Journal of Applied Analysis and Computation
Volume 12, Number 1, February 2022, 153–178
Website:http://www.jaac-online.com
DOI:10.11948/20210094

STATIONARY DISTRIBUTION AND CONTROL STRATEGY OF A STOCHASTIC DENGUE MODEL WITH SPATIAL DIFFUSION*

Kangkang Chang[1], Qimin Zhang[1,†] and Huaimin Yuan[2]

Abstract In this paper, we establish a dengue model, which is described by the spatial diffusion and Brownian motion, and discuss the stationary distribution and optimal control of the stochastic dengue model. At first, we show the existence of the global positive solution by constructing Lyapunov function. The sufficient conditions are given for the existence and uniqueness of stationary distribution of the positive solution. Subsequently, we introduce the control strategy, namely, decrease the infected individual and spray mosquito insecticides. The first order necessary conditions are derived for the existence of optimal control by applying Pontryagins maximum principle. Finally, numerical simulations are introduced to confirm the analytical results. The simulation results verified the existence of stationary distribution, and there are certain differences in the solutions of the stationary distribution in different spaces. The influence of different noise intensity on the stationary distribution and the effect of different control strategy for stochastic dengue fever.

Keywords Stationary distribution, control strategy, Stochastic dengue model, spatial diffusion.

MSC(2010) 35E20, 37N35, 93E03.

1. Introduction

As we all know, there is a long history of models for the spread of mosquito-borne infection, including both spatial and stochastic models [1,6,7,13,14,21,25–27,30,31]. Dengue fever is an infectious disease transmitted by mosquitoes. The disease breaks out in different areas almost every year. Since January 2020, some countries in the world have successively broken out dengue fever epidemics of varying degrees, such as Brazil, Singapore, Laos, Malaysia, Pakistan and India, as of 30 May, 558,767 dengue cases (including 357 deaths) had been confirmed in Brazil (International Travel Health Advisory Network), by 17 August, Singapore had reported a total of

†The corresponding author. Email: zhangqimin64@sina.com(Q. Zhang)
1School of Mathematics and Statistics, Ningxia University, Yinchuan, 750021, China
2School of Information and Engineering, Ningxia University, Yinchuan, 750021, China
*The authors were supported by the Ningxia Natural Science Foundation (No. 2020AAC03067) and National Natural Science Foundation of China (No. 12161068).

25,053 suspected cases (including 20 deaths) (International Travel Health Advisory Network) of dengue fever, Malaysia had 78,303 dengue cases (including 127 deaths) as of 3 October (International Travel Health Advisory Network). From the above data, we can see that dengue fever still poses a great threat to human public health. Therefore, it is necessary to study the dynamic behavior of dengue virus and how to control the spread of dengue disease.

Mathematical models have become an important tool in understanding the transmission mechanisms of dengue epidemic. As far as we know there are different forms of dengue models, here we consider the following dengue model [3]:

$$\begin{cases} \frac{dS_H}{dt} = \mu_h N_H - \mu S_H - \frac{\beta_H b}{N_H+m} S_H I_V, \\ \frac{dI_H}{dt} = \frac{\beta_H b}{N_H+m} S_H I_V - (\mu + \gamma_H) I_H, \\ \frac{dR_H}{dt} = \gamma_H I_H - \mu R_H, \\ \frac{dS_V}{dt} = A - \nu S_V - \frac{\beta_V b}{N_H+m} S_V I_H, \\ \frac{dI_V}{dt} = \frac{\beta_V b}{N_H+m} S_V I_H - \nu I_V, \\ S_H > 0, I_H > 0, R_H > 0, S_V > 0, I_V > 0, \end{cases} \tag{1.1}$$

where $S_H(t)$, $I_H(t)$ and $R_H(t)$ represent the human population densities of susceptible, infectious and recovery, at time t, respectively. N_H is the human population and $N_H = S_H + I_H + R_H$. $S_V(t)$ and $I_V(t)$ represent the mosquitoes population densities of susceptible and infectious, respectively. m represents the densities of alternative hosts. μ_h is the birth rate of human. Natural death rate of human is given by μ. γ_H denotes the recovery rate of human. A represents the recruitment rate of mosquitoes. ν is the nature death rate of mosquitoes. b is the mosquitoes biting rate. β_H denotes the transmission rate of dengue to the human from the mosquito. β_V represents the transmission rate of dengue to the mosquito from human.

About the system (1.1), a lot of work on dynamics behavior was discovered [8, 23, 24, 29]. For example, Tridip Sardar et al. given a compartmental model of dengue transmission with memory and analyzed the stability of equilibrium [23]. Jean Jules Tewa et al. [24] studied the stability analysis of disease-free and endemic cases. Min Zhu et al. [29] considered a dengue fever with nonlocal incidence and free boundaries, and proposed spatial-temporal risk index and analyzed the relationship between different model variables according to the risk index. In order to reveal the effect of stochastic noise, some scholars have introduced stochastic processes based on Brownian motion into system (1.1). For instance, Wei Sun et al. [22] introduced the stochastic perturbations into a deterministic compartmental model for a dengue, and obtained that stochastic perturbation can improve the stability of the disease-free equilibrium point. Q. Liu et al. [12] studied the dynamical behavior of a stochastic SIR-SI dengue model. They obtained sufficient conditions for the existence of an stationary distribution and extinction of the diseases.

Besides, when dengue breaks outbreak, it imposes a huge financial burden which includes unemployment and medical expenses. Hence, from epidemiological and economic point of view, how to control the spread of dengue fever is a meaningful question. To solve these problems, the optimal control strategy is proposed. For example, Ahmed Abdelrazec et al. [2] introduced larval populations into system (1.1), and investigated the impact of limited public health resources on dengue transmission and control. Helena Sofia Rodriguesa et al. [20] used vaccination as control strategy, and showed the optimal control in two different ways. Helena Sofia Rodrigues et al. [19] discussed mosquito population control strategies, and showed that the implementation of controls has a positive impact on reducing the number of infected. However, references [2,19,20] used a single control strategy, and did not consider the effects of space diffusion. As far as we know, the range of humans and mosquitoes is not fixed, so the dengue virus can spread from one region to another when humans and mosquitoes are in different spatial locations. Hence, it makes sense to consider spatial diffusion.

Based on the above analysis, in this paper, we propose a new dengue model, which is described by the spatial diffusion and Brownian motion, and analyze the stationary distribution and optimal control of the model. Here, we adopt a mixed control strategy, namely, treat the infected individual and spray mosquito insecticides, and we compare the effects of different strategies for infected. The innovations of this paper are: (1) we introduce spatial diffusion and Brownian motion into the system (1.1), and the resulting model is an extension of the previous literature; (2) we present the positive solution of stationary distribution for stochastic dengue model with spatial diffusion; meanwhile, the sufficient conditions are given for the existence and uniqueness of stationary distribution of the solution; (3) we analyze how to control the dengue epidemic, namely, treat the infected individual and spray mosquito insecticides. The necessary conditions for the effective control of dengue model are derived.

The structure of the article is as follows. We propose a new model with the spatial diffusion and Brownian motion and give the preliminary knowledge in next Section. In Section 3, we first present the positive solution of stationary distribution. Then, the sufficient conditions are given for the existence and uniqueness of stationary distribution of the solution to the dengue model. In Section 4, we analyze the optimal control of system; the first order necessary conditions are derived for the existence of optimal control by applying Pontryagins maximum principle. In Section 5, numerical simulation is given to prove the theoretical results. In Section 6 concluding remarks are given.

2. Model and preliminary knowledge

In this section, we introduce the spatial diffusion and the Brownian motion into system (1.1). In addition, in order to prove the latter theory, we give the preliminary knowledge. Based on System (1.1), we first describe the following two cases:

(1) Due to the rapid development of modern means of transportation, such as aircraft, high-speed rail, etc. This increases the rate of movement of people, which increases the risk of contracting the virus, and also allows people who are already infected to quickly carry the virus to another area.

(2) Although the distance of mosquitoes flying alone is relatively short, the rapid development of transportation also creates convenient conditions for the cross-regional transmission of mosquitoes.

Hence, in view of the spread of the infected area and the two conditions mentioned above. We get the following model:

$$\begin{cases} \frac{\partial S_H}{\partial t} = d_1 \triangle S_H + \mu_h N_H - \mu S_H - \frac{\beta_H b}{N_H + m} S_H I_V, & x \in \Omega, t > 0 \\ \frac{\partial I_H}{\partial t} = d_2 \triangle I_H + \frac{\beta_H b}{N_H + m} S_H I_V - (\mu + \gamma_H) I_H, & x \in \Omega, t > 0 \\ \frac{\partial R_H}{\partial t} = d_3 \triangle R_H + \gamma_H I_H - \mu R_H, & x \in \Omega, t > 0 \\ \frac{\partial S_V}{\partial t} = d_4 \triangle S_V + A - \nu S_V - \frac{\beta_V b}{N_H + m} S_V I_H, & x \in \Omega, t > 0 \\ \frac{\partial I_V}{\partial t} = d_5 \triangle I_V + \frac{\beta_V b}{N_H + m} S_V I_H - \nu I_V, & x \in \Omega, t > 0 \end{cases} \tag{2.1}$$

with boundary condition

$$\frac{\partial S_H}{\partial \nu} = \frac{\partial I_H}{\partial \nu} = \frac{\partial R_H}{\partial \nu} = \frac{\partial S_V}{\partial \nu} = \frac{\partial I_V}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \tag{2.2}$$

and initial condition

$$\begin{aligned} &S_H(x,0) = S_{H,0}(x), I_H(x,0) = I_{H,0}(x), R_H(x,0) = R_{H,0}(x), \\ &S_V(x,0) = S_{V,0}(x), I_V(x,0) = I_{V,0}(x), x \in \Omega. \end{aligned} \tag{2.3}$$

$S_H(x,t)$, $I_H(x,t)$ and $R_H(x,t)$ represent the human population densities of susceptible, infectious and recovery, at location x and time t. N_H is the human population and $N_H = S_H + I_H + R_H$. $S_V(x,t)$ and $I_V(x,t)$ represent the mosquitoes population densities of susceptible and infectious, respectively. d_1, d_2, d_3 represent the diffusion coefficient of susceptible, infectious and recovery for human population, respectively. d_4, d_5 denote the diffusion coefficient of susceptible and infectious for mosquitoes population, respectively. Ω is a bounded domain with smooth boundary.

The death of humans and mosquitoes can be influenced by external environmental factors, such as outbreaks of disease and other accidents in humans. For mosquitoes, sudden changes in the weather, such as a sudden drop in temperature or high temperature and less rain can affect the survival rate of mosquito larvae, as well as mosquito control also can affect the survival of mosquitoes. Hence, we assume the death rate is a random process. Next, we introduce the Gaussian white noise $\eta_i(t)$ (with zero mean and unit covariance) and Brownian motion $B_i(t)$ (in fact, white noise is often defined as the informal derivative of a Brownian motion [10, Definition 11.6.3]), which has the following form:

$$\mu = \mu + \xi_1 \eta_1(t), \quad \nu = \nu + \xi_2 \eta_2(t) \ \ and \ \ dB_i(t) = \eta_i(t) dt \ \ (i = 1, 2), \tag{2.4}$$

ξ_i is the intensity of noise. Substituting (2.4) into system (2.1) implies the following stochastic system

$$\begin{cases} dS_H = [d_1 \triangle S_H + \mu_h N_H - \mu S_H - \frac{\beta_H b}{N_H + m} S_H I_V] dt - \xi_1(t) S_H dB_1(t), \\ dI_H = [d_2 \triangle I_H + \frac{\beta_H b}{N_H + m} S_H I_V - \gamma_H I_H - \mu I_H] dt - \xi_1(t) I_H dB_1(t), \\ dR_H = [d_3 \triangle R_H + \gamma_H I_H - \mu R_H] dt - \xi_1(t) R_H dB_1(t), \\ dS_V = [d_4 \triangle S_V + A - \nu S_V - \frac{\beta_V b}{N_H + m} S_V I_H] dt - \xi_2(t) S_V dB_2(t), \\ dI_V = [d_5 \triangle I_V + \frac{\beta_V b}{N_H + m} S_V I_H - \nu I_V] dt - \xi_2(t) I_V dB_2(t), \end{cases} \tag{2.5}$$

with boundary condition

$$\frac{\partial S_H}{\partial \nu} = \frac{\partial I_H}{\partial \nu} = \frac{\partial R_H}{\partial \nu} = \frac{\partial S_V}{\partial \nu} = \frac{\partial I_V}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0,$$

and initial condition

$$\begin{aligned}&S_H(x,0)=S_{H,0}(x), I_H(x,0)=I_{H,0}(x), R_H(x,0)=R_{H,0}(x),\\&S_V(x,0)=S_{V,0}(x), I_V(x,0)=I_{V,0}(x), x\in\Omega.\end{aligned}$$

Let $\mathbb{B}$ be a linear operator defined by

$$\mathbb{B}\begin{pmatrix}S_H\\I_H\\R_H\\S_V\\I_V\end{pmatrix}=\begin{pmatrix}d_1\triangle S_H\\d_2\triangle I_H\\d_3\triangle R_H\\d_4\triangle S_V\\d_5\triangle I_V\end{pmatrix}. \tag{2.6}$$

Then, we define a nonlinear operator $\mathbb{C}$ by

$$\mathbb{C}\begin{pmatrix}S_H\\I_H\\R_H\\S_V\\I_V\end{pmatrix}=\begin{pmatrix}\mu_h N_H-\mu S_H-\frac{\beta_H b}{N_H+m}S_H I_V-\xi_1(t)S_H\dot{B}_1(t)\\\frac{\beta_H b}{N_H+m}S_H I_V-\gamma_H I_H-\mu I_H-\xi_1(t)I_H\dot{B}_1(t)\\\gamma_H I_H-\mu R_H-\xi_1(t)R_H\dot{B}_1(t)\\A-\nu S_V-\frac{\beta_V b}{N_H+m}S_V I_H-\xi_2(t)S_V\dot{B}_2(t)\\\frac{\beta_V b}{N_H+m}S_V I_H-\nu I_V-\xi_2(t)I_V\dot{B}_2(t)\end{pmatrix}. \tag{2.7}$$

Let $\mathcal{W}(x,t)=(S_H(x,t),I_H(x,t),R_H(x,t),S_V(x,t),I_V(x,t))^T$; together with (2.6) and (2.7), system (2.5) is rewritten as the following abstract Cauchy problem

$$\frac{d}{dt}\mathcal{W}(x,t)=\mathbb{B}\mathcal{W}(x,t)+\mathbb{C}\mathcal{W}(x,t). \tag{2.8}$$

Let

$$\begin{aligned}V=&\mathcal{H}^1(\Omega)\\=&\{\phi\mid\phi\in L^2(\Omega),\frac{\partial\phi}{\partial x}\in L^2(\Omega),\ \text{where } \tfrac{\partial\phi}{\partial x} \text{ are generalized partial derivatives}\},\end{aligned}$$

$V^{'}=\mathcal{H}^{-1}(\Omega)$ is the dual space of V; we denote by $\|\cdot\|$ the norm in V, by $\langle\cdot,\cdot\rangle$ the duality product between V and V'. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},P)$ be a complete probability space, and $B_i(t),(i=1,2)$ defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},P)$, $R^5_+=(x_1,x_2,x_3,x_4,x_5)\in R^5, x_i>0,(i=1,2,3,4,5)$. Next, we introduce a lemma that gives a criterion for the existence of an ergodic stationary distribution to system (2.5).

Notation

$$\overline{g}=\sup_{t\to\infty}g(t),\underline{g}=\inf_{t\to\infty}g(t), \tag{2.9}$$

here, $g(t)$is a continuous bounded function. In next section, we prove the stationary distribution of system (2.5).

3. Stationary distribution

In order to get the conclusion, we first give a lemma.

Lemma 3.1. *For any initial data* $(S_{H,0}, I_{H,0}, R_{H,0}, S_{V,0}, I_{V,0})$*, the solution* $\mathcal{W}(x,t) = (S_H(x,t), I_H(x,t), R_H(x,t), S_V(x,t), I_V(x,t))$ *of system (2.5), satisfies that*

$$\limsup_{t\to\infty}(S_H(x,t)+I_H(x,t)+R_H(x,t)+S_V(x,t)+I_V(x,t))<\infty.$$

Proof. Let

$$N(t)=\int_\Omega [S_H(x,t)+I_H(x,t)+R_H(x,t)+S_V(x,t)+I_V(x,t)]dx,$$

by (2.5), we have

$$\begin{aligned}
\frac{\partial N(t)}{\partial t} &= \int_\Omega [\frac{\partial}{\partial t}S_H(x,t)+\frac{\partial}{\partial t}I_H(x,t)+\frac{\partial}{\partial t}R_H(x,t)+\frac{\partial}{\partial t}S_V(x,t)+\frac{\partial}{\partial t}I_V(x,t)]dx\\
&= \int_\Omega [d_1\triangle S_H+\mu_h N_H-\mu S_H-\frac{\beta_H b}{N_H+m}S_H I_V-\xi_1(t)S_H\dot{B}_1(t)+d_2\triangle I_H\\
&\quad +\frac{\beta_H b}{N_H+m}S_H I_V-\gamma_H I_H-\mu I_H-\xi_1(t)I_H\dot{B}_1(t)+d_3\triangle R_H+\gamma_H I_H-\mu R_H\\
&\quad -\xi_1(t)R_H\dot{B}_1(t)+d_4\triangle S_V+A-\nu S_V-\frac{\beta_V b}{N_H+m}S_V I_H-\xi_2(t)S_V\dot{B}_2(t)\\
&\quad +d_5\triangle I_V+\frac{\beta_V b}{N_H+m}S_V I_H-\nu I_V-\xi_2(t)I_V\dot{B}_2(t)]dx.
\end{aligned}$$

Next, we continue our process.

$$\begin{aligned}
\frac{\partial N(t)}{\partial t} \le& d_1\int_{\partial\Omega}(\frac{\partial}{\partial\nu}S_H(x,t))dx+d_2\int_{\partial\Omega}(\frac{\partial}{\partial\nu}I_H(x,t))dx+d_3\int_{\partial\Omega}(\frac{\partial}{\partial\nu}R_H(x,t))dx\\
&+d_4\int_{\partial\Omega}(\frac{\partial}{\partial\nu}S_V(x,t))dx+d_5\int_{\partial\Omega}(\frac{\partial}{\partial\nu}I_V(x,t))dx\\
&+\int_\Omega(\mu_h N_H+A-\mu S_H(x,t)-\mu I_H(x,t)-\mu R_H(x,t))-\nu S_V(x,t)\\
&-\nu I_V(x,t)-\xi_1(t)S_H\dot{B}_1(t)-\xi_1(t)I_H\dot{B}_1(t)-\xi_1(t)R_H\dot{B}_1(t)\\
&-\xi_2(t)S_V\dot{B}_2(t)-\xi_2(t)I_V\dot{B}_2(t))dx\\
\le&(\mu_h N_H+A)|\Omega|-BN(t)-\int_\Omega(\xi_1(t)S_H\dot{B}_1(t)+\xi_1(t)I_H\dot{B}_1(t)\\
&+\xi_1(t)R_H\dot{B}_1(t)+\xi_2(t)S_V\dot{B}_2(t)+\xi_2(t)I_V\dot{B}_2(t))dx,
\end{aligned}$$

where $|\Omega|$ denotes the volume of Ω, $B=\min\{\mu,\nu\}$. X(t) is the solution to the following stochastic differential equation

$$\begin{cases}
dX(t)=[(\mu_h N_H+A)-BX(t)]dt-\int_\Omega \xi_1(s)S_H(x,s)dxdB_1(s)\\
\qquad -\int_\Omega \xi_1(s)I_H(x,s)dxdB_1(s)-\int_\Omega \xi_1(s)R_H(x,s)dxdB_1(s)\\
\qquad -\int_\Omega \xi_2(s)S_V(x,s)dxdB_2(s)-\int_\Omega \xi_2(s)I_V(x,s)dxdB_2(s),\\
X(0)=N(0).
\end{cases}\tag{3.1}$$

We get the solution to equation (3.1) in the following form

$$X(T)=\frac{\mu_h N_H+A}{B}+(X(0)-\frac{\mu_h N_H+A}{B})e^{-Bt}+F(t),$$

where

$$\begin{aligned}F(t)=&-\int_0^t e^{-B(t-s)}\int_\Omega \xi_1(s)S_H(x,s)dxdB_1(s)\\&-\int_0^t e^{-B(t-s)}\int_\Omega \xi_2(s)I_H(x,s)dxdB_1(s)\\&-\int_0^t e^{-B(t-s)}\int_\Omega \xi_1(s)R_H(x,s)dxdB_1(s)\\&-\int_0^t e^{-B(t-s)}\int_\Omega \xi_2(s)S_V(x,s)dxdB_2(s)\\&-\int_0^t e^{-B(t-s)}\int_\Omega \xi_1(s)I_V(x,s)dxdB_2(s).\end{aligned}$$

F(t) is a continuous local martingale with M(0) =0, a.s. On the basis of stochastic comparison theorem, we obtain $N(t)\le X(t)$, a.s.

Define $X(t)=X(0)+G(t)-U(t)+F(t)$, $G(t)=\frac{\mu_h N_H+A}{B}(1-e^{-Bt})$ and $U(t)=X(0)(1-e^{-Bt})$. Obviously, G(t) and U(t) are continuous adapted increasing processes on $t\ge 0$ with G(0) =U(0) =0. In terms of the nonnegative semimartingale convergence theorem [15], we have $\lim_{t\to\infty}X(t)<\infty$ a.s. Therefore, $\limsup_{t\to\infty}N(t)<\infty$, a.s. The proof is completed. □

Next, we prove that there exists a unique positive solution of system (2.5).

Theorem 3.1. *For any initial data $(S_{H,0},I_{H,0},R_{H,0},S_{V,0},I_{V,0})>0$, there exists a unique positive solution $(S_H(x,t),I_H(x,t),R_H(x,t),S_V(x,t),I_V(x,t))$ of system (6) for $t>0$ on Ω.*

Proof. Since the coefficients of model (2.5) satisfy the local Lipschitz condition, there is a unique local solution on $t\in[0,\tau_e)$, where τ_e is the explosion time. Let $l_0>1$ be sufficiently large for

$$\frac{1}{l_0}\le\min_{0<t<\tau_e}|\mathcal{W}(x,t)|\le\max_{0<t<\tau_e}|\mathcal{W}(x,t)|\le l_0,$$

For each integer $l>l_0$, define the stopping time

$$\begin{aligned}\tau_l=&\inf\{t\in[0,\tau_e]:\min(S_H(x,t),I_H(x,t),R_H(x,t),S_V(x,t),I_V(x,t))\le\frac{1}{l}\\&\text{or }\max(S_H(x,t),I_H(x,t),R_H(x,t),S_V(x,t),I_V(x,t))\ge l\}.\end{aligned}$$

Let $\inf\emptyset=\infty$ ($\emptyset$ represents the empty set). τ_l is increasing as $l\to\infty$. Let $\tau_\infty=\lim_{l\to\infty}\tau_l$, then $\tau_\infty<\tau_e$ a.s. In the following, we need to show $\tau_\infty=\infty$ a.s. Therefore, according to Itô¡¯s formula, we have

$$\begin{aligned}&d(\|S_H(x,t)\|^2+\|I_H(x,t)\|^2+\|R_H(x,t)\|^2+\|S_v(x,t)\|^2+\|I_v(x,t)\|^2)\\=&\{2\langle S_H(x,t),d_1\triangle S_H+\mu_h N_H-\mu S_H-\frac{\beta_H b}{N_H+m}S_H I_V\rangle+2\langle I_H(x,t),d_2\triangle I_H\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\beta_H b}{N_H+m} S_H I_V - \gamma_H I_H - \mu I_H\rangle + 2\langle R_H(x,t), d_3 \triangle R_H + \gamma_H I_H - \mu R_H\rangle \\
&+ 2\langle S_V(x,t), d_4 \triangle S_V + A - \nu S_V - \frac{\beta_V b}{N_H+m} S_V I_H\rangle + 2\langle I_V(x,t), d_5 \triangle I_V \\
&+ \frac{\beta_V b}{N_H+m} S_V I_H - \nu I_V\rangle + \xi_1^2(t)\|S_H(x,t)\|^2 + \xi_1^2(t)\|I_H(x,t)\|^2 + \xi_1^2(t)\|R_H(x,t)\|^2 \\
&+ \xi_2^2(t)\|S_V(x,t)\|^2 + \xi_2^2(t)\|I_V(x,t)\|^2\}dt + 2\langle S_H(x,t), -\xi_1 S_H(x,t) dB_1(t)\rangle \\
&+ 2\langle I_H(x,t), -\xi_1 I_H(x,t) dB_1(t)\rangle + 2\langle R_H(x,t), -\xi_1 R_H(x,t) dB_1(t)\rangle + 2\langle S_V(x,t), \\
&- \xi_2 S_V(x,t) dB_2(t)\rangle + 2\langle I_V(x,t), -\xi_2 I_V(x,t) dB_2(t)\rangle. \qquad (3.2)
\end{aligned}
$$

Now, let $l > l_0$ and $T > 0$; we can integrate both sides of (3.2) from 0 to $\tau_l \wedge T$ and take expectations to get

$$
\begin{aligned}
&E[\|S_H(x,\tau_l \wedge T)\|^2 + \|I_H(x,\tau_l \wedge T)\|^2 + \|R_H(x,\tau_l \wedge T)\|^2 + \|S_V(x,\tau_l \wedge T)\|^2 \\
&+ \|I_V(x,\tau_l \wedge T)\|^2] - (\|S_{H,0}\|^2 + \|I_{H,0}\|^2 + \|R_{H,0}\|^2 + \|S_{V,0}\|^2 + \|I_{V,0}\|^2) \\
\le & E\int_0^{\tau_l \wedge T} \{-2\langle \nabla S_H(x,s), d_1 \nabla S_H(x,s)\rangle + 2\langle \mu_h N_H, S_H(x,s)\rangle - 2\langle \nabla I_H(x,s), \\
&d_2 I_H(x,s)\rangle + 2\langle I_H(x,s), \frac{\beta_H b}{N_H+m} S_H I_V\rangle - 2\langle \nabla R_H(x,s), d_3 \nabla R_H(x,s)\rangle \\
&+ 2\langle R_h(x,s), \gamma_H I_H\rangle - 2\langle \nabla S_V(x,s), d_4 \nabla S_V(x,s)\rangle + 2\langle S_V(x,s), A\rangle \\
&- 2\langle \nabla I_V(x,s), d_5 I_V(x,s)\rangle + 2\langle I_V(x,s), \frac{\beta_V b}{N_H+m} S_V I_H\rangle + \xi_1^2\|S_H(x,s)\|^2 \\
&+ \xi_1^2 I_H(x,s) + \xi_1^2 R_H(x,s) + \xi_2^2 S_V(x,s) + \xi_2^2 I_V(x,s)\} ds \\
\le & E\int_0^{\tau_l \wedge T} \{-2d_1\lambda_0\|S_H(x,s)\|^2 + 2\langle \mu_h N_H, S_H(x,s)\rangle - 2d_2\lambda_0\|I_H(x,s)\|^2 \\
&+ 2\langle I_H(x,s), \frac{\beta_H b}{N_H+m} S_H I_V\rangle - 2d_3\lambda_0\|R_h(x,s)\|^2 + 2\langle R_h(x,s), \gamma_H I_H\rangle \\
&- 2d_4\lambda_0\|S_V(x,s)\|^2 + 2\langle S_V(x,s), A\rangle - 2d_5\lambda_0\|I_V(x,s)\|^2 + 2\langle I_V(x,s), \\
&\frac{\beta_V b}{N_H+m} S_V I_H\rangle + \xi_1^2\|S_H(x,s)\|^2 + \xi_1^2\|I_H(x,s)\|^2 + \xi_1^2\|R_H(x,s)\|^2 \\
&+ \xi_2^2\|S_V(x,s)\|^2 + \xi_2^2\|I_V(x,s)\|^2\} ds,
\end{aligned}
$$

where $\lambda_0 = \inf_{u\in\mathcal{H}} \|\nabla u(x,s)\|^2/\|u(x,s)\|^2$.

Then according to Lemma 3.1 and fundamental inequality, we have

$$
\begin{aligned}
&E[\|S_H(x,\tau_l \wedge T)\|^2 + \|I_H(x,\tau_l \wedge T)\|^2 + \|R_H(x,\tau_l \wedge T)\|^2 + \|S_V(x,\tau_l \wedge T)\|^2 \\
&+ \|I_V(x,\tau_l \wedge T)\|^2] \\
\le & (\|S_{H,0}\|^2 + \|I_{H,0}\|^2 + \|R_{H,0}\|^2 + \|S_{V,0}\|^2 + \|I_{V,0}\|^2) + E\int_0^{\tau_l \wedge T} \{-2d_1\lambda_0 \\
&\|S_H(x,s)\|^2 + \mu_h^2 N_H^2 + \|S_H(x,s)\|^2 - 2d_2\lambda_0\|I_H(x,s)\|^2 + \|I_H(x,s)\|^2 \\
&+ \frac{\beta_H^2 b^2}{(N_H+m)^2} M_1^2\|I_V\|^2 - 2d_3\lambda_0\|R_h(x,s)\|^2 + \|R_h(x,s)\|^2 + \gamma_H^2\|I_H\|^2 - 2d_4 \\
&\lambda_0\|S_V(x,s)\|^2 + A^2 + \|S_V(x,s)\|^2 - 2d_5\lambda_0\|I_V(x,s)\|^2 + \|I_V(x,s)\|^2 + \frac{\beta_V^2 b^2}{(N_H+m)^2}
\end{aligned}
$$

$$M_1^2\|I_H\|^2+\xi_1^2\|S_H(x,s)\|^2+\xi_1^2 I_H(x,s)+\xi_1^2 R_H(x,s)+\xi_2^2 S_V(x,s)+\xi_2^2 I_V(x,s)\}ds$$
$$\le M_2+M_3E\int_0^{\tau_l\wedge T}\{\|S_H(x,s)\|^2+\|I_H(x,s)\|^2+\|R_H(x,s)\|^2+\|S_V(x,s)\|^2$$
$$+\|I_V(x,s)\|^2\},$$

where

$$M_2=\|S_{H,0}\|^2+\|I_{H,0}\|^2+\|R_{H,0}\|^2+\|S_{V,0}\|^2+\|I_{V,0}\|^2+(\mu_h^2N_H^2+A^2)\tau_l,$$

$$M_3=\max\{(1-2d_1\lambda_0+\xi_1^2),(1-2d_2\lambda_0+\gamma_H^2+\xi_1^2+\frac{\beta_V^2b^2M_1^2}{(N_H+m)^2}),(1+\xi_1^2),$$
$$(1-2d_4\lambda_0+\xi_2^2),(1-2d_5\lambda_0+\xi_2^2+\frac{\beta_V^2b^2M_1^2}{(N_H+m)^2})\}.$$

By the Gronwall inequality

$$\begin{aligned}&E[\|S_H(x,\tau_l\wedge T)\|^2+\|I_H(x,\tau_l\wedge T)\|^2+\|R_H(x,\tau_l\wedge T)\|^2\\&+\|S_V(x,\tau_l\wedge T)\|^2+\|I_V(x,\tau_l\wedge T)\|^2]\le M_2e^{M_3T}.\end{aligned}\tag{3.3}$$

By (3.3), $l\to\infty$ means that

$$E[\|S_H(x,T)\|^2+\|I_H(x,T)\|^2+\|R_H(x,T)\|^2+\|S_V(x,T)\|^2+\|I_V(x,T)\|^2]\le M_2e^{M_3T}.$$

Define

$$\begin{aligned}\lambda_l=&\inf_{\|\mathcal{W}(t)\|>l,0<t<\infty}(\|S_H(x,t)\|^2+\|I_H(x,t)\|^2+\|R_H(x,t)\|^2\\&+\|S_V(x,t)\|^2+\|I_V(x,t)\|^2),\ \text{for any } l>l_0.\end{aligned}\tag{3.4}$$

Combine (3.3) and (3.4) to get

$$\lambda_lP(\tau_l\le T)\le M_2e^{M_3T}.$$

Since $\lim_{l\to\infty}\lambda_l=\infty$, in the above inequality, let $l\to\infty$, we can get $P(\tau_\infty\le T)=0$, namely,

$$P(\tau_l\ge T)=1.$$

This proof is completed. The above theorem represents the system (2.5) exists a unique global solution. Next, we prove the boundness of the solution for system (2.5). □

Theorem 3.2. *For any $\kappa>0$, we have*

$$\max\{E\sup_{0\le t\le T}\|S_H(x,t)\|^k,E\sup_{0\le t\le T}\|I_H(x,t)\|^k,E\sup_{0\le t\le T}\|R_H(x,t)\|^k,$$
$$E\sup_{0\le t\le T}\|S_V(x,t)\|^k,E\sup_{0\le t\le T}\|I_V(x,t)\|^k\}\le M_\kappa,$$

where $M_\kappa$is a constant that depends on κ and the parameters in model (2.5). The proof is shown in Appendix A.

Next, we give the sufficient conditions which are the existence and uniqueness of stationary distribution of the solution to the dengue model.

Definition 3.1 ([11]). A stationary distribution for $\mathcal{W}(x,t)$, $t \geq 0$, of Eq. (2.5) is defined as a probability measure $\lambda \in P(\mathcal{H})$ satisfying

$$\lambda(f) = \lambda(P_t f), t > 0,$$

here

$$\lambda(f) := \int_{\mathcal{H}} f(\psi)\lambda(d\psi), P_t f(\psi) := Ef(\mathcal{W}(x,t,\psi)), f \in C_b(\mathcal{H}).$$

For $\lambda_1, \lambda_2 \in P(\mathcal{H})$, define a metric on $P(\mathcal{H})$ by

$$d(\lambda_1, \lambda_2) = \sup_{f \in \mathcal{A}} |\int_{\mathcal{H}} f(\psi)\lambda_1(d\psi) - f(\varphi)\lambda_2(d\varphi)|,$$

where

$$\mathcal{A} := \{f : \mathcal{H} \to R, |f(\psi) - f(\varphi)| \leq |\psi - \varphi|_{\mathcal{H}},\ \psi,\ \varphi \in \mathcal{H} \text{ and } |f(\cdot)| \leq 1\}.$$

$P(\mathcal{H})$ is complete under the metric $d(\cdot,\cdot)$. Therefore, we have the following lemma

Lemma 3.2. *For any bounded subset B of $\mathcal{H}$, $m \geq 1$, we have*

$$(i) \lim_{t\to\infty} \sup_{\psi,\varphi \in B} E\|\mathcal{W}(x,t,\psi) - \mathcal{W}(x,t,\varphi)\|^m_{\mathcal{H}} = 0;$$

$$(ii) \lim_{t\to\infty} \sup_{\psi \in B} E\|\mathcal{W}(x,t,\psi)\|^m_{\mathcal{H}} < \infty.$$

Proof. The Theorem 3.2 is equal to condition (ii) in Lemma 3.2; and the condition (i) in Lemma 3.2 is proved in Theorem 3.3, hence, we are not prove. □

Theorem 3.3. *For system (2.5), if there is a constant $\eta > 0$, $k > 1$, such that $\eta + M_5 > 0$, then, there exists a unique stationary distribution $\lambda \in P(\mathcal{H})$ for $\mathcal{W}(x,t) = (S_H(x,t), I_H(x,t), R_H(x,t), S_V(x,t), I_V(x,t)), t \geq 0$.*

The proof is shown in Appendix B.

Above, we prove the existence and uniqueness of the stationary distribution of the system (2.5). Next, considering the optimal control problem, we derive the first order necessary conditions for the existence of optimal control by applying Pontryagins maximum principle.

4. Optimal control problem

In this section, we show an optimal control problem of system (2.5) by implementing both treatment for infected individuals and the spray insecticide for mosquito. We aim to reduce the infected individuals, while keeping the cost to apply the control at the minimum level. Next, we introduce the control variables.

(1) Reducing the number of dengue infected individuals. To increase awareness and understanding of the disease and reduce the risk of infection by educating people on prevention methods of dengue fever. We use u_1 as the treatment control of decrease dengue infection individuals.

(2) Reducing the number of mosquito. In our daily life, we use insecticides or repellents to kill mosquitoes. We use u_2 as a control variable against the mosquito. Thus, we study the following optimal control problem of system (2.5).

$$\begin{cases} dS_H = [d_1\triangle S_H + \mu_h N_H - \mu S_H - \dfrac{\beta_H b}{N_H+m} S_H I_V]dt - \xi_1(t) S_H dB_1(t), \\ dI_H = [d_2\triangle I_H + \dfrac{\beta_H b}{N_H+m} S_H I_V - \delta_1 u_1 I_H - \gamma_H I_H - \mu I_H]dt - \xi_1(t) I_H dB_1(t), \\ dR_H = [d_3\triangle R_H + \delta_1 u_1 I_H + \gamma_H I_H - \mu R_H]dt - \xi_1(t) R_H dB_1(t), \\ dS_V = [d_4\triangle S_V + A - \delta_2 u_2 S_V - \nu S_V - \dfrac{\beta_V b}{N_H+m} S_V I_H]dt - \xi_2(t) S_V dB_2(t), \\ dI_V = [d_5\triangle I_V + \dfrac{\beta_V b}{N_H+m} S_V I_H - \delta_2 u_2 I_V - \nu I_V]dt - \xi_2(t) I_V dB_2(t), \end{cases} \tag{4.1}$$

where δ_1 and δ_2 represent the effectiveness of control variable u_1 and u_2, representably. Next, the objective function is as follows:

$$J(u_1,u_2) = E\{\int_0^T \int_\Omega L(I_H(x,t),u_1(t),u_2(t))dxdt + \int_\Omega h(I_H(x,T))dx\}, \tag{4.2}$$

here, $L(I_H(x,t),u_1,u_2) = B_1 I_H(x,t) + \frac{1}{2}B_2 u_1^2 + \frac{1}{2}B_3 u_2^2$, $h(I_H(x,T))$ is a function of infective individual at terminal time T. Note that we only consider minimizing the number of infected individuals. $B_i, (i=1,2,3)$ are weight for I_H, u_1 and u_2. Our goal is to point out an optimal control pair (u_1^*, u_2^*) such that

$$J(u_1^*,u_2^*) = \min_{u_1,u_2\in U} J(u_1,u_2),$$

where the control set U is considered as

$$\begin{aligned} U = &\{u_1(x,t), u_2(x,t) \text{ are Lebesgue measurable}, \\ &0 \le u_1(x,t), u_2(x,t) \le 1, t\in[0,T]\}. \end{aligned} \tag{4.3}$$

Next using the results in [4, 28], the existence of the optimal control pair is given.

Theorem 4.1. *There exists an optimal control pair (u_1^*, u_2^*), for $J(u_1^*,u_2^*) = \min_{u_1,u_2\in U} J(u_1,u_2)$.*

Proof. The objective functional (4.2) is convex respect to the control variables u_1, u_2 as the control and state variables are all nonnegative. The convexity and closedness of the admissible control set U can also be indicated by the definition in (4.3). Therefore, the optimal control is bounded and the necessary conditions for the existence of the optimal control u_1^*, u_2^* are satisfied. The theorem is proved. □

Further, we figure out the optimal control, by constructing the Hamiltonian function [28] and using Pontryagin's maximum principle [18].

Theorem 4.2. *There exist a first-order adjoint process $(p(x,t), q(x,t))$, satisfying* *the following adjoint equation:*

$$
\begin{cases}
dp_1(x,t) = &- \{[d_1\triangle - \mu - \dfrac{\beta_H b}{N_H+m} I_V]p_1(x,t) + \dfrac{\beta_H b}{N_H+m} I_V p_2(x,t) \\
& - \xi_1(t) q_1(x,t)\}dt + q_1(x,t)dB_1(t), \\
dp_2(x,t) = &- \{[d_2\triangle - \delta_1 u_1 - \gamma_H - \mu]p_2(x,t) - \dfrac{\beta_V b}{N_H+m} S_V p_4(t)(x,t) \\
& + [\delta_1 u_1 + \gamma_H]p_3(x,t) + \dfrac{\beta_V b}{N_H+m} S_V p_5(x,t) - \xi_1(t) q_2(x,t) \\
& + B_1\}dt + q_2(x,t)dB_1(t), \\
dp_3(x,t) = &- \{[d_3\triangle - \mu - (\mu_0]p_3(x,t) - \xi_1(t)q_3(x,t)\}dt + q_3(x,t)dB_1(t), \\
dp_4(x,t) = &- \{[d_4\triangle - \delta_2 u_2 - \nu - \dfrac{\beta_V b}{N_H+m} I_H]p_4(x,t) \dfrac{\beta_V b}{N_H+m} I_H p_5(t) \\
& - \xi_2(t) q_4(x,t)\}dt + q_4(x,t)dB_1(t), \\
dp_5(x,t) = &- \{- \dfrac{\beta_H b}{N_H+m} S_H p_1(x,t) + \dfrac{\beta_H b}{N_H+m} S_H p_2(x,t) - \nu p_5(x,t) \\
& (d_5\triangle - \delta_2 u_2)p_5(x,t) - \xi_2(t) q_5(x,t)\}dt + q_5(x,t)dB_1(t), \\
p_1(x,T) = & h_{S_H}(S_H(x,T)), p_2(x,T) = h_{I_H}(I_H(x,T)), p_3(x,T) = h_{R_H}(R_H(x,T)), \\
p_4(x,T) = & h_{S_V}(S_V(x,T)), p_5(x,T) = h_{I_V}(I_V(x,T)).
\end{cases}
\tag{4.4}
$$

Furthermore, the optimal control is given as follows:

$$
\begin{cases}
u_1^* = \min\{\max\{\frac{(p_2-p_3)\delta_1 I_H}{B_2}, 0\}, 1\}, \\
u_2^* = \min\{\max\{\frac{\delta_2 S_V p_4 + \delta_2 I_V p_5}{B_3}, 0\}, 1\}.
\end{cases}
\tag{4.5}
$$

Proof. We define H as the Hamiltonian, and we can get equation (4.4) by direct calculating. Moreover, The u_1^* and u_2^* are obtained by using the optimality conditions $\frac{\partial H}{\partial u_1} = 0$ and $\frac{\partial H}{\partial u_2} = 0$, respectively. Hence, we have $u_1 = \frac{(p_2-p_3)\delta_1 I_H}{B_2}$, $u_2 = \frac{\delta_2 S_V p_4 + \delta_2 I_V p_5}{B_3}$.

According to control set (4.5), we obtain

$$
u_1^* = \begin{cases}
0, \text{ if } \dfrac{(p_2-p_3)\delta_1 I_H}{B_2} < 0, \\
\dfrac{(p_2-p_3)\delta_1 I_H}{B_2}, \text{ if } 0 \le \dfrac{(p_2-p_3)\delta_1 I_H}{B_2} \le 1, \\
1, \text{ if } \dfrac{(p_2-p_3)\delta_1 I_H}{B_2} > 1,
\end{cases}
$$

$$
u_2^* = \begin{cases}
0, \text{ if } \dfrac{\delta_2 S_V p_4 + \delta_2 I_V p_5}{B_3} < 0, \\
\dfrac{\delta_2 S_V p_4 + \delta_2 I_V p_5}{B_3}, \text{ if } 0 \le \dfrac{\delta_2 S_V p_4 + \delta_2 I_V p_5}{B_3} \le 1, \\
1, \text{ if } \dfrac{\delta_2 S_V p_4 + \delta_2 I_V p_5}{B_3} > 1.
\end{cases}
$$

Hence the optimal value of the functional can be obtained. □

5. Numerical simulations

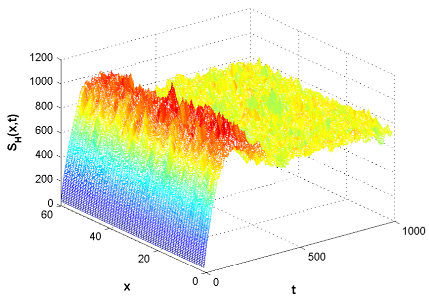
5.1. Numerical simulations of stationary distribution

In order to better understand our results, we present the numerical simulation in this section. Based on the Milstein's method [5], the discrete form of system (2.5) is given as follows and give the algorithm process.

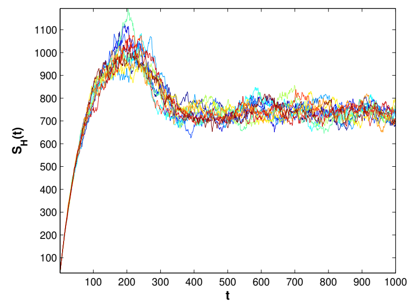
$$\begin{aligned}
S_{H(i,j+1)} =& S_{H(i,j)} + [d_1 \frac{S_{H(i+1,j)} - 2S_{H(i,j)} + S_{H(i-1,j)}}{(\triangle x)^2} + \mu_h N_H - \mu S_{H(i,j)} \\
& - \frac{\beta_H b}{N_H + m} S_{H(i,j)} I_{V(i,j)}]\triangle t - \xi_1 S_{H(i,j)} \varsigma_j - \frac{\xi_1^2}{2} S_{H(i,j)}^2 (\varsigma_j^2 - 1)\triangle t, \\
I_{H(i,j+1)} =& I_{H(i,j)} + [d_2 \frac{I_{H(i+1,j)} - 2I_{H(i,j)} + I_{H(i-1,j)}}{(\triangle x)^2} + \frac{\beta_H b}{N_H + m} S_{H(i,j) I_{V(i,j)}} \\
& - \gamma_H I_{H(i,j)} - \mu I_{H(i,j)}]\triangle t - \xi_1 I_{H(i,j)} \varsigma_j - \frac{\xi_1^2}{2} I_{H(i,j)}^2 (\varsigma_j^2 - 1)\triangle t, \\
R_{H(i,j+1)} =& R_{H(i,j)} + [d_3 \frac{R_{H(i+1,j)} - 2R_{H(i,j)} + R_{H(i-1,j)}}{(\triangle x)^2} + \gamma_H I_{H(i,j)} \\
& - \mu R_{H(i,j)}]\triangle t - \xi_1 R_{H(i,j)} \varsigma_j - \frac{\xi_1^2}{2} R_{H(i,j)}^2 (\varsigma_j^2 - 1)\triangle t, \\
S_{V(i,j+1)} =& S_{V(i,j)} + [d_4 \frac{S_{V(i+1,j)} - 2S_{V(i,j)} + S_{V(i-1,j)}}{(\triangle x)^2} + A - \nu S_{V(i,j)} \\
& - \frac{\beta_V b}{N_H + m} S_{V(i,j)} I_{H(i,j)}]\triangle t - \xi_2 S_{V(i,j)} \varsigma_j - \frac{\xi_2^2}{2} S_{V(i,j)}^2 (\varsigma_j^2 - 1)\triangle t, \\
I_{V(i,j+1)} =& I_{V(i,j)} + [d_5 \frac{I_{V(i+1,j)} - 2I_{V(i,j)} + I_{V(i-1,j)}}{(\triangle x)^2} + \frac{\beta_V b}{N_H + m} S_{V(i,j)} I_{H(i,j)} \\
& - \nu I_{V(i,j)}]\triangle t - \xi_2 I_{V(i,j)} \varsigma_j - \frac{\xi_2^2}{2} I_{V(i,j)}^2 (\varsigma_j^2 - 1)\triangle t,
\end{aligned}$$

where $\varsigma_j, (j = 1, 2, 3)$ are independent Gaussian random variables $N(0, 1)$. The parameter values are chosen as follows:
$\beta_H = 0.75$ [16,17], $b = 0.5, N_H + m = 55, \mu_h N_H = 14, \mu = 0.125$, $\beta_V = 0.5$ [16,17], $\gamma_H = 1.4$ [9], $A = 5, \nu = 0.8$ [9], $\xi_1 = 0.01, \xi_2 = 0.01$, $d_1 = 0.015, d_2 = 0.020, d_3 = 0.015, d_4 = d_5 = 0.01$.

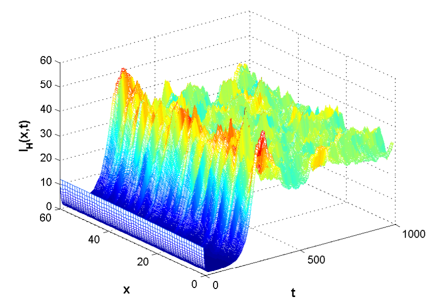
Next, we verify the existence of the stationary distribution from the perspective of numerical simulation. In figure 1, the left column shows the paths of $S_H(x,t), I_H(x,t), R_H(x,t), S_V(x,t)$ *and* $I_V(x,t)$, respectively. We can see that the existence of stationary distribution of the solution for system (2.5). The right column displays the profile map of $S_H(x,t), I_H(x,t), R_H(x,t), S_V(x,t)$ *and* $I_V(x,t)$. The curves with different colors represent the changes in time in different spaces, and we can see that there are certain differences in the solutions of the stationary distribution in different spaces. In figure 2, we give the histograms of $S_H(x,t), I_H(x,t)$, $R_H(x,t), S_V(x,t)$ *and* $I_V(x,t)$, respectively, we can know that there is a stationary distribution. In figure 3, we give the effect of different noise intensity for the stationary distribution of system (2.5) depended on time; when $\varepsilon = 0$, we can see amplitude of fluctuation is slight, but with the increase of ε, the amplitude of fluctuation becomes larger and larger. Hence, we can conclude that the distribution tends to be stable for lower intensity of volatility.


S_H(x,t)
0 200 400 600 800 1000 1200
x
0 20 40 60
t
0 500 1000

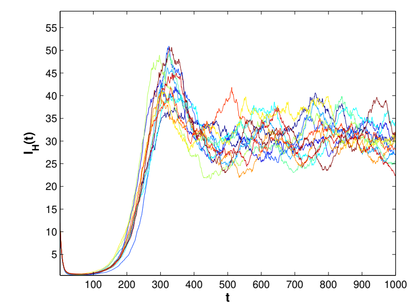
(a)


S_H(t)
100 200 300 400 500 600 700 800 900 1000 1100
t
100 200 300 400 500 600 700 800 900 1000

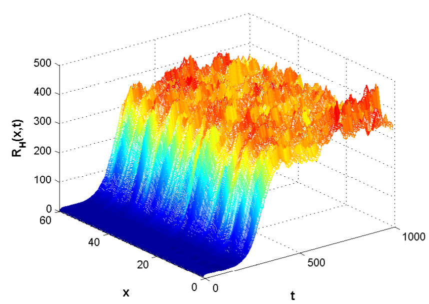
(b)


I_H(x,t)
0 10 20 30 40 50 60
x
0 20 40 60
t
0 500 1000

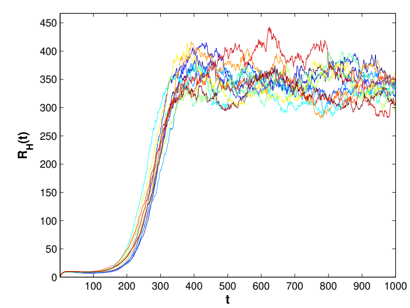
(c)


I_H(t)
5 10 15 20 25 30 35 40 45 50 55
t
100 200 300 400 500 600 700 800 900 1000

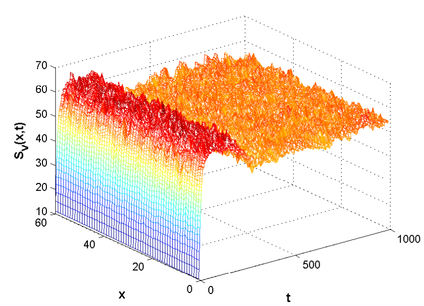
(d)


R_H(x,t)
0 100 200 300 400 500
x
0 20 40 60
t
0 500 1000

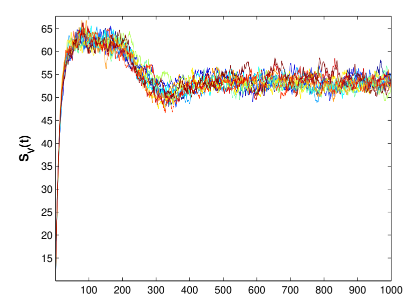
(e)


R_H(t)
0 50 100 150 200 250 300 350 400 450
t
100 200 300 400 500 600 700 800 900 1000

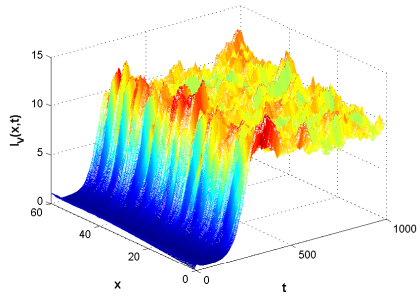
(f)


S_V(x,t)
10 20 30 40 50 60 70
x
0 20 40 60
t
0 500 1000

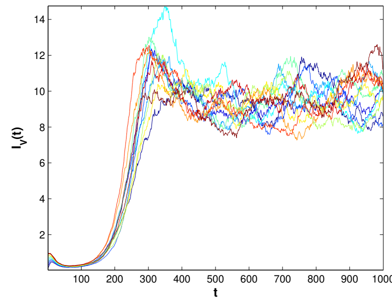
(g)


S_V(t)
15 20 25 30 35 40 45 50 55 60 65
t
100 200 300 400 500 600 700 800 900 1000

(h)

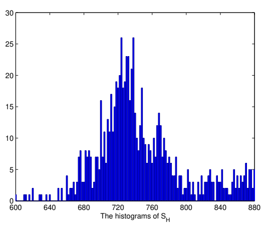


(i)

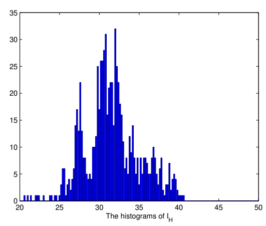


(j)

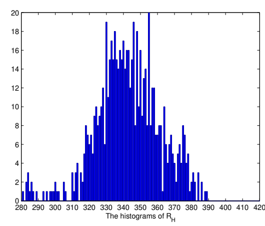
Figure 1. The solutions are observed in system (2.5) under initial conditions $(S_{H,0}(x), I_{H,0}(x), R_{H,0}(x), S_{V,0}(x), I_{V,0}(x)) = (33 + \sin\frac{\pi x}{60}, 11 + \sin\frac{\pi x}{60}, 0, 10 + \sin\frac{\pi x}{60}, \sin\frac{\pi x}{60})$



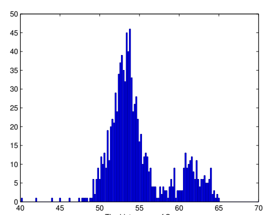
(a)



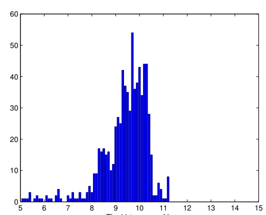
(b)



(c)

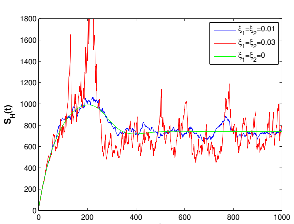


(d)

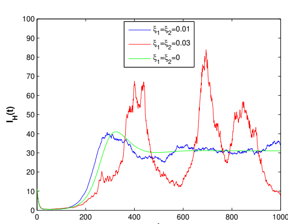


(e)

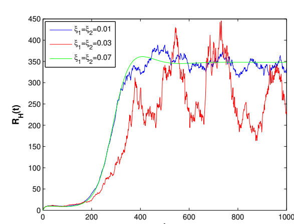
Figure 2. The histograms of S_H, I_H, R_H, S_V, I_V



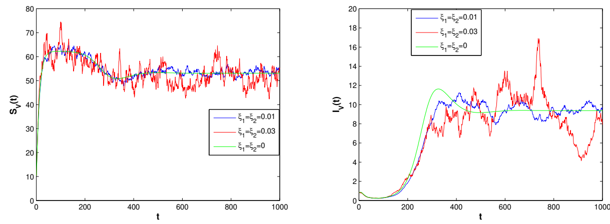
(a)



(b)



(c)

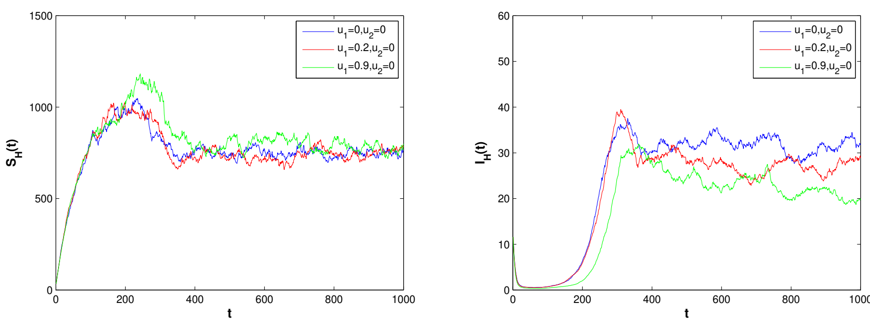


(d) (e)

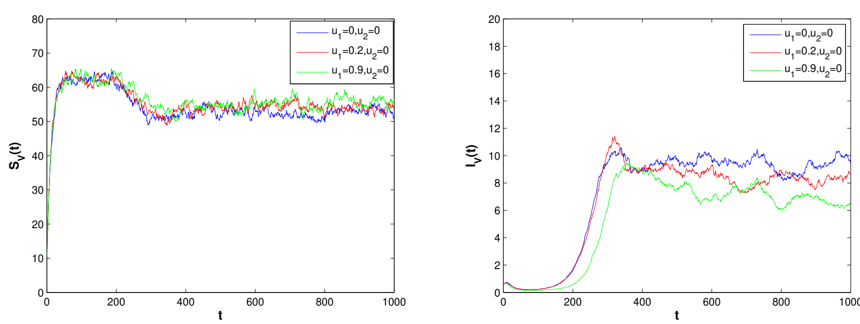
Figure 3. The evolution of a single path of S_H, I_H, R_H, S_V, I_V for different noise intensity

5.2. Numerical simulations of the optimal control

In this section, we compare the effects of different control intensity on dengue transmission, and by using the Milsteins method [5] to discrete the system (4.1). In Figures 4 and 5, we can see that infected individuals and mosquito populations decrease when a control strategy is introduced; however, we can observe that the number of individuals infected by humans and mosquitoes gradually decreases when the two control strategies is introduced, and as the control is increased, the number of individuals infected by humans and mosquitoes has greatly decreased from Figure 6.

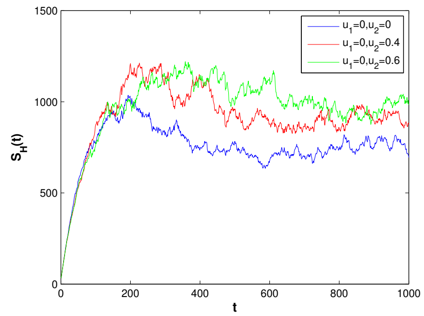


(a) (b)

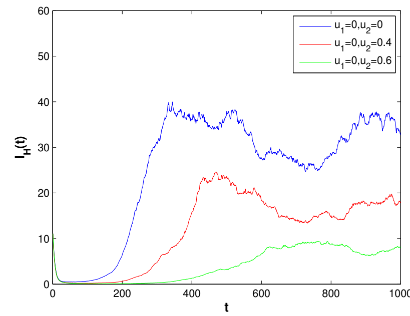


(c) (d)

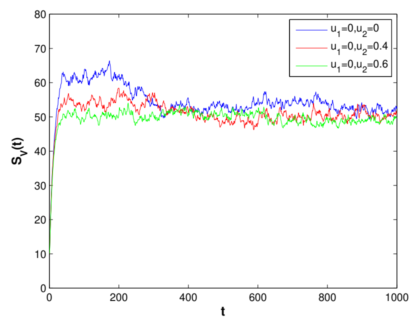
Figure 4. The effects of treating infected individuals



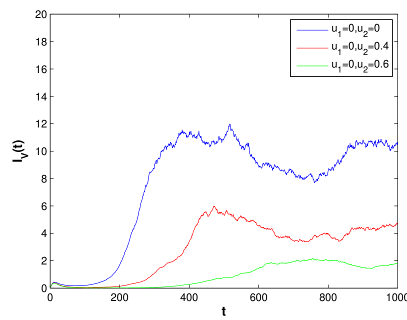
(a)



(b)

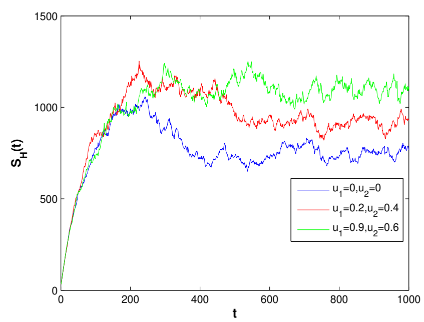


(c)

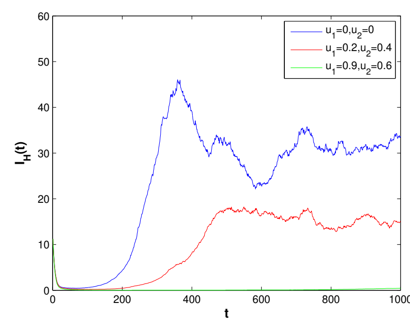


(d)

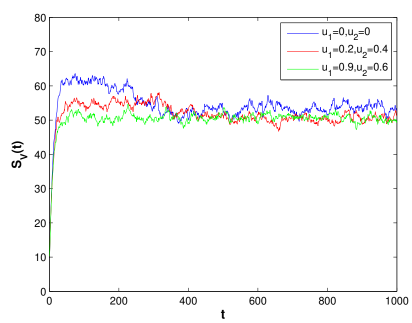
Figure 5. The effects of mosquito control



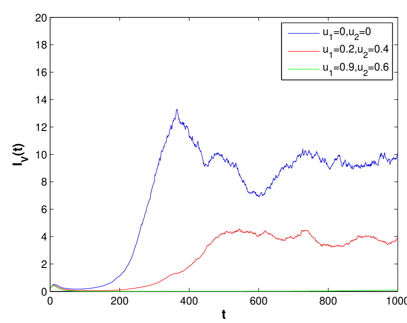
(a)



(b)

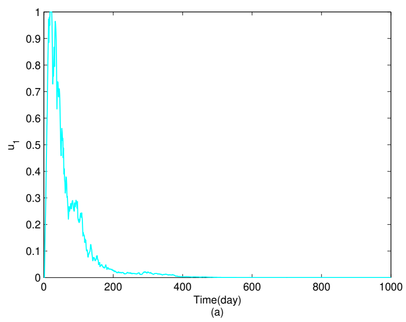


(c)

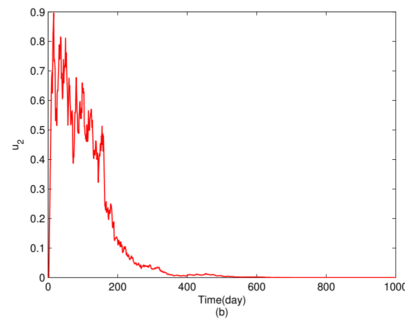


(d)

Figure 6. The effects of both control measures



(a)



(b)

Figure 7. The graph for the two controls

6. Concluding remarks

Here, we proposed and analyzed a stochastic dengue model combine with spatial diffusion. In the first part, we analyzed the stationary distribution of system (2.5). The positive solution of stationary distribution is given by Theorem 3.1, the sufficient conditions are obtained for the existence and uniqueness of stationary distribution of the solution by virtue of Theorem 3.3. Subsequently, we introduced the control strategy, namely, decrease the number of infected individual and spray mosquito insecticides. The necessary conditions for the effective control of system (4.1) are derived by applying Pontryagin's maximum principle. We have considered the effects of different control strategy, when the two control strategies is introduced, the number of infected individuals decreases sharply. Finally, it is still a problem worth further discussion, because the system may be disturbed by other random factors, such as impulsive perturbations, Markov switching, *Lévy* jumps, etc. We will explore these issues in our future work.

Acknowledgments. The research is supported in part by the Ningxia Natural Science Foundation (No. 2020AAC03067) and National Natural Science Foundation of China (No. 12161068).

Conflict of Interest. The authors declare that they have no conflict of interest.

APPENDIX A: THE PROOF OF THEOREM 3.2

Proof. First, we consider $\kappa > 1$. By applying the $It\hat{o}'s$ formula and taking exception, we have

$$
\begin{aligned}
&E\sup_{0\le t\le T}\|S_H(x,s)\|^k+\|I_H(x,s)\|^k+\|R_H(x,s)\|^k+\|S_V(x,s)\|^k+\|I_V(x,s)\|^k]\\
&-E\sup_{0\le t\le T}(\|S_{H,0}\|^k+\|I_{H,0}\|^k+\|R_{H,0}\|^k+\|S_{V,0}\|^k+\|I_{V,0}\|^k)\\
=&E\sup_{0\le t\le T}\int_0^t\{\kappa\|S_H(x,s)\|^{\kappa-2}\langle S_H(x,s),d_1\triangle S_H+\mu_h N_H-\mu S_H-\frac{\beta_H b}{N_H+m}S_H I_V\rangle\\
&+\kappa\|I_H(x,s)\|^{\kappa-2}\langle I_H(x,s),d_2\triangle I_H+\frac{\beta_H b}{N_H+m}S_H I_V-\gamma_H I_H-\mu I_H\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\kappa\|R_H(x,s)\|^{\kappa-2}\langle R_H(x,s), d_3\triangle R_H+\gamma_H I_H-\mu R_H\rangle+\kappa\|S_V(x,s)\|^{\kappa-2}\langle S_V(x,s),\\
&d_4\triangle S_V+A-\nu S_V-\frac{\beta_V b}{N_H+m}S_V I_H\rangle+\kappa\|I_V(x,s)\|^{\kappa-2}\langle I_V(x,s), d_5\triangle I_V\\
&+\frac{\beta_V b}{N_H+m}S_V I_H-\nu I_V\rangle+\frac{1}{2}k(k-1)\xi_1^2(s)\|S_H(x,s)\|^k+\frac{1}{2}k(k-1)\xi_1^2(s)\\
&\|I_H(x,s)\|^k+\frac{1}{2}k(k-1)\xi_1^2(s)\|R_H(x,s)\|^k+\frac{1}{2}k(k-1)\xi_2^2(s)\|S_V(x,s)\|^k\\
&+\frac{1}{2}k(k-1)\xi_2^2(s)\|I_V(x,s)\|^k\}ds-E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|S_H(x,s)\|^k dB_1(s)\\
&-E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|I_H(x,s)\|^k dB_1(s)-E\sup_{0\le t\le T}\int_0^t \xi_1(s)\|R_H(x,s)\|^k dB_1(s)\\
&-E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|S_V(x,s)\|^k dB_2(s)-E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|I_V(x,s)\|^k dB_2(s).
\end{aligned}
$$

Let's just zoom in and out for the above equality, and we get

$$
\begin{aligned}
&E\sup_{0\le t\le T}\|S_H(x,s)\|^k+\|I_H(x,s)\|^k+\|R_H(x,s)\|^k+\|S_V(x,s)\|^k+\|I_V(x,s)\|^k]\\
&-E\sup_{0\le t\le T}(\|S_{H,0}\|^k+\|I_{H,0}\|^k+\|R_{H,0}\|^k+\|S_{V,0}\|^k+\|I_{V,0}\|^k)\\
\le &E\int_0^t\{-\kappa\|S_H(x,s)\|^{\kappa-2}\langle\nabla S_H(x,s), d_1\nabla S_H(x,s)\rangle+\kappa\|S_H(x,s)\|^{\kappa-2}\\
&\langle\mu_h N_H, S_H(x,s)\rangle-\kappa\|I_H(x,s)\|^{\kappa-2}\langle\nabla I_H(x,s), d_2 I_H(x,s)\rangle+\kappa\|I_H(x,s)\|^{\kappa-2}\\
&\langle I_H(x,s), \frac{\beta_H b}{N_H+m}S_H I_V\rangle-\kappa\|R_H(x,s)\|^{\kappa-2}\langle\nabla R_H(x,s), d_3\nabla R_H(x,s)\rangle\\
&+\kappa\|R_H(x,s)\|^{\kappa-2}\langle R_h(x,s),\gamma_H I_H\rangle-\kappa\|S_V(x,s)\|^{\kappa-2}\langle\nabla S_V(x,s), d_4\nabla S_V(x,s)\rangle\\
&+\kappa\|S_V(x,s)\|^{\kappa-2}\langle S_V(x,s), A\rangle-\kappa\|I_V(x,s)\|^{\kappa-2}\langle\nabla I_V(x,s), d_5 I_V(x,s)\rangle\\
&+\kappa\|I_V(x,s)\|^{\kappa-2}\langle I_V(x,s),\frac{\beta_V b}{N_H+m}S_V I_H\rangle+\frac{1}{2}k(k-1)\xi_1^2\|S_H(x,s)\|^k\\
&+\frac{1}{2}k(k-1)\xi_1^2\|I_H(x,s)\|^k+\frac{1}{2}k(k-1)\xi_1^2\|R_H(x,s)\|^k+\frac{1}{2}k(k-1)\xi_2^2\|S_V(x,s)\|^k\\
&+\frac{1}{2}k(k-1)\xi_2^2\|I_V(x,s)\|^k\}ds-E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|S_H(x,s)\|^k dB_1(s)\\
&-E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|I_H(x,s)\|^k dB_1(s)-E\sup_{0\le t\le T}\int_0^t \xi_1(s)\times\|R_H(x,s)\|^k dB_1(s)\\
&-E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|S_V(x,s)\|^k dB_2(s)-E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|I_V(x,s)\|^k dB_2(s)\\
\le &E\int_0^t\{-kd_1\lambda_0\|S_H(x,s)\|^k+k\mu_h N_H\|S_H(x,s)\|^{k-1}-kd_2\lambda_0\|I_H(x,s)\|^k\\
&+k\frac{\beta_H b}{N_H+m}M_1\|I_H(x,s)\|^{k-1}\|I_V(x,s)\|-kd_3\lambda_0\|R_h(x,s)\|^k\\
&+k\gamma_H\|R_h(x,s)\|^{k-1}\|I_H\|-kd_4\lambda_0\|S_V(x,s)\|^k+kA\|S_V(x,s)\|^{k-1}\\
&-kd_5\lambda_0\|I_V(x,s)\|^k+k\frac{\beta_V b}{N_H+m}M_1\|I_V\|^{k-1}\|I_H\|+\frac{1}{2}k(k-1)\xi_1^2\|S_H(x,s)\|^k
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}k(k-1)\xi_1^2\|I_H(x,s)\|^k+\frac{1}{2}k(k-1)\xi_1^2\|R_H(x,s)\|^k+\frac{1}{2}k(k-1)\xi_2^2\|S_V(x,s)\|^k\\
&+\frac{1}{2}k(k-1)\xi_2^2\|I_V(x,s)\|^k\}ds-E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|S_H(x,s)\|^k dB_1(s)\\
&-E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|I_H(x,s)\|^k dB_1(s)-E\sup_{0\le t\le T}\int_0^t \xi_1(s)\|R_H(x,s)\|^k dB_1(s)\\
&-E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|S_V(x,s)\|^k dB_2(s)-E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|I_V(x,s)\|^k dB_2(s).
\end{aligned}
$$

Using the Young inequality and Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
&E\sup_{0\le t\le T}\|S_H(x,s)\|^k+\|I_H(x,s)\|^k+\|R_H(x,s)\|^k+\|S_V(x,s)\|^k+\|I_V(x,s)\|^k]\\
\le&E\sup_{0\le t\le T}(\|S_{H,0}\|^k+\|I_{H,0}\|^k+\|R_{H,0}\|^k+\|S_{V,0}\|^k+\|I_{V,0}\|^k)+E\sup_{0\le t\le T}\int_0^t\{\mu_h^kN_H^k\\
&+A^k+(-kd_1\lambda_0+k-1+\frac{1}{2}k(k-1)\xi_1^2)\|S_H(x,s)\|^k+(-kd_2\lambda_0+k-1+\gamma_H^k\\
&+\frac{\beta_V^kb^kM_1^k}{(N_H+m)^k}+\frac{1}{2}k(k-1)\xi_1^2)\|I_H(x,s)\|^k+(-kd_3\lambda_0+k-1+\frac{1}{2}k(k-1)\xi_1^2)\\
&\|R_H(x,s)\|^k+(-kd_4\lambda_0+k-1+\frac{1}{2}k(k-1)\xi_2^2)\|S_V(x,s)\|^k+(-kd_5\lambda_0+k-1\\
&+\frac{\beta_H^kb^kM_1^k}{(N_H+m)^k}+\frac{1}{2}k(k-1)\xi_2^2)\|I_V(x,s)\|^k\}ds+E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|S_H(x,s)\|^k dB_1(s)\\
&+E\sup_{0\le t\le T}\int_0^t k\xi_1(s)\|I_H(x,s)\|^k dB_1(s)+E\sup_{0\le t\le T}\int_0^t \xi_1(s)\|R_H(x,s)\|^k dB_1(s)\\
&+E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|S_V(x,s)\|^k dB_2(s)+E\sup_{0\le t\le T}\int_0^t k\xi_2(s)\|I_V(x,s)\|^k dB_2(s)\\
\le&E\sup_{0\le t\le T}(\|S_{H,0}\|^k+\|I_{H,0}\|^k+\|R_{H,0}\|^k+\|S_{V,0}\|^k+\|I_{V,0}\|^k)+(\mu_h^kN_H^k+A^k)T\\
&+M_4E\sup_{0\le t\le T}\int_0^t\{\|S_H(x,s)\|^\kappa+\|I_H(x,s)\|^\kappa+\|R_H(x,s)\|^\kappa+\|S_V(x,s)\|^\kappa\\
&+\|I_V(x,s)\|^\kappa\}ds+E\sup_{0\le t\le T}\|S_H(x,s)\|^{\kappa/2}(\int_0^t\kappa^2\xi_1^2(s)\|S_H(x,s)\|^k ds)^{1/2}\\
&+E\sup_{0\le t\le T}\|I_H(x,s)\|^{\kappa/2}(\int_0^t\kappa^2\xi_1^2(s)\times\|I_H(x,s)\|^k ds)^{1/2}+E\sup_{0\le t\le T}\|R_H(x,s)\|^{\kappa/2}\\
&(\int_0^t\kappa^2\xi_1^2(s)\|R_H(x,s)\|^k ds)^{1/2}+E\sup_{0\le t\le T}\|S_V(x,s)\|^{\kappa/2}(\int_0^t\kappa^2\xi_2^2(s)\|S_V(x,s)\|^k ds)^{1/2}\\
&+E\sup_{0\le t\le T}\|I_V(x,s)\|^{\kappa/2}(\int_0^t\kappa^2\xi_2^2(s)\|I_V(x,s)\|^k ds)^{1/2}+2E\int_0^t k^2(\xi_1^2\|S_H(x,s)\|^k\\
&+\xi_1^2\|I_H(x,s)\|^k+\xi_1^2\|R_H(x,s)\|^k+\xi_2^2\|S_V(x,s)\|^k+\xi_2^2\|I_V(x,s)\|^k)ds\\
\le&E\sup_{0\le t\le T}(\|S_{H,0}\|^k+\|I_{H,0}\|^k+\|R_{H,0}\|^k+\|S_{V,0}\|^k+\|I_{V,0}\|^k)+(\mu_h^kN_H^k+A^k)T\\
&+M_4E\sup_{0\le t\le T}\int_0^t\{\|S_H(x,s)\|^\kappa+\|I_H(x,s)\|^\kappa+\|R_H(x,s)\|^\kappa+\|S_V(x,s)\|^\kappa
\end{aligned}
$$

$$
\begin{aligned}
&+\|I_V(x,s)\|^{\kappa}\}ds+\frac{1}{2}E\sup_{0\le t\le T}(\|S_H(x,s)\|^{\kappa}+\|I_H(x,s)\|^{\kappa}+\|R_H(x,s)\|^{\kappa}\\
&+\|S_V(x,s)\|^{\kappa}+\|I_V(x,s)\|^{\kappa})\\
\le&2E\sup_{0\le t\le T}(\|S_{H,0}\|^{k}+\|I_{H,0}\|^{k}+\|R_{H,0}\|^{k}+\|S_{V,0}\|^{k}+\|I_{V,0}\|^{k})+2(\mu_h^kN_H^k+A^k)T\\
&+M_4E\sup_{0\le t\le T}\int_0^t\{\|S_H(x,s)\|^{\kappa}+\|I_H(x,s)\|^{\kappa}+\|R_H(x,s)\|^{\kappa}\\
&+\|S_v(x,s)\|^{\kappa}+\|I_V(x,s)\|^{\kappa}\}ds,
\end{aligned}
$$

where

$$
\begin{aligned}
M_4=&\max\{-kd_1\lambda_0+k-1+\frac{1}{2}k(k-1)\xi_1^2,-kd_2\lambda_0+k-1+\gamma_H^k+\frac{\beta_V^kb^kM_1^k}{(N_H+m)^k}\\
&+\frac{1}{2}k(k-1)\xi_1^2,-kd_3\lambda_0+k-1+\frac{1}{2}k(k-1)\xi_1^2,-kd_4\lambda_0+k-1+\frac{1}{2}k(k-1)\xi_2^2,\\
&-kd_5\lambda_0+k-1+\frac{\beta_H^kb^kM_1^k}{(N_H+m)^k}+\frac{1}{2}k(k-1)\xi_2^2\}.
\end{aligned}
$$

Based on the Gronwall inequality, we obtained

$$
\begin{aligned}
&E\sup_{0\le t\le T}\{\|S_H(x,s)\|^{\kappa}+\|I_H(x,s)\|^{\kappa}+\|R_H(x,s)\|^{\kappa}+\|S_H(x,s)\|^{\kappa}+\|I_H(x,s)\|^{\kappa}\}\\
\le&(2E(\|S_{H,0}\|^{\kappa}+\|I_{H,0}\|^{\kappa}+\|R_{H,0}\|^{\kappa}+\|S_{v,0}\|^{\kappa}+\|I_{V,0}\|^{\kappa})\\
&+2(\mu_h^kN_H^k+A^k)T)e^{2M_4T}:=M_{\kappa}.
\end{aligned}
$$

For $0<\kappa<1$, based on the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
&E\sup_{0\le t\le T}\{\|S_H(x,s)\|^{\kappa}+\|I_H(x,s)\|^{\kappa}+\|R_H(x,s)\|^{\kappa}+\|S_V(x,s)\|^{\kappa}+\|I_V(x,s)\|^{\kappa}\}\\
\le&(E1^{\frac{2}{2-\kappa}})^{1-\kappa/2}\{E(\sup_{0\le t\le T}\{\|S_H(x,s)\|^{\kappa}+\|I_H(x,s)\|^{\kappa}+\|R_H(x,s)\|^{\kappa}+\|S_V(x,s)\|^{\kappa}\\
&+\|I_V(x,s)\|^{\kappa}\}^{\frac{\kappa}{2}}:=M_{\kappa}.
\end{aligned}
$$

This proof is completed. □

APPENDIX B: THE PROOF OF THEOREM 3.3

Proof. The Theorem 3.2 is equal to condition (ii) in Lemma 3.2. In order to complete proof, we only need to verify that condition (i) is valid. Next, we consider the difference of two mild solutions of (2.5) with distinct initial data $\psi,\varphi\in\Omega$

$$
e(x,t)=\begin{pmatrix}e_1(x,t,\psi,\varphi)\\e_2(x,t,\psi,\varphi)\\e_3(x,t,\psi,\varphi)\\e_4(x,t,\psi,\varphi)\\e_5(x,t,\psi,\varphi)\end{pmatrix}=\begin{pmatrix}S_H(x,t,\psi)-S_H(x,t,\varphi)\\I_H(x,t,\psi)-I_H(x,t,\varphi)\\R_H(x,t,\psi)-R_H(x,t,\varphi)\\S_V(x,t,\psi)-S_V(x,t,\varphi)\\I_V(x,t,\psi)-I_V(x,t,\varphi)\end{pmatrix},\tag{6.1}
$$

with $\|e(x,t,\psi,\varphi)\|^\kappa = \|e_1(x,t,\psi,\varphi)\|^\kappa + \|e_2(x,t,\psi,\varphi)\|^\kappa + \|e_3(x,t,\psi,\varphi)\|^\kappa + \|e_4(x,t,\psi,\varphi)\|^\kappa + \|e_5(x,t,\psi,\varphi)\|^\kappa$, By Lemma 3.1 and $It\hat{o}'s$ formula, we have

$$\begin{aligned}
&d(e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa)\\
=&\eta e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa dt + e^{\eta t}\{\kappa\|e_1(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_1(x,t,\psi,\varphi), d_1\triangle e_1(x,t,\psi,\varphi)\\
&-\mu e_1(x,t,\psi.\varphi) - \frac{\beta_H b}{N_H+m}(S_H(x,t,\psi)I_V(x,t,\psi) - S_H(x,t,\varphi)I_V(x,t,\varphi))\rangle dt\\
&+\kappa\|e_2(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_2(x,t,\psi,\varphi), d_2\triangle e_2(x,t,\psi,\varphi) + \frac{\beta_H b}{N_H+m}(S_H(x,t,\psi)\\
&I_V(x,t,\psi) - S_H(x,t,\varphi)I_V(x,t,\varphi)) - \mu e_2(x.t,\psi,\varphi)\rangle dt + \kappa\|e_3(x,t,\psi,\varphi)\|^{\kappa-2}\\
&\langle e_3(x,t,\psi,\varphi), d_3\triangle e_3(x,t,\psi,\varphi) - \mu e_3(x,t,\psi,\varphi)\rangle dt + \kappa\|e_4(x,t,\psi,\varphi)\|^{\kappa-2}\\
&\langle e_4(x,t,\psi,\varphi), d_4\triangle e_4(x,t,\psi,\varphi) - \nu e_4(x,t,\psi,\varphi) - \frac{\beta_V b}{N_H+m}(S_V(x,t,\psi)I_H(x,t,\psi)\\
&-S_V(x,t,\varphi)I_H(x,t,\varphi))\rangle dt + \kappa\|e_5(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_5(x,t,\psi,\varphi), d_5\triangle e_5(x,t,\psi,\varphi)\\
&+\frac{\beta_V b}{N_H+m}(S_V(x,t,\psi)I_H(x,t,\psi) - S_V(x,t,\varphi)I_H(x,t,\varphi)) - \nu e_5(x.t,\psi,\varphi)\rangle dt\\
&+\frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_1(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_2(x,t,\psi,\varphi)\|^k dt\\
&+\frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_3(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\|e_4(x,t,\psi,\varphi)\|^k dt\\
&+\frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\|e_5(x,t,\psi,\varphi)\|^k dt + \kappa\|e_1(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_1(x,t,\psi,\varphi),\\
&-\xi_1(t)\cdot\triangle e_1(x,t,\psi,\varphi)dB_1(t)\rangle + \kappa\|e_2(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_2(x,t,\psi,\varphi),\\
&-\xi_1(t)\triangle e_2(x,t,\psi,\varphi)dB_1(t)\rangle + \kappa\|e_3(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_3(x,t,\psi,\varphi),\\
&-\xi_1(t)\triangle e_3(x,t,\psi,\varphi)dB_1(t)\rangle\} + \kappa\|e_4(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_4(x,t,\psi,\varphi),\\
&-\xi_2(t)\triangle e_4(x,t,\psi,\varphi)dB_2(t)\rangle + \kappa\|e_5(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_5(x,t,\psi,\varphi),\\
&-\xi_2(t)\triangle e_5(x,t,\psi,\varphi)dB_2(t)\rangle.
\end{aligned}$$

Next, we have a further transformation to $(S_H(x,t,\psi)I_V(x,t,\psi) - S_H(x,t,\varphi)I_V(x,t,\varphi))$ and $(S_V(x,t,\psi)I_H(x,t,\psi) - S_V(x,t,\varphi)I_H(x,t,\varphi))$ by virtue of equality (6.1), and get the following equality

$$\begin{aligned}
&d(e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa)\\
=&\eta e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa dt + e^{\eta t}\{\kappa\|e_1(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_1(x,t,\psi,\varphi), d_1\triangle e_1(x,t,\psi,\varphi)\\
&-\mu e_1(x,t,\psi.\varphi) - \frac{\beta_H b}{N_H+m}(S_H(x,t,\psi)e_5(x,t,\psi,\varphi) + e_1(x,t,\psi,\varphi)I_V(x,t,\varphi))\rangle dt\\
&+\kappa\|e_2(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_2(x,t,\psi,\varphi), d_2\triangle e_2(x,t,\psi,\varphi) + \frac{\beta_H b}{N_H+m}(S_H(x,t,\psi)\\
&e_5(x,t,\psi,\varphi) + e_1(x,t,\psi,\varphi)I_V(x,t,\varphi)) - \mu e_2(x.t,\psi,\varphi)\rangle dt + \kappa\|e_3(x,t,\psi,\varphi)\|^{\kappa-2}\\
&\langle e_3(x,t,\psi,\varphi), d_3\triangle e_3(x,t,\psi,\varphi) - \mu e_3(x,t,\psi,\varphi)\rangle dt + \kappa\|e_4(x,t,\psi,\varphi)\|^{\kappa-2}\\
&\langle e_4(x,t,\psi,\varphi), d_4\triangle e_4(x,t,\psi,\varphi) - \nu e_4(x,t,\psi,\varphi) - \frac{\beta_V b}{N_H+m}\langle S_V(x,t,\psi)\cdot\\
&e_2(x,t,\psi,\varphi) + e_4(x,t,\psi,\varphi)I_H(x,t,\varphi))\rangle dt + \kappa\|e_5(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_5(x,t,\psi,\varphi),\\
&d_5\triangle e_5(x,t,\psi,\varphi) + \frac{\beta_V b}{N_H+m}\langle S_V(x,t,\psi)e_2(x,t,\psi,\varphi) + e_4(x,t,\psi,\varphi)I_H(x,t,\varphi))
\end{aligned}$$

$$
\begin{aligned}
&- \nu e_5(x.t,\psi,\varphi)\rangle dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_1(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\\
&\|e_2(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_3(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\\
&\|e_4(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\|e_5(x,t,\psi,\varphi)\|^k dt + \kappa\|e_1(x,t,\psi,\varphi)\|^{\kappa-2}\\
&\langle e_1(x,t,\psi,\varphi), -\xi_1(t)\triangle e_1(x,t,\psi,\varphi)dB_1(t)\rangle + \kappa\|e_2(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_2(x,t,\psi,\varphi),\\
&- \xi_1(t)\triangle e_2(x,t,\psi,\varphi)dB_1(t)\rangle + \kappa\|e_3(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_3(x,t,\psi,\varphi), -\xi_1(t)\\
&\triangle e_3(x,t,\psi,\varphi)dB_1(t)\rangle\} + \kappa\|e_4(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_4(x,t,\psi,\varphi), -\xi_2(t)\triangle e_4(x,t,\psi,\varphi)\\
&dB_2(t)\rangle + \kappa\|e_5(x,t,\psi,\varphi)\|^{\kappa-2}\langle e_5(x,t,\psi,\varphi), -\xi_2(t)\triangle e_5(x,t,\psi,\varphi)dB_2(t)\rangle.
\end{aligned}
$$

Let's just zoom in and out for the above equality, and we get

$$
\begin{aligned}
&d(e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa)\\
\le&\eta e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa dt + \kappa e^{\eta t}\{-\|e_1(x,t,\psi,\varphi)\|^{k-2}\langle\nabla e_1(x,t,\psi,\varphi), d_1\nabla e_1(x,t,\psi,\varphi)\rangle\\
&-\|e_2(x,t,\psi,\varphi)\|^{k-2}\langle\nabla e_2(x,t,\psi,\varphi), d_2\nabla e_2(x,t,\psi,\varphi)\rangle + \frac{\beta_H b}{N_H+m}\|e_2(x,t,\psi,\varphi)\|^{k-2}\\
&\langle e_2(x,t,\psi,\varphi), S_H(x,t,\psi)e_5(x,t,\psi,\varphi) + e_1(x,t,\psi,\varphi)I_V(x,t,\varphi)\rangle - \|e_3(x,t,\psi,\varphi)\|^{k-2}\\
&\langle\nabla e_3(x,t,\psi,\varphi), d_3\nabla e_3(x,t,\psi,\varphi)\rangle - \|e_4(x,t,\psi,\varphi)\|^{k-2}\langle\nabla e_4(x,t,\psi,\varphi), d_4\\
&\nabla e_4(x,t,\psi,\varphi)\rangle - \|e_5(x,t,\psi,\varphi)\|^{k-2}\langle\nabla e_5(x,t,\psi,\varphi), d_5\nabla e_5(x,t,\psi,\varphi)\rangle + \frac{\beta_V b}{N_H+m}\\
&\|e_5(x,t,\psi,\varphi)\|^{k-2}\langle e_5(x,t,\psi,\varphi), S_V(x,t,\psi)e_2(x,t,\psi,\varphi) + e_4(x,t,\psi,\varphi)I_H(x,t,\varphi)\rangle\}dt\\
&+ \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_1(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_2(x,t,\psi,\varphi)\|^k dt\\
&+ \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_3(x,t,\psi,\varphi)\|^K dt + \frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\|e_4(x,t,\psi,\varphi)\|^k dt\\
&+ \frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\|e_5(x,t,\psi,\varphi)\|^k dt\\
\le&\eta e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa dt + \kappa e^{\eta t}\{-d_1\lambda_0\|e_1(x,t,\psi,\varphi)\|^k - d_2\lambda_0\|e_2(x,t,\psi,\varphi)\|^k\\
&+ \frac{\beta_H bM_1}{N_H+m}\|e_2(x,t,\psi,\varphi)\|^{k-1}(\|e_5(x,t,\psi,\varphi)\| + \|e_1(x,t,\psi,\varphi)\|) - d_3\lambda_0\|e_3(x,t,\psi,\varphi)\|^k\\
&- d_4\lambda_0\|e_4(x,t,\psi,\varphi)\|^k + \frac{\beta_V bM_1}{N_H+m}\|e_5(x,t,\psi,\varphi)\|^{k-1}(\|e_2(x,t,\psi,\varphi)\|\\
&+ \|e_4(x,t,\psi,\varphi)\|)\}dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_1(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\\
&\|e_2(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_1^2(t)\|e_3(x,t,\psi,\varphi)\|^K dt + \frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\\
&\|e_4(x,t,\psi,\varphi)\|^k dt + \frac{1}{2}\kappa(\kappa-1)\xi_2^2(t)\|e_5(x,t,\psi,\varphi)\|^k dt.
\end{aligned}
$$

Integrating on both sides of the above inequality and taking expectations, then, applying the Young inequality, we get

$$
\begin{aligned}
&E[e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa]\\
\le&\|e(x,0,\psi,\varphi)\|^\kappa + E\int_0^t e^{\eta s}\{\eta\|e(x,s,\psi,\varphi)\|^\kappa + (-kd_1\lambda_0 + \frac{(\beta_H bM_1)^k}{(N_H+m)^k}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\kappa(\kappa-1)\xi_1^2)\|e_1(x,t,\psi,\varphi)\|^\kappa + (-kd_2\lambda_0 + 2k - 2 + \frac{(\beta_V bM_1)^k}{(N_H+m)^k} + \\
&\frac{1}{2}\kappa(\kappa-1)\xi_1^2)\|e_2(x,t,\psi,\varphi)\|^\kappa + (-kd_3\lambda_0 + \frac{1}{2}\kappa(\kappa-1)\xi_1^2)\|e_3(x,t,\psi,\varphi)\|^\kappa \\
&+ (-kd_4\lambda_0 + \frac{(\beta_V bM_1)^k}{(N_H+m)^k} + \frac{1}{2}\kappa(\kappa-1)\xi_2^2)\|e_4(x,t,\psi,\varphi)\|^\kappa \\
&+ (-kd_5\lambda_0 + 2k - 2 + \frac{(\beta_H bM_1)^k}{(N_H+m)^k} + \frac{1}{2}\kappa(\kappa-1)\xi_2^2)\|e_5(x,t,\psi,\varphi)\|^\kappa.
\end{aligned}
$$

Next, we take the supremum on both sides of the above inequality

$$
\begin{aligned}
&E \sup_{0\le t\le T}[e^{\eta t}\|e(x,t,\psi,\varphi)\|^\kappa] \\
\le &\|e(x,0,\psi,\varphi)\|^\kappa + E \sup_{0\le t\le T}(\eta + M_5)\int_0^t e^{\eta s}\|e(x,s,\psi,\varphi)\|^\kappa ds,
\end{aligned}
\tag{6.2}
$$

here

$$
\begin{aligned}
M_5 = &\max\{-kd_1\lambda_0 + \frac{(\beta_H bM_1)^k}{(N_H+m)^k} + \frac{1}{2}\kappa(\kappa-1)\xi_1^2, -kd_2\lambda_0 + 2k - 2 + \frac{(\beta_V bM_1)^k}{(N_H+m)^k} \\
&+ \frac{1}{2}\kappa(\kappa-1)\xi_1^2, -kd_3\lambda_0 + \frac{1}{2}\kappa(\kappa-1)\xi_1^2, -kd_4\lambda_0 + \frac{(\beta_V bM_1)^k}{(N_H+m)^k} + \frac{1}{2}\kappa(\kappa-1)\xi_2^2, \\
&- kd_5\lambda_0 + 2k - 2 + \frac{(\beta_H bM_1)^k}{(N_H+m)^k} + \frac{1}{2}\kappa(\kappa-1)\xi_2^2\}.
\end{aligned}
$$

Based on the Gronwall inequality, we obtain

$$
\|e(x,t,\psi,\varphi)\|^\kappa \le \|e(x,0,\psi,\varphi)\|^\kappa e^{-(\eta+M_5)t},
$$

thereby

$$
\lim_{t\to\infty} E\|e(x,t,\psi,\varphi)\|^\kappa = 0.
$$

Therefore, condition (i) in Lemma 3.2 holds, thus existing a stationary distribution for system (2.5). Next, we prove the uniqueness of stationary distribution, assume that $\bar{\lambda}$ is also a stationary distribution to $\mathcal{W}(x,t)$. There exists a constant $M > 0$, we can get the following result

$$
|\lambda(f) - \bar{\lambda}(f)| \le \int_{\mathcal{H}\times\mathcal{H}} |P_t f(\psi) - P_t f(\varphi)|\lambda(d\psi)\bar{\lambda}(d\varphi) \le Me^{-\eta t},
$$

when $t \to \infty$, we can get the uniqueness of stationary distribution. □

References

[1] S. Aniţa and V. Capasso, *Regional Control for Spatially Structured Mosquito Borne Epidemics,* Vietnam Journal of Mathematics, 2021, 49(1), 21–35.

[2] A. Abdelrazec, J. BšŠlair, C. Shan and H. Zhu, *Modeling the spread and control of dengue with limited public health resources,* Mathematical Biosciences, 2016, 271, 136–145.

[3] L. Esteva and C. Vargas, *Analysis of a dengue disease transmission model,* Mathematical Biosciences, 1998, 150, 131–151.

[4] W. H. Fleming, and R. W. Rishel, *Deterministic and Stochastic Optimal Control,* Springer, New York, NY, USA, 1975.

[5] D. J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations,* SIAM review, 2001, 43(3), 525–546.

[6] L. Hu, M. Tang, Z. Wu, Z. Xi and J. Yu, *The threshold infection level for Wolbachia invasion in random environments,* Journal of Differential Equations, 2019, 266(7), 4377–4393.

[7] M. Iftikhar, A. Sohail and N. Ahmad, *Detrministic and stochastic analysis of dengue spread model,* Biomedical Engineering: Applications. Basis and Communications, 2019, 31(03), Article ID 1950008.

[8] M. A. Khan, A. Khan, A. Elsonbaty and A. A. Elsadany, *Modeling and simulation results of a fractional dengue model,* The European Physical Journal Plus, 2019, 134(8), 379–393.

[9] T. K. Kar and S. Jana, *Application of three controls optimally in a vector-borne disease-a mathematical study,* Communications in Nonlinear Science and Numerical Simulation, 2013, 18(10), 2868–2884.

[10] H. Kuo, *Introduction to Stochastic Integration,* Springer, 2006.

[11] K. Liu, *Stationary Distributions of Second Order Stochastic Evolution Equations with Memory in Hilbert Spaces.* Stochastic Processes and their Applications, 2020, 130, 366–393.

[12] Q. Liu, D. Jiang and T. Hayat, *Stationary distribution and extinction of a stochastic dengue epidemic model,* Journal of the Franklin Institute, 2018, 355, 8891–8914.

[13] E. Massaro, D. Kondor and C. Ratti, *Assessing the interplay between human mobility and mosquito borne diseases in urban environments,* Scientific reports, 2019, 9(1), 1–13.

[14] R. C. Mittal, R. Goel and N. Ahlawat, *An Efficient Numerical Simulation of a Reaction-Diffusion Malaria Infection Model using B-splines Collocation,* Chaos Solitons and Fractals, 2021, 143, Article ID 110566.

[15] X. Mao, *Stochastic Differential Equations and Applications (Second Edition),* Horwood Publishing, Chichester, Horwood, 2007.

[16] E. A. Newton and P. Reiter, *A model of the transmission of dengue fever with an evaluation of the impact of ultra-low volume (ULV) insecticide applications on dengue epidemics,* The American journal of tropical medicine and hygiene, 1992, 47(6), 709–720.

[17] C. Paupy, B. Ollomo, B. Kamgang, S. Moutailler, D. Rousset, M. Demanou, J. Hervé, E. Leroy and F. Simard, *Comparative role of Aedes albopictus and Aedes aegypti in the emergence of dengue and chikungunya in central Africa,* Vector-Borne and Zoonotic Diseases, 2010, 10(3), 259–266.

[18] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The Maximum Principle. The Mathematical Theory of Optimal Processes,* New York: John Wiley and Sons, 1962.

[19] H. S. Rodriguesa, M. Teresab, T. Monteirob and D. F. Torresc, *Seasonality effects on dengue: basic reproduction number, sensitivity analysis and optimal*

control, Mathematical Methods in the Applied Sciences, 2016, 39(16), 4671–4679.

[20] H. S. Rodriguesa, M. Teresab, T. Monteirob and D. F. Torresc, *Optimal Control of a Dengue Epidemic Model with Vaccination,* American Institute of Physics, 2011, 1389(1), 1232–1235.

[21] L. Sedda, B. M. Taylor, A. E. Eiras, J. T. Marques and R. J. Dillon, *Using the intrinsic growth rate of the mosquito population improves spatio-temporal dengue risk estimation,* Acta. Tropica, 2020, 208, Article ID 105519.

[22] W. Sun, L. Xue and X. Yan, *Stability of a dengue epidemic model with independent stochastic perturbations,* Journal of Mathematical Analysis and Applications, 2018, 468, 998–1017.

[23] T. Sardar, S. Rana and J. Chattopadhyay, *A mathematical model of dengue transmission with memory,* Commun Nonlinear Sci Numer Simulat, 2015, 22, 511–525.

[24] J. J. Tewa, J. L. Dimi and S. Bowong, *Lyapunov functions for a dengue disease transmission model,* Chaos Solitons and Fractals, 2009, 39, 936–941.

[25] M. Umar, M. A. Z. Raja, Z. Sabir, A. S. Alwabli and M. Shoaib, *A stochastic computational intelligent solver for numerical treatment of mosquito dispersal model in a heterogeneous environment,* The European Physical Journal Plus, 2020, 135(7), 1–23.

[26] P. J. Witbooi, G. J. Abiodun, G. J. V. Schalkwyk and I. H. I. Ahmed, *Stochastic modeling of a mosquito-borne disease,* Advances in Difference Equations, 2020, 1, 1–15.

[27] Y. Xue, L. Hu and L. Nie, *Modelling the Wolbachia Strains for Dengue Fever Virus Control in the Presence of Seasonal Fluctuation,* Journal of Nonlinear Modeling and Analysis, 2021, 3(2), 209–224.

[28] J. Yong and X. Zhou, *Stochastic Control: Hamiltonian Systems and HJB Equations,* Springer, 1999.

[29] M. Zhu, Z. Lin and L. Zhang, *Spatial-temporal risk index and transmission of a nonlocal dengue model,* Nonlinear Analysis: Real World Applications, 2020, 53, Article ID 103076.

[30] L. Zhang and S. Wang, *A time-periodic and reaction-diffusion Dengue fever model with extrinsic incubation period and crowding effects,* Nonlinear Analysis: Real World Applications, 2020, 51, Article ID 102988.

[31] M. Zhang and Z. Lin, *The diffusive model for aedes aegypti mosouito on a periodically evolving domain,* Dynamical Systems Series B, 2019, 24(9), 4703–4720.