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A GENERAL DECAY RESULT FOR A VON KARMAN EQUATION WITH MEMORY AND ACOUSTIC BOUNDARY CONDITIONS*

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Abstract We study a viscoelastic von Karman equation of memory type with acoustic boundary conditions. Utilizing some properties of convex functions and the perturbed energy method, we build a general decay result when the kernel function k is a very general type. This work extends and complements some previous decay results of solutions for von von Karman equations.

Keywords Von Karman equation, general decay, convex function, acoustic boundary condition.

MSC(2010) 35B40, 35L70, 74D99.

1. Introduction

In this work, we are concerned with a von Karman equation with memory and acoustic boundary conditions

$$u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - \int_0^t k(t-s)\Delta^2 u(s)ds = [u, v] \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

$$\Delta^2 v = -[u, u] \text{ in } \Omega \times (0, \infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, \infty), \tag{1.3}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0 \times (0, \infty), \tag{1.4}$$

$$\mathcal{B}_1 u - \mathcal{B}_1\Big(\int_0^t k(t-s)u(s)ds\Big) = 0 \text{ on } \Gamma_1 \times (0, \infty), \tag{1.5}$$

$$\mathcal{B}_2 u - \alpha\frac{\partial u_{tt}}{\partial \nu} - \mathcal{B}_2\Big(\int_0^t k(t-s)u(s)ds\Big) = -r(x)y_t \text{ on } \Gamma_1 \times (0, \infty), \tag{1.6}$$

$$u_t + p(x)y_t + q(x)y = 0 \text{ on } \Gamma_1 \times (0, \infty), \tag{1.7}$$

$$u(0) = u_0, \ \ u_t(0) = u_1 \text{ in } \Omega, \ \ y(0) = y_0 \text{ on } \Gamma_1, \tag{1.8}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $meas(\Gamma_0) > 0$, $meas(\Gamma_1) > 0$, $x = (x_1, x_2) \in \Omega \cup \partial\Omega$, and

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$\nu = (\nu_1, \nu_2)$ is the outward unit normal vector on $\partial\Omega$, the von Karman bracket $[\cdot, \cdot]$ is defined as

$$[\varphi, \phi] = \varphi_{x_1x_1}\phi_{x_2x_2} + \varphi_{x_2x_2}\phi_{x_1x_1} - 2\varphi_{x_1x_2}\phi_{x_1x_2},$$
$$\mathcal{B}_1\phi = \Delta\phi + (1-\mu)B_1\phi, \;\; \mathcal{B}_2\phi = \frac{\partial\Delta\phi}{\partial\nu} + (1-\mu)\frac{\partial B_2\phi}{\partial\tau},$$

here μ is Poisson's ratio with $0 < \mu < \frac{1}{2}$,

$$B_1\phi = 2\nu_1\nu_2\phi_{x_1x_2} - \nu_1^2\phi_{x_2x_2} - \nu_2^2\phi_{x_1x_1},$$
$$B_2\phi = (\nu_1^2 - \nu_2^2)\phi_{x_1x_2} + \nu_1\nu_2(\phi_{x_2x_2} - \phi_{x_1x_1}).$$

And, the thickness α of the plate is positive, $r > 0, p \geq p_0 > 0, q > 0$ are essentially bounded, and the kernel $k : [0,\infty) \to (0,\infty)$ is a non-increasing differentiable function with $1 - \int_0^\infty k(s)ds := l > 0$. The von Karman equations (1.1)-(1.8) model a nonlinear elastic plate by describing the transversal displacement u and the Airy-stress function v . Von Karman equation also arises in many applications such as bifurcation theory, shells, and etc.

One of the main concern in the study of viscoelastic problems is to establish more general and explicit decay rates of solutions by imposing minimal assumptions on the kernel function k. And many stability results have been established [4,8,16–18,21]. For instance, Messaoudi [16] showed decay estimates of exponential type for viscoelastic wave equations when k fulfills

$$k'(t) \leq -\zeta(t)k(t), \tag{1.9}$$

where ζ is positive, differentiable, and non-increasing. Messaoudi and Al-Khulaifi [17] proved a decay result of general type for a quasilinear viscoelastic wave equation when

$$k'(t) \leq -\zeta(t)k^q(t), \tag{1.10}$$

here $1 \leq q < \frac{3}{2}$. And then, a question 'Can the range of parameter q be expanded from $1 \leq q < \frac{3}{2}$ to $1 \leq q < 2$?' was raised. Pursued the ideas introduced by Lasiecka and Tataru [13] and Jin etc [11], Mustafa [21] answered for the question. He obtained more generalized and explicit decay rates for viscoelastic wave equations by endowing the following new assumption

$$k'(t) \leq -\zeta(t)K(k(t)), \tag{1.11}$$

where K is an increasing convex function meeting some conditions. He explained that (1.10) with $1 \leq q < 2$ is only a special case of (1.11). For the recent articles associated with the assumption (1.11), we mention the works [9,10,14,15]. In the present article, we are interested in a new general decay estimate of solutions to the viscoelastic von Karman system (1.1)-(1.8). For physical application of acoustic boundary conditions, we refer [2,3]. We also cite [7,12,22,28,29] and references therein for works involving such boundary conditions. Many authors discussed on the stability for von Karman systems with dissipative effects [5,6,19,20,23,25–27]. Among those, the authors of [19,20,23] derived exponential decay results when k has the property (1.9) with $\zeta(t) = \zeta$. Park etc [26] established a decay result of exponential type to problem (1.1)-(1.8) when k satisfies (1.9) and $\int_0^\infty k(s)ds < \frac{1}{2}$. Park [25] showed an arbitrary energy estimate for a von Karman equation with

Dirichlet boundary condition. Based on these articles, we extend and complement the result of [26] under the condition (1.11).

Here is the outline of this paper. In section 2, we give some materials such as notations, hypothesis, and auxiliary formulas. In section 3, we derive a general decay result by utilizing some properties of convex functions and the multiplier method.

2. Preliminaries

We let

$$V = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \Gamma_0\},\ W = \{\phi \in H^2(\Omega) : \phi = \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_0\},$$
$$(\varphi, \phi) = \int_\Omega \varphi(x)\phi(x)dx,\ ||\varphi||^2 = (\varphi, \varphi),$$

and

$$(\varphi, \phi)_{\Gamma_1} = \int_{\Gamma_1} \varphi(x)\phi(x)d\Gamma,\ ||\varphi||^2_{\Gamma_1} = (\varphi, \varphi)_{\Gamma_1}.$$

From now on, if there is no ambiguity, we omit the variables t and x. $||\cdot||_X$ denotes the norm of a Banach space X. The bilinear form $b(\cdot, \cdot)$ is defined by

$$b(\varphi, \phi) = \varphi_{x_1x_1}\phi_{x_1x_1} + \varphi_{x_2x_2}\phi_{x_2x_2} + \mu(\varphi_{x_1x_1}\phi_{x_2x_2} + \varphi_{x_2x_2}\phi_{x_1x_1}) + 2(1-\mu)\varphi_{x_1x_2}\phi_{x_1x_2}. \tag{2.1}$$

For $(\varphi, \phi) \in (H^4(\Omega) \cap W) \times W$, we know

$$\int_\Omega (\Delta^2\varphi)\phi dx = \int_\Omega b(\varphi, \phi)dx - (\mathcal{B}_1\varphi, \frac{\partial \phi}{\partial \nu})_{\Gamma_1} + (\mathcal{B}_2\varphi, \phi)_{\Gamma_1}. \tag{2.2}$$

Due to $\Gamma_0 \neq \emptyset$, it is well known ([5]) that

$$c_1||\phi||^2_{H^2(\Omega)} \le \int_\Omega b(\phi, \phi)dx \le c_2||\phi||^2_{H^2(\Omega)} \quad \text{for some } c_1, c_2 > 0. \tag{2.3}$$

Let C_p, C_{p,Γ_1} and C_s be the imbedding constants with

$$\begin{aligned} &||\varphi||^2 \le C_p \int_\Omega b(\varphi, \varphi)dx,\ ||\varphi||^2_{\Gamma_1} \le C_{p,\Gamma_1} \int_\Omega b(\varphi, \varphi)dx, \\ &||\nabla\varphi||^2 \le C_s \int_\Omega b(\varphi, \varphi)dx,\ \forall \varphi \in W. \end{aligned} \tag{2.4}$$

From (2.1), it is seen (see [26])

$$\int_\Omega b(\varphi, \phi)dx \le \delta||\varphi||^2_{H^2(\Omega)} + \frac{5}{8\delta}||\phi||^2_{H^2(\Omega)} \text{ for all } \delta > 0,$$

which gives

$$\int_\Omega b(\varphi, \phi)dx \le \delta \int_\Omega b(\varphi, \varphi)dx + \frac{5}{8c_1^2\delta} \int_\Omega b(\phi, \phi)dx \quad \text{for all } \delta > 0. \tag{2.5}$$

Based on the arguments of [19, 22, 27], we get the existence result. For every $(u_0, u_1, y_0) \in (H^4(\Omega) \cap W) \times (H^3(\Omega) \cap W) \times L^2(\Gamma_1)$, there exists a solution (u, y) of problem (1.1)-(1.8) verifying

$$\begin{aligned} &u \in L^\infty(0,T; H^4(\Omega) \cap W),\ u_t \in L^\infty(0,T; H^3(\Omega) \cap V),\\ &r^{\frac{1}{2}} y \in L^\infty(0,T; L^2(\Gamma_1)),\ r^{\frac{1}{2}} y' \in L^2(0,T; L^2(\Gamma_1)). \end{aligned}$$

Now, we endow some hypothesis on k to derive our desired decay result.
(A) We assume that the kernel k verifies

$$k'(t) \le -\zeta(t) K(k(t)) \quad \text{for all } t \ge 0, \tag{2.6}$$

where $K : (0,\infty) \to (0,\infty)$ is a C^1-function, which is either linear or strictly increasing and strictly convex C^2-function on $(0, \epsilon]$, $\epsilon \le k(0)$, $K(0) = K'(0) = 0$, and ζ is positive, differentiable, and non-increasing.

Some examples of the function k satisfying (**A**) are provided by Mustafa [21].

3. A general decay result

Throughout this work, we set

$$\begin{aligned} (k\Box\varphi)(t) &= \int_0^t k(t-s)||\varphi(t) - \varphi(s)||^2 ds,\\ (k\Box\partial^2\varphi)(t) &= \int_0^t k(t-s) \int_\Omega b(\varphi(t) - \varphi(s), \varphi(t) - \varphi(s)) dx ds, \end{aligned}$$

and

$$h_\beta(t) = \beta k(t) - k'(t) \ \text{ and } \ C_\beta = \int_0^\infty \frac{k^2(s)}{h_\beta(s)} ds.$$

Let the energy of the solution to (1.1)-(1.8) as

$$\begin{aligned} E(t) =& \frac{1}{2}\Big\{||u_t||^2 + \alpha||\nabla u_t||^2 + \Big(1 - \int_0^t k(s)ds\Big)\int_\Omega b(u,u)dx \\ &+ \frac{1}{2}||\Delta v||^2 + (k\Box\partial^2 u) + ||\sqrt{rq}y||^2_{\Gamma_1}. \end{aligned} \tag{3.1}$$

Taking the inner product (1.1) with u_t in $L^2(\Omega)$, applying (1.2)-(1.8), and taking advantage of relation

$$\begin{aligned} \int_\Omega \int_0^t k(t-s) b(u(s), u_t) ds dx =& -\frac{1}{2}\frac{d}{dt}\Big[(k\Box\partial^2 u) - \Big(\int_0^t k(s)ds\Big)\int_\Omega b(u,u)dx\Big] \\ &-\frac{1}{2}k(t)\int_\Omega b(u,u)dx + \frac{1}{2}(k'\Box\partial^2 u), \end{aligned} \tag{3.2}$$

we get ([26])

$$E'(t) = -||\sqrt{rp}y_t||^2_{\Gamma_1} - \frac{1}{2}k(t)\int_\Omega b(u,u)dx + \frac{1}{2}k'\Box\partial^2 u \le 0. \tag{3.3}$$

As in [21, 26], we define

$$L(t) = ME(t) + M_1\Phi(t) + M_2\Psi(t),$$

where

$$\Phi(t) = (u_t, u) + \alpha(\nabla u_t, \nabla u) + \frac{1}{2}||\sqrt{rp}y||^2_{\Gamma_1} + (u, ry)_{\Gamma_1}$$

and

$$\Psi(t) = -\int_0^t k(t-s)(u(t)-u(s), u_t)ds - \alpha\int_0^t k(t-s)(\nabla u(t) - \nabla u(s), \nabla u_t)ds.$$

Lemma 3.1. *For every $\beta > 0$ and $\gamma > 0$, it fulfills*

$$\begin{aligned}\Phi'(t) \leq\, & ||u_t||^2 + \alpha||\nabla u_t||^2 - \Big(\frac{l}{2} - \gamma\Big)\int_\Omega b(u,u)dx - ||\Delta v||^2 \\ & + \frac{5c_2}{4c_1^3 l}C_\beta(h_\beta \Box \partial^2 u) - ||\sqrt{rq}y||^2_{\Gamma_1} + \frac{C_{p,\Gamma_1}||r||^2_\infty}{\gamma}||y_t||^2_{\Gamma_1}. \qquad (3.4)\end{aligned}$$

Proof. From (1.1)-(1.8), one sees ([26])

$$\begin{aligned}\Phi'(t) =\, & ||u_t||^2 + \alpha||\nabla u_t||^2 - \Big(1 - \int_0^t k(s)ds\Big)\int_\Omega b(u,u)dx - ||\Delta v||^2 \\ & + \int_0^t k(t-s)\int_\Omega b(u(s)-u(t), u)dxds + 2(u, ry_t)_{\Gamma_1} - ||\sqrt{rq}y||^2_{\Gamma_1}. \qquad (3.5)\end{aligned}$$

Applying (2.3) and Hölder inequality, we get

$$\begin{aligned}& \int_\Omega b(\int_0^t k(t-s)(u(t)-t(s))ds, \int_0^t k(t-s)(u(t)-u(s))ds)dx \\ & \leq c_2||\int_0^t k(t-s)(u(t)-u(s))ds||^2_{H^2(\Omega)} \\ & \leq c_2\Big(\int_0^t \frac{k^2(s)}{h_\beta(s)}ds\Big)\int_0^t h_\beta(t-s)||u(t)-u(s)||^2_{H^2(\Omega)}dx \\ & \leq \frac{c_2 C_\beta}{c_1}(h_\beta \Box \partial^2 u). \qquad (3.6)\end{aligned}$$

Utilizing (2.5) and (3.6), we have

$$\begin{aligned}& \int_0^t k(t-s)\int_\Omega b(u(s)-u(t), u)dxdx \\ & \leq \delta\int_\Omega b(u,u)dx + \frac{5}{8c_1^2\delta}\int_\Omega b(\int_0^t k(t-s)(u(s)-u(t)ds, \int_0^t k(t-s)(u(s)-u(t)ds)dx \\ & \leq \delta\int_\Omega b(u,u)dx + \frac{5c_2C_\beta}{8c_1^3\delta}(h_\beta\Box\partial^2 u). \qquad (3.7)\end{aligned}$$

By (2.4),

$$2(u, ry_t)_{\Gamma_1} \leq 2||r||_\infty||u||_{\Gamma_1}||y_t||_{\Gamma_1} \leq \gamma\int_\Omega b(u,u)dx + \frac{C_{p,\Gamma_1}||r||^2_\infty}{\gamma}||y_t||^2_{\Gamma_1}. \quad (3.8)$$

Combining (3.7)-(3.8) with (3.5) and putting $\delta = \frac{l}{2}$, we get (3.4). □

Lemma 3.2. *For every $\beta > 0$ and $0 < \eta < 1$, it holds*

$$\Psi'(t) \le -\alpha\Big(\int_0^t k(s)ds - \eta\Big)||\nabla u_t||^2 - \Big(\int_0^t k(s)ds - \eta\Big)||u_t||^2 + \eta||y_t||^2_{\Gamma_1} + \frac{C(1+C_\beta)}{\eta}(h_\beta \Box \partial^2 u) + 2\eta\int_\Omega b(u,u)dx \quad \text{for some } C > 0. \tag{3.9}$$

Proof. By (1.1)-(1.8), we find ([26])

$$\begin{aligned}\Psi'(t) =& -\alpha\Big(\int_0^t k(s)ds\Big)||\nabla u_t||^2 - \Big(\int_0^t k(s)ds\Big)||u_t||^2 - \Big(ry_t, \int_0^t k(t-s)(u(t)-u(s))ds\Big)_{\Gamma_1} \\ & - \Big(\int_0^t k(t-s)(u(t)-u(s))ds, [u,v]\Big) - \Big(\int_0^t k'(t-s)(u(t)-u(s))ds, u_t\Big) \\ & - \alpha\Big(\int_0^t k'(t-s)(\nabla u(t) - \nabla u(s))ds, \nabla u_t\Big) \\ & + \int_\Omega b(\int_0^t k(t-s)(u(t)-u(s))ds, \int_0^t k(t-s)(u(t)-u(s))ds)dx \\ & + \Big(1 - \int_0^t k(s)ds\Big)\int_0^t k(t-s)\int_\Omega b(u, u(t)-u(s))dxds \\ :=& -\Big(\int_0^t k(s)ds\Big)||u_t||^2 - \alpha\Big(\int_0^t k(s)ds\Big)||\nabla u_t||^2 + \sum_{i=1}^6 D_i. \end{aligned} \tag{3.10}$$

For every $\eta > 0$,

$$\begin{aligned}|D_1| &\le \eta||y_t||^2_{\Gamma_1} + \frac{||r||^2_\infty}{4\eta}||\int_0^t k(t-s)(u(t)-u(s))ds||^2_{\Gamma_1} \\ &\le \eta||y_t||^2_{\Gamma_1} + \frac{||r||^2_\infty}{4\eta}\Big(\int_0^t \frac{k^2(s)}{h_\beta(s)}ds\Big)\int_{\Gamma_1}\int_0^t h_\beta(t-s)(u(t)-u(s))^2 dsd\Gamma \\ &\le \eta||y_t||^2_{\Gamma_1} + \frac{||r||^2_\infty C_{p,\Gamma_1} C_\beta}{4\eta}(h_\beta \Box \partial^2 u)\end{aligned}$$

and

$$\begin{aligned}|D_2| &\le a||u||_{H^2(\Omega)}||v||_{W^{2,\infty}(\Omega)}||\int_0^t k(t-s)(u(t)-u(s))ds|| \\ &\le \frac{a||v||_{W^{2,\infty}(\Omega)}}{\sqrt{c_1}}\sqrt{\int_\Omega b(u,u)dx}\sqrt{C_\beta(h_\beta \Box u)} \\ &\le \eta\int_\Omega b(u,u)dx + \frac{a^2||v||^2_{W^{2,\infty}(\Omega)}C_\beta C_p}{4c_1\eta}(h_\beta \Box \partial^2 u),\end{aligned}$$

here the Karman bracket property $||[u,v]|| \le a||u||_{H^2(\Omega)}||v||_{W^{2,\infty}(\Omega)}$ (see p. 270 in [6]) and (2.3) are applied. Recalling $k' = \beta k - h_\beta$, we get

$$\begin{aligned}|D_3| &\le \eta||u_t||^2 + \frac{1}{4\eta}||\int_0^t k'(t-s)(u(t)-u(s))ds||^2 \\ &\le \eta||u_t||^2 + \frac{1}{2\eta}\Big(||\int_0^t h_\beta(t-s)(u(t)-u(s))ds||^2\end{aligned}$$

$$+\beta^2||\int_0^t k(t-s)(u(t)-u(s))ds||^2\Big)$$
$$\le \eta||u_t||^2+\frac{1}{2\eta}\Big\{\Big(\int_0^t h_\beta(s)ds\Big)(h_\beta\Box u)+\beta^2 C_\beta(h_\beta\Box u)\Big\}$$
$$\le \eta||u_t||^2+\frac{(\beta(1-l)+k(0))C_p}{2\eta}(h_\beta\Box\partial^2 u)+\frac{\beta^2 C_\beta C_p}{2\eta}(h_\beta\Box\partial^2 u)$$

and

$$|D_4|\le \eta\alpha||\nabla u_t||^2+\frac{\alpha(\beta(1-l)+k(0))C_s}{2\eta}(h_\beta\Box\partial^2 u)+\frac{\alpha\beta^2 C_\beta C_s}{2\eta}(h_\beta\Box\partial^2 u).$$

Noting (3.6) and (3.7) with $\delta=\eta$, we observe

$$|D_5|\le \frac{c_2 C_\beta}{\eta c_1}(h_\beta\Box\partial^2 u)\quad \text{for } 0<\eta<1$$

and

$$|D_6|\le \eta\int_\Omega b(u,u)dx+\frac{5c_2C_\beta}{8c_1^3\eta}(h_\beta\Box\partial^2 u).$$

These inequalities of $D_i(1\le i\le 6)$ and (3.10) complete the proof. □

Lemma 3.3. *Set* $f(t)=\int_t^\infty k(s)ds$. *Then, the following function*

$$\Xi(t)=\int_\Omega\int_0^t f(t-s)b(u(s),u(s))dsdx$$

verifies

$$\Xi'(t)\le\Big(1+\frac{5}{2c_1^2}\Big)(1-l)\int_\Omega b(u,u)dx-\frac{1}{2}(k\Box\partial^2 u). \tag{3.11}$$

Proof. Direct calculation and (2.5) with $\delta=\frac{5}{2c_1^2}$ supply

$$\begin{aligned}
\Xi'(t)&=f(0)\int_\Omega b(u,u)dx-\int_\Omega\int_0^t k(t-s)b(u(s),u(s))dsdx\\
&=\Big(\int_0^\infty k(s)ds\Big)\int_\Omega b(u(t),u(t))dx-\int_\Omega\int_0^t k(t-s)b(u(t)-u(s),u(t)-u(s))dsdx\\
&\quad-\int_\Omega\int_0^t k(t-s)b(u,u)dsdx-2\int_\Omega\int_0^t k(t-s)b(u(t),u(s)-u(t))dsdx\\
&=\Big(\int_0^\infty k(s)ds\Big)\int_\Omega b(u,u)dx-(k\Box\partial^2 u)-\Big(\int_0^t k(s)ds\Big)\int_\Omega b(u,u)dx\\
&\quad-2\int_\Omega\int_0^t k(t-s)b(u(t),u(s)-u(t))dsdx\\
&\le\Big(\int_t^\infty k(s)ds+2\delta\int_0^t k(s)ds\Big)\int_\Omega b(u,u)dx-\Big(1-\frac{5}{4c_1^2\delta}\Big)(k\Box\partial^2 u)\\
&=-\frac{1}{2}(k\Box\partial^2 u)+\Big(1+\frac{5}{2c_1^2}\Big)(1-l)\int_\Omega b(u,u)dx.
\end{aligned}$$

□

Lemma 3.4. *For $t \geq t^* = k^{-1}(\epsilon)$, it fulfills*

$$L'(t) \leq \frac{1}{4}k\Box\partial^2 u - ||u_t||^2 - \alpha||\nabla u_t||^2 - ||\Delta v||^2 - ||\sqrt{rq}y||^2_{\Gamma_1}$$
$$-(2+\frac{5}{2c_1^2})(1-l)\int_\Omega b(u,u)dx. \tag{3.12}$$

Moreover, the energy $E(t)$ is equivalent to $L(t)$.

Proof. From (3.3), (3.4), (3.9), and $k' = \beta k - h_\beta$,

$$\begin{aligned}
L'(t) \leq & -\frac{M}{2}k(t)\int_\Omega b(u,u)dx + \frac{M\beta}{2}(k\Box\partial^2 u) - \Big(Mp_0 - M_2\eta - \frac{M_1||r||^2_\infty C_{p,\Gamma_1}}{\gamma}\Big)||y_t||^2_{\Gamma_1} \\
& - \Big\{M_2\Big(\int_0^t k(s)ds - \eta\Big) - M_1\Big\}\alpha||\nabla u_t||^2 - \Big\{M_2\Big(\int_0^t k(s)ds - \eta\Big) - M_1\Big\}||u_t||^2 \\
& - \Big\{M_1\Big(\frac{l}{2} - \gamma\Big) - 2M_2\eta\Big\}\int_\Omega b(u,u)dx - M_1||\Delta v||^2 - M_1||\sqrt{rq}y||^2_{\Gamma_1} \\
& - \Big\{\frac{M}{2} - \frac{5M_1c_2C_\beta}{4c_1^2 l} - \frac{M_2C(1+C_\beta)}{\eta}\Big\}(h_\beta\Box\nabla u).
\end{aligned}$$

The assumption (**A**) ensures the existence of $t^* > 0$ with $k(t^*) = \epsilon$. Selecting $\gamma = \frac{l}{4}$, $\eta = \frac{l}{4M_2}$ and denoting $k^* = \int_0^{t^*} k(s)ds$, we read

$$\begin{aligned}
L'(t) \leq & \frac{M\beta}{2}(k\Box\partial^2 u) - \Big(Mp_0 - \frac{l}{4} - \frac{4M_1||r||^2_\infty C_{p,\Gamma_1}}{l}\Big)||y_t||^2_{\Gamma_1} \\
& - \Big\{M_2k^* - \frac{l}{4} - M_1\Big\}\alpha||\nabla u_t||^2 - \Big\{M_2k^* - \frac{l}{4} - M_1\Big\}||u_t||^2 \\
& - \Big\{\frac{M_1 l}{4} - \frac{l}{2}\Big\}\int_\Omega b(u,u)dx - M_1||\Delta v||^2 - M_1||\sqrt{rq}y||^2_{\Gamma_1} \\
& - \Big\{\frac{M}{4} - \frac{4M_2^2C}{l} + \frac{M}{4} - C_\beta\Big(\frac{4M_2^2C}{l} + \frac{5M_1c_2}{4c_1^2 l}\Big)\Big\}(h_\beta\Box\nabla u) \quad \text{for } t \geq t^*.
\end{aligned} \tag{3.13}$$

Once $M_1 > 1$ is fixed so that

$$\frac{M_1 l}{4} - \frac{l}{2} > \Big(2 + \frac{5}{2c_1^2}\Big)(1-l), \tag{3.14}$$

we choose $M_2 > 0$ with

$$M_2k^* - \frac{l}{4} - M_1 > 1. \tag{3.15}$$

Since $\frac{\beta k^2(s)}{h_\beta(s)} < k(s)$, by the arguments of [21], $\lim_{\beta\to 0^+}\beta C_\beta = \lim_{\beta\to 0^+}\int_0^\infty \frac{\beta k^2(s)}{h_\beta(s)}ds = 0$. So, there exists $0 < \beta_0 < 1$ such that

$$\beta C_\beta < \frac{1}{8(\frac{5c_2M_1}{4c_1^2 l} + \frac{4CM_2^2}{l})} \quad \text{for } \beta < \beta_0. \tag{3.16}$$

Now, we pick $\beta = \frac{1}{2M}$ and $M > 0$ appropriately large to get

$$\beta = \frac{1}{2M} < \beta_0, \ Mp_0 - \frac{l}{4} - \frac{4M_1||r||^2_\infty C_{p,\Gamma_1}}{l} > 0, \ \frac{M}{4} - \frac{4CM_2^2}{l} > 0. \tag{3.17}$$

From (3.16), we also observe that

$$\frac{M}{4} - C_\beta\Big(\frac{5c_2M_1}{4c_1^2 l} + \frac{4CM_2^2}{l}\Big) = \frac{M}{4} - \frac{1}{8\beta} = \frac{M}{4} - \frac{M}{4} = 0. \tag{3.18}$$

Adapting (3.14), (3.15), (3.17), (3.18) to (3.13), the inequality (3.12) is proved. Furthermore, the equivalence $L(t) \sim E(t)$ can be proved as Lemma 3.1 of [26]. □

Theorem 3.1. *Under the assumption* (**A**), *it holds*

$$E(t) \le C_0 \tilde{K}^{-1}\Big(\omega \int_{k^{-1}(\epsilon)}^{t} \zeta(s)ds\Big) \quad \text{for } t \ge t^*,$$

where $\omega > 0$, $C_0 > 0$, *and*

$$\tilde{K}(s) = \int_s^\epsilon \frac{1}{\tau K'(\tau)} d\tau. \tag{3.19}$$

Proof. When K is linear, the proof can be found in [26]. Thus, we only consider for the case of K is nonlinear by applying the ideas of [14, 21, 24]. The continuity of k and ζ with respect to t provides the existence of $c_3, c_4 > 0$ satisfying

$$c_3 \le \zeta(t)K(k(t)) \le c_4 \quad \text{for } t \in [0, t^*].$$

Moreover,

$$k'(t) \le -\zeta(t)K(k(t)) \le -c_3 \le -\frac{c_3}{k(0)}k(t) \quad \text{for } t \in [0, t^*]. \tag{3.20}$$

The estimate $L'(t)$ in (3.12) guarantees

$$L'(t) \le -\rho E(t) + \frac{5}{4}(k\Box\partial^2 u), \tag{3.21}$$

where $\rho = \min\{2, 2(1-l)\Big(2 + \frac{5}{2c_1^2}\Big)\}$. This, (3.20), and (3.3) give

$$\begin{aligned}
L'(t) &\le -\rho E(t) + \frac{5}{4}(k\Box\partial^2 u) \\
&= -\rho E(t) + \frac{5}{4}\int_0^{t^*} k(s)\int_\Omega b(u(t)-u(t-s), u(t)-u(t-s))dxds \\
&\quad + \frac{5}{4}\int_{t^*}^{t} k(s)\int_\Omega b(u(t)-u(t-s), u(t)-u(t-s))dxds \\
&\le -\rho E(t) - \frac{5k(0)}{4c_3}\int_0^{t^*} k'(s)\int_\Omega b(u(t)-u(t-s), u(t)-u(t-s))dxds \\
&\quad + \frac{5}{4}\int_{t^*}^{t} k(s)\int_\Omega b(u(t)-u(t-s), u(t)-u(t-s))dxds \\
&\le -\rho E(t) - \frac{5k(0)}{4c_3}(k'\Box\partial^2 u)(t) \\
&\quad + \frac{5}{4}\int_{t^*}^{t} k(s)\int_\Omega b(u(t)-u(t-s), u(t)-u(t-s))dxds \\
&\le -\rho E(t) - \frac{5k(0)}{2c_3}E'(t)
\end{aligned}$$

$$+\frac{5}{4}\int_{t^*}^t k(s)\int_\Omega b(u(t)-u(t-s),u(t)-u(t-s))dxds,\ t\geq t^*.$$

Setting

$$F(t)=L(t)+\frac{5k(0)}{2c_3}E(t),$$

we get

$$F'(t)\leq -\rho E(t)+\frac{5}{4}\int_{t^*}^t k(s)\int_\Omega b(u(t)-u(t-s),u(t)-u(t-s))dxds\quad\text{for } t\geq t^*. \tag{3.22}$$

On the other hand, (3.11) and (3.12) gives

$$\begin{aligned}(L(t)+\Xi(t))' &\leq -\frac{1}{4}(k\Box\partial^2 u)-||u_t||^2-\alpha||\nabla u_t||^2-||\Delta v||^2-||\sqrt{rq}y||^2_{\Gamma_1}\\ &\quad -(1-l)\int_\Omega b(u,u)dx\\ &\leq -\min\{\frac{1}{2},2(1-l)\}E(t),\end{aligned}\tag{3.23}$$

and hence

$$\begin{aligned}\int_{t^*}^t E(s)ds &\leq -\frac{1}{\min\{\frac{1}{2},2(1-l)\}}\int_{t^*}^t(L'(s)+\Xi'(s))ds\\ &\leq \frac{L(t^*)+\Xi(t^*)}{\min\{\frac{1}{2},2(1-l)\}}\quad\text{for all } t\geq t^*.\end{aligned}$$

Thus, we obtain

$$0<\int_0^\infty E(s)ds=\int_0^{t^*}E(s)ds+\int_{t^*}^t E(s)ds<\infty. \tag{3.24}$$

Next, we put

$$\Gamma(t):=m\int_{t^*}^t\int_\Omega b(u(t)-u(t-s),u(t)-u(t-s))dxds$$

and

$$\chi(t):=-\int_{t^*}^t k'(s)\int_\Omega b(u(t)-u(t-s),u(t)-u(t-s))dxds.$$

Thanks to (3.24), we can select $0<m<1$ such that

$$\Gamma(t)<1\quad\text{for } t\geq t^*. \tag{3.25}$$

Moreover, (3.3) implies

$$\chi(t)\leq -(k'\Box\partial^2 u)(t)\leq -2E'(t). \tag{3.26}$$

Applying (**A**), the formula $K(\lambda w)\leq\lambda K(w)$ for $0\leq\lambda\leq 1$ and $w\in(0,\epsilon]$, Jensen's inequality, and the fact $m\int_{t^*}^t k(s)\int_\Omega b(u(t)-u(t-s),u(t)-u(t-s))dxds<\epsilon$, we infer

$$\chi(t)=-\frac{1}{m\Gamma(t)}\int_{t^*}^t\Gamma(t)k'(s)m\int_\Omega b(u(t)-u(t-s),u(t)-u(t-s))dxds$$

$$\begin{aligned}
&\geq \frac{1}{m\Gamma(t)} \int_{t^*}^{t} \Gamma(t)\zeta(s)K(k(s))m \int_{\Omega} b(u(t)-u(t-s),u(t)-u(t-s))dxds \\
&\geq \frac{\zeta(t)}{m\Gamma(t)} \int_{t^*}^{t} K\Big(\Gamma(t)k(s)\Big)m \int_{\Omega} b(u(t)-u(t-s),u(t)-u(t-s))dxds \\
&\geq \frac{\zeta(t)}{m} K\Big(m \int_{t^*}^{t} k(s) \int_{\Omega} b(u(t)-u(t-s),u(t)-u(t-s))dxds\Big) \\
&= \frac{\zeta(t)}{m} \overline{K}\Big(m \int_{t^*}^{t} k(s) \int_{\Omega} b(u(t)-u(t-s),u(t)-u(t-s))dxds\Big), \qquad (3.27)
\end{aligned}$$

where $\overline{K}$ is an extension of K, which is strictly increasing and strictly convex C^2 on $(0,\infty)$. Thus,

$$\int_{t^*}^{t} k(s) \int_{\Omega} b(u(t)-u(t-s),u(t)-u(t-s))dxds \leq \frac{1}{m}\overline{K}^{-1}\Big(\frac{m\chi(t)}{\zeta(t)}\Big).$$

Substituting this into (3.22), we find

$$F'(t) \leq -\rho E(t) + \frac{5}{4m}\overline{K}^{-1}\Big(\frac{m\chi(t)}{\zeta(t)}\Big) \quad \text{for } t \geq t^*. \qquad (3.28)$$

On the other hand, the convex function K has the properties

$$wz \leq K^*(w) + K(z) \quad \text{for } w, z \geq 0 \qquad (3.29)$$

and

$$K^*(w) = w(K')^{-1}(w) - K((K')^{-1}(w)) \quad \text{for } w \geq 0, \qquad (3.30)$$

where K^* is the conjugate function of K (see [1]).

Let $0 < \theta < \min\{\epsilon, \frac{4\rho m E(0)}{5}\}$, $\mathcal{E}(t) = \frac{E(t)}{E(0)}$, and $c_5 > 0$. Since $\overline{K}'(s) > 0$, $\overline{K}''(s) > 0$, $E'(t) \leq 0$, and $\overline{K}(0) = \overline{K}'(0) = 0$, we find from (3.28), (3.29), and (3.30) that

$$\begin{aligned}
&\Big[\overline{K}'(\theta\mathcal{E}(t))F(t) + c_5E(t)\Big]' \\
&\leq -\rho\overline{K}'(\theta\mathcal{E}(t))E(t) + \frac{5}{4m}\overline{K}'(\theta\mathcal{E}(t))\overline{K}^{-1}\Big(\frac{m\chi(t)}{\zeta(t)}\Big) + c_5E'(t) \\
&\leq -\rho\overline{K}'(\theta\mathcal{E}(t))E(t) + \frac{5}{4m}\overline{K}^*\Big(\overline{K}'(\theta\mathcal{E}(t))\Big) + \frac{5\chi(t)}{4\zeta(t)} + c_5E'(t) \\
&\leq -\rho\overline{K}'(\theta\mathcal{E}(t))E(t) + \frac{4\theta}{5m}\mathcal{E}(t)\overline{K}'(\theta\mathcal{E}(t)) + \frac{5\chi(t)}{4\zeta(t)} + c_5E'(t) \\
&= -\rho E(0)K'(\theta\mathcal{E}(t))\mathcal{E}(t) + \frac{5\theta}{4m}\mathcal{E}(t)K'(\theta\mathcal{E}(t)) + \frac{5\chi(t)}{4\zeta(t)} + c_5E'(t), \qquad (3.31)
\end{aligned}$$

where we used $\theta\mathcal{E}(t) < \epsilon$ in the last equality. Considering (3.31) and (3.26), we have

$$\begin{aligned}
&\Big[\zeta(t)\Big\{\overline{K}'(\theta\mathcal{E}(t))F(t) + c_5E(t)\Big\} + \frac{5}{2}E(t)\Big]' \\
&\leq -\rho E(0)\zeta(t)K'(\theta\mathcal{E}(t))\mathcal{E}(t) + \frac{5\theta}{4m}\zeta(t)\mathcal{E}(t)K'(\theta\mathcal{E}(t)) + \frac{5\chi(t)}{4} + c_5\zeta(t)E'(t) + \frac{5}{2}E'(t)
\end{aligned}$$

$$\leq -\rho E(0)\zeta(t)K'(\theta\mathcal{E}(t))\mathcal{E}(t) + \frac{5\theta}{4m}\zeta(t)\mathcal{E}(t)K'(\theta\mathcal{E}(t)) + c_5\zeta(t)E'(t)$$
$$\leq -\zeta(t)\Big(\rho E(0) - \frac{5\theta}{4m}\Big)K'(\theta\mathcal{E}(t))\mathcal{E}(t)$$
$$= -c_6\zeta(t)K_0(\mathcal{E}(t)) \quad \text{for } t \geq t^*, \tag{3.32}$$

where $c_6 = \rho E(0) - \frac{5\theta}{4m}$ and

$$K_0(s) = sK'(\theta s). \tag{3.33}$$

We also note

$$c_7 E(t) \leq \zeta(t)\Big\{\overline{K}'(\theta\mathcal{E}(t))F(t) + c_5 E(t)\Big\} + \frac{5}{2}E(t) \leq c_8 E(t).$$

Finally, setting

$$\mathcal{L}(t) = \frac{\zeta(t)\Big\{\overline{K}'(\theta\mathcal{E}(t))F(t) + c_5 E(t)\Big\} + \frac{5}{2}E(t)}{c_8 E(0)}, \tag{3.34}$$

we see that

$$\mathcal{L}(t) \leq \mathcal{E}(t) \leq 1. \tag{3.35}$$

Because K_0 is increasing on $(0,1]$, we deduce from (3.32), (3.34), and (3.35)

$$\mathcal{L}'(t) \leq -c_9\zeta(t)K_0(\mathcal{L}(t)) \quad \text{for } t \geq t^*, \tag{3.36}$$

where $c_9 = \frac{c_6}{c_8 E(0)}$. Integrating this over (t^*, t) and employing the integration by substitution, we get

$$\int_{t^*}^{t} c_9\zeta(s)ds \leq \int_{t}^{t^*} \frac{\mathcal{L}'(s)}{K_0(\mathcal{L}(s))}ds = \int_{t}^{t^*} \frac{\mathcal{L}'(s)}{\mathcal{L}(s)K'(\theta\mathcal{L}(s))}ds = \int_{\theta\mathcal{L}(t)}^{\theta\mathcal{L}(t^*)} \frac{1}{sK'(s)}ds$$
$$\leq \int_{\theta\mathcal{L}(t)}^{\epsilon} \frac{1}{sK'(s)}ds = \tilde{K}(\theta\mathcal{L}(t)), \tag{3.37}$$

here $\tilde{K}$ is the function defined in (3.19). Owing to $\tilde{K}$ is strictly decreasing on $(0,\epsilon]$, we conclude, for some $\omega > 0$,

$$\mathcal{L}(t) \leq \frac{1}{\theta}\tilde{K}^{-1}\Big(\omega\int_{t^*}^{t}\zeta(s)ds\Big) \quad \text{for } t \geq t^*.$$

This completes the proof. □

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