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APPROXIMATION OF INTEGRABLE FUNCTIONS BY GENERALIZED DE LA VALLÉE POUSSIN MEANS OF THE POSITIVE ORDER

Xhevat Z. Krasniqi[1,†], Włodzimierz Łenski[2] and Bogdan Szal[2]

Abstract In this paper several results pertaining to the pointwise and normwise approximation of integrable functions by generalized de la Vallée Poussin means of positive order are presented. The pointwise estimates of the considered deviation in terms of pointwise moduli of continuity based on the Lebesgue points and points of differentiability of indefinite integral are obtained. Some results of L. Leindler and the second author are generalized.

Keywords Rate of approximation, summability of Fourier series, de la Vallée-Poussin mean, monotone sequence.

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1. Introduction

Let L^p $(1 \leq p < +\infty)$ be the class of all 2π- periodic real-valued functions integrable in the Lebesgue sense with p-th power and let C be the class of all 2π- periodic real-valued continuous functions with the norms

$$\|f\|_X = \begin{cases} \left(\int_T |f(t)|^p dt\right)^{1/p}, & \text{when } X = L^p, \\ \max_{t\in T} |f(t)|, & \text{when } X = C, \end{cases}$$

where $T := [-\pi, \pi]$.

Consider its trigonometric Fourier series

$$Sf(x) := \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

with the partial sums $S_k f$.

The generalized Vallée-Poussin means of a given sequence $(S_k f)$ is defined in [7]. Let $\lambda := (\lambda_n)$ be a monotone non-decreasing sequence of integers such that $\lambda_0 = 0$, $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leq 1$ for $n \in \mathbb{N}$.

†The corresponding author. Email: xhevat.krasniqi@uni-pr.edu (Xhevat Z. Krasniqi)
1University of Prishtina "Hasan Prishtina", Faculty of Education, Department of Mathematics and Informatics, Avenue "Mother Theresa", Prishtinë 10000, Republic of Kosova
2University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics, 65-516 Zielona Góra, ul. Szafrana 4a, Poland

We introduce the generalized de la Vallée-Poussin means of the order $\gamma > -1$

$$V_n^\gamma f = \frac{1}{C_{\lambda_n}^\gamma} \sum_{k=n-\lambda_n}^{n} C_{n-k}^{\gamma-1} S_k f = \frac{1}{C_{\lambda_n}^\gamma} \sum_{k=0}^{\lambda_n} C_k^{\gamma-1} S_{n-k} f$$

and modified one

$$\begin{aligned} W_n^\gamma f &= \frac{\sum_{k=n-\lambda_n}^{n} C_{n-k}^{\gamma-1} S_k f + \sum_{k=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1} S_k f}{C_{\lambda_n}^\gamma + (n-\lambda_n) C_{\lambda_n}^{\gamma-1}} \\ &= \frac{\sum_{k=0}^{n} C_{n-k}^{\gamma-1} S_k f - \sum_{k=0}^{n-\lambda_n-1} \left(C_{n-k}^{\gamma-1} - C_{\lambda_n}^{\gamma-1} \right) S_k f}{C_{\lambda_n}^\gamma + (n-\lambda_n) C_{\lambda_n}^{\gamma-1}}, \end{aligned}$$

where $C_0^\gamma = 1$ and $C_n^\gamma = \binom{n+\gamma}{\gamma} = \frac{(\gamma+1)(\gamma+2)\dots(\gamma+n)}{n!}$ for $n \in \mathbb{N}$. In special case $C_n^{-1} = 0$ for $n \in \mathbb{N}$, $C_n^0 = 1$ and $C_1^\gamma = \gamma + 1$ for $n \in \mathbb{N} \cup \{0\}$ (see [12, Chapter III]). We also define the Cesàro means $\sigma_n^\gamma f = \frac{1}{C_n^\gamma} \sum_{k=0}^{n} C_{n-k}^{\gamma-1} S_k f$ with $\gamma > -1$ (the Fejér means $\sigma_n f = \sigma_n^1 f$) and it is easily to seen that $W_n^1 f = \sigma_n^1 f$ but in case $n = \lambda_n$ we have $V_n^\gamma f = W_n^\gamma f = \sigma_n^\gamma f$ for $\gamma > 0$.

By the Dirichlet formula $D_n(t) = \frac{1}{2} + \sum_{\mu=1}^{n} \cos \mu t = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}$ we have, at the point x,

$$S_n f(x) = \frac{1}{\pi} \int_0^\pi [f(t+x) + f(t-x)] D_n(t) dt,$$

whence

$$V_n^\gamma f(x) - f(x) = \frac{1}{\pi} \int_0^\pi \varphi_x(t) V_n^\gamma(t) dt$$

and

$$W_n^\gamma f(x) - f(x) = \frac{1}{\pi} \int_0^\pi \varphi_x(t) W_n^\gamma(t) dt$$

where

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

More precisely,

$$V_n^\gamma(t) = \sum_{k=n-\lambda_n}^{n} \frac{C_{n-k}^{\gamma-1} \sin\left(k+\frac{1}{2}\right)t}{2C_{\lambda_n}^\gamma \sin\frac{t}{2}} = \sum_{k=0}^{\lambda_n} \frac{C_k^{\gamma-1} \sin\left(n-k+\frac{1}{2}\right)t}{2C_{\lambda_n}^\gamma \sin\frac{t}{2}}$$

and

$$\begin{aligned} W_n^\gamma(t) &= \frac{\sum_{k=n-\lambda_n}^{n} C_{n-k}^{\gamma-1} \sin\left(k+\frac{1}{2}\right)t + \sum_{k=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1} \sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)} \\ &= \frac{\sum_{k=0}^{\lambda_n} C_k^{\gamma-1} \sin\left(n-k+\frac{1}{2}\right)t + \sum_{k=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1} \sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)}. \end{aligned}$$

As a measure of approximation of $f(x)$ by $V_n^\gamma f(x)$ or $W_n^\gamma f(x)$ we will use the pointwise moduli of continuity of f in the space L^1 defined by the formulas:

$$\Omega_x f(\delta) = \frac{1}{\delta} \int_0^\delta |\varphi_x(t)| \, dt, \text{ where } \delta > 0,$$

based on the definition of the Lebesgue points of f (L-*points*),

$$w_x f(\delta) = \sup_{0<h\leq\delta} \frac{1}{\delta}\left|\int_0^h \varphi_x(t)dt\right|, \text{ where } \delta > 0,$$

based on the definition of the points of differentiability of the indefinite integral of f (D-*points*) and the classical modulus of continuity of f in the space X defined by the formula:

$$\omega f(\delta)_X := \sup_{|t|\leq\delta} \|\varphi_\cdot(t)\|_X, \text{ where } \delta \geq 0.$$

We will also use the notations

$$\text{Lip}\ (\omega, X) = \{f \in X : \omega f(\delta)_X = O(\omega(\delta))\},$$

where ω is a function of modulus of continuity type, i.e. nondecreasing continuous functions having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$, and

$$\text{Lip}\ (\delta^\alpha, X) = \{f \in X : \omega f(\delta)_X = O(\delta^\alpha),\ 0 < \alpha \leq 1\}.$$

The reason of consideration of two similar means $V_n^\gamma f$ and $W_n^\gamma f$ explain the following Remarks.

Remark 1.1. We also note that the relation

$$w_x f(\delta) = o_x(1) \text{ as } \delta \to 0^+$$

is more general than the condition

$$\Omega_x f(\delta) = o_x(1) \text{ as } \delta \to 0^+.$$

We can check this for the function

$$f(u) = \begin{cases} -\frac{1}{2}\sin\frac{1}{u} & \text{for } -1 \leq u < 0, \\ \frac{1}{2}\sin\frac{1}{u} & \text{for } 0 < u \leq 1, \\ 0 & \text{for } u = 0 \text{ and } |u| > 1. \end{cases}$$

In [10, p. 34] there are proved that

$$\lim_{\lambda\to 0+} \frac{1}{\lambda}\int_0^\lambda \varphi_0(u)\,du = 0 \quad \text{and} \quad \lim_{\lambda\to 0+} \frac{1}{\lambda}\int_0^\lambda |\varphi_0(u)|\,du \geq \frac{2}{\pi^2},$$

where $\varphi_0(u) = \sin\frac{1}{u}$.

Remark 1.2. It is well-known (see [2,6]) that the $(C,1)$ means do not tend to f at the $D-points$ of f but the (C,γ) means with $\gamma > 1$ do. In this paper these facts will be presented in the approximation version with the quantity $w_x f$ as a measure of such approximation.

Throughout this paper we write $u \ll v$ if there exists a positive constant K, such that $u \leq Kv$ not the same at each occurrences and $\sum_a^b = 0$ when $a > b$.

Regarding to summability of Fourier series Leindler proved, among others, the following two theorems.

Theorem 1.1 ([7]). *Let $f \in L^1$. If $\lambda_n \to \infty$ and conditions*

$$\int_{1/(n+1)}^{1/(\lambda_n+1)} \frac{|\varphi_x(t)|}{t} dt = o_x(1), \quad n\int_0^{1/(n+1)} |\varphi_x(t)| dt = o_x(1)$$

are satisfied, then

$$V_n^1 f(x) = \frac{1}{\lambda_n + 1} \sum_{k=n-\lambda_n}^{n} S_k f\,(x) \to f(x), \quad (\lambda_n, n \to \infty),$$

for a.e. x.

Theorem 1.2 ([7]). *If $f \in Lip\,(\delta^\alpha, C)$, then*

$$|V_n^1 f(x) - f(x)| \ll \begin{cases} \frac{1}{(\lambda_n+1)^\alpha}, & \text{when } 0 < \alpha < 1 \\ \frac{1+\ln(\lambda_n+1)}{1+\lambda_n}, & \text{when } \alpha = 1 \end{cases}$$

holds true uniformly in x.

On the other hand, Łenski proved, among others, the next two theorems (the second one also Kranz).

Theorem 1.3 ([8]). *If $f \in L^1$, then*

$$|\sigma_n f(x) - f(x)| \ll \frac{1}{n+1} \sum_{k=0}^{n} \Omega_x f\left(\frac{\pi}{k+1}\right), \ (n = 0, 1, \ldots),$$

holds for all real x.

Theorem 1.4 ([5] $\gamma \in (1, 2)$, [8] $\gamma = 2$). *If $f \in L^1$, then*

$$|\sigma_n^\gamma f(x) - f(x)| \ll \frac{1}{(n+1)^{\gamma-1}} \sum_{k=0}^{n} \frac{1}{(k+1)^{2-\gamma}} w_x f\left(\frac{\pi}{k+1}\right), \ (1 < \gamma \leq 2),$$

holds for all real x.

Similar problems was also examined among others by Bustamante [1], Jafarov [4], Ghodadra & Fulop [3], Rovenska & Novikov [9] and Totur & Canak [11].

The purpose of the paper it is to prove the mentioned results with the generalized de la Vallée-Poussin means of the order $\gamma > 0$.

2. Lemmas

Lemma 2.1. *The following formulas hold:*
(i) For $n = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{n} \sin\left(k + \frac{1}{2}\right) t = \frac{\sin^2 \frac{(n+1)t}{2}}{\sin \frac{t}{2}} = \frac{1 - \cos\left(n+1\right)t}{2 \sin \frac{t}{2}},$$

$$\sum_{k=0}^{n} \cos\left(k + \frac{1}{2}\right) t = \frac{\sin \frac{(n+1)t}{2} \cos \frac{(n+1)t}{2}}{\sin \frac{t}{2}} = \frac{\sin\left(n+1\right)t}{2 \sin \frac{t}{2}}.$$

(ii) For $m=0,1,2,\ldots,\ n\in\mathbb{R}$,

$$\sum_{k=0}^{m}\sin\left(n-k+\frac{1}{2}\right)t=\frac{\cos\left(n-m\right)t-\cos\left(n+1\right)t}{2\sin\frac{t}{2}},$$
$$\sum_{k=0}^{m}\cos\left(n-k+\frac{1}{2}\right)t=-\frac{\sin\left(n-m\right)t-\sin\left(n+1\right)t}{2\sin\frac{t}{2}}.$$

Proof. (i) Using the equalities

$$\sum_{k=0}^{n}\sin kt=\frac{\sin\frac{nt}{2}\sin\frac{(n+1)t}{2}}{\sin\frac{t}{2}}\quad\text{and}\quad\sum_{k=0}^{n}\cos kt=\frac{\sin\frac{(n+1)t}{2}\cos\frac{nt}{2}}{\sin\frac{t}{2}}$$

we obtain

$$\begin{aligned}\sum_{k=0}^{n}\sin\left(k+\frac{1}{2}\right)t&=\cos\frac{t}{2}\sum_{k=0}^{n}\sin kt+\sin\frac{t}{2}\sum_{k=0}^{n}\cos kt\\&=\cos\frac{t}{2}\frac{\sin\frac{nt}{2}\sin\frac{(n+1)t}{2}}{\sin\frac{t}{2}}+\sin\frac{t}{2}\frac{\sin\frac{(n+1)t}{2}\cos\frac{nt}{2}}{\sin\frac{t}{2}}\\&=\frac{\sin\frac{(n+1)t}{2}\left(\cos\frac{t}{2}\sin\frac{nt}{2}+\sin\frac{t}{2}\cos\frac{nt}{2}\right)}{\sin\frac{t}{2}}\\&=\frac{\sin^2\frac{(n+1)t}{2}}{\sin\frac{t}{2}}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n}\cos\left(k+\frac{1}{2}\right)t&=\cos\frac{t}{2}\sum_{k=0}^{n}\cos kt-\sin\frac{t}{2}\sum_{k=0}^{n}\sin kt\\&=\cos\frac{t}{2}\frac{\sin\frac{(n+1)t}{2}\cos\frac{nt}{2}}{\sin\frac{t}{2}}-\sin\frac{t}{2}\frac{\sin\frac{nt}{2}\sin\frac{(n+1)t}{2}}{\sin\frac{t}{2}}\\&=\frac{\sin\frac{(n+1)t}{2}\left(\cos\frac{t}{2}\cos\frac{nt}{2}-\sin\frac{t}{2}\sin\frac{nt}{2}\right)}{\sin\frac{t}{2}}\\&=\frac{\sin\frac{(n+1)t}{2}\cos\frac{(n+1)t}{2}}{\sin\frac{t}{2}}.\end{aligned}$$

(ii) Using (i) we obtain

$$\begin{aligned}&\sum_{k=0}^{m}\sin\left(n-k+\frac{1}{2}\right)t\\&=\sum_{k=0}^{m}\sin\left[(n+1)-\left(k+\frac{1}{2}\right)\right]t\\&=\sin\left(n+1\right)t\sum_{k=0}^{m}\cos\left(k+\frac{1}{2}\right)t-\cos\left(n+1\right)t\sum_{k=0}^{m}\sin\left(k+\frac{1}{2}\right)t\\&=\frac{\sin\left(n+1\right)t\sin\left(m+1\right)t}{2\sin\frac{t}{2}}-\frac{\cos\left(n+1\right)t\left(1-\cos\left(m+1\right)t\right)}{2\sin\frac{t}{2}}\end{aligned}$$

$$= \frac{\cos{(n-m)\,t} - \cos{(n+1)\,t}}{2\sin\frac{t}{2}}$$

and

$$\begin{aligned}
&\sum_{k=0}^{m} \cos\left(n-k+\frac{1}{2}\right)t \\
=&\sum_{k=0}^{m} \cos\left[(n+1)-\left(k+\frac{1}{2}\right)\right]t \\
=&\cos{(n+1)\,t}\sum_{k=0}^{m}\cos\left(k+\frac{1}{2}\right)t + \sin{(n+1)\,t}\sum_{k=0}^{m}\sin\left(k+\frac{1}{2}\right)t \\
=&\frac{\cos{(n+1)\,t}\sin{(m+1)\,t}}{2\sin\frac{t}{2}} + \frac{\sin{(n+1)\,t}\,(1-\cos{(m+1)\,t})}{2\sin\frac{t}{2}} \\
=&\frac{-\sin{(n-m)\,t} + \sin{(n+1)\,t}}{2\sin\frac{t}{2}}.
\end{aligned}$$

Hence, the proof is done. □

Lemma 2.2. *The following inequalities hold true:*
(i) $|V_n^\gamma(t)| \ll n+1$ *for* $\gamma > 0,\ 0 < t \leq \pi$,
(ii) $|V_n^\gamma(t)| \ll \begin{cases} \frac{1}{t^2(\lambda_n+1)} & \textit{for } \gamma \geq 1,\ 0 < t \leq \pi, \\ \frac{1}{t^{1+\gamma}(\lambda_n+1)^\gamma} & \textit{for } 0<\gamma<1,\ \frac{\pi}{\lambda_n+1} \leq t \leq \pi, \end{cases}$
(iii) $|V_n^\gamma(t)| \ll \frac{1}{t}$ *for* $\gamma > 0,\ 0 < t \leq \pi$.

Proof. (i) We can write,

$$V_n^\gamma(t) = \frac{1}{2\sin\frac{t}{2}C_{\lambda_n}^\gamma}\sum_{k=0}^{\lambda_n} C_k^{\gamma-1}\sin\left(n-k+\frac{1}{2}\right)t,$$

whence

$$\begin{aligned}
|V_n^\gamma(t)| &\leq \frac{1}{2\sin\frac{t}{2}C_{\lambda_n}^\gamma}\sum_{k=0}^{\lambda_n} C_k^{\gamma-1}\,(2n-2k+1)\sin\frac{t}{2} \\
&\leq \frac{(2n+1)}{2C_{\lambda_n}^\gamma}\sum_{k=0}^{\lambda_n} C_k^{\gamma-1} \leq n+1.
\end{aligned}$$

(ii) Summation by parts and Lemma 2.1 (ii) give

$$\begin{aligned}
V_n^\gamma(t) &= \sum_{k=n-\lambda_n}^{n} \frac{C_{n-k}^{\gamma-1}\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}C_{\lambda_n}^\gamma} = \sum_{k=0}^{\lambda_n}\frac{C_k^{\gamma-1}\sin\left(n-k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}C_{\lambda_n}^\gamma} \\
&= \sum_{m=0}^{\lambda_n-1}\left(C_m^{\gamma-1}-C_{m+1}^{\gamma-1}\right)\sum_{k=0}^{m}\frac{\sin\left(n-k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}C_{\lambda_n}^\gamma} + C_{\lambda_n}^{\gamma-1}\sum_{k=0}^{\lambda_n}\frac{\sin\left(n-k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}C_{\lambda_n}^\gamma} \\
&= \sum_{m=0}^{\lambda_n-1} C_{m+1}^{\gamma-2}\frac{\cos{(n+1)\,t}-\cos{(n-m)\,t}}{4\sin^2\frac{t}{2}C_{\lambda_n}^\gamma} - C_{\lambda_n}^{\gamma-1}\frac{\cos{(n+1)\,t}-\cos{(n-\lambda_n)\,t}}{4\sin^2\frac{t}{2}C_{\lambda_n}^\gamma}
\end{aligned}$$

$$= \sum_{m=0}^{\lambda_n-1} C_{m+1}^{\gamma-2} \frac{\sin\frac{(m+1)t}{2}\sin\frac{(2n-m+1)t}{2}}{2\sin^2\frac{t}{2}C_{\lambda_n}^{\gamma}} - C_{\lambda_n}^{\gamma-1}\frac{\sin\frac{(\lambda_n+1)t}{2}\sin\frac{(2n-\lambda_n+1)t}{2}}{2\sin^2\frac{t}{2}C_{\lambda_n}^{\gamma}},$$

whence, when $\gamma \geq 1$,

$$|V_n^{\gamma}(t)| \leq \frac{\pi^2}{2t^2 C_{\lambda_n}^{\gamma}}\left\{\sum_{m=0}^{\lambda_n-1} C_{m+1}^{\gamma-2} + C_{\lambda_n}^{\gamma-1}\right\} \leq \frac{\pi^2}{t^2 C_{\lambda_n}^{\gamma}} 2C_{\lambda_n}^{\gamma-1} \ll \frac{1}{t^2(\lambda_n+1)}.$$

Moreover

$$\begin{aligned} V_n^{\gamma}(t) &= \frac{1}{2\sin\frac{t}{2}C_{\lambda_n}^{\gamma}} \sum_{k=0}^{\lambda_n} C_k^{\gamma-1}\sin\left(n-k+\frac{1}{2}\right)t \\ &= Im\left\{\frac{\exp\left[i\left(n+\frac{1}{2}\right)t\right]}{2\sin\frac{t}{2}C_{\lambda_n}^{\gamma}}\sum_{k=0}^{\lambda_n} C_k^{\gamma-1}\exp(-ikt)\right\} \\ &= Im\left\{\frac{\exp\left[i\left(n+\frac{1}{2}\right)t\right]}{2\sin\frac{t}{2}C_{\lambda_n}^{\gamma}}\left[\frac{1}{(1-e^{-it})^{\gamma}} - \sum_{k=\lambda_n+1}^{\infty} C_k^{\gamma-1}\exp(-ikt)\right]\right\} \end{aligned}$$

whence, by [12, Chapter III (5.7) and I (2.2)],

$$|V_n^{\gamma}(t)| \ll \frac{1}{t^{1+\gamma}(\lambda_n+1)^{\gamma}} + \frac{1}{t^2(\lambda_n+1)} \ll \frac{1}{t^{1+\gamma}(\lambda_n+1)^{\gamma}},$$

when $0 < \gamma < 1$ and $\frac{\pi}{\lambda_n+1} \leq t \leq \pi$.

(iii) Using the definition, we obtain

$$\begin{aligned} |V_n^{\gamma}(t)| &= \left|\frac{1}{2\sin\frac{t}{2}C_{\lambda_n}^{\gamma}} \sum_{m=n-\lambda_n}^{n} C_{n-m}^{\gamma-1}\sin\left(m+\frac{1}{2}\right)t\right| \\ &\leq \frac{1}{2\sin\frac{t}{2}C_{\lambda_n}^{\gamma}} \sum_{m=n-\lambda_n}^{n} C_{n-m}^{\gamma-1} \leq \frac{\pi}{2t}. \end{aligned}$$

The proof is done. □

Lemma 2.3. *The following inequalities hold true:*

(i) $|W_n^{\gamma}(t)| \ll n+1$ *for* $\gamma > 0,\ 0 < t \leq \pi$,

(ii) $|W_n^{\gamma}(t)| \ll \begin{cases} \frac{1}{t^2(\lambda_n+1)} & \text{for } \gamma \geq 1,\ 0 < t \leq \pi, \\ \frac{1}{t^{1+\gamma}(\lambda_n+1)^{\gamma}} & \text{for } 0 < \gamma < 1,\ \frac{\pi}{\lambda_n+1} \leq t \leq \pi, \end{cases}$

(iii) $|W_n^{\gamma}(t)| \ll \frac{1}{t}$ *for* $\gamma > 0,\ 0 < t \leq \pi$.

(iv) $|\frac{d}{dt}W_n^{\gamma}(t)| \ll \begin{cases} \frac{1}{(\lambda_n+1)t^3} & \text{for } \gamma \geq 2,\ 0 < t \leq \pi, \\ \frac{1}{t^{1+\gamma}(\lambda_n+1)^{\gamma-1}} & \text{for } 1 < \gamma < 2,\ \frac{\pi}{\lambda_n+1} \leq t \leq \pi, \end{cases}$

(v) $|\frac{d}{dt}W_n^{\gamma}(t)| \ll (n+1)^2$ *for* $\gamma > 0,\ 0 < t \leq \pi$,

(vi) $|\frac{d}{dt}W_n^{\gamma}(t)| \ll \frac{1}{(n+1)t^3} + \frac{1}{t^2}$ *for* $\gamma > 0,\ 0 < t \leq \pi$.

Proof. (i) We can write

$$W_n^{\gamma}(t) = \frac{\sum_{m=0}^{\lambda_n} C_m^{\gamma-1}\sin\left(n-m+\frac{1}{2}\right)t + \sum_{m=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1}\sin\left(m+\frac{1}{2}\right)t}{2\sin\frac{t}{2}\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)},$$

whence

$$\begin{aligned} |W_n^\gamma(t)| &\le |V_n^\gamma(t)| + \frac{\sum_{m=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1} \left|\sin\left(m+\frac{1}{2}\right)t\right|}{2\sin\frac{t}{2}\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)} \\ &\le |V_n^\gamma(t)| + \frac{\sum_{m=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1} (2m+1)\sin\frac{t}{2}}{2\sin\frac{t}{2}\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)} \\ &\ll n+1+\frac{(n-\lambda_n)(2n-2\lambda_n-1)C_{\lambda_n}^{\gamma-1}}{2\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)} \\ &\le n+1+\frac{(n-\lambda_n)(2n-2\lambda_n-1)C_{\lambda_n}^{\gamma-1}}{2(n-\lambda_n)C_{\lambda_n}^{\gamma-1}} \\ &= n+1+\left(n-\lambda_n-\frac{1}{2}\right) \ll n+1. \end{aligned}$$

(ii) Since

$$|W_n^\gamma(t)| \le |V_n^\gamma(t)| + \frac{C_{\lambda_n}^{\gamma-1}}{2\sin\frac{t}{2}C_{\lambda_n}^\gamma}\left|\sum_{m=0}^{n-\lambda_n-1}\sin\left(m+\frac{1}{2}\right)t\right|,$$

by Lemma 2.1 (i), it is easily to see that

$$|W_n^\gamma(t)| \ll |V_n^\gamma(t)| + \frac{1}{2\sin^2\frac{t}{2}(\lambda_n+1)}\sin^2\frac{(n-\lambda_n)t}{2} \le |V_n^\gamma(t)| + \frac{\pi^2}{2t^2(\lambda_n+1)}.$$

Hence by Lemma 2.2 we get (ii).

(iii) Using the estimate

$$\begin{aligned} |W_n^\gamma(t)| &\le |V_n^\gamma(t)| + \frac{\sum_{m=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1} \left|\sin\left(m+\frac{1}{2}\right)t\right|}{2\sin\frac{t}{2}\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)} \\ &\le |V_n^\gamma(t)| + \frac{1}{2\sin\frac{t}{2}(n-\lambda_n)}\left|\sum_{m=0}^{n-\lambda_n-1}\sin\left(m+\frac{1}{2}\right)t\right|, \end{aligned}$$

we obtain

$$|W_n^\gamma(t)| \le |V_n^\gamma(t)| + \frac{1}{2\sin\frac{t}{2}(n-\lambda_n)}\left|\sum_{m=0}^{n-\lambda_n-1} 1\right| \le |V_n^\gamma(t)| + \frac{\pi(n-\lambda_n)}{2t(n-\lambda_n)},$$

whence (iii) follows.

(iv) Similarly to the proof of Lemma 2.2,

$$\begin{aligned} &\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)W_n^\gamma(t) \\ &= \frac{\sum_{m=0}^{\lambda_n-1}\left(C_m^{\gamma-1} - C_{m+1}^{\gamma-1}\right)\sum_{k=0}^{m}\sin\left(n-k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} \end{aligned}$$

$$+\frac{C_{\lambda_n}^{\gamma-1}\sum_{k=0}^{\lambda_n}\sin\left(n-k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}+\frac{C_{\lambda_n}^{\gamma-1}\sum_{k=0}^{n-\lambda_n-1}\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$
$$=S_1(t)+S_2(t)+S_3(t).$$

Next, by Lemma 2.1 (ii),

$$S_1(t)=\sum_{m=0}^{\lambda_n-1}C_{m+1}^{\gamma-2}\left[\frac{\cos(n+1)t-\cos(n-m)t}{4\sin^2\frac{t}{2}}\right],$$
$$S_2(t)=C_{\lambda_n}^{\gamma-1}\left[\frac{\cos(n-\lambda_n)t-\cos(n+1)t}{4\sin^2\frac{t}{2}}\right]$$

and

$$S_3(t)=C_{\lambda_n}^{\gamma-1}\left[\frac{1-\cos(n-\lambda_n)t}{4\sin^2\frac{t}{2}}\right],$$

whence

$$\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)W_n^{\gamma}(t)$$
$$=\sum_{m=0}^{\lambda_n-1}C_{m+1}^{\gamma-2}\left[\frac{\cos(n+1)t-\cos(n-m)t}{4\sin^2\frac{t}{2}}\right]+C_{\lambda_n}^{\gamma-1}\left[\frac{1-\cos(n+1)t}{4\sin^2\frac{t}{2}}\right]$$
$$=\sum_{m=0}^{\lambda_n}C_m^{\gamma-2}\frac{\cos(n+1)t}{4\sin^2\frac{t}{2}}-C_0^{\gamma-2}\frac{\cos(n+1)t}{4\sin^2\frac{t}{2}}$$
$$-\sum_{m=0}^{\lambda_n-1}C_{m+1}^{\gamma-2}\frac{\cos(n-m)t}{4\sin^2\frac{t}{2}}+C_{\lambda_n}^{\gamma-1}\frac{1}{4\sin^2\frac{t}{2}}-C_{\lambda_n}^{\gamma-1}\frac{\cos(n+1)t}{4\sin^2\frac{t}{2}}$$
$$=\frac{C_{\lambda_n}^{\gamma-1}}{4\sin^2\frac{t}{2}}-C_0^{\gamma-2}\frac{\cos(n+1)t}{4\sin^2\frac{t}{2}}-\sum_{m=1}^{\lambda_n}C_m^{\gamma-2}\frac{\cos(n+1-m)t}{4\sin^2\frac{t}{2}}$$
$$=\frac{C_{\lambda_n}^{\gamma-1}}{4\sin^2\frac{t}{2}}-\sum_{m=0}^{\lambda_n}C_m^{\gamma-2}\frac{\cos(n+1-m)t}{4\sin^2\frac{t}{2}}.$$

Summation by parts once more and Lemma 2.1 (ii), with $n+\frac{1}{2}$ instead of n, give

$$\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)W_n^{\gamma}(t)-\frac{C_{\lambda_n}^{\gamma-1}}{4\sin^2\frac{t}{2}}$$
$$=-\sum_{m=0}^{\lambda_n-1}\left(C_m^{\gamma-2}-C_{m+1}^{\gamma-2}\right)\sum_{k=0}^{m}\frac{\cos(n+1-k)t}{4\sin^2\frac{t}{2}}-C_{\lambda_n}^{\gamma-2}\sum_{k=0}^{\lambda_n}\frac{\cos(n+1-k)t}{4\sin^2\frac{t}{2}}$$
$$=\sum_{m=0}^{\lambda_n-1}\frac{C_{m+1}^{\gamma-3}}{8\sin^3\frac{t}{2}}\left[-\sin\left(n-m+\frac{1}{2}\right)t+\sin\left(n+\frac{3}{2}\right)t\right]$$
$$-\frac{C_{\lambda_n}^{\gamma-2}}{8\sin^3\frac{t}{2}}\left[-\sin\left(n-\lambda_n+\frac{1}{2}\right)t+\sin\left(n+\frac{3}{2}\right)t\right]$$
$$=-\sum_{m=0}^{\lambda_n}\frac{C_m^{\gamma-3}}{8\sin^3\frac{t}{2}}\sin\left(n-m+\frac{3}{2}\right)t+\frac{C_0^{\gamma-3}}{8\sin^3\frac{t}{2}}\sin\left(n-0+\frac{3}{2}\right)t$$

$$+\sin\left(n+\frac{3}{2}\right)t\sum_{m=0}^{\lambda_n}\frac{C_m^{\gamma-3}}{8\sin^3\frac{t}{2}}-\sin\left(n+\frac{3}{2}\right)t\frac{C_0^{\gamma-3}}{8\sin^3\frac{t}{2}}$$
$$+\frac{C_{\lambda_n}^{\gamma-2}}{8\sin^3\frac{t}{2}}\sin\left(n-\lambda_n+\frac{1}{2}\right)t-\frac{C_{\lambda_n}^{\gamma-2}}{8\sin^3\frac{t}{2}}\sin\left(n+\frac{3}{2}\right)t$$
$$=-\sum_{m=0}^{\lambda_n}\frac{C_m^{\gamma-3}}{8\sin^3\frac{t}{2}}\sin\left(n-m+\frac{3}{2}\right)t+\frac{C_{\lambda_n}^{\gamma-2}}{8\sin^3\frac{t}{2}}\sin\left(n-\lambda_n+\frac{1}{2}\right)t.$$

Hence

$$\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)\frac{d}{dt}W_n^{\gamma}(t)$$
$$=\frac{-2C_{\lambda_n}^{\gamma-1}\frac{1}{2}\cos\frac{t}{2}}{4\sin^3\frac{t}{2}}$$
$$-\sum_{m=0}^{\lambda_n}C_m^{\gamma-3}\left[\frac{\left(n+\frac{3}{2}-m\right)\cos\left(n+\frac{3}{2}-m\right)t\sin^3\frac{t}{2}-\frac{3}{2}\sin\left(n+\frac{3}{2}-m\right)t\sin^2\frac{t}{2}\cos\frac{t}{2}}{8\sin^6\frac{t}{2}}\right]$$
$$+C_{\lambda_n}^{\gamma-2}\left[\frac{\left(n-\lambda_n+\frac{1}{2}\right)\cos\left(n-\lambda_n+\frac{1}{2}\right)t\sin^3\frac{t}{2}-\frac{3}{2}\sin\left(n-\lambda_n+\frac{1}{2}\right)t\ \sin^2\frac{t}{2}\cos\frac{t}{2}}{8\sin^6\frac{t}{2}}\right]$$

and therefore

$$\left|\frac{d}{dt}W_n^{\gamma}(t)\right|\ll\frac{C_{\lambda_n}^{\gamma-1}+\left(n+\frac{3}{2}+\lambda_n\right)\sum_{m=0}^{\lambda_n}\left|C_m^{\gamma-3}\right|+\left|C_{\lambda_n}^{\gamma-2}\right|\left(n+\lambda_n+\frac{1}{2}\right)}{\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)t^3}$$
$$\ll\frac{1}{(\lambda_n+1)\,t^3}+\frac{2(n+1)C_{\lambda_n}^{\gamma-2}}{(n+1)C_{\lambda_n}^{\gamma-1}t^3}\ll\frac{1}{(\lambda_n+1)\,t^3},\ \text{when } \gamma\geq 2,$$

since

$$C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}$$
$$=\frac{(\gamma+1)\,(\gamma+2)\,...\,(\gamma+\lambda_n)}{\lambda_n!}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}$$
$$=\frac{\gamma\,(\gamma+1)\,(\gamma+2)\,...\,(\gamma+\lambda_n-1)}{\lambda_n!}\frac{(\gamma+\lambda_n)}{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}$$
$$=C_{\lambda_n}^{\gamma-1}\left(\gamma+\frac{\lambda_n}{\gamma}+n-\lambda_n\right)=(n-\lambda_n\left(1-\frac{1}{\gamma}\right)+1)C_{\lambda_n}^{\gamma-1}$$
$$\gg(n+1)C_{\lambda_n}^{\gamma-1}\geq(\lambda_n+1)C_{\lambda_n}^{\gamma-1},\ \text{when } \gamma>1.$$

Finally, when $1<\gamma<2$,

$$\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)W_n^{\gamma}(t)$$
$$=\frac{\sum_{m=0}^{\lambda_n}C_m^{\gamma-1}\sin\left(n-m+\frac{1}{2}\right)t+\sum_{m=0}^{n-\lambda_n-1}C_{\lambda_n}^{\gamma-1}\sin\left(m+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$
$$=Im\left\{\frac{\exp\left[i\left(n+\frac{1}{2}\right)t\right]}{2\sin\frac{t}{2}}\sum_{m=0}^{\lambda_n}C_m^{\gamma-1}\exp\left(-imt\right)\right\}$$

$$+C_{\lambda_n}^{\gamma-1} Im\left\{\frac{\exp\frac{-it}{2}}{2\sin\frac{t}{2}}\sum_{m=0}^{n-\lambda_n-1} C_m^{1-1}\exp\left(-imt\right)\right\},$$

whence, by [12, Chapter'XI, (1.10)],

$$\left|\frac{d}{dt}W_n^{\gamma}(t)\right| \ll \frac{1}{C_{\lambda_n}^{\gamma-1}t^{\gamma+1}}.$$

(v) It is evident

$$\begin{aligned}
&\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)\frac{d}{dt}W_n^{\gamma}(t)\\
&=\sum_{m=0}^{\lambda_n} C_m^{\gamma-1}\frac{d}{dt}\left(\frac{1}{2}+\sum_{\mu=1}^{m}\cos\mu t\right)+C_{\lambda_n}^{\gamma-1}\sum_{m=0}^{n-\lambda_n-1}\frac{d}{dt}\left(\frac{1}{2}+\sum_{\mu=1}^{m}\cos\mu t\right)\\
&=\sum_{m=0}^{\lambda_n} C_m^{\gamma-1}\left(-\sum_{\mu=1}^{m}\mu\sin\mu t\right)+C_{\lambda_n}^{\gamma-1}\sum_{m=0}^{n-\lambda_n-1}\left(-\sum_{\mu=1}^{m}\mu\sin\mu t\right),
\end{aligned}$$

whence

$$\begin{aligned}
\left|\frac{d}{dt}W_n^{\gamma}(t)\right| &\le \frac{\left(\lambda_n+1\right)^2 C_{\lambda_n}^{\gamma}+C_{\lambda_n}^{\gamma-1}(n-\lambda_n)^3}{C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}}\\
&\ll \left(n+1\right)^2.
\end{aligned}$$

(vi) Using the following formula from the proof of Lemma 2.3 (iv),

$$\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)W_n^{\gamma}(t)=\frac{C_{\lambda_n}^{\gamma-1}}{4\sin^2\frac{t}{2}}-\sum_{m=0}^{\lambda_n} C_m^{\gamma-2}\frac{\cos\left(n+1-m\right)t}{4\sin^2\frac{t}{2}}$$

we obtain

$$\begin{aligned}
&\left(C_{\lambda_n}^{\gamma}+(n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)\frac{d}{dt}W_n^{\gamma}(t)\\
&=\frac{-2C_{\lambda_n}^{\gamma-1}\frac{1}{2}\cos\frac{t}{2}}{4\sin^3\frac{t}{2}}\\
&-\sum_{m=0}^{\lambda_n} C_m^{\gamma-2}\frac{-\left(n+1-m\right)\sin\left(n+1-m\right)t\sin^2\frac{t}{2}-\sin\frac{t}{2}\cos\frac{t}{2}\cos\left(n+1-m\right)t}{4\sin^4\frac{t}{2}},
\end{aligned}$$

whence

$$\left|\frac{d}{dt}W_n^{\gamma}(t)\right| \ll \frac{C_{\lambda_n}^{\gamma-1}}{(n+1)C_{\lambda_n}^{\gamma-1}t^3}+\frac{(n+1)\,C_{\lambda_n}^{\gamma-1}t^2}{(n+1)C_{\lambda_n}^{\gamma-1}t^4}+\frac{C_{\lambda_n}^{\gamma-1}t}{(n+1)C_{\lambda_n}^{\gamma-1}t^4}=\frac{2}{(n+1)t^3}+\frac{1}{t^2}.$$

The Lemma is proved. □

3. Main Results

Firstly we prove the following theorem.

Theorem 3.1. *Let $f \in L^1$. If $n/\lambda_n = O(1)$, then*

$$|V_n^\gamma f(x) - f(x)| \ll \begin{cases} \frac{1}{n+1}\sum_{k=0}^{n} \Omega_x f\left(\frac{\pi}{k+1}\right), & (\gamma \geq 1), \\ \frac{1}{(n+1)^\gamma}\sum_{k=1}^{n} \frac{1}{(k+1)^{1-\gamma}} \Omega_x f\left(\frac{\pi}{k+1}\right), & (0 < \gamma < 1). \end{cases}$$

Proof. The desired inequality, when $\gamma \geq 1$, follows by Lemma 2.2 and the calculations

$$\begin{aligned}
&|V_n^\gamma f(x) - f(x)| \\
&\leq \frac{1}{\pi}\left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right) |\varphi_x(t) V_n^\gamma(t)|\, dt \\
&\ll \frac{1}{\pi}(n+1)\int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\, dt + \frac{1}{\pi(\lambda_n+1)}\int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|}{t^2} dt \\
&\leq \frac{2}{\pi(n+2)}\sum_{k=0}^{n}(k+1)\int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\, dt + \frac{1}{\pi(\lambda_n+1)}\int_{\frac{\pi}{n+1}}^{\pi} \frac{d}{dt}\left(\int_0^t |\varphi_x(u)|\, du\right)\frac{dt}{t^2} \\
&\leq \frac{2}{n+2}\sum_{k=0}^{n}\frac{k+1}{\pi}\int_0^{\frac{\pi}{k+1}} |\varphi_x(t)|\, dt + \frac{1}{\pi(\lambda_n+1)}\left[\frac{1}{t^2}\int_0^t |\varphi_x(u)|\, du\right]_{\frac{\pi}{n+1}}^{\pi} \\
&\quad + \frac{2}{\pi(\lambda_n+1)}\int_{\frac{\pi}{n+1}}^{\pi}\left(\int_0^t |\varphi_x(u)|\, du\right)\frac{dt}{t^3} \\
&\leq \frac{2}{n+2}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) + \frac{1}{\pi^2(\lambda_n+1)}\Omega_x f(\pi) \\
&\quad - \frac{n+1}{\pi^2(\lambda_n+1)}\Omega_x f\left(\frac{\pi}{n+1}\right) + \frac{2}{\pi^3(\lambda_n+1)}\int_1^{n+1}\left(v\int_0^{\pi/v} |\varphi_x(u)|\, du\right) dv \\
&\ll \frac{1}{n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) + \frac{1}{\lambda_n+1}\Omega_x f(\pi) + \Omega_x f\left(\frac{\pi}{n+1}\right) \\
&\quad + \frac{2}{\pi^3(\lambda_n+1)}\sum_{k=1}^{n}\int_k^{k+1}\left(v\int_0^{\pi/v} |\varphi_x(u)|\, du\right) dv \\
&\leq \frac{2}{n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) + \frac{1}{\lambda_n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) \\
&\quad + \frac{4}{\pi^3(\lambda_n+1)}\sum_{k=1}^{n}\left(k\int_0^{\pi/k} |\varphi_x(u)|\, du\right) \\
&\leq \frac{2}{n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) + \frac{1}{\lambda_n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) \\
&\quad + \frac{4}{\pi^2(\lambda_n+1)}\sum_{k=1}^{n}\Omega_x f\left(\frac{\pi}{k}\right) \ll \frac{1}{n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right).
\end{aligned}$$

Similarly, in case $0 < \gamma < 1$,

$$|V_n^\gamma f(x) - f(x)|$$

$$\ll \frac{n+1}{\pi}\int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\,dt + \frac{1}{\pi(\lambda_n+1)^\gamma}\int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|}{t^{1+\gamma}}dt$$

$$\le \frac{1+\gamma}{\pi(n+1)^\gamma}\sum_{k=0}^{n}(k+1)^\gamma \int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\,dt + \frac{1}{\pi(\lambda_n+1)^\gamma}\left[\frac{1}{t^{1+\gamma}}\int_0^t |\varphi_x(u)|\,du\right]_{\frac{\pi}{n+1}}^{\pi}$$

$$+ \frac{1+\gamma}{\pi(\lambda_n+1)^\gamma}\int_{\frac{\pi}{n+1}}^{\pi}\left(\int_0^t |\varphi_x(u)|\,du\right)\frac{dt}{t^{2+\gamma}}$$

$$\le \frac{1+\gamma}{\pi(n+1)^\gamma}\sum_{k=0}^{n}(k+1)^\gamma \int_0^{\frac{\pi}{k+1}} |\varphi_x(t)|\,dt + \frac{1}{\pi^{1+\gamma}(\lambda_n+1)^\gamma}\Omega_x f(\pi)$$

$$- \frac{(n+1)^\gamma}{\pi^{1+\gamma}(\lambda_n+1)^\gamma}\Omega_x f\left(\frac{\pi}{n+1}\right) + \frac{1+\gamma}{\pi^{2+\gamma}(\lambda_n+1)^\gamma}\int_1^{n+1}\left(v^\gamma\int_0^{\pi/v}|\varphi_x(u)|\,du\right)dv$$

$$\ll \frac{1}{(n+1)^\gamma}\sum_{k=0}^{n}\frac{1}{(k+1)^{1-\gamma}}\Omega_x f\left(\frac{\pi}{k+1}\right) + \frac{1}{(\lambda_n+1)^\gamma}\Omega_x f(\pi) + \Omega_x f\left(\frac{\pi}{n+1}\right)$$

$$+ \frac{1}{(\lambda_n+1)^\gamma}\sum_{k=1}^{n}\int_k^{k+1}\left(v^\gamma\int_0^{\pi/v}|\varphi_x(u)|\,du\right)dv$$

$$\le \frac{2}{(n+1)^\gamma}\sum_{k=0}^{n-1}\frac{1}{(k+1)^{1-\gamma}}\Omega_x f\left(\frac{\pi}{k+1}\right) + \frac{1}{(\lambda_n+1)^\gamma}\Omega_x f(\pi)$$

$$+ \frac{1}{(\lambda_n+1)^\gamma}\sum_{k=1}^{n}\left((2k)^\gamma\int_0^{\pi/k}|\varphi_x(u)|\,du\right)$$

$$\le \frac{2}{(n+1)^\gamma}\sum_{k=0}^{n}\frac{1}{(k+1)^{1-\gamma}}\Omega_x f\left(\frac{\pi}{k+1}\right) + \frac{1}{(\lambda_n+1)^\gamma}\Omega_x f(\pi)$$

$$+ \frac{2^\gamma\pi}{(\lambda_n+1)^\gamma}\sum_{k=1}^{n}\frac{1}{k^{1-\gamma}}\Omega_x f\left(\frac{\pi}{k}\right)$$

$$\ll \frac{1}{(n+1)^\gamma}\sum_{k=0}^{n}\frac{1}{(k+1)^{1-\gamma}}\Omega_x f\left(\frac{\pi}{k+1}\right).$$

Thus the proof is done. □

In the next theorem we will use the modulus $w_x f$ as a measure of approximation by $W_n^\gamma f$ and therefore the natural assumption, by [2,6], is $\gamma > 1$.

Theorem 3.2. *Let $f \in L^1$. If $n/\lambda_n = O(1)$ and $\frac{1}{\delta}\left|\int_0^\delta \varphi_x(t)dt\right| = O_x(1)$, when $\delta \to 0+$, then*

$$|W_n^\gamma f(x) - f(x)| \ll \begin{cases} \frac{1}{n+1}\sum_{m=0}^{n} w_x f\left(\frac{\pi}{m+1}\right), & (\gamma \ge 2), \\ \frac{1}{(n+1)^{\gamma-1}}\sum_{m=0}^{n}\frac{w_x f\left(\frac{\pi}{m+1}\right)}{(m+1)^{2-\gamma}}, & (1 < \gamma < 2). \end{cases}$$

Proof. It is clear that, by the definition, $\lim_{t\to 0+} tW_n^\gamma(t) = 0$ and by the assumption on φ_x we can obtain

$$\lim_{t\to 0+} W_n^\gamma(t)\int_0^t \varphi_x(s)ds = 0,$$

whence

$$
\begin{aligned}
&W_n^\gamma f(x) - f(x)\\
=&\frac{1}{\pi}\int_0^\pi \varphi_x(t) W_n^\gamma(t)dt = \frac{1}{\pi}\int_0^\pi \frac{d}{dt}\left(\int_0^t \varphi_x(s)ds\right) W_n^\gamma(t)dt\\
=&\frac{1}{\pi} W_n^\gamma(\pi)\int_0^\pi \varphi_x(s)ds - \frac{1}{\pi}\lim_{t\to 0+} W_n^\gamma(t)\int_0^t \varphi_x(s)ds\\
&-\frac{1}{\pi}\left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^\pi\right)\left(\int_0^t \varphi_x(s)ds\right)\frac{d}{dt}W_n^\gamma(t)dt\\
=&\frac{1}{\pi} W_n^\gamma(\pi)\int_0^\pi \varphi_x(s)ds - \frac{1}{\pi}\left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^\pi\right)\left(\int_0^t \varphi_x(s)ds\right)\frac{d}{dt}W_n^\gamma(t)dt.
\end{aligned}
$$

Since

$$
|W_n^\gamma(\pi)| = \left|\frac{\sum_{m=0}^{\lambda_n} C_m^{\gamma-1}(-1)^{n-m} + \sum_{m=0}^{n-\lambda_n-1} C_{\lambda_n}^{\gamma-1}(-1)^m}{2\left(C_{\lambda_n}^\gamma + (n-\lambda_n)C_{\lambda_n}^{\gamma-1}\right)}\right| \ll \frac{1}{n+1},
$$

therefore Lemma 2.3 gives

$$
\begin{aligned}
&|W_n^\gamma f(x) - f(x)|\\
\ll&\begin{cases}
\frac{1}{(n+1)\pi^2} w_x f(\pi) + \frac{1}{\pi}(n+1)^2\int_0^{\frac{\pi}{n+1}} t w_x f(t)\,dt + \frac{1}{\lambda_n+1}\int_{\frac{\pi}{n+1}}^\pi \frac{w_x f(t)}{t^2}dt\\
\frac{1}{(n+1)\pi^2} w_x f(\pi) + \frac{1}{\pi}(n+1)^2\int_0^{\frac{\pi}{n+1}} t w_x f(t)\,dt + \frac{1}{(\lambda_n+1)^{\gamma-1}}\int_{\frac{\pi}{n+1}}^\pi \frac{w_x f(t)}{t^\gamma}dt
\end{cases}\\
\ll&\begin{cases}
\frac{w_x f(\pi)}{n+1} + w_x f\left(\frac{\pi}{n+1}\right) + \frac{1}{n+1}\sum_{m=0}^n w_x f\left(\frac{\pi}{m+1}\right), & (\gamma \ge 2),\\
\frac{w_x f(\pi)}{n+1} + w_x f\left(\frac{\pi}{n+1}\right) + \frac{1}{(n+1)^{\gamma-1}}\sum_{m=0}^n \frac{w_x f\left(\frac{\pi}{m+1}\right)}{(m+1)^{2-\gamma}}, & (1<\gamma<2),
\end{cases}
\end{aligned}
$$

whence the desired result follows. □

Now we prove the two theorems of the Leindler type.

Theorem 3.3. *Let $f \in L^1$ and $\gamma > 0$. If λ_n tends to infinity and conditions*

$$
\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}} \frac{|\varphi_x(t)|}{t}dt = o_x(1) \text{ and } (n+1)\int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|dt = o_x(1)
$$

are satisfied, then $V_n^\gamma f(x)$ and $W_n^\gamma f(x)$ converge to $f(x)$, for a.e. x.

Proof. We can write, by the definitions of $V_n^\gamma f$ and $W_n^\gamma f$,

$$
\begin{aligned}
&\begin{matrix} V_n^\gamma f(x)\\ W_n^\gamma f(x)\end{matrix} - f(x)\\
=&\frac{1}{\pi}\int_0^\pi \varphi_x(t)\begin{matrix} V_n^\gamma(x)\\ W_n^\gamma(x)\end{matrix}\,dt\\
=&\frac{1}{\pi}\int_0^{\frac{\pi}{n+1}} \varphi_x(t)\begin{matrix} V_n^\gamma(x)\\ W_n^\gamma(x)\end{matrix}\,dt + \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}} \varphi_x(t)\begin{matrix} V_n^\gamma(x)\\ W_n^\gamma(x)\end{matrix}\,dt
\end{aligned}
$$

$$+\frac{1}{\pi}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\varphi_x(t)\begin{array}{c}V_n^{\gamma}(x)\\ W_n^{\gamma}(x)\end{array}dt:=\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3.$$

Using Lemma 2.2 (i) or 2.3 (i)

$$|\mathbf{I}_1|\le\frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}|\varphi_x(t)|\begin{array}{c}|V_n^{\gamma}(t)|\\ |W_n^{\gamma}(t)|\end{array}dt\ll(n+1)\int_0^{\frac{\pi}{n+1}}|\varphi_x(t)|dt=o_x(1)$$

and by Lemma 2.2 (iii) or 2.3 (iii)

$$|\mathbf{I}_2|\le\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}|\varphi_x(t)|\begin{array}{c}|V_n^{\gamma}(t)|\\ |W_n^{\gamma}(t)|\end{array}dt\ll\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{|\varphi_x(t)|}{t}=o_x(1).$$

The estimate of $|\mathbf{I}_3|$ is similar to the proof of Theorem 3.1. Indeed, by Lemma 2.2 (ii) or 2.3 (ii)

$$\begin{aligned}|\mathbf{I}_3|&\le\frac{1}{\pi}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}|\varphi_x(t)|\begin{array}{c}|V_n^{\gamma}(t)|\\ |W_n^{\gamma}(t)|\end{array}dt\ll\frac{1}{\lambda_n+1}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt\\ &\ll\frac{1}{\lambda_n+1}\left(\left[\frac{1}{t^2}\int_0^t|\varphi_x(u)|du\right]_{\frac{\pi}{\lambda_n+1}}^{\pi}+\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{1}{t^3}\int_0^t|\varphi_x(u)|dudt\right)\\ &\ll\frac{1}{\lambda_n+1}\left[\frac{1}{\pi^2}\int_0^{\pi}|\varphi_x(u)|du+\sum_{k=1}^{\lambda_n}\left(\frac{k}{\pi}\int_0^{\pi/k}|\varphi_x(u)|\,du\right)\right]\\ &=o_x(1),\ \text{when }\gamma\ge1\end{aligned}$$

and

$$\begin{aligned}|\mathbf{I}_3|&\le\frac{1}{\pi}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}|\varphi_x(t)|\begin{array}{c}|V_n^{\gamma}(t)|\\ |W_n^{\gamma}(t)|\end{array}dt\ll\frac{1}{(\lambda_n+1)^{\gamma}}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^{1+\gamma}}dt\\ &\le\frac{1}{(\lambda_n+1)^{\gamma}}\left[\frac{1}{t^{1+\gamma}}\int_0^t|\varphi_x(u)|\,du\right]_{\frac{\pi}{\lambda_n+1}}^{\pi}+\frac{1+\gamma}{(\lambda_n+1)^{\gamma}}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\left(\int_0^t|\varphi_x(u)|\,du\right)\frac{dt}{t^{2+\gamma}}\\ &\ll\frac{1}{(\lambda_n+1)^{\gamma}}\left[\frac{1}{\pi^{1+\gamma}}\int_0^{\pi}|\varphi_x(u)|\,du\right]+\frac{1+\gamma}{(\lambda_n+1)^{\gamma}}\sum_{k=1}^{n}k^{\gamma}\left(\int_0^{\pi/k}|\varphi_x(u)|\,du\right)\\ &=o_x(1),\ \text{when }0<\gamma<1.\end{aligned}$$

Thus the proof is completed. □

Theorem 3.4. *Let $f\in L^1$ and $\gamma>1$. If conditions*

$$\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\left|\int_0^t\varphi_x(s)ds\right|}{t^2}dt=o_x(1),\ \ (\lambda_n\to\infty)\ \ \text{and}\ \frac{1}{t}\int_0^t\varphi_x(s)ds=o_x(1),\ \ (t\to0+)$$

are satisfied, then $W_n^{\gamma}f(x)$ converges to $f(x)$, for a.e. x.

Proof. Similar to the proof of Theorem 3.2

$$\begin{aligned}
W_n^\gamma f(x) - f(x) &= \frac{1}{\pi}\int_0^\pi \varphi_x(t) W_n^\gamma(t)dt = \frac{1}{\pi}\int_0^\pi \frac{d}{dt}\left(\int_0^t \varphi_x(s)ds\right) W_n^\gamma(t)dt \\
&= \frac{1}{\pi} W_n^\gamma(\pi)\int_0^\pi \varphi_x(s)ds - \frac{1}{\pi}\lim_{t\to 0+} W_n^\gamma(t)\int_0^t \varphi_x(s)ds \\
&\quad - \frac{1}{\pi}\int_0^\pi \left(\int_0^t \varphi_x(s)ds\right)\frac{d}{dt}W_n^\gamma(t)dt \\
&= \frac{1}{\pi} W_n^\gamma(\pi)\int_0^\pi \varphi_x(s)ds \\
&\quad - \frac{1}{\pi}\left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}} + \int_{\frac{\pi}{\lambda_n+1}}^{\pi}\right)\left(\int_0^t \varphi_x(s)ds\right)\frac{d}{dt}W_n^\gamma(t)dt \\
&= \frac{1}{\pi} W_n^\gamma(\pi)\int_0^\pi \varphi_x(s)ds - (\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3)
\end{aligned}$$

and

$$\frac{1}{\pi}\left|W_n^\gamma(\pi)\right|\left|\int_0^\pi \varphi_x(s)ds\right| \le \frac{1}{n+1}\frac{1}{\pi}\left|\int_0^\pi \varphi_x(s)ds\right| = o_x(1).$$

Using Lemma 2.3 (v)

$$\begin{aligned}
|\mathbf{J}_1| &\le \frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\left|\int_0^t \varphi_x(s)ds\right|\left|\frac{d}{dt}W_n^\gamma(t)\right|dt \ll (n+1)^2\int_0^{\frac{\pi}{n+1}}\left|\int_0^t \varphi_x(s)ds\right|dt \\
&= o_x(1)\,(n+1)^2\int_0^{\frac{\pi}{n+1}} tdt = o_x(1)
\end{aligned}$$

and by Lemma 2.3 (vi)

$$\begin{aligned}
|\mathbf{J}_2| &\le \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\left|\int_0^t \varphi_x(s)ds\right|\left|\frac{d}{dt}W_n^\gamma(t)\right|dt \\
&\ll \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\left|\int_0^t \varphi_x(s)ds\right|\left[\frac{1}{(n+1)t^3} + \frac{1}{t^2}\right]dt \ll \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\left|\int_0^t \varphi_x(s)ds\right|}{t^2}dt = o_x(1).
\end{aligned}$$

By Lemma 2.3 (iv)

$$\begin{aligned}
|\mathbf{J}_3| &\le \frac{1}{\pi}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\left|\int_0^t \varphi_x(s)ds\right|\left|\frac{d}{dt}W_n^\gamma(t)\right|dt \\
&\ll \begin{cases} \frac{1}{\lambda_n+1}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{1}{t^3}\left|\int_0^t \varphi_x(s)ds\right|dt, & (\gamma \ge 2)\,, \\ \frac{1}{(\lambda_n+1)^{\gamma-1}}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{1}{t^{1+\gamma}}\left|\int_0^t \varphi_x(s)ds\right|dt, & (1<\gamma<2)\,. \end{cases}
\end{aligned}$$

By the assumption, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } t < \delta, \text{ then } \frac{1}{t}\left|\int_0^t \varphi_x(s)ds\right| < \epsilon.$$

Therefore

$$\frac{1}{\lambda_n+1}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{1}{t^3}\left|\int_0^t \varphi_x(s)ds\right|dt$$

$$= \frac{1}{\lambda_n+1}\left(\int_{\frac{\pi}{\lambda_n+1}}^{\delta} + \int_{\delta}^{\pi}\right)\frac{1}{t^3}\left|\int_0^t \varphi_x(s)ds\right| dt$$
$$< \frac{\epsilon}{\lambda_n+1}\int_{\frac{\pi}{\lambda_n+1}}^{\delta}\frac{dt}{t^2} + \frac{1}{\lambda_n+1}\int_0^{\pi}|\varphi_x(s)|ds\int_{\delta}^{\pi}\frac{dt}{t^3} \ll \epsilon$$

and analogously

$$\frac{1}{(\lambda_n+1)^{\gamma-1}}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{1}{t^{1+\gamma}}\left|\int_0^t \varphi_x(s)ds\right| dt \ll \epsilon.$$

Thus desired result follows analogously as in the proof of Theorem 3.3. □

Remark 3.1. If $f \in C$, then there exists an $s_t \in (0,t)$ such that $\int_0^t \varphi_x(s)ds = t\varphi_x(s_t)$ and therefore

$$\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\int_0^t \varphi_x(s)ds}{t^2}dt = \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\varphi_x(s_t)}{t}dt.$$

We can also note the following estimates

$$\left|\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\varphi_x(t)}{t}dt\right|$$
$$= \left|\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{1}{t}\frac{d}{dt}\left(\int_0^t \varphi_x(s)ds\right)dt\right|$$
$$= \left|\left[\frac{1}{t}\int_0^t \varphi_x(s)ds\right]_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}} + \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\int_0^t \varphi_x(s)ds}{t^2}dt\right|$$
$$\le \frac{\lambda_n+1}{\pi}\left|\int_0^{\frac{\pi}{\lambda_n+1}}\varphi_x(s)ds\right| + \frac{n+1}{\pi}\left|\int_0^{\frac{\pi}{n+1}}\varphi_x(s)ds\right| + \left|\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\int_0^t \varphi_x(s)ds}{t^2}dt\right|$$

and

$$\left|\int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{\varphi_x(t)}{t}dt\right| \le w_x f\left(\frac{\pi}{\lambda_n+1}\right) + w_x f\left(\frac{\pi}{n+1}\right) + \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{w_x f(t)}{t}dt.$$

Theorem 3.5. *If $f \in L^1$, then*

$$|V_n^{\gamma} f(x) - f(x)|$$
$$\ll \begin{cases} \sum_{k=\lambda_n}^{n}\frac{1}{k+1}\Omega_x f(\frac{\pi}{k+1})_X + \frac{1}{\lambda_n+1}\sum_{k=0}^{\lambda_n}\Omega_x f(\frac{\pi}{k+1})_X, & (\gamma \ge 1), \\ \sum_{k=\lambda_n}^{n}\frac{1}{k+1}\Omega_x f(\frac{\pi}{k+1})_X + \frac{1}{(\lambda_n+1)^{\gamma}}\sum_{k=0}^{\lambda_n}\frac{1}{(k+1)^{1-\gamma}}\Omega_x f(\frac{\pi}{k+1})_X, & (0<\gamma<1). \end{cases}$$

Proof. By the proof of Theorem 3.3

$$|V_n^{\gamma} f(x) - f(x)|$$
$$\ll \begin{cases} (n+1)\int_0^{\frac{\pi}{n+1}}|\varphi_x(t)|dt + \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{|\varphi_x(t)|}{t}dt + \frac{1}{\lambda_n+1}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt, & (\gamma \ge 1), \\ (n+1)\int_0^{\frac{\pi}{n+1}}|\varphi_x(t)|dt + \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}}\frac{|\varphi_x(t)|}{t}dt + \frac{1}{(\lambda_n+1)^{\gamma}}\int_{\frac{\pi}{\lambda_n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^{1+\gamma}}dt, & (0<\gamma<1). \end{cases}$$

The result follows by more precise estimates, similar to the proof of Theorem 3.1. □

Theorem 3.6. *If $f \in L^1$, then*

$$|W_n^\gamma f(x) - f(x)| \ll \begin{cases} \sum_{k=\lambda_n}^{n} \frac{1}{k+1} w_x f(\frac{\pi}{k+1})_X + \frac{1}{\lambda_n+1} \sum_{k=0}^{\lambda_n} w_x f(\frac{\pi}{k+1})_X, & (\gamma \geq 2), \\ \sum_{k=\lambda_n}^{n} \frac{1}{k+1} w_x f(\frac{\pi}{k+1})_X + \frac{1}{(\lambda_n+1)^\gamma} \sum_{k=0}^{\lambda_n} \frac{1}{(k+1)^{1-\gamma}} w_x f(\frac{\pi}{k+1})_X, & (1 < \gamma < 2). \end{cases}$$

Proof. By the proof of Theorem 3.4

$$|W_n^\gamma f(x) - f(x)| \ll \begin{cases} \frac{w_x f(\pi)}{n+1} + w_x f\left(\frac{\pi}{n+1}\right) + \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}} \frac{w_x f(t)}{t} dt + \frac{1}{\lambda_n+1} \int_{\frac{\pi}{\lambda_n+1}}^{\pi} \frac{t w_x f(t)}{t^3} dt, & (\gamma \geq 2), \\ \frac{w_x f(\pi)}{n+1} + w_x f\left(\frac{\pi}{n+1}\right) + \int_{\frac{\pi}{n+1}}^{\frac{\pi}{\lambda_n+1}} \frac{w_x f(t)}{t} dt + \frac{1}{(\lambda_n+1)^{\gamma-1}} \int_{\frac{\pi}{\lambda_n+1}}^{\pi} \frac{t w_x f(t)}{t^{2+\gamma}} dt, & (1<\gamma<2). \end{cases}$$

The result follows by more precise estimates, similar to the proof of Theorem 3.1. □

4. Corollaries and Remarks

We can observe evident remark:

Remark 4.1. If $f \in X$, then

$$\|w.f(\delta)\|_X \leq \|\Omega.f(\delta)\|_X \leq \omega f(\delta)_X.$$

Using this remark we will easily derive the following corollaries.

Corollary 4.1. *If $f \in Lip(\omega, X)$, then*

$$\begin{array}{l} \|V_n^\gamma f(\cdot) - f(\cdot)\|_X \\ \|W_n^\gamma f(\cdot) - f(\cdot)\|_X \end{array} \ll \begin{cases} \sum_{k=\lambda_n}^{n} \frac{1}{k+1} \omega f(\frac{\pi}{k+1})_X + \frac{1}{\lambda_n+1} \sum_{k=0}^{\lambda_n} \omega f(\frac{\pi}{k+1})_X, & (\gamma \geq 1), \\ \sum_{k=\lambda_n}^{n} \frac{1}{k+1} \omega f(\frac{\pi}{k+1})_X + \frac{1}{(\lambda_n+1)^\gamma} \sum_{k=0}^{\lambda_n} \frac{1}{(k+1)^{1-\gamma}} \omega f(\frac{\pi}{k+1})_X, & (0 < \gamma < 1), \end{cases}$$

holds true.

Corollary 4.2. *If $f \in Lip(\delta^\alpha, X)$, then*

$$\begin{array}{l} \|V_n^\gamma f(\cdot) - f(\cdot)\|_X \\ \|W_n^\gamma f(\cdot) - f(\cdot)\|_X \end{array} \ll \begin{cases} \frac{1}{(\lambda_n+1)^\alpha}, & (0 < \alpha < 1 \leq \gamma \text{ and } 0 < \alpha < \gamma < 1), \\ \frac{\ln(\lambda_n+1)}{\lambda_n+1}, & (\alpha = 1 \leq \gamma), \\ \frac{\ln(\lambda_n+1)}{(\lambda_n+1)^\gamma}, & (0 < \alpha = \gamma \leq 1), \\ \frac{1}{(\lambda_n+1)^\gamma}, & (0 < \gamma < \alpha \leq 1), \end{cases}$$

holds true.

Finally, we have:

Remark 4.2. As the special cases, Theorem 3.3 and Corollary 4.2 with $\gamma = 1$ give the Leindler results [7] (see Theorems 1.1 and 1.2, respectively) but when $\lambda_n = n$ Theorems 3.1 and 3.2 give Theorems 1.3 and 1.4, respectively.

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