} - r_{2'}}},$$

we get the second type of unbounded solution of system (2.8)

$$\phi_{u_2}(\xi) = \sqrt{r_{2'} + \frac{(r_{3'} - r_{2'})(1 + \exp(\sqrt{2\alpha_1(r_{3'} - r_{2'})}\xi))^2}{(1 - \exp(\sqrt{2\alpha_1(r_{3'} - r_{2'})}\xi))^2}}, \quad \xi > 0.$$

(iii) Consider the third type of unbounded orbits, for example unbounded orbit Π_3, shown in Fig. 4(a), whose energy is higher than energy of the center $\widetilde{P}_1$, but lower than energy of saddle $\widetilde{P}_3$. Assuming that $r_{4'}$, $r_{5'}$ and $r_{6'}$ satisfy the constraint condition $0 < r_{4'} < r_{5'} < r_{6'} < \psi < +\infty$ and choosing $\psi_0 = \psi(0) = +\infty$, it can be expressed by

$$-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_{4'})(\psi - r_{5'})(\psi - r_{6'})}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{\psi}^{+\infty} \frac{d\psi}{\sqrt{(\psi - r_{4'})(\psi - r_{5'})(\psi - r_{6'})}} = g \cdot sn^{-1}(\sqrt{\frac{r_{6'} - r_{4'}}{\psi - r_{4'}}}, k),$$

where $g = \dfrac{2}{\sqrt{r_{6'} - r_{4'}}}$, $k^2 = \dfrac{r_{5'} - r_{4'}}{r_{6'} - r_{4'}}$, we get the third type of unbounded solution of system (2.8)

$$\phi_{u_3}(\xi) = \sqrt{r_{4'} + \frac{r_{6'} - r_{4'}}{sn^2(\sqrt{\dfrac{\alpha_1(r_{6'} - r_{4'})}{2}}\xi)}}, \quad 0 < \xi < \xi_{1''},$$

where $\xi_{1''} = \sqrt{\dfrac{8}{\alpha_1(r_{6'} - r_{4'})}} \cdot \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{r_{5'} - r_{4'}}{r_{6'} - r_{4'}} \cdot \sin^2\theta}}$.

(iv) Consider the fourth type of unbounded orbits, for example Π_4, shown in Fig. 4(a), whose energy is equal to energy of the center $\widetilde{P}_2$. Assuming that $r_{7'}$ and $r_{8'}$ satisfy the constraint condition $0 < r_{7'} < r_{8'} < \psi < +\infty$ and choosing $\psi_0 = \psi(0) = +\infty$, we have

$$-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi - r_{7'})^2(\psi - r_{8'})}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{\psi}^{+\infty} \frac{d\psi}{(\psi - r_{7'})\sqrt{\psi - r_{8'}}} = \frac{1}{\sqrt{r_{8'} - r_{7'}}}(\pi - 2\arctan\sqrt{\frac{\psi - r_{8'}}{r_{8'} - r_{7'}}}),$$

we obtain the fourth type of unbounded solution of system (2.8)

$$\phi_{u_4}(\xi) = \sqrt{r_{8'} + (r_{8'} - r_{7'}) \cdot \cot^2(\sqrt{\frac{\alpha_1(r_{8'} - r_{7'})}{2}}\xi)}, \quad 0 < \xi < \xi_{2''},$$

where $\xi_{2''} = \sqrt{\dfrac{2}{\alpha_1(r_{8'} - r_{7'})}} \cdot \pi$.

(v) Consider the fifth type of unbounded orbits, for example Π_5, shown in Fig. 4(a), whose energy is lower than energy of the center $\widetilde{P}_2$. It is not difficult to obtain the fifth type of unbounded solution as follows

$$\phi_{u_5}(\xi)=\sqrt{r_{9'}-\sqrt[4]{\frac{2\alpha_1 r_{9'}^2+(\alpha_1+2\alpha_2)r_{9'}+4h_2}{\alpha_1}}+\frac{2\sqrt[4]{\frac{2\alpha_1 r_{9'}^2+(\alpha_1+2\alpha_2)r_{9'}+4h_2}{\alpha_1}}}{1-cn(\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{9'}^2+(\alpha_1+2\alpha_2)r_{9'}+4h_2}{\alpha_1}}\xi)}},$$

where $0<r_{9'}<\psi<+\infty$, $h_2\in(0,h(\widetilde{P}_2))$, $0<\xi<\xi_{3''}$ and

$$\xi_{3''}=\frac{4}{\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{9'}^2+(\alpha_1+2\alpha_2)r_{9'}+4h_2}{\alpha_1}}}\cdot\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{\sqrt{8\alpha_1^2 r_{9'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{9'}+16\alpha_1 h_2}-(2\alpha_2+3\alpha_1 r_{9'})}{\sqrt{32\alpha_1^2 r_{9'}^2+16\alpha_1(\alpha_1+\alpha_2)r_{9'}+64\alpha_1 h_2}}\cdot\sin^2\theta}}.$$

II. When $\alpha_1>0$, $\alpha_2<0$ and $\alpha_3=-\dfrac{4\alpha_2^3}{27\alpha_1^2}$, we consider two subcases in this case.

(i) Consider the first type of unbounded orbits Ω^1 and Ω_1 shown in Fig. 4(b), whose energy is equal to energy of the cusp $\widetilde{P}_5$. Choosing $\psi_0=\psi(0)=+\infty$, we have

$$-\int_{+\infty}^{\psi}\frac{d\psi}{\sqrt{2\alpha_1}\sqrt{(\psi+\frac{2\alpha_2}{3\alpha_1})^2(\psi+\frac{2\alpha_2}{3\alpha_1})}}=\int_0^{\xi}d\xi,\quad \xi>0.$$

Noting that

$$\int_{\psi}^{+\infty}\frac{d\psi}{(\psi+\frac{2\alpha_2}{3\alpha_1})\sqrt{\psi+\frac{2\alpha_2}{3\alpha_1}}}=\frac{2}{\sqrt{\psi+\frac{2\alpha_2}{3\alpha_1}}},$$

we get the sixth type of unbounded solution of system (2.8)

$$\phi_{u_6}(\xi)=\sqrt{-\frac{2\alpha_2}{3\alpha_1}+\frac{2}{\alpha_1\xi^2}},\quad \xi>0.$$

(ii) Consider other unbounded orbits, for example Ω_2 and Ω_3, shown in Fig. 4(b), they can be expressed uniformly. Assuming that $r_{10'}$ satisfies the constraint condition $0<r_{10'}<\psi<+\infty$, $h_3\neq h(\widetilde{P}_5)$ and choosing $\psi_0=\psi(0)=+\infty$, we obtain

$$\phi_{u_7}(\xi)=\sqrt{r_{10'}-\sqrt[4]{\frac{2\alpha_1 r_{10'}^2+(\alpha_1+2\alpha_2)r_{10'}+4h_3}{\alpha_1}}+\frac{2\sqrt[4]{\frac{2\alpha_1 r_{10'}^2+(\alpha_1+2\alpha_2)r_{10'}+4h_3}{\alpha_1}}}{1-cn(\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{10'}^2+(\alpha_1+2\alpha_2)r_{10'}+4h_3}{\alpha_1}}\xi)}},$$

where $0<\xi<\xi_{4''}$ and

$$\xi_{4''}=\frac{4}{\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{10'}^2+(\alpha_1+2\alpha_2)r_{10'}+4h_3}{\alpha_1}}}$$

$$\cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{\sqrt{8\alpha_1^2 r_{10'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{10'}+16\alpha_1 h_3}-(2\alpha_2+3\alpha_1 r_{10'})}{\sqrt{32\alpha_1^2 r_{10'}^2+16\alpha_1(\alpha_1+\alpha_2)r_{10'}+64\alpha_1 h_3}}\cdot \sin^2\theta}}.$$

III. Simlarly, when $\alpha_1 > 0$, $\alpha_2 \geq 0$, one can check that the unbounded solution $\phi_{u_8}(\xi)$ has the same form as solution $\phi_{u_1}(\xi)$. We get the corresponding unbounded solution of system (2.8)

$$\phi_{u_8}(\xi) = \sqrt{r_{11'} - \sqrt[4]{\frac{2\alpha_1 r_{11'}^2+(\alpha_1+2\alpha_2)r_{11'}+4h_4}{\alpha_1}} + \frac{2\sqrt[4]{\frac{2\alpha_1 r_{11'}^2+(\alpha_1+2\alpha_2)r_{11'}+4h_4}{\alpha_1}}}{1-cn(\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{11'}^2+(\alpha_1+2\alpha_2)r_{11'}+4h_4}{\alpha_1}}\xi)}},$$

where $0 < r_{11'} < \psi < +\infty$, h_4 is a real number, $0 < \xi < \xi_{5''}$ and

$$\xi_{5''} = \frac{4}{\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{11'}^2+(\alpha_1+2\alpha_2)r_{11'}+4h_4}{\alpha_1}}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{\sqrt{8\alpha_1^2 r_{11'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{11'}+16\alpha_1 h_4}-(2\alpha_2+3\alpha_1 r_{11'})}{\sqrt{32\alpha_1^2 r_{11'}^2+16\alpha_1(\alpha_1+\alpha_2)r_{11'}+64\alpha_1 h_4}}\cdot \sin^2\theta}}.$$

4. Exact traveling wave solutions of Eq. (1.1)

In order to get the type I traveling wave solution of the GKMN equation, it only needs to substitute the solution $p(\xi)$ of system (2.3) into the formula (2.1). Here we list them in the Appendix 1 for the sake of simplicity.

But, it is not so easy to get the type II traveling wave solution of the GKMN equation. It needs us to substitute the solutions $\phi(\xi)$ of system (2.8) into the ODE (2.7) to solve $\varphi(\xi)$, and then plug $\phi(\xi)$ and $\varphi(\xi)$ into the formula (2.5)

(S1) Noting that

$$\phi_{b_1}(\xi) = \sqrt{r_1 + (r_2 - r_1)sn^2(\sqrt{\frac{\alpha_1(r_3-r_1)}{2}}\xi)}, \quad -T_{0'} < \xi < T_{0'},$$

and

$$\int \frac{du}{1+k\cdot sn^2(u)} = \frac{u}{2} + \frac{1}{2(1+k)}\tan^{-1}[(1+k)tn(u)\cdot nd(u)],$$

we have

$$\begin{aligned}\varphi_{b_1}(\xi) &= \int \left(\frac{e}{am\phi_{b_1}^2(\xi)} + \frac{c}{2am}\right)d\xi \\ &= \frac{e+cr_1}{2amr_1}\xi + \frac{r_1}{2r_2}\sqrt{\frac{2}{\alpha_1(r_3-r_1)}} \\ &\quad \cdot \tan^{-1}\left(\frac{r_2}{r_1}tn(\sqrt{\frac{\alpha_1(r_3-r_1)}{2}}\xi)\cdot nd(\sqrt{\frac{\alpha_1(r_3-r_1)}{2}}\xi)\right) + C_1,\end{aligned}$$

where C_1 is a constant.

Thus we obtain the final solution $q_1(x,y,t) = \phi_{b_1}(\xi)\exp(\varphi_{b_1}(\xi) - \mu t)i)$.

(S2) Noting that

$$\phi_{b_2}(\xi) = \sqrt{r_4 + \frac{(r_5 - r_4)(1 - \exp(\sqrt{2\alpha_1(r_5 - r_4)}\xi))^2}{(1 + \exp(\sqrt{2\alpha_1(r_5 - r_4)}\xi))^2}}, \quad -\infty < \xi < +\infty,$$

we have

$$\begin{aligned}\varphi_{b_2}(\xi) &= \int \left(\frac{e}{am\phi_{b_2}^2(\xi)} + \frac{c}{2am}\right)d\xi \\ &= \frac{2e\sqrt{2\alpha_1(r_5 - r_4)} + cr_5}{2amr_5}\xi + \frac{(r_5 - r_4)e}{8amr_5\sqrt{r_4(r_5 - r_4)}} \\ &\quad \cdot \arctan\left(\frac{r_5 - r_4}{2\sqrt{r_4(r_5 - r_4)}}\exp(\sqrt{2\alpha_1(r_5 - r_4)}\xi) + \frac{r_5 - r_4}{2\sqrt{r_4(r_5 - r_4)}} - \frac{r_5}{2r_4}\right) + C_2,\end{aligned}$$

where C_2 is a constant.

Thus we obtain the final solution $q_2(x,y,t) = \phi_{b_2}(\xi)\exp(\varphi_{b_2}(\xi) - \mu t)i)$.

(S3) Noting that

$$\begin{aligned}\phi_{b_3}(\xi) = &\sqrt{r_6 + \frac{(2\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)(r_7 - r_6)sn^2(\sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}}\xi)}{\alpha_1(r_7 - r_6)sn^2(\sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}}\xi) - (2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2)}}, \\ &- T_{1'} < \xi < T_{1'},\end{aligned}$$

and

$$\int \frac{du}{1 \pm k \cdot sn(u)} = \frac{1}{k'^2}[E(u) + k(1 \pm k \cdot sn(u)) \cdot cd(u)],$$

where $k' = \sqrt{1 - k^2}$, we have

$$\begin{aligned}\varphi_{b_3}(\xi) &= \int \left(\frac{e}{am\phi_{b_3}^2(\xi)} + \frac{c}{2am}\right)d\xi \\ &= \left(\frac{e\alpha_1}{am(3\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)}\sqrt{-\frac{2}{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}} + \frac{c}{2am}\right)\xi \\ &\quad + \frac{e(3\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2 - \alpha_1)\sqrt{-2(2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2)}}{am(3\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)(2\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)} \\ &\quad \cdot \left(E(\sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}}\xi) + \sqrt{\frac{\alpha_1 r_7 - \alpha_1 r_6}{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}}\right. \\ &\quad \left. \cdot cd(\sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}}\xi)\right) + C_3,\end{aligned}$$

where C_3 is a constant.

Thus we obtain the final solution $q_3(x,y,t) = \phi_{b_3}(\xi)\exp(\varphi_{b_3}(\xi) - \mu t)i)$.

(S4) Noting that

$$\phi_{u_1}(\xi)=\sqrt{r_{1'}-\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}+\frac{2\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}}{1-cn(\sqrt{2\alpha_1}\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}\xi)}},$$

and

$$\int\frac{1-cn(u)}{1+\alpha\cdot cn(u)}du=\frac{1}{\alpha}[-u+\frac{1}{1-\alpha}(\Pi(\varphi,\frac{\alpha^2}{\alpha^2-1},k)-\alpha\cdot f_1)],$$

where $0<\xi<\xi_{0''}$, $\varphi=am(u)$ and $f_1=\sqrt{\frac{1-\alpha^2}{k^2+k'^2\alpha^2}}\cdot\tan^{-1}[\sqrt{\frac{k^2+k'^2\alpha^2}{1-\alpha^2}}\cdot sd(u)]$, $k'=\sqrt{1-k^2}$, we have

$$\begin{aligned}
&\varphi_{u_1}(\xi)\\
&=\int\left(\frac{e}{am\phi_{u_1}^2(\xi)}+\frac{c}{2am}\right)d\xi\\
&=\left(\frac{e}{am(r_{1'}-\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}})}+\frac{c}{2am}\right)\xi\\
&\quad+\frac{e(\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}+r_{1'})}{2amr_{1'}\sqrt[4]{8\alpha_1^2r_{1'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{1'}+16\alpha_1h_1}(\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}-r_{1'})}\\
&\quad\cdot\Pi[am(\sqrt[4]{8\alpha_1^2r_{1'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{1'}+16\alpha_1h_1}\xi),\\
&\quad-\frac{(\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}-r_{1'})^2}{4\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}r_{1'}},k]\\
&\quad-\frac{e}{amr_{1'}\sqrt[4]{8\alpha_1^2r_{1'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{1'}+16\alpha_1h_1}}\\
&\quad\cdot\sqrt{\frac{\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}r_{1'}}{4r_{1'}k^2\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}+(\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}-r_{1'})^2}}\\
&\quad\cdot\tan^{-1}\left(\sqrt{k^2+\frac{(\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}-r_{1'})^2}{4\sqrt[4]{\frac{2\alpha_1 r_{1'}^2+(\alpha_1+2\alpha_2)r_{1'}+4h_1}{\alpha_1}}r_{1'}}}\right.\\
&\quad\left.\cdot sd(\sqrt[4]{8\alpha_1^2r_{1'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{1'}+16\alpha_1h_1}\xi)\right)+C_4,
\end{aligned}$$

where $k^2=\dfrac{\sqrt{8\alpha_1^2r_{1'}^2+4\alpha_1(\alpha_1+2\alpha_2)r_{1'}+16\alpha_1h_1}-(2\alpha_2+3\alpha_1r_{1'})}{\sqrt{32\alpha_1^2r_{1'}^2+16\alpha_1(\alpha_1+\alpha_2)r_{1'}+64\alpha_1h_1}}$, C_4 is a con-

stant.

Thus we obtain the final solution $q_4(x,y,t)=\phi_{u_1}(\xi)\exp(\varphi_{u_1}(\xi)-\mu t)i)$. Similar calculation can be applied to the solutions $\phi_{u_5}(\xi)$, $\phi_{u_7}(\xi)$ and $\phi_{u_8}(\xi)$, we ignore them here for simplicity.

(S5) Noting that

$$\phi_{u_2}(\xi)=\sqrt{r_{2'}+\frac{(r_{3'}-r_{2'})(1+\exp(\sqrt{2\alpha_1(r_{3'}-r_{2'})}\xi))^2}{(1-\exp(\sqrt{2\alpha_1(r_{3'}-r_{2'})}\xi))^2}},\quad \xi>0,$$

we have

$$\begin{aligned}
&\varphi_{u_2}(\xi)\\
&=\int\left(\frac{e}{am\phi_{u_2}^2(\xi)}+\frac{c}{2am}\right)d\xi\\
&=\frac{4e(r_{2'}-r_{3'})}{amr_{3'}\sqrt{2\alpha_1(r_{3'}-r_{2'})}}\ln(r_{3'}\sqrt{2\alpha_1(r_{3'}-r_{2'})}\xi+r_{3'}-2r_{2'})\\
&\quad+\frac{4e(r_{2'}-r_{3'})^2}{amr_{3'}\sqrt{2\alpha_1(r_{3'}-r_{2'})}(r_{3'}\exp(\sqrt{2\alpha_1(r_{3'}-r_{2'})}\xi)+r_{3'}-2r_{2'})}+\frac{c}{2am}\xi+C_5,
\end{aligned}$$

where C_5 is a constant.

Thus we obtain the final solution $q_5(x,y,t)=\phi_{u_2}(\xi)\exp(\varphi_{u_2}(\xi)-\mu t)i)$.

(S6) Noting that

$$\phi_{u_3}(\xi)=\sqrt{r_{4'}+\frac{r_{6'}-r_{4'}}{sn^2(\sqrt{\frac{\alpha_1(r_{6'}-r_{4'})}{2}}\xi)}},\quad 0<\xi<\xi_{1''},$$

and

$$\int\frac{du}{1+k\cdot sn^2(u)}=\frac{u}{2}+\frac{1}{2(1+k)}\cdot\tan^{-1}[(1+k)\cdot tn(u)\cdot nd(u)],$$

we have

$$\begin{aligned}
\varphi_{u_3}(\xi)&=\int\left(\frac{e}{am\phi_{u_3}^2(\xi)}+\frac{c}{2am}\right)d\xi\\
&=\frac{e+cr_{4'}}{2amr_{4'}}\xi-\frac{e(r_{6'}-r_{4'})}{4amr_{4'}^2r_{6'}}\sqrt{\frac{2}{\alpha_1(r_{6'}-r_{4'})}}\\
&\quad\cdot\tan^{-1}\left(\frac{r_{6'}}{r_{6'}-r_{4'}}\cdot tn(\sqrt{\frac{\alpha_1(r_{6'}-r_{4'})}{2}}\xi)\cdot nd(\sqrt{\frac{\alpha_1(r_{6'}-r_{4'})}{2}}\xi)\right)+C_6,
\end{aligned}$$

where C_6 is a constant.

Thus we obtain the final solution $q_6(x,y,t)=\phi_{u_3}(\xi)\exp(\varphi_{u_3}(\xi)-\mu t)i)$.

(S7) Noting that

$$\phi_{u_4}(\xi)=\sqrt{r_{8'}+(r_{8'}-r_{7'})\cdot\cot^2(\sqrt{\frac{\alpha_1(r_{8'}-r_{7'})}{2}}\xi)},\quad 0<\xi<\xi_{2''},$$

we have

$$\begin{aligned}\varphi_{u_4}(\xi) &= \int \left(\frac{e}{am\phi_{u_4}^2(\xi)} + \frac{c}{2am}\right)d\xi \\ &= \frac{2e + cr_{7'}}{2amr_{7'}}\xi - \frac{e}{amr_{7'}}\sqrt{\frac{2}{\alpha_1 r_{8'}}} \\ &\quad \cdot \arctan\left(\sqrt{\frac{r_{8'}}{r_{8'} - r_{7'}}} \cdot \tan(\sqrt{\frac{\alpha_1(r_{8'} - r_{7'})}{2}}\xi)\right) + C_7,\end{aligned}$$

where C_7 is a constant.

Thus we obtain the final solution $q_7(x, y, t) = \phi_{u_4}(\xi)\exp(\varphi_{u_4}(\xi) - \mu t)i)$.

(S8) Noting that

$$\phi_{u_6}(\xi) = \sqrt{-\frac{2\alpha_2}{3\alpha_1} + \frac{2}{\alpha_1\xi^2}}, \quad \xi > 0,$$

we have

$$\begin{aligned}\varphi_{u_6}(\xi) &= \int \left(\frac{e}{am\phi_{u_6}^2(\xi)} + \frac{c}{2am}\right)d\xi \\ &= \frac{c\alpha_2 - 3e\alpha_1}{2am\alpha_2}\xi - \frac{9e\alpha_1^2}{4am\alpha_2^2}\sqrt{-\frac{3}{\alpha_2}} \cdot \arctan(\sqrt{-\frac{\alpha_2}{3}}\xi) + C_8,\end{aligned}$$

where C_8 is a constant.

Thus we obtain the final solution $q_8(x, y, t) = \phi_{u_6}(\xi)\exp(\varphi_{u_6}(\xi) - \mu t)i)$.

5. Discussion and conclusion

In this paper, by using the dynamical system method, we study two kinds of traveling wave systems of the GKMN equation and obtain all type I and type II traveling wave solutions of it. Especially, some new solutions q_{b_7}, $q_{u_\iota}(\iota = 2, 2^{'}, 4..8, 4^{'}..8^{'})$ and $q_j(j = 1..8)$ have not been reported before, which not only help ones to understand the complicated physical phenomena described by the model further, but also can be used to verify the correctness of the numerical solutions. In particular, we can generate more solutions of Eq. (1.1) by using of these new solutions. For example, when $a = -1$ and $b = -2$, one can consider these new solutions as "seeds" and apply the perturbation $(n; M)$-fold Darboux transformation $\widetilde{\varphi} = T(\lambda)\varphi$ and $\widetilde{q}_{N-1} = q_0 - 2B^{(N-1)}$ mentioned in [20] to construct more new solutions. In addition, this method is an effective way to deal with traveling waves of a PDE and can be used in other PDE models.

Appendix 1

Table 1. Type I traveling wave solutions of Eq. (1.1)

$\kappa b(a\kappa\omega - r)$	$\frac{\kappa b}{am}$	traveling wave solution	p	ξ
$\kappa b(a\kappa\omega - r) > 0$	$\frac{\kappa b}{am} < 0$	$q_{b_1}(x,y,t) = \left(p_1 + \frac{(p_2-p_1)(p_3-p_1)}{(p_3-p_1)-(p_3-p_2)sn^2(\sqrt{\frac{-\kappa b(p_4-p_2)(p_3-p_1)}{4am}}\xi)}\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$p_1 < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_2 < p < p_3 < \sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_4$	$-T_0 < \xi < T_0$
		$q_{b_2}(x,y,t) = \left(\frac{p_6-p_5}{2}\tanh(\frac{p_6-p_5}{2}\sqrt{-\frac{\kappa b}{am}}\xi)\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$-\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} = p_5 < p < p_6 = \sqrt{\frac{a\kappa\omega-r}{2\kappa b}}$	$-\infty < \xi < +\infty$
		$q_{b_{2'}}(x,y,t) = \left(\frac{p_6-p_5}{2}\tanh(\frac{p_5-p_6}{2}\sqrt{-\frac{\kappa b}{am}}\xi)\right)\cdot\exp((\kappa x+\omega y-rt)i)$		
		$q_{u_0}(x,y,t) = \sqrt[4]{-\frac{2amh_0}{\kappa b}}\cdot\sqrt{-1+\frac{2}{1-cn(2\sqrt[4]{-\frac{2\kappa bh_0}{am}}\xi)}}\cdot\exp((\kappa x+\omega y-rt)i)$	$0 < p < +\infty$	$0 < \xi < \xi_0$
		$q_{u_1}(x,y,t) = \sqrt{\frac{a\kappa\omega-r}{2\kappa b}}\cdot\left(1+\frac{2}{\exp(\sqrt{\frac{2(r-a\kappa\omega)}{am}}\xi)-1}\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$0 < \sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p < +\infty$	$\xi > 0$
		$q_{u_{1'}}(t,x,y) = -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}}\cdot\left(1+\frac{2}{\exp(\sqrt{\frac{2(r-a\kappa\omega)}{am}}\xi)-1}\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$-\infty < p < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < 0$	$\xi > 0$
		$q_{u_2}(x,y,t) = \frac{p_{4'}}{sn(p_{4'}\sqrt{-\frac{\kappa b}{am}}\xi)}\cdot\exp((\kappa x+\omega y-rt)i)$	$p_{1'} < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_{2'} < 0 < p_{3'} < \sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_{4'} < p < +\infty$	$0 < \xi < \xi_1$
		$q_{u_{2'}}(x,y,t) = \frac{p_{1'}}{sn(p_{1'}\sqrt{-\frac{\kappa b}{am}}\xi)}\cdot\exp((\kappa x+\omega y-rt)i)$	$-\infty < p < p_{1'} < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_{2'} < 0 < p_{3'} < \sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_{4'}$	$0 < \xi < \xi_{1'}$
		$q_{u_3}(x,y,t) = p_{5'}\csc(p_{5'}\sqrt{-\frac{\kappa b}{am}}\xi)\cdot\exp((\kappa x+\omega y-rt)i)$	$0 < \sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_{5'} < p < +\infty$	$0 < \xi < \xi_2$
		$q_{u_{3'}}(x,y,t) = p_{5'}\csc(-p_{5'}\sqrt{-\frac{\kappa b}{am}}\xi)\cdot\exp((\kappa x+\omega y-rt)i)$	$-\infty < p < p_{5'} < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < 0$	$0 < \xi < \xi_{2'}$
		$q_{u_4}(x,y,t) = \sqrt{-\frac{\kappa bp_{6'}^2 - a\kappa\omega + r}{\kappa b} + \frac{2\kappa bp_{6'}^2 - a\kappa\omega + r}{\kappa bsn^2(\sqrt{-\frac{2\kappa bp_{6'}^2 - a\kappa\omega + r}{am}}\xi)}}\cdot\exp((\kappa x+\omega y-rt)i)$	$0 < \sqrt{\frac{a\kappa\omega-r}{\kappa b}} < p_{6'} < p < +\infty$	$0 < \xi < \xi_3$
		$q_{u_{4'}}(x,y,t) = -\sqrt{-\frac{\kappa bp_{6'}^2 - a\kappa\omega + r}{\kappa b} + \frac{2\kappa bp_{6'}^2 - a\kappa\omega + r}{\kappa bsn^2(\sqrt{-\frac{2\kappa bp_{6'}^2 - a\kappa\omega + r}{am}}\xi)}}\cdot\exp((\kappa x+\omega y-rt)i)$	$-\infty < p < -p_{6'} < -\sqrt{\frac{a\kappa\omega-r}{\kappa b}} < 0$	$0 < \xi < \xi_{3'}$
	$\frac{\kappa b}{am} > 0$	$q_{b_3}(x,y,t) = \left(p_{10} - \frac{(p_{10}-p_8)(p_{10}-p_7)}{(p_{10}-p_8)+(p_8-p_7)sn^2(\sqrt{\frac{\kappa b}{am}}\frac{\sqrt{(p_{10}-p_8)(p_9-p_7)}}{2}\xi)}\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$p_7 < p < p_8 < 0 < p_9 < \sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_{10}$	$-T_1 < \xi < T_1$
		$q_{b_{3'}}(x,y,t) = \left(p_8 + \frac{(p_9-p_8)(p_{10}-p_8)}{(p_{10}-p_8)-(p_{10}-p_9)sn^2(\sqrt{\frac{\kappa b}{am}}\frac{\sqrt{(p_{10}-p_8)(p_9-p_7)}}{2}\xi)}\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$p_7 < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_8 < 0 < p_9 < p < p_{10}$	$-T_1 < \xi < T_1$
		$q_{b_4}(x,y,t) = \left(\frac{-2p_{11}\exp(\sqrt{\frac{\kappa b}{am}}p_{11}\lvert\xi\rvert)}{\exp(2\sqrt{\frac{\kappa b}{am}}p_{11}\lvert\xi\rvert)+1}\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$-p_{11} < p < 0 < \sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < p_{11}$	$-\infty < \xi < +\infty$
		$q_{b_{4'}}(x,y,t) = \left(\frac{2p_{11}\exp(\sqrt{\frac{\kappa b}{am}}p_{11}\lvert\xi\rvert)}{\exp(2\sqrt{\frac{\kappa b}{am}}p_{11}\lvert\xi\rvert)+1}\right)\cdot\exp((\kappa x+\omega y-rt)i)$	$-p_{11} < -\sqrt{\frac{a\kappa\omega-r}{2\kappa b}} < 0 < p < p_{11}$	$-\infty < \xi < +\infty$
		$q_{b_5}(x,y,t) = \sqrt{\frac{(\kappa bp_{12}^2 - a\kappa\omega + r)p_{12}^2 sn^2(\sqrt{\frac{2\kappa bp_{12}^2 - a\kappa\omega + r}{am}}\xi)}{2\kappa bp_{12}^2 - a\kappa\omega + r - \kappa bp_{12}^2 sn^2(\sqrt{\frac{2\kappa bp_{12}^2 - a\kappa\omega + r}{am}}\xi)}}\cdot\exp((\kappa x+\omega y-rt)i)$	$-p_{12} < p < p_{12}$	$-T_2 < \xi < T_2$

$\kappa b(a\kappa\omega - r)$	$\frac{\kappa b}{am}$	traveling wave solution	p	ξ
$\kappa b(a\kappa\omega - r) < 0$	$\frac{\kappa b}{am} < 0$	$q_{u_5}(x,y,t) = \sqrt[4]{-\frac{2amh_{0'}}{\kappa b}} \cdot \sqrt{-1 + \frac{2}{1-cn(2\sqrt[4]{-\frac{2\kappa bh_{0'}}{am}}\xi)}} \cdot \exp((\kappa x + \omega y - rt)i)$	$0 < p < +\infty$	$0 < \xi < \xi_4$
		$q_{u_6}(x,y,t) = 2\sqrt{\frac{r-a\kappa\omega}{\kappa b}} \cdot \frac{\exp(\sqrt{\frac{a\kappa\omega-r}{am}}\xi)}{\exp(2\sqrt{\frac{a\kappa\omega-r}{am}}\xi)-1} \cdot \exp((\kappa x + \omega y - rt)i)$	$0 < p < +\infty$	$\xi > 0$
		$q_{u_{6'}}(x,y,t) = 2\sqrt{\frac{r-a\kappa\omega}{\kappa b}} \cdot \frac{\exp(\sqrt{\frac{a\kappa\omega-r}{am}}\xi)}{1-\exp(2\sqrt{\frac{a\kappa\omega-r}{am}}\xi)} \cdot \exp((\kappa x + \omega y - rt)i)$	$-\infty < p < 0$	$\xi > 0$
		$q_{u_8}(x,y,t) = \sqrt{-\frac{\kappa bp_{7'}^2 - a\kappa\omega + r}{\kappa b} + \frac{2\kappa bp_{7'}^2 - a\kappa\omega + r}{\kappa bsn^2(\sqrt{-\frac{2\kappa bp_{7'}^2 - a\kappa\omega + r}{am}}\xi)}} \cdot \exp((\kappa x + \omega y - rt)i)$	$0 < p_{7'} < p < +\infty$	$0 < \xi < \xi_5$
		$q_{u_{8'}}(x,y,t) = -\sqrt{-\frac{\kappa bp_{7'}^2 - a\kappa\omega + r}{\kappa b} + \frac{2\kappa bp_{7'}^2 - a\kappa\omega + r}{\kappa bsn^2(\sqrt{-\frac{2\kappa bp_{7'}^2 - a\kappa\omega + r}{am}}\xi)}} \cdot \exp((\kappa x + \omega y - rt)i)$	$-\infty < p < -p_{7'} < 0$	$0 < \xi < \xi_{5'}$
	$\frac{\kappa b}{am} > 0$	$q_{b_6}(x,y,t) = \sqrt{\frac{(\kappa bp_{13}^2 - a\kappa\omega + r)p_{13}^2 sn^2(\sqrt{\frac{2\kappa bp_{13}^2 - a\kappa\omega + r}{am}}\xi)}{2\kappa bp_{13}^2 - a\kappa\omega + r - \kappa bp_{13}^2 sn^2(\sqrt{\frac{2\kappa bp_{13}^2 - a\kappa\omega + r}{am}}\xi)}} \cdot \exp((\kappa x + \omega y - rt)i)$	$-p_{13} < p < p_{13}$	$-T_3 < \xi < T_3$
$\kappa b(a\kappa\omega - r) = 0$	$\frac{\kappa b}{am} < 0$	$q_{u_{5'}}(x,y,t) = \sqrt{\frac{2}{1-cn(8h_{0'}\xi)} - 1} \cdot \exp((\kappa x + \omega y - rt)i)$	$0 < p < +\infty$	$0 < \xi < \xi_{4'}$
		$q_{u_7}(x,y,t) = \sqrt{-\frac{am}{\kappa b}} \cdot \frac{1}{\xi} \cdot \exp((\kappa x + \omega y - rt)i)$	$0 < p < +\infty$	$\xi > 0$
		$q_{u_{7'}}(x,y,t) = -\sqrt{-\frac{am}{\kappa b}} \cdot \frac{1}{\xi} \cdot \exp((\kappa x + \omega y - rt)i)$	$-\infty < p < 0$	$\xi > 0$
		$q_{u_9}(x,y,t) = \sqrt{-\frac{\kappa bp_{8'}^2}{\kappa b} + \frac{2\kappa bp_{8'}^2}{\kappa bsn^2(\sqrt{-\frac{2\kappa bp_{8'}^2}{am}}\xi)}} \cdot \exp((\kappa x + \omega y - rt)i)$	$0 < p_{8'} < p < +\infty$	$0 < \xi < \xi_6$
		$q_{u_{9'}}(x,y,t) = -\sqrt{-\frac{\kappa bp_{8'}^2}{\kappa b} + \frac{2\kappa bp_{8'}^2}{\kappa bsn^2(\sqrt{-\frac{2\kappa bp_{8'}^2}{am}}\xi)}} \cdot \exp((\kappa x + \omega y - rt)i)$	$-\infty < p < -p_{8'} < 0$	$0 < \xi < \xi_{6'}$
	$\frac{\kappa b}{am} > 0$	$q_{b_7}(x,y,t) = \sqrt{\frac{p_{14}^2 sn^2(\sqrt{\frac{2\kappa bp_{14}^2}{am}}\xi)}{2 - sn^2(\sqrt{\frac{2\kappa bp_{14}^2}{am}}\xi)}} \cdot \exp((\kappa x + \omega y - rt)i)$	$-p_{14} < p < p_{14}$	$-T_4 < \xi < T_4$

References

[1] A. Biswas, Y. Yıldırım, E. Yaşar, Q. Zhou, S. P. Moshokoa and M. Belic, *Optical soliton perturbation with quadratic-cubic nonlinearity using a couple of strategic algorithms*, Chin. J. Phys., 2018, 56(5), 1990–1998.

[2] S. N. Chow and J. K. Hale, *Methods of bifurcation theory*, Springer Science & Business Media, 2012.

[3] M. Ekici, A. Sonmezoglu, A. Biswas and M. R. Belic, *Optical solitons in (2+1)-dimensions with Kundu-Mukherjee-Naskar equation by extended trial function scheme*, Chin. J. Phys., 2019, 57, 72–77.

[4] B. Ghanbari, H. Günerhan, O. A. İlhan and H. M. Baskonus, *Some new families of exact solutions to a new extension of nonlinear Schrödinger equation*, Phys. Scripta., 2020, 95(7), Article ID: 075208.

[5] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, Springer, New York, 1983.

[6] A. Jhangeer, A. R. Seadawy, F. Ali and A. Ahmed, *New complex waves of perturbed Schrödinger equation with Kerr law nonlinearity and Kundu-Mukherjee-Naskar equation*, Results Phys., 2020, 16, Article ID: 102816.

[7] N. A. Kudryashov, *General solution of traveling wave reduction for the Kundu-Mukherjee-Naskar model*, Optik., 2019, 186, 22–27.

[8] A. Kundu, *Novel hierarchies & hidden dimensions in integrable field models: theory & application*, Journal of Physics: Conference Series, 2014, 482, Article ID: 012022.

[9] A. Kundu and A. Mukherjee, *Novel integrable higher-dimensional nonlinear Schrödinger equation: properties, solutions, applications*, 2013, arXiv:1305.4023.

[10] A. Kundu, A. Mukherjee and T. Naskar, *Modelling rogue waves through exact dynamical lump soliton controlled by ocean currents*, Proc. R. Soc. Lond. Ser. A Math. Phys., 2014, 470(2164), Article ID: 20130576.

[11] D. Kumar, G. C. Paul, T. Biswas, A. R. Seadawy, R. Baowali, M. Kamal and H. Rezazadeh, *Optical solutions to the Kundu-Mukherjee-Naskar equation: mathematical and graphical analysis with oblique wave propagation*, Phys. Scripta., 2020, 96(2), Article ID: 025218.

[12] A. Mukherjee, M. Janaki and A. Kundu, *A new (2+1) dimensional integrable evolution equation for an ion acoustic wave in a magnetized plasma*, Phys. Plasmas., 2015, 22(7), Article ID: 072302.

[13] W. Peng, S. Tian and T. Zhang, *Optical solitons, complexitons and power series solutions of a (2+1)-dimensional nonlinear Schrödinger equation*, Modern Phys. Lett. B., 2018, 32(28), Article ID: 1850336.

[14] D. Qiu, Y. Zhang and J. He, *The rogue wave solutions of a new (2+1)-dimensional equation*, Commun. Nonlinear Sci. Numer. Simul., 2016, 30(1–3), 307–315.

[15] S. T. R. Rizvi, I. Afzal and K. Ali, *Dark and singular optical solitons for Kundu-Mukherjee-Naskar model*, Modern Phys. Lett. B., 2020, 34(6), Article ID: 2050074.

[16] H. Rezazadeh, A. Kurt, A. Tozar, O. Tasbozan and S. Mirhosseini-Alizamini, *Wave behaviors of Kundu-Mukherjee-Naskar model arising in optical fiber communication systems with complex structure*, Opt. Quant. Electon., 2021. DOI:10.21203/rs.3.rs-168692/v1.

[17] S. Singh, A. Mukherjee, K. Sakkaravarthi and K. Murugesan, *Higher dimensional localized and periodic wave dynamics in an integrable (2+1)-dimensional deep water oceanic wave model*, Wave. Random. Complex., 2021, 1–20. DOI:10.1080/17455030.2021.1874621.

[18] T. A. Sulaiman and H. Bulut, *The new extended rational SGEEM for construction of optical solitons to the (2+1)-dimensional Kundu-Mukherjee-Naskar model*, Appl. Math. Nonlinear Sci., 2019, 4(2), 513–521.

[19] R. Talarposhti, P. Jalili, H. Rezazadeh, B. Jalili, D. Ganji, W. Adel and A. Bekir, *Optical soliton solutions to the (2+1)-dimensional Kundu-Mukherjee-Naskar equation*, Internat. J. Modern Phys. B., 2020, 34(11), Article ID: 2050102.

[20] X. Wen, *Higher-order rational solutions for the (2+1)-dimensional KMN equation*, Proc. Rom. Acad. Ser. A Math. Phys., 2017, 18(3), 191–198.

[21] Y. Yıldırım, *Optical soltons to Kundu-Mukherjee-Naskar model with trial equation approach*, Optik., 2019, 183, 1061–1065.

[22] Y. Yıldırım, *Optical soltons to Kundu-Mukherjee-Naskar model with modified simple equation approach*, Optik., 2019, 184, 247–252.

[23] Y. Yıldırım, *Optical soltons to Kundu-Mukherjee-Naskar model in birefringent fibers with trial equation approach*, Optik., 2019, 183, 1026–1031.

[24] Y. Yıldırım, *Optical solitons to Kundu-Mukherjee-Naskar model in birefringent fibers with modified equation approach*, Optik., 2019, 184, 121–127.

[25] Y. Yıldırım and M. Mirzazadeh, *Optical pulses with Kundu-Mukherjee-Naskar model in fiber communication systems*, Chinese J. Phys., 2020, 64, 183–193.

[26] Z. Zhang, T. Ding, W. Huang and Z. Dong, *Qualitative Theory of Differential Equations*, American Mathematical Society, Providence, RI, USA, 1992.