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DYNAMICS ANALYSIS OF THREE-SPECIES REACTION-DIFFUSION SYSTEM VIA THE MULTIPLE SCALE PERTURBATION METHOD*

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Abstract In this paper, the general analysis of spatiotemporal dynamics of three-species reaction-diffusion system induced by Turing bifurcation is given. Firstly, by employing the Routh-Hurwitz criterion the conditions for Turing bifurcation in three-species reaction-diffusion equations are obtained. Secondly, through the tool of the multiple scale perturbation method the amplitude equations of Turing patterns are also given. Finally, we take a three-species predator-prey model as an example to illustrate the application of these general theoretical results, and meanwhile carry out many numerical simulations to depict spots pattern, stripes pattern and labyrinthine pattern and demonstrate the validity of these theories.

Keywords Three-species reaction-diffusion system, spatiotemporal dynamics, multiple scale perturbation method.

MSC(2010) 35K57, 35B10.

1. Introduction

Since the Turing bifurcation was proposed by Turing [24] in 1952, it was favored by more and more scholars such as mathematicians [25], biologists [1, 9], chemists [8, 15], physicists [7, 12] and ecologists [13, 14] and so on. In recent decades, many researchers had focused their attention on mathematical models to understand the mechanism of the Turing bifurcation which can generate all kinds of spatial patterns. For example, Baurmann et al. [2] had investigated the emergence of spatiotemporal patterns of the spatially extended predator-prey systems; Zhang et al. [28] had studied spatial dynamics of the Beddington-DeAngelis predator-prey model by the tool of Turing instability; Song and Jiang [20] had found that periodic spot, stripe, labyrinth, and gap patterns exist in the Mussel-Algae model near the Turing-Hopf bifurcation point. They demonstrated that the Turing bifurcation, which leads to the different spatial patterns to occur, is that a homogeneous equilibrium point can

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be stable to homogeneous perturbations, but unstable to certain spatially varying perturbations.

These mathematical models, which were employed to investigate the mechanism of generating different spatial patterns, mainly include two species or substances. However, since biological and chemical systems usually involve more than two dynamically independent species, it is necessary to establish tree-species reaction-diffusion equations to reflect spatial distributions of species in our real world. For example, Parshad et al. [16] investigated the long time dynamics of a three-species food chain model and White and Gilligan [26] also studied the spatial heterogeneity of three-species system involving hosts, parasites and hyperparasite.

Although Qian [17], Satnoianu [19], Mukherjee [11], Song [22, 23], Jiang [3] and Zhang [27] have given the method of determining the parameter space for Turing bifurcation, to our best knowledge, there are few systematic theories about the Turing bifurcation including conditions for Turing bifurcation and the amplitude equations of Turing patterns of the general three-species reaction-diffusion equations. In this paper, we mainly study the general theories of Turing bifurcation of the three-species reaction-diffusion equations which can be described by the following form

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + f(u, v, w), \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + g(u, v, w), \\ \dfrac{\partial w}{\partial t} = d_3 \Delta w + h(u, v, w), \end{cases} \tag{1.1}$$

where $u = u(t, x)$, $v = v(t, x)$ and $w = w(t, x)$ for $t \in [0, +\infty)$ and $x \in \Omega \subset \mathbb{R}^2$ represent the density of three species at time t and at position x, $d_1 > 0$, $d_2 > 0$ and $d_3 > 0$ mean the diffusion coefficients of the species u, v and w, respectively, $\Delta = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ is Laplcian operator about the spatial variables X and Y, the functions $f(u, v, w), g(u, v, w)$ and $h(u, v, w)$ denote the kinetic reaction among the species u, v and w, and are all infinite order differentiable functions about variables u, v and w. Besides, system (1.1) has the following initial conditions

$$u(0, x) = \phi(x), v(0, x) = \psi(x), w(0, x) = \eta(x),$$

for $x \in \Omega$, and the homogeneous Neumann boundary conditions

$$\frac{\partial u(t, x)}{\partial n} = \frac{\partial v(t, x)}{\partial n} = \frac{\partial w(t, x)}{\partial n} = 0,$$

where ϕ, ψ and η are positive continuous functions about the variable x, n is the outward unit normal vector of the smooth boundary $\partial\Omega$.

In this paper, not only the conditions of occurrence of Turing bifurcation of three-species reaction-diffusion equations are given but also the amplitude equations of the different spatial patterns are presented by the multiple scale perturbation method. The obtained results would be very useful to analyze the spatiotemporal dynamics of many biological and ecological models [16, 26].

The paper is structured as follows. In Section 2, employing the Routh-Hurwitz criterion, we mainly give the conditions for occurrence of Turing bifurcation of system (1.1). The amplitude equations of Turing patterns are also obtained by using the tool of the multiple scale perturbation method in Section 3. In Section

4, in order to illustrate the applications of these theoretical results a three-species predator-prey model is introduced to obtain the conditions for Turing bifurcation and the amplitude equations of spatial Turing patterns. The corresponding many numerical simulations are carried out to demonstrate these theories of the Turing bifurcation. Finally, some conclusions are drawn in Section 5.

2. Conditions for Turing bifurcation

In order to get the conditions for the Turing bifurcation of the general three-species reaction-diffusion equations (1.1), we will take the diffusion coefficient d_1 as the bifurcation parameter and employ the Routh-Hurtiwz criterion to analyze the properties of one of the positive equilibrium points of equations (1.1).

Supposing that

$$\begin{cases} f(u,v,w)=0, \\ g(u,v,w)=0, \\ h(u,v,w)=0, \end{cases} \tag{2.1}$$

we will get one of the positive equilibrium points $E^* = (u^*, v^*, w^*)$. Near the positive equilibrium point E^* we need to rewrite equations (1.1) with the vector form as follows:

$$\frac{\partial U(t,x)}{\partial t} = D\Delta U(t,x) + L(\alpha)U(t,x) + N(U(t,x)), \tag{2.2}$$

where

$$U(t,x) = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, L(\alpha) = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix},$$

$$N(U(t,x)) = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}, l_{11} = \frac{\partial f(u,v,w)}{\partial u}|_{u=u^*,v=v^*,w=w^*},$$

$$l_{12} = \frac{\partial f(u,v,w)}{\partial v}|_{u=u^*,v=v^*,w=w^*},\ l_{13} = \frac{\partial f(u,v,w)}{\partial w}|_{u=u^*,v=v^*,w=w^*},$$

$$l_{21} = \frac{\partial g(u,v,w)}{\partial u}|_{u=u^*,v=v^*,w=w^*},\ l_{22} = \frac{\partial g(u,v,w)}{\partial v}|_{u=u^*,v=v^*,w=w^*},$$

$$l_{23} = \frac{\partial g(u,v,w)}{\partial w}|_{u=u^*,v=v^*,w=w^*},\ l_{31} = \frac{\partial h(u,v,w)}{\partial u}|_{u=u^*,v=v^*,w=w^*},$$

$$l_{32} = \frac{\partial h(u,v,w)}{\partial v}|_{u=u^*,v=v^*,w=w^*},\ l_{33} = \frac{\partial h(u,v,w)}{\partial w}|_{u=u^*,v=v^*,w=w^*},$$

and $N_1 = N_1(u,v,w)$, $N_2 = N_2(u,v,w)$ and $N_3 = N_3(u,v,w)$ represent the quadratic, cubic and even more higher order terms about u, v and w. We will analysis the linear stability of this positive equilibrium point E^* of system (2.2) to get the conditions for Turing bifurcation of system (1.1).

Let

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \exp(\lambda t + i\mathrm{k} \cdot r), \tag{2.3}$$

where c_1, c_2 and c_3 are coefficients, $\mathrm{k} = (k_X, k_Y)$, $\mathrm{k} \cdot \mathrm{k} = k^2$, k_X and k_Y respectively denote the wave numbers in the X-direction and Y-direction, and λ is the growth rate of perturbation in time t, i is the imaginary unit and $i^2 = -1$, $r = (X, Y)$ is the spatial vector in two-dimensional space. Thus, substituting equations (2.3) into the linear part of system (2.2), we have the characteristic equation

$$\lambda^3 + b_1(k^2)\lambda^2 + b_2(k^2)\lambda + b_3(k^2) = 0, \tag{2.4}$$

where $k^2 = k_X^2 + k_Y^2$, and

$$\begin{aligned} b_1(k^2) &= (d_1 + d_2 + d_3)k^2 - l_{11} - l_{22} - l_{33}, \\ b_2(k^2) &= (d_1k^2 - l_{11})(d_2k^2 - l_{22}) + (d_1k^2 - l_{11})(d_3k^2 - l_{33}) + (d_2k^2 - l_{22})(d_3k^2 - l_{33}) \\ &\quad - l_{12}l_{21} - l_{13}l_{31}, \\ b_3(k^2) &= (d_1k^2 - l_{11})(d_2k^2 - l_{22})(d_3k^2 - l_{33}) + l_{12}l_{21}(l_{33} - d_3k^2) + l_{13}l_{31}(l_{22} - d_2k^2). \end{aligned}$$

In the following our work, we will give the conditions for the Turing bifurcation of system (2.2) by using the Routh-Hurwtiz criterion to the characteristic equation (2.4).

Firstly, in the absence of diffusion of system (2.2), the characteristic equation (2.4) can be transformed into the following form

$$\lambda^3 + b_1(0)\lambda^2 + b_2(0)\lambda + b_3(0) = 0, \tag{2.5}$$

where

$$\begin{aligned} b_1(0) &= -l_{11} - l_{22} - l_{33}, \\ b_2(0) &= l_{11}l_{22} + l_{11}l_{33} + l_{22}l_{33} - l_{12}l_{21} - l_{13}l_{31}, \\ b_3(0) &= l_{22}l_{13}l_{31} + l_{33}l_{12}l_{21} - l_{11}l_{22}l_{33}. \end{aligned}$$

By using the Routh-Hurwitz criterion, the necessary and sufficient conditions for the characteristic equation (2.5) without diffusion to have the negative real part are

$$\begin{cases} (i) b_1(0) > 0, \\ (ii) b_3(0) > 0, \\ (iii) b_1(0)b_2(0) - b_3(0) > 0. \end{cases} \tag{2.6}$$

Secondly, introducing diffusion d_1, d_2, and d_3, we will give the conditions for occurring Turing bifurcation of system (2.2). Through the Routh-Hurwitz criterion we know that the Turing bifurcation occur only if one of the following conditions is violated:

$$\begin{cases} (i) b_1(k^2) > 0, \\ (ii) b_3(k^2) > 0, \\ (iii) b_1(k^2)b_2(k^2) - b_3(k^2) > 0. \end{cases}$$

Case 1. when $b_1(k^2) > 0$.
Because of $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $k^2 > 0$ and $b_1(0) > 0$, they lead to $b_1(k^2) > 0$. That is to say, the condition of $b_1(k^2) > 0$ is not violated.

Case 2. when $b_3(k^2) > 0$.
Let $H_2(k^2) = b_3(k^2)$ and $z = k^2$, then we have

$$H_2(z) = h_{23}z^3 + h_{22}z^2 + h_{21}z + h_{20},$$

where

$$\begin{aligned} h_{23} &= d_1d_2d_3, \\ h_{22} &= -d_1d_2l_{33} - d_1d_3l_{22} - d_2d_3l_{11}, \\ h_{21} &= d_1l_{22}l_{33} + d_2l_{11}l_{33} + d_3l_{11}l_{22} - d_1l_{23}l_{32} - d_2l_{13}l_{31} - d_3l_{12}l_{21}, \\ h_{20} &= l_{11}l_{23}l_{32} + l_{12}l_{21}l_{33} + l_{13}l_{31}l_{22} - l_{11}l_{22}l_{33} - l_{12}l_{23}l_{31} - l_{13}l_{21}l_{32}. \end{aligned}$$

Noticing that the coefficient $d_1d_2d_3$ of the first term $H_2(z)$ is always positive, we have the following results about $H_2(z)$:
(i) Because of $z = k^2$, $H_2(z) \to +\infty$ when $z \to +\infty$.
(ii) The first order derivative of $H_2(z)$ about z is

$$\frac{\mathrm{d}H_2(z)}{\mathrm{d}z} = 3h_{23}z^2 + 2h_{22}z + h_{21},$$

which is a quadratic expression in z and has the following two roots

$$z_{21} = \frac{-h_{22} + \sqrt{h_{22}^2 - 3h_{23}h_{21}}}{3h_{23}}$$

and

$$z_{22} = \frac{-h_{22} - \sqrt{h_{22}^2 - 3h_{23}h_{21}}}{3h_{23}}.$$

The second derivative of $H_2(z)$ about z is

$$\frac{\mathrm{d}^2H_2(z)}{\mathrm{d}z^2} = 6h_{23}z + 2h_{22}.$$

If the condition $b_3(k^2) > 0$ is violated, from the above analyses we can obtain the following conclusion about the Turing bifurcation of system (2.2).

Conclusion 1. The necessary and sufficient conditions for violating the conditions $b_3(k^2) > 0$ are alternative as follows:

$$\begin{aligned} &(i) h_{22}^2 - 3h_{23}h_{21} < 0, H_2(0) = h_{20} \le 0; \\ &(ii) h_{22}^2 - 3h_{23}h_{21} \ge 0, z_{2,min} = z_{21}, H_2(z_{21}) \le 0. \end{aligned}$$

Case 3. when $b_1(k^2)b_2(k^2) - b_3(k^2) > 0$.
Let $H_3(k^2) = b_1(k^2)b_2(k^2)$ and $z = k^2$, then we have

$$H_3(z) = h_{33}z^3 + h_{32}z^2 + h_{31}z + h_{30},$$

where

$$\begin{aligned} h_{33} &= (d_1 + d_2 + d_3)(d_1d_2 + d_1d_3 + d_2d_3), \\ h_{32} &= -(d_1 + d_2 + d_3)(d_1l_{22} + d_1l_{33} + d_2l_{11} + d_2l_{33} + d_3l_{11} + d_3l_{22}) \\ &\quad - (l_{11} + l_{22} + l_{33})(d_1d_2 + d_1d_3 + d_2d_3), \\ h_{31} &= (d_1 + d_2 + d_3)(l_{11}l_{22} + l_{11}l_{33} + l_{22}l_{33} - l_{12}l_{21} - l_{13}l_{31} - l_{23}l_{32}) \\ &\quad + (l_{11} + l_{22} + l_{33})(d_1l_{22} + d_1l_{33} + d_2l_{11} + d_2l_{33} + d_3l_{11} + d_3l_{22}), \\ h_{30} &= -(l_{11} + l_{22} + l_{33})(l_{11}l_{22} + l_{11}l_{33} + l_{22}l_{33} - l_{12}l_{21} - l_{13}l_{31} - l_{23}l_{32}). \end{aligned}$$

Let $H_4(z) = H_4(k^2) = b_1(k^2)b_2(k^2) - b_3(k^2) = H_3(z) - H_2(z)$, then

$$H_4(z) = h_{43}z^3 + h_{42}z^2 + h_{41}z + h_{40},$$

where $h_{43} = h_{33} - h_{23}$, $h_{42} = h_{32} - h_{22}$, $h_{41} = h_{31} - h_{21}$, $h_{40} = h_{30} - h_{20}$. Noticing that the coefficient $(d_1 + d_2 + d_3)(d_1d_2 + d_2d_3 + d_3d_1)$ of the first term $H_4(z)$ is always positive. Similarly, we can obtain the following properties about $H_4(z)$:
(i) Because of $z = k^2$, $H_2(z) \to +\infty$ when $z \to +\infty$.
(ii) The first order derivative of $H_4(z)$ about z is

$$\frac{\mathrm{d}H_4(z)}{\mathrm{d}z} = 3h_{43}z^2 + 2h_{42}z + h_{41},$$

which is a quadratic expression in z and has the following two roots

$$z_{41} = \frac{-h_{42} + \sqrt{h_{42}^2 - 3h_{43}h_{41}}}{3h_{43}}$$

and

$$z_{42} = \frac{-h_{42} - \sqrt{h_{42}^2 - 3h_{43}h_{41}}}{3h_{43}}.$$

The second derivative of $H_2(z)$ about z is

$$\frac{\mathrm{d}^2H_4(z)}{\mathrm{d}z^2} = 6h_{43}z + 2h_{42}.$$

If the condition $b_1(k^2)b_2(k^2) - b_3(k^2) > 0$ is violated, from the above analyses, we can also get the following conclusion.

Conclusion 2. The necessary and sufficient conditions for violating the conditions $b_1(k^2)b_2(k^2) - b_3(k^2) > 0$ are alternative as follows:

$$\begin{aligned} &(i) h_{42}^2 - 3h_{43}h_{41} < 0, H_4(0) = H_{40} \le 0, \\ &(ii) h_{42}^2 - 3h_{43}h_{41} \ge 0, z_{4,min} = z_{41}, H_4(z_{41}) \le 0. \end{aligned}$$

Collecting conclusions 1 and 2, we can obtain the following theorem above the Turing bifurcation of system (2.2).

Theorem 2.1. *The necessary and sufficient conditions for Turing bifurcation of system (2.2) are as follows:*

$$(i)h_{22}^2 - 3h_{23}h_{21} > 0, (ii)z_{2,min} = z_{21} > 0,$$
$$(iii)H_2(z_{2,min}) = H_2(z_{21}) < 0, (iv)h_{42}^2 - 3h_{43}h_{41} \geq 0,$$
$$(v)z_{4,min} = z_{41}, H_4(z_{4,min}) = H_4(z_{41}) > 0, (vi)b_1(0) > 0,$$
$$(vii)b_3(0) > 0, (viii)b_1(0)b_2(0) - b_3(0) > 0.$$

or

$$(i)h_{22}^2 - 3h_{23}h_{21} > 0, (ii)z_{2,min} = z_{21} > 0,$$
$$(iii)H_2(z_{2,min}) = H_2(z_{21}) < 0, (iv)h_{42}^2 - 3h_{43}h_{41} < 0,$$
$$(v)h_{40} > 0, (vi)b_1(0) > 0,$$
$$(vii)b_3(0) > 0, (viii)b_1(0)b_2(0) - b_3(0) > 0.$$

3. Amplitude Equations of Turing patterns via the Multiple scale method

In this section, we will employ the multiple scale perturbation method to obtain the amplitude equations of Turing patterns because close to the Turing bifurcation threshold the dynamics of system (2.2) varies very slowly. By selecting the diffusion coefficient d_1 as a bifurcation parameter and denoting $d_1 = \alpha$, all kinds of different patterns are investigated as the parameter α varies. In order to get the amplitude equations of Turing patterns, we mainly divide into four steps in the following work.

Step 1. we need to rewrite system as Taylor series to three order terms as follows:

$$\frac{\partial U}{\partial t} = L_c U + (\alpha - \alpha_c) M \Delta U + N_2(U) + N_3(U), \tag{3.1}$$

where

$$U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad L_c = \begin{pmatrix} f_u + \alpha_c \Delta & f_v & f_w \\ g_u & g_v + d_2 \Delta & g_w \\ h_u & h_v & h_w + d_3 \Delta \end{pmatrix},$$

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N_2(U) = \frac{1}{2} \begin{pmatrix} F_{uvw}^2 \\ G_{uvw}^2 \\ H_{uvw}^2 \end{pmatrix}, \ N_3(U) = \frac{1}{6} \begin{pmatrix} F_{uvw}^3 \\ G_{uvw}^3 \\ H_{uvw}^3 \end{pmatrix}$$

and

$$f_u = l_{11}, f_v = l_{12}, f_w = l_{13}, g_u = l_{21}, g_v = l_{22}, g_w = l_{23}, h_u = l_{31}, h_v = l_{32}, h_w = l_{33},$$
$$F_{uvw}^2 = f_{uu}u^2 + f_{vv}v^2 + f_{ww}w^2 + 2f_{uv}uv + 2f_{uw}uw + 2f_{vw},$$
$$G_{uvw}^2 = g_{uu}u^2 + g_{vv}v^2 + g_{ww}w^2 + 2g_{uv}uv + 2g_{uw}uw + 2g_{vw},$$

$$H^2_{uvw} = h_{uu}u^2 + h_{vv}v^2 + h_{ww}w^2 + 2h_{uv}uv + 2h_{uw}uw + 2h_{vw},$$
$$F^3_{uvw} = 3f_{uuv}u^2v + 3f_{uuw}u^2w + 3f_{vvu}v^2u + 3f_{vvw}v^2w + 3f_{wwu}w^2u + 3f_{wwv}w^2v + 6f_{uvw}uvw + f_{uuu}u^3 + f_{vvv}v^3 + f_{www}w^3,$$
$$G^3_{uvw} = 3g_{uuv}u^2v + 3g_{uuw}u^2w + 3g_{vvu}v^2u + 3g_{vvw}v^2w + 3g_{wwu}w^2u + 3g_{wwv}w^2v + 6g_{uvw}uvw + g_{uuu}u^3 + g_{vvv}v^3 + g_{www}w^3,$$
$$H^3_{uvw} = 3h_{uuv}u^2v + 3h_{uuw}u^2w + 3h_{vvu}v^2u + 3h_{vvw}v^2w + 3h_{wwu}w^2u + 3h_{wwv}w^2v + 6h_{uvw}uvw + h_{uuu}u^3 + h_{vvv}v^3 + h_{www}w^3,$$
$$f_{uu} = \frac{\partial^2 f}{\partial u^2}|_{u=u^*,v=v^*,w=w^*},\quad f_{vv} = \frac{\partial^2 f}{\partial v^2}|_{u=u^*,v=v^*,w=w^*},$$
$$f_{ww} = \frac{\partial^2 f}{\partial w^2}|_{u=u^*,v=v^*,w=w^*},$$
$$f_{uv} = \frac{\partial^2 f}{\partial u \partial v}|_{u=u^*,v=v^*,w=w^*},\quad f_{uw} = \frac{\partial^2 f}{\partial u \partial w}|_{u=u^*,v=v^*,w=w^*},$$
$$f_{vw} = \frac{\partial^2 f}{\partial v \partial w}|_{u=u^*,v=v^*,w=w^*},$$
$$g_{uu} = \frac{\partial^2 g}{\partial u^2}|_{u=u^*,v=v^*,w=w^*},\quad g_{vv} = \frac{\partial^2 g}{\partial v^2}|_{u=u^*,v=v^*,w=w^*},$$
$$g_{ww} = \frac{\partial^2 g}{\partial w^2}|_{u=u^*,v=v^*,w=w^*},$$
$$g_{uv} = \frac{\partial^2 g}{\partial u \partial v}|_{u=u^*,v=v^*,w=w^*},\quad g_{uw} = \frac{\partial^2 g}{\partial u \partial w}|_{u=u^*,v=v^*,w=w^*},$$
$$g_{vw} = \frac{\partial^2 g}{\partial v \partial w}|_{u=u^*,v=v^*,w=w^*},$$
$$h_{uu} = \frac{\partial^2 h}{\partial u^2}|_{u=u^*,v=v^*,w=w^*},\quad h_{vv} = \frac{\partial^2 h}{\partial v^2}|_{u=u^*,v=v^*,w=w^*},$$
$$h_{ww} = \frac{\partial^2 h}{\partial w^2}|_{u=u^*,v=v^*,w=w^*},$$
$$h_{uv} = \frac{\partial^2 h}{\partial u \partial v}|_{u=u^*,v=v^*,w=w^*},\quad h_{uw} = \frac{\partial^2 h}{\partial u \partial w}|_{u=u^*,v=v^*,w=w^*},$$
$$h_{vw} = \frac{\partial^2 h}{\partial v \partial w}|_{u=u^*,v=v^*,w=w^*},$$
$$f_{uuu} = \frac{\partial^3 f}{\partial u^3}|_{u=u^*,v=v^*,w=w^*},\quad f_{uuv} = \frac{\partial^3 f}{\partial u^2 \partial v}|_{u=u^*,v=v^*,w=w^*},$$
$$f_{uvv} = \frac{\partial^3 f}{\partial u \partial v^2}|_{u=u^*,v=v^*,w=w^*},\quad f_{vvv} = \frac{\partial^3 f}{\partial v^3}|_{u=u^*,v=v^*,w=w^*},$$
$$g_{uuu} = \frac{\partial^3 g}{\partial u^3}|_{u=u^*,v=v^*,w=w^*},\quad g_{uuv} = \frac{\partial^3 g}{\partial u^2 \partial v}|_{u=u^*,v=v^*,w=w^*},$$
$$g_{uvv} = \frac{\partial^3 g}{\partial u \partial v^2}|_{u=u^*,v=v^*,w=w^*},\quad g_{vvv} = \frac{\partial^3 g}{\partial v^3}|_{u=u^*,v=v^*,w=w^*},$$
$$h_{uuu} = \frac{\partial^3 h}{\partial u^3}|_{u=u^*,v=v^*,w=w^*},\quad h_{uuv} = \frac{\partial^3 h}{\partial u^2 \partial v}|_{u=u^*,v=v^*,w=w^*},$$
$$h_{uvv} = \frac{\partial^3 h}{\partial u \partial v^2}|_{u=u^*,v=v^*,w=w^*},\quad h_{vvv} = \frac{\partial^3 h}{\partial v^3}|_{u=u^*,v=v^*,w=w^*}.$$

Step 2. according to the multiple scale perturbation method, defining

$$\begin{cases} u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots, \\ v = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots, \\ w = \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \cdots, \\ t = t_0 + \epsilon t_1 + \epsilon^2 t_2 + \cdots, \\ \alpha = \alpha_c + \epsilon \alpha_1 + \epsilon^2 \alpha_2 + \cdots, \end{cases} \tag{3.2}$$

and substituting equations (3.2) into equation (3.1), we will get the following results about ϵ, ϵ^2 and ϵ^3, respectively.

Step 3. noting that $\frac{\partial u_1}{\partial t_0} = 0$, $\frac{\partial v_1}{\partial t_0} = 0$ and $\frac{\partial w_1}{\partial t_0} = 0$, for the first order of ϵ, the solution of linear problem of (3.1) is

$$L_c \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = 0. \tag{3.3}$$

Since L_c is the linear operator of system (3.1) at the Turing bifurcation threshold, $(u_1\ \ v_1\ \ w_1)^T$ is the linear combination of the eigenvectors corresponding to the eigenvalue 0. Solving equation (3.3), we can get

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ 1 \end{pmatrix} \sum_{j=1}^{3} W_j \exp(i\mathrm{k}_j \cdot r) + \text{c.c.}, \tag{3.4}$$

where $\delta_1 = \frac{f_v g_w - f_w (g_v - d_2 k_c^2)}{(f_u - \alpha_c k_c^2)(g_v - d_2 k_c^2) - f_v g_u}$, $\delta_2 = \frac{-g_w - g_u \delta_1}{g_v - d_2 k_c^2}$, W_j is the amplitude of mode $\exp(i\mathrm{k}_j \cdot r)(j = 1, 2, 3)$ and c.c. denotes the conjugate of the former terms.

For the second order of ϵ^2,

$$L_c \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \frac{\partial}{\partial t_1} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \alpha_1 M \Delta \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} F_{uvw1}^2 \\ G_{uvw1}^2 \\ H_{uvw1}^2 \end{pmatrix}, \tag{3.5}$$

where

$$\begin{aligned} F_{uvw1}^2 &= f_{uu} u_1^2 + f_{vv} v_1^2 + f_{ww} w_1^2 + 2 f_{uv} u_1 v_1 + 2 f_{uw} u_1 w_1 + 2 f_{vw} v_1 w_1, \\ G_{uvw1}^2 &= g_{uu} u_1^2 + g_{vv} v_1^2 + g_{ww} w_1^2 + 2 g_{uv} u_1 v_1 + 2 g_{uw} u_1 w_1 + 2 g_{vw} v_1 w_1, \\ H_{uvw1}^2 &= h_{uu} u_1^2 + h_{vv} v_1^2 + h_{ww} w_1^2 + 2 h_{uv} u_1 v_1 + 2 h_{uw} u_1 w_1 + 2 h_{vw} v_1 w_1. \end{aligned}$$

Let the right hand of equations (3.5) as follows

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \frac{\partial}{\partial t_1}\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \alpha_1 M\Delta \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} F_{uvw1}^2 \\ G_{uvw1}^2 \\ H_{uvw1}^2 \end{pmatrix}. \tag{3.6}$$

In order to satisfy Fredholm solvability conditions, the vector function on the right hand of equation (3.5) need to be orthogonal to the eigenvectors of the zero eigenvalue of L_c^* which is the adjoint of L_c and has the following form

$$L_c^* = \begin{pmatrix} f_u + \alpha_c\Delta & g_u & h_u \\ f_v & g_v + d_2\Delta & h_v \\ f_w & g_w & h_w + d_3\Delta \end{pmatrix}.$$

The eigenvectors of the operator L_c^* are

$$\begin{pmatrix} 1 \\ \gamma_2 \\ \gamma_1 \end{pmatrix} \exp(-i\mathrm{k}_j \cdot r) + c.c.(j = 1, 2, 3),$$

where $\gamma_1 = \frac{f_v g_u - (f_u - \alpha_c k_c^2)(g_v - d_2 k_c^2)}{h_v g_u + h_u(g_v - d_2 k_c^2)}, \gamma_2 = \frac{\alpha_c k_c^2 - f_u - h_u\gamma_1}{g_u}$. The orthogonality condition is given by

$$\begin{pmatrix} 1 & \gamma_2 & \gamma_1 \end{pmatrix} \begin{pmatrix} F_x^j \\ F_y^j \\ F_z^j \end{pmatrix} = 0, \tag{3.7}$$

where F_x^j, F_y^j and F_z^j represent the coefficients of $\exp(i\mathrm{k}_j \cdot r)$ term in F_x, F_y and F_z of equation (3.6)$(j = 1, 2, 3)$, respectively. Taking the $\exp(i\mathrm{k}_1 \cdot r)$ as an example and connecting equations (3.4), we have

$$\begin{pmatrix} F_x^1 \\ F_y^1 \\ F_z^1 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ 1 \end{pmatrix} \frac{\partial W_1}{\partial t_1} + \alpha_1 k_c^2 M \begin{pmatrix} \delta_1 \\ \delta_2 \\ 1 \end{pmatrix} W_1 - \begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix} \overline{W}_2\, \overline{W}_3, \tag{3.8}$$

where

$$f_2 = f_{uu}\delta_1^2 + f_{vv}\delta_2^2 + f_{ww} + 2f_{uv}\delta_1\delta_2 + 2f_{uw}\delta_1 + 2f_{vw}\delta_2,$$
$$g_2 = g_{uu}\delta_1^2 + g_{vv}\delta_2^2 + g_{ww} + 2g_{uv}\delta_1\delta_2 + 2g_{uw}\delta_1 + 2g_{vw}\delta_2,$$
$$h_2 = h_{uu}\delta_1^2 + h_{vv}\delta_2^2 + h_{ww} + 2h_{uv}\delta_1\delta_2 + 2h_{uw}\delta_1 + 2h_{vw}\delta_2.$$

Using the Fredholm solvability condition (3.7) and connecting equation (3.8) we get

$$C\frac{\partial W_1}{\partial t_1} = -k_c^2\alpha_1\delta_1 W_1 + (f_2 + \gamma_2 g_2 + \gamma_1 h_2)\overline{W}_2\, \overline{W}_3, \tag{3.9}$$

where $C = \delta_1 + \gamma_2\delta_2 + \gamma_1$. Similarly, taking the coefficients of $\exp(ik_2 \cdot r)$ and $\exp(ik_3 \cdot r)$, we also obtain the following results

$$\begin{cases} C\dfrac{\partial W_2}{\partial t_1} = -k_c^2\alpha_1\delta_1 W_2 + (f_2 + \gamma_2 g_2 + \gamma_1 h_2)\overline{W}_1\ \overline{W}_3, \\ C\dfrac{\partial W_3}{\partial t_1} = -k_c^2\alpha_1\delta_1 W_3 + (f_2 + \gamma_2 g_2 + \gamma_1 h_2)\overline{W}_2\ \overline{W}_1. \end{cases} \tag{3.10}$$

Solution of equation (3.5) has the following form

$$\begin{aligned} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} &= \begin{pmatrix} X_{00} \\ Y_{00} \\ Z_{00} \end{pmatrix} + \sum_{j=1}^{3} \begin{pmatrix} X_j \\ Y_j \\ Z_j \end{pmatrix} \exp(i\mathrm{k}_j \cdot r) + \begin{pmatrix} X_{jj} \\ Y_{jj} \\ Z_{jj} \end{pmatrix} \exp(2i\mathrm{k}_j \cdot r) \\ &+ \begin{pmatrix} X_{12} \\ Y_{12} \\ Z_{12} \end{pmatrix} \exp(i(\mathrm{k}_1 - \mathrm{k}_2) \cdot r) + \begin{pmatrix} X_{23} \\ Y_{23} \\ Z_{23} \end{pmatrix} \exp(i(\mathrm{k}_2 - \mathrm{k}_3) \cdot r) \\ &+ \begin{pmatrix} X_{31} \\ Y_{31} \\ Z_{31} \end{pmatrix} \exp(i(\mathrm{k}_3 - \mathrm{k}_1) \cdot r) + \text{c.c.}, \end{aligned} \tag{3.11}$$

where the calculations formulas of $X_{00}, Y_{00}, Z_{00}, X_j, Y_j, Z_j (j = 1, 2, 3), X_{11}, Y_{11}, Z_{11}, X_{jj}, Y_{jj}, Z_{jj} (j = 1, 2, 3), X_{12}, Y_{12}, Z_{12}, X_{23}, Y_{23}, Z_{23}, X_{31}, Y_{31}, Z_{31}$ see Appendix A.

For the third order of ϵ^3

$$L_c \begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} = \tfrac{\partial}{\partial t_1} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} + \tfrac{\partial}{\partial t_2} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \alpha_1 M\Delta \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} - \alpha_2 M\Delta \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \begin{pmatrix} n_{x\epsilon^3} \\ n_{y\epsilon^3} \\ n_{z\epsilon^3} \end{pmatrix}, \tag{3.12}$$

where the calculation formulas of $n_{x\epsilon^3}, n_{y\epsilon^3}, n_{z\epsilon^3}$ see Appendix B.

Let the right hand of equation (3.12) as follows

$$\begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix} = \tfrac{\partial}{\partial t_1} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} + \tfrac{\partial}{\partial t_2} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \alpha_1 M\Delta \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} - \alpha_2 M\Delta \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} - \begin{pmatrix} n_{x\epsilon^3} \\ n_{y\epsilon^3} \\ n_{z\epsilon^3} \end{pmatrix}. \tag{3.13}$$

Collecting the coefficients for $\exp(i\mathrm{k}_1 \cdot r)$ from the right hand of equation (3.13) we find

$$\begin{pmatrix} \delta_1(\frac{\partial Z_1}{\partial t_1}+\frac{\partial W_1}{\partial t_2}) \\ \delta_2(\frac{\partial Z_1}{\partial t_1}+\frac{\partial W_1}{\partial t_2}) \\ (\frac{\partial Z_1}{\partial t_1}+\frac{\partial W_1}{\partial t_2}) \end{pmatrix} + \alpha_1 k_c^2 \begin{pmatrix} \delta_1 \\ 0 \\ 0 \end{pmatrix} Z_1 + \alpha_2 k_c^2 \begin{pmatrix} \delta_1 \\ 0 \\ 0 \end{pmatrix} W_1 - \begin{pmatrix} G_{2x}^{k_1} \\ G_{2y}^{k_1} \\ G_{2z}^{k_1} \end{pmatrix} - \frac{1}{6} \begin{pmatrix} G_{3x}^{k_1} \\ G_{3y}^{k_1} \\ G_{3z}^{k_1} \end{pmatrix}, \tag{3.14}$$

where the expressions of $G_{2x}^{k_1}, G_{2y}^{k_1}, G_{2z}^{k_1}, G_{3x}^{k_1}, G_{3y}^{k_1}, G_{3z}^{k_1}$ see Appendix C.

Using the Fredholm orthogonality condition to equation (3.14), it can be transformed into

$$C(\frac{\partial W_1}{\partial t_2}+\frac{\partial Z_1}{\partial t_1})=EZ_1+FW_1+H(\overline{W}_2\overline{Z}_3+\overline{W}_3\overline{Z}_2)+[G_1|W_1|^2+G_2(|W_2|^2+|W_3|^2)]W_1, \tag{3.15}$$

where the values of C, E, F, H, G_1 and G_2 see Appendix D.

Similarly, the other two equations can also be obtained by the transformation of the subscript of W and Z.

$$\begin{cases} C(\frac{\partial W_2}{\partial t_2}+\frac{\partial V_2}{\partial t_1}) =EV_2+FW_2+H(\overline{W}_3\overline{V}_1+\overline{W}_1\overline{V}_3)+[G_1|W_2|^2 \\ \qquad\qquad\qquad + G_2(|W_3|^2+|W_1|^2)]W_2, \\ C(\frac{\partial W_3}{\partial t_2}+\frac{\partial V_3}{\partial t_1}) =EV_3+FW_3+H(\overline{W}_1\overline{V}_2+\overline{W}_2\overline{V}_1)+[G_1|W_3|^2 \\ \qquad\qquad\qquad + G_2(|W_1|^2+|W_2|^2)]W_3. \end{cases} \tag{3.16}$$

Step 4. the amplitude of A_j $(j=1,2,3)$ can be expanded as

$$A_j=\epsilon W_j+\epsilon^2 Z_j+0(\epsilon^2). \tag{3.17}$$

By the expression of A_j $(j=1,2,3)$ of (3.17) and combining equations (3.9), (3.10), (3.15) and (3.16), we can get the amplitude equation corresponding to A_j $(j=1,2,3)$ as

$$\begin{cases} \tau_0\frac{\partial A_1}{\partial t}=\mu A_1+h\overline{A}_2\overline{A}_3-[g_1|A_1|^2+g_2(|A_2|^2+|A_3|^2)]A_1, \\ \tau_0\frac{\partial A_2}{\partial t}=\mu A_1+h\overline{A}_2\overline{A}_3-[g_1|A_2|^2+g_2(|A_1|^2+|A_3|^2)]A_2, \\ \tau_0\frac{\partial A_3}{\partial t}=\mu A_1+h\overline{A}_2\overline{A}_3-[g_1|A_3|^2+g_2(|A_2|^2+|A_2|^2)]A_3, \end{cases} \tag{3.18}$$

where

$$\tau_0=\frac{C}{\alpha_c\delta_1k_c^2},\quad \mu=\frac{\alpha_c-\alpha}{\alpha_c},\quad h=\frac{H}{\alpha_c\delta_1k_c^2},\quad g_1=-\frac{G_1}{\alpha_c\delta_1k_c^2},\quad g_2=-\frac{G_2}{\alpha_c\delta_1k_c^2}.$$

By the linear stability analysis of the above amplitude equations (3.18), these amplitude equations may be expressed as

$$A_j=\rho_j\exp(i\theta_j),\ (j=1,2,3),$$

where ρ_j and θ_j respectively denote the mode and the corresponding phase angle. Substituting them into equations (3.18) and separating the real and imaginary parts yields, we also obtain the following differential equations of the real variables

$$\begin{cases} \tau_0 \dfrac{\partial \theta}{\partial t} = -h \dfrac{\rho_1^2\rho_2^2 + \rho_1^2\rho_3^2 + \rho_2^2\rho_3^2}{\rho_1\rho_2\rho_3} \sin\theta, \\ \tau_0 \dfrac{\partial \rho_1}{\partial t} = \mu\rho_1 + h\rho_2\rho_3 \cos\theta - g_1\rho_1^3 - g_2(\rho_2^2 + \rho_3^2)\rho_1, \\ \tau_0 \dfrac{\partial \rho_2}{\partial t} = \mu\rho_2 + h\rho_1\rho_3 \cos\theta - g_1\rho_2^3 - g_2(\rho_1^2 + \rho_3^2)\rho_2, \\ \tau_0 \dfrac{\partial \rho_3}{\partial t} = \mu\rho_3 + h\rho_1\rho_2 \cos\theta - g_1\rho_3^3 - g_2(\rho_1^2 + \rho_2^2)\rho_3, \end{cases}$$

where $\theta = \theta_1 + \theta_2 + \theta_3$.

To summary the above analyses, the following results can be obtained about system (2.2).

Theorem 3.1. *(i)The homogeneous stationary state, given by*

$$\rho_1 = \rho_2 = \rho_3$$

is stable for $\mu < \mu_2 = 0$ and unstable for $\mu > \mu_2 = 0$.

(ii) Stripe patterns S is also given by

$$\rho_1 = \sqrt{\frac{\mu}{g_1}} \text{ and } \rho_2 = \rho_3 = 0,$$

which is stable for $\mu > \mu_3 = \frac{h^2 g_1}{(g_2-g_1)^2}$ and unstable $\mu < \mu_3 = \frac{h^2 g_1}{(g_2-g_1)^2}$.

(iii) Hexagon patterns H_0 are H_π are given by

$$\rho_1 = \rho_2 = \rho_3 = \frac{|h| \pm \sqrt{h^2 + 4(g_1 + 2g_2)\mu}}{2(g_1 + 2g_2)},$$

with $\theta = 0$ or $\theta = \pi$, and exist when $\mu > \mu_1 = \frac{-h^2}{4(g_2-g_1)}$. The solution $\rho^+ = \frac{|h|+\sqrt{h^2+4(g_1+2g_2)\mu}}{2(g_1+2g_2)}$ is stable only for $\mu < \mu_4 = \frac{h^2(2g_1+g_2)}{(g_2-g_1)^2}$, and the solution $\rho^- = \frac{|h|-\sqrt{h^2+4(g_1+2g_2)\mu}}{2(g_1+2g_2)}$ is always unstable.

(iv) The mixed states are given by

$$\rho_1 = \frac{|h|}{g_2 - g_1}, \ \rho_2 = \rho_3 = \sqrt{\frac{\mu - g_1\rho_1^2}{g_1 + g_2}},$$

which exist for $g_2 > g_1$ and $\mu > \mu_3$, and is always unstable.

4. Applications

In this section we mainly consider the following three-species prey-predator model with ratio-dependent functional responses proposed by Hsu et al. [6]

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u(1-u) - \dfrac{a_1 uv}{u+v}, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v - c_1 v + \dfrac{b_1 u(t,x) v}{u+v} - \dfrac{a_2 vw}{v+w}, \\ \dfrac{\partial w}{\partial t} = d_3\Delta w + \dfrac{b_2 vw}{v+w} - c_2 w, \end{cases} \tag{4.1}$$

where $u = u(t, x), v = v(t, x)$ and $w = w(t, x)$ respectively present the population densities of prey at lowest level of the food chain, intermediate predator that prey upon u, and top predator that prey upon v at time t and position x. All parameters are positive constants, where a_1 and a_2 denote the maximum ingestion rates of intermediate predator v and top predator w; b_1 and b_2 are the conversion factors of prey to intermediate predator and the top predator; c_1 and c_2 mean the mortality rates of the intermediate predator and the top predator, respectively.

Through the calculations we know that when the conditions $0 < a_1 b_1(b_1 - c_1) < b_1 b_2 + a_1 a_2(b_2 - c_2)$ and $0 < a_2(b_2 - c_2) < b_2(b_1 - c_1)$ are satisfied system (4.1) has the one positive real equilibrium point $E^* = (u^*, v^*, w^*)$ which corresponds the coexistence state, where $u^* = \frac{b_1 b_2 - a_1 b_2(b_1 - c_1) + a_1 a_2(b_2 - c_2)}{b_1 b_2}$, $w^* = \frac{b_2 - c_2}{c_2}$, $v^* = \frac{b_2(b_1 - c_1) - a_2(b_2 - c_2)}{b_2 c_1 + a_2(b_2 - c_2)} u^*$. According to the kinetic reaction functions $f(u, v, w) = u(1 - u) - \frac{a_1 uv}{u+v}$, $g(u, v, w) = -c_1 v + \frac{b_1 uv}{u+v} - \frac{a_2 vw}{v+w}$ and $h(u, v, w) = \frac{b_2 vw}{v+w} - c_2 w$, and connecting the theoretical analysis of the equilibrium point E^* in the second section and the results of Theorem 1, we will obtain the red region of Turing bifurcation space which is spanned by parameters d_1 and a_1, see Figure 1. And the other parameters are fixed $a_2 = 0.52, b_1 = 1.5, b_2 = 2.0, c_1 = 0.61, c_2 = 1.05, d_2 = 0.2, d_3 = 1$.

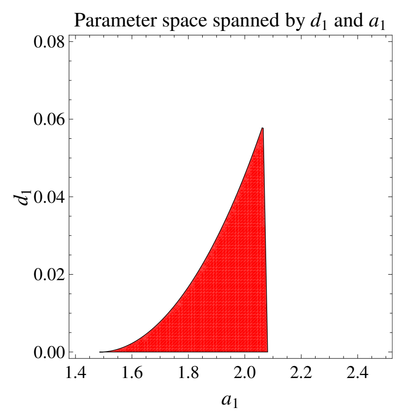

Figure 1. The red domain denotes the parameter space of Turing bifurcation spanned by d_1 and a_1 according to Theorem 1.

According to Fig. 1, if we fix the value of $a_1 = 2.0$ and take the diffusion coefficient d_1 as the bifurcation parameter, we can get the critical Turing bifurcation value of d_1 and denote $d_{1T} = 0.0456$. when $d_1 = 0.045$, using the theories of multiple scale method, we can calculate the following values μ, μ_1, μ_2, μ_3 and μ_4 in Theorem 2. In this case, system (4.1) will have the stable spots pattern because of $\mu_1 < \mu < \mu_3$. Similarly, when $d_1 = 0.01$, we can also calculate the value μ, μ_1, μ_2, μ_3 and μ_4 in Theorem 2. In this case, because of $\mu_4 < \mu$ system (4.1) will have the stable stipes pattern by taking different initial conditions. According to the results of the above analyses, we know that when we take the value of $d_1 = 0.045$, system (4.1) has the spots pattern. Employing the numerical calculation method in [4] and taking the initial state as the perturbation of the nontrivial equilibrium point E^*, $\Omega = [0, 50] \times [0, 50]$, $\Delta x = 0.25$ and $\Delta t = 0.01$, we plot the time evolution of spatial pattern for the prey u at the different time $t = 0, 10, 1000, 100000$, see Figure 2. Similarly, when we take the value of $d_1 = 0.01$, system (4.1) will have the stripes pattern by selecting the initial conditions as the periodical perturbations.

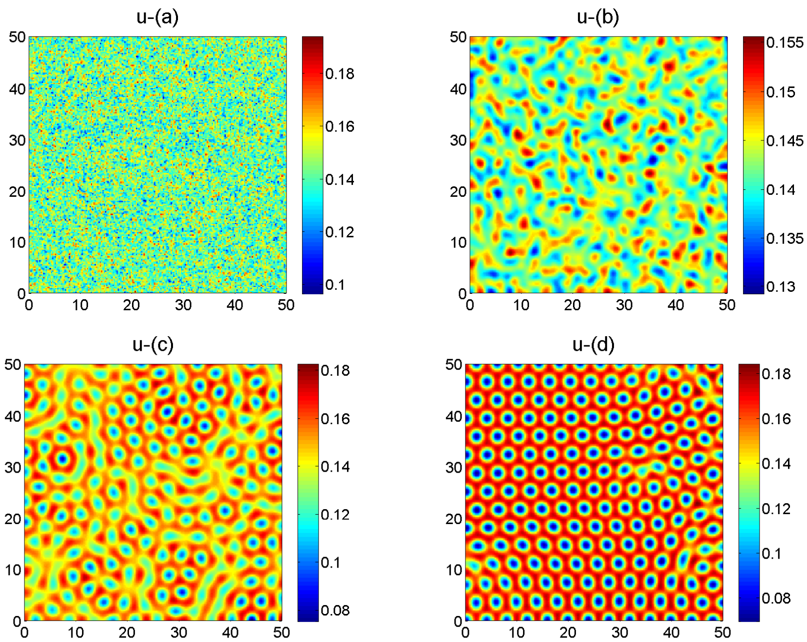

Figure 2. The time evolution plots of the density of the prey u at the different $t = 0$, $t = 10, t = 1000, t = 100000$ respectively when the diffusion coefficient $d_1 = 0.045$.

The corresponding time evolution of spatial patterns for the prey u at the different time $t = 0, 10, 1000, 100000$, respectively, see Figure 3.

In addition, when we take the value of $d_1 = 0.02$, system (4.1) will have the labyrinthine pattern by selecting the random or periodical initial perturbations. The corresponding time evolution of spatial patterns for the prey u at the different time $t = 0, 10, 1000, 100000$, respectively, see Figures 4-5.

5. Conclusions

In this paper, we have given the systematical theories of the Turing bifurcation of the general three-species reaction-diffusion equations, including the conditions for Turing bifurcation and the corresponding amplitude equations which describe the sufficient criteria of occurrence of the different Turing patterns. These obtained theoretical results have generalized the applications of Turing bifurcation theories in the two-species reaction-diffusion system [10, 18, 21].

In addition, we employ the obtained results to investigate a three-species prey-predator model with ratio-dependent functional responses. Through the theoretical analysis and numerical simulation method we have obtained the rich spatiotemporal dynamics compared with [6]. We not only give the bifurcation parameter space of this model but also illustrate these different spatial patterns such as spots pattern, stripes pattern and labyrinthine pattern as the diffusion coefficient varies.

These theoretical results which we obtain can be directly used in many concrete biological and ecological models such as [5, 16, 26] to investigate effect of the diffusive coefficients on the spatial pattern formation such as spots pattern and stripes

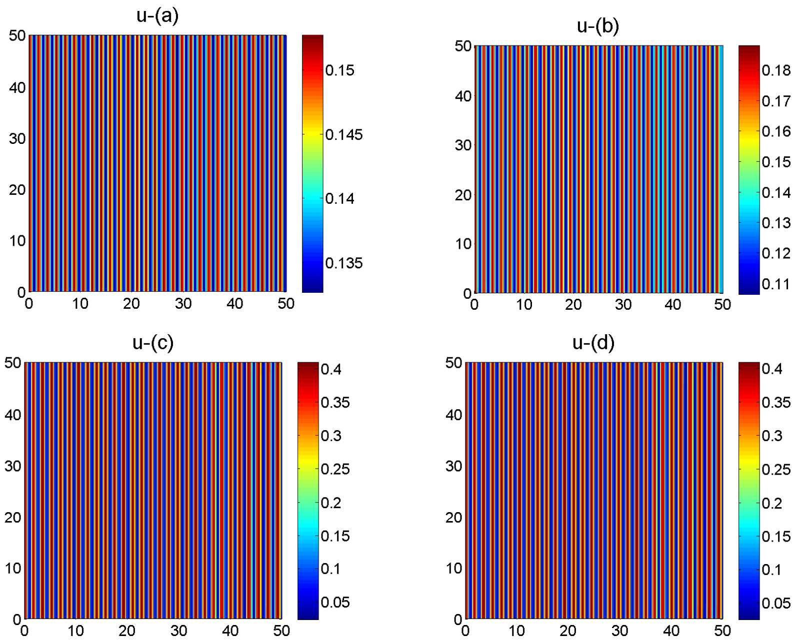

Figure 3. The time evolution plots of the density of the prey u at the different $t = 0$, $t = 10, t = 1000, t = 100000$ respectively when the diffusion coefficient $d_1 = 0.01$.

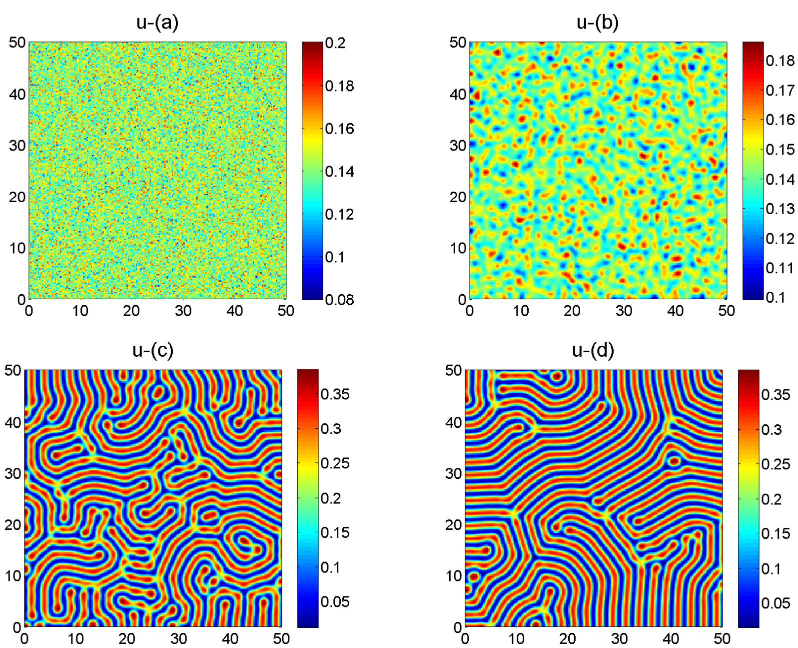

Figure 4. The time evolution plots of the density of the prey u at the different $t = 0$, $t = 10, t = 1000, t = 100000$ respectively when the diffusion coefficient $d_1 = 0.02$.

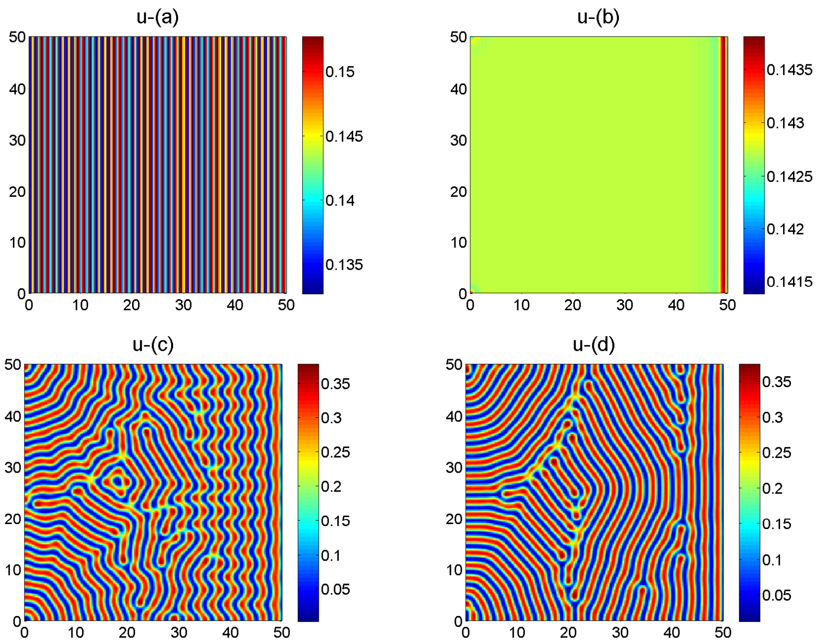

Figure 5. The time evolution plots of the density of the prey u at the different $t = 0$, $t = 10, t = 1000, t = 100000$ respectively when the diffusion coefficient $d_1 = 0.02$.

pattern. we have given the theories of the Turing bifurcation of the general three-species reaction-diffusion equations, including the conditions for Turing bifurcation and the corresponding amplitude equations which describe the different kinds of Turing patterns. That is to say, it is very convenient to employ the theoretical methods and results to obtain the numerical simulations of Turing patterns of a three-species food chain model.

Acknowledgments

We would like to thank the reviewers for their valuable comments and suggestions, which can significantly improve the quality of our paper indeed.

Appendices

In this section, some detailed presentations in Section 3 are given as follows.

Appendix A

$$\begin{pmatrix} X_{11} \\ Y_{11} \\ Z_{11} \end{pmatrix} = \begin{pmatrix} h_{x11} \\ h_{y11} \\ h_{z11} \end{pmatrix} W_1^2, \quad \begin{pmatrix} X_{00} \\ Y_{00} \\ Z_{00} \end{pmatrix} = \begin{pmatrix} h_{x00} \\ h_{y00} \\ h_{z00} \end{pmatrix} (W_1 + W_2 + W_3),$$

$$\begin{pmatrix} X_{12} \\ Y_{12} \\ Z_{13} \end{pmatrix} = \begin{pmatrix} h_{x*} \\ h_{y*} \\ h_{z*} \end{pmatrix} W_1 \overline{W}_2, \quad X_j = \delta_1 Z_j, \quad Y_j = \delta_2 Z_j, \qquad \begin{pmatrix} h_{x11} \\ h_{y11} \\ h_{z11} \end{pmatrix} = -\frac{S}{2} \begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix},$$

$$\begin{pmatrix} h_{x00} \\ h_{y00} \\ h_{z00} \end{pmatrix} = -P \begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix}, \quad \begin{pmatrix} h_{x*} \\ h_{y*} \\ h_{z*} \end{pmatrix} = -Q \begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix},$$

$$P = \begin{pmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{pmatrix}^{-1}, S = \begin{pmatrix} f_u - 4\alpha_c k_c^2 & f_v & f_w \\ g_u & g_v - 4d_2 k_c^2 & g_w \\ h_u & h_v & h_w - 4d_3 k_c^2 \end{pmatrix}^{-1},$$

$$Q = \begin{pmatrix} f_u - 3\alpha_c k_c^2 & f_v & f_w \\ g_u & g_v - 3d_2 k_c^2 & g_w \\ h_u & h_v & h_w - 3d_3 k_c^2 \end{pmatrix}^{-1}.$$

Appendix B

$$\begin{aligned}
n_{x\epsilon^3} &= n_{2x\epsilon^3} + \frac{1}{6} n_{3x\epsilon^3}, n_{y\epsilon^3} = n_{2y\epsilon^3} + \frac{1}{6} n_{3y\epsilon^3}, n_{z\epsilon^3} = n_{2z\epsilon^3} + \frac{1}{6} n_{3z\epsilon^3}, \\
n_{2x\epsilon^3} &= f_{uu} u_1 u_2 + f_{vv} v_1 v_2 + f_{ww} w_1 w_2 + f_{uv}(u_1 v_2 + v_1 u_2) \\
&\quad + f_{uw}(u_1 w_2 + w_1 u_2) + f_{vw}(v_1 w_2 + w_1 v_2), \\
n_{3x\epsilon^3} &= f_{uuu} u_1^3 + f_{vvv} v_1^3 + f_{www} w_1^3 + 3 f_{uuv} u_1^2 v_1 + 3 f_{uuw} u_1^2 w_1 + 3 f_{vvu} v_1^2 w_1 \\
&\quad + 3 f_{vvw} v_1^2 w_1 + 3 f_{wwu} w_1^2 u_1 + 3 f_{wwv} w_1^2 v_1 + 6 f_{uvw} u_1 v_1 w_1, \\
n_{2y\epsilon^3} &= g_{uu} u_1 u_2 + g_{vv} v_1 v_2 + g_{ww} w_1 w_2 + g_{uv}(u_1 v_2 + v_1 u_2) \\
&\quad + g_{uw}(u_1 w_2 + w_1 u_2) + g_{vw}(v_1 w_2 + w_1 v_2), \\
n_{3y\epsilon^3} &= g_{uuu} u_1^3 + g_{vvv} v_1^3 + g_{www} w_1^3 + 3 g_{uuv} u_1^2 v_1 + 3 g_{uuw} u_1^2 w_1 + 3 g_{vvu} v_1^2 w_1 \\
&\quad + 3 g_{vvw} v_1^2 w_1 + 3 g_{wwu} w_1^2 u_1 + 3 g_{wwv} w_1^2 v_1 + 6 g_{uvw} u_1 v_1 w_1, \\
n_{2z\epsilon^3} &= h_{uu} u_1 u_2 + h_{vv} v_1 v_2 + h_{ww} w_1 w_2 + h_{uv}(u_1 v_2 + v_1 u_2) \\
&\quad + h_{uw}(u_1 w_2 + w_1 u_2) + h_{vw}(v_1 w_2 + w_1 v_2), \\
n_{3z\epsilon^3} &= h_{uuu} u_1^3 + h_{vvv} v_1^3 + h_{www} w_1^3 + 3 h_{uuv} u_1^2 v_1 + 3 h_{uuw} u_1^2 w_1 + 3 h_{vvu} v_1^2 w_1 \\
&\quad + 3 h_{vvw} v_1^2 w_1 + 3 h_{wwu} w_1^2 u_1 + 3 h_{wwv} w_1^2 v_1 + 6 h_{uvw} u_1 v_1 w_1.
\end{aligned}$$

Appendix C

$$\begin{aligned}
G_{2x}^{k_1} &= f_{uu} G_{xu_1u_2}^{k_1} + f_{vv} G_{xv_1v_2}^{k_1} + f_{ww} G_{xw_1w_2}^{k_1} + f_{uv}(G_{xu_1v_2}^{k_1} + G_{xv_1u_2}^{k_1}) \\
&\quad + f_{uw}(G_{xu_1w_2}^{k_1} + G_{xw_1u_2}^{k_1}) + f_{vw}(G_{xv_1w_2}^{k_1} + G_{xw_1v_2}^{k_1}),
\end{aligned}$$

$$
\begin{aligned}
G_{2y}^{k_1} &= g_{uu}G_{yu_1u_2}^{k_1} + g_{vv}G_{yv_1v_2}^{k_1} + g_{ww}G_{yw_1w_2}^{k_1} + g_{uv}(G_{yu_1v_2}^{k_1} + G_{yv_1u_2}^{k_1}) \\
&\quad + g_{uw}(G_{yu_1w_2}^{k_1} + G_{yw_1u_2}^{k_1}) + g_{vw}(G_{yv_1w_2}^{k_1} + G_{yw_1v_2}^{k_1}), \\
G_{2z}^{k_1} &= h_{uu}G_{zu_1u_2}^{k_1} + h_{vv}G_{zv_1v_2}^{k_1} + h_{ww}G_{zw_1w_2}^{k_1} + h_{uv}(G_{zu_1v_2}^{k_1} + G_{zv_1u_2}^{k_1}) \\
&\quad + h_{uw}(G_{zu_1w_2}^{k_1} + G_{zw_1u_2}^{k_1}) + h_{vw}(G_{zv_1w_2}^{k_1} + G_{zw_1v_2}^{k_1}), \\
G_{3x}^{k_1} &= f_{uuu}G_{xu_1u_1u_1}^{k_1} + f_{vvv}G_{xv_1v_1v_1v_1}^{k_1} + f_{www}G_{xw_1w_1w_1}^{k_1} + 3f_{uuv}G_{xu_1u_1v_1}^{k_1} \\
&\quad + 3f_{uuw}G_{xu_1u_1w_1}^{k_1} + 3f_{uvv}G_{xu_1v_1v_1}^{k_1} + 3f_{vvw}G_{xv_1v_1w_1}^{k_1} + 3f_{uww}G_{xu_1w_1w_1}^{k_1} \\
&\quad + 3f_{vww}G_{xv_1w_1w_1}^{k_1} + 6f_{uvw}G_{xu_1v_1w_1}^{k_1}, \\
G_{3y}^{k_1} &= g_{uuu}G_{yu_1u_1u_1}^{k_1} + g_{vvv}G_{yv_1v_1v_1v_1}^{k_1} + g_{www}G_{yw_1w_1w_1}^{k_1} + 3g_{uuv}G_{yu_1u_1v_1}^{k_1} \\
&\quad + 3g_{uuw}G_{yu_1u_1w_1}^{k_1} + 3g_{uvv}G_{yu_1v_1v_1}^{k_1} + 3g_{vvw}G_{yv_1v_1w_1}^{k_1} + 3g_{uww}G_{yu_1w_1w_1}^{k_1} \\
&\quad + 3g_{vww}G_{yv_1w_1w_1}^{k_1} + 6g_{uvw}G_{yu_1v_1w_1}^{k_1}, \\
G_{3z}^{k_1} &= h_{uuu}G_{zu_1u_1u_1}^{k_1} + h_{vvv}G_{zv_1v_1v_1v_1}^{k_1} + h_{www}G_{zw_1w_1w_1}^{k_1} + 3h_{uuv}G_{zu_1u_1v_1}^{k_1} \\
&\quad + 3h_{uuw}G_{zu_1u_1w_1}^{k_1} + 3h_{uvv}G_{zu_1v_1v_1}^{k_1} + 3h_{vvw}G_{zv_1v_1w_1}^{k_1} + 3h_{uww}G_{zu_1w_1w_1}^{k_1} \\
&\quad + 3h_{vww}G_{zv_1w_1w_1}^{k_1} + 6h_{uvw}G_{zu_1v_1w_1}^{k_1}, \\
G_{xu_1u_2}^{k_1} &= \delta_1[X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1], \\
G_{xv_1v_2}^{k_1} &= \delta_2[Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1], \\
G_{xw_1w_2}^{k_1} &= [Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1], \\
G_{xu_1v_2}^{k_1} &= \delta_1[Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1], \\
G_{xv_1u_2}^{k_1} &= \delta_2[X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1], \\
G_{xu_1w_2}^{k_1} &= \delta_1[Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1], \\
G_{xw_1u_2}^{k_1} &= [X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1], \\
G_{xv_1w_2}^{k_1} &= \delta_2[Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1], \\
G_{xw_1v_2}^{k_1} &= [Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1], \\
G_{yu_1u_2}^{k_1} &= \delta_1[X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1], \\
G_{yv_1v_2}^{k_1} &= \delta_2[Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2) \\
&\qquad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1], \\
G_{yw_1w_2}^{k_1} &= [Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1],\\
G^{k_1}_{yu_1v_2} &= \delta_1[Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1],\\
G^{k_1}_{yv_1u_2} &= \delta_2[X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1],\\
G^{k_1}_{yu_1w_2} &= \delta_1[Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1],\\
G^{k_1}_{yw_1u_2} &= [X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1],\\
G^{k_1}_{yv_1w_2} &= \delta_2[Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1],\\
G^{k_1}_{yw_1v_2} &= [Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1],\\
G^{k_1}_{zu_1u_2} &= \delta_1[X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1],\\
G^{k_1}_{zv_1v_2} &= \delta_2[Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1],\\
G^{k_1}_{zw_1w_2} &= [Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1],\\
G^{k_1}_{zu_1v_2} &= \delta_1[Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1],\\
G^{k_1}_{zv_1u_2} &= \delta_2[X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1],\\
G^{k_1}_{zu_1w_2} &= \delta_1[Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1],\\
G^{k_1}_{zw_1u_2} &= [X_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_1(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{x*}(|W_2|^2 + |W_3|^2)W_1 + h_{x11}|W_1|^2W_1],\\
G^{k_1}_{zv_1w_2} &= \delta_2[Z_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + (\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{z*}(|W_2|^2 + |W_3|^2)W_1 + h_{z11}|W_1|^2W_1],\\
G^{k_1}_{zw_1v_2} &= [Y_{00}(|W_1|^2 + |W_2|^2 + |W_3|^2)W_1 + \delta_2(\overline{W}_2\overline{Z}_3 + \overline{W}_3\overline{Z}_2)\\
&\quad + h_{y*}(|W_2|^2 + |W_3|^2)W_1 + h_{y11}|W_1|^2W_1],\\
G^{k_1}_{xu_1u_1u_1} &= 3\delta_1^3W_1(|W_1|^2 + 2|W_2|^2 + 2|W_3|^2),\\
G^{k_1}_{xv_1v_1v_1} &= 3\delta_2^3W_1(|W_1|^2 + 2|W_2|^2 + 2|W_3|^2),\\
G^{k_1}_{xw_1w_1w_1} &= 3W_1(|W_1|^2 + 2|W_2|^2 + 2|W_3|^2),\\
G^{k_1}_{xu_1u_1v_1} &= 3\delta_1^2\delta_2W_1(|W_1|^2 + 2|W_2|^2 + 2|W_3|^2),\\
G^{k_1}_{xu_1u_1w_1} &= 3\delta_1^2W_1(|W_1|^2 + 2|W_2|^2 + 2|W_3|^2),
\end{aligned}
$$

$$
\begin{aligned}
G^{k_1}_{xu_1v_1v_1} &= 3\delta_1\delta_2^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{xv_1v_1w_1} &= 3\delta_2^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{xw_1w_1u_1} &= 3\delta_1 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{xw_1w_1v_1} &= 3\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{xu_1v_1w_1} &= 3\delta_1\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yu_1u_1u_1} &= 3\delta_1^3 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yv_1v_1v_1} &= 3\delta_2^3 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yw_1w_1w_1} &= 3W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yu_1u_1v_1} &= 3\delta_1^2\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yu_1u_1w_1} &= 3\delta_1^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yu_1v_1v_1} &= 3\delta_1\delta_2^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yv_1v_1w_1} &= 3\delta_2^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yw_1w_1u_1} &= 3\delta_1 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yw_1w_1v_1} &= 3\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{yu_1v_1w_1} &= 3\delta_1\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zu_1u_1u_1} &= 3\delta_1^3 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zv_1v_1v_1} &= 3\delta_2^3 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zw_1w_1w_1} &= 3W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zu_1u_1v_1} &= 3\delta_1^2\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zu_1u_1w_1} &= 3\delta_1^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zu_1v_1v_1} &= 3\delta_1\delta_2^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zv_1v_1w_1} &= 3\delta_2^2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zw_1w_1u_1} &= 3\delta_1 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zw_1w_1v_1} &= 3\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2),\\
G^{k_1}_{zu_1v_1w_1} &= 3\delta_1\delta_2 W_1(|W_1|^2+2|W_2|^2+2|W_3|^2).
\end{aligned}
$$

Appendix D

$$
\begin{aligned}
C &= \delta_1+\gamma_2\delta_2+\gamma_1,\\
E &= -\alpha_1\delta_1 k_c^2,\\
F &= -\alpha_2\delta_1 k_c^2,\\
H &= f_2+\gamma_2 g_2+\gamma_1 h_2,\\
G_1 &= G_{1x}+\gamma_2 G_{1y}+\gamma_1 G_{1z},\\
G_{1x} &= (X_{00}+h_{x11})(\delta_1 f_{uu}+\delta_2 f_{uv}+f_{uw})+(Y_{00}+h_{y11})(\delta_1 f_{uv}+\delta_2 f_{vv}+f_{vw})\\
&\quad +(Z_{00}+h_{z11})(f_{ww}+\delta_1 f_{uw}+\delta_2 f_{vw})+\frac{1}{2}\delta_1^3 f_{uuu}+\frac{1}{2}\delta_2^3 f_{vvv}+\frac{1}{2}f_{www}+\frac{3}{2}\delta_1^2\delta_2 f_{uuv}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{2}\delta_1^2 f_{uuw} + \frac{3}{2}\delta_2^2\delta_1 f_{vvu} + \frac{3}{2}\delta_2^2 f_{vvw} + \frac{3}{2}\delta_1 f_{wwu} + \frac{3}{2}\delta_2 f_{wwv} + 3\delta_1\delta_2 f_{uvw},\\
G_{1y} =& (X_{00} + h_{x11})(\delta_1 g_{uu} + \delta_2 g_{uv} + g_{uw}) + (Y_{00} + h_{y11})(\delta_1 g_{uv} + \delta_2 g_{vv} + g_{vw})\\
&+ (Z_{00} + h_{z11})(g_{ww} + \delta_1 g_{uw} + \delta_2 g_{vw}) + \frac{1}{2}\delta_1^3 g_{uuu} + \frac{1}{2}\delta_2^3 g_{vvv} + \frac{1}{2} g_{www}\\
&+ \frac{3}{2}\delta_1^2\delta_2 g_{uuv} + \frac{3}{2}\delta_1^2 g_{uuw} + \frac{3}{2}\delta_2^2\delta_1 g_{vvu} + \frac{3}{2}\delta_2^2 g_{vvw} + \frac{3}{2}\delta_1 g_{wwu}\\
&+ \frac{3}{2}\delta_2 g_{wwv} + 3\delta_1\delta_2 g_{uvw},\\
G_{1z} =& (X_{00} + h_{x11})(\delta_1 h_{uu} + \delta_2 h_{uv} + h_{uw}) + (Y_{00} + h_{y11})(\delta_1 h_{uv} + \delta_2 h_{vv} + h_{vw})\\
&+ (Z_{00} + h_{z11})(h_{ww} + \delta_1 h_{uw} + \delta_2 h_{vw}) + \frac{1}{2}\delta_1^3 h_{uuu} + \frac{1}{2}\delta_2^3 h_{vvv} + \frac{1}{2} h_{www}\\
&+ \frac{3}{2}\delta_1^2\delta_2 h_{uuv} + \frac{3}{2}\delta_1^2 h_{uuw} + \frac{3}{2}\delta_2^2\delta_1 h_{vvu} + \frac{3}{2}\delta_2^2 h_{vvw} + \frac{3}{2}\delta_1 h_{wwu}\\
&+ \frac{3}{2}\delta_2 h_{wwv} + 3\delta_1\delta_2 h_{uvw},\\
G_2 =& G_{2x} + \gamma_2 G_{2y} + \gamma_1 G_{2z},\\
G_{2x} =& (X_{00} + h_{x*})(\delta_1 f_{uu} + \delta_2 f_{uv} + f_{uw}) + (Y_{00} + h_{y*})(\delta_1 f_{uv} + \delta_2 f_{vv} + f_{vw})\\
&+ (Z_{00} + h_{z*})(f_{ww} + \delta_1 f_{uw} + \delta_2 f_{vw}) + \delta_1^3 f_{uuu} + \delta_2^3 f_{vvv} + f_{www} + 3\delta_1^2\delta_2 f_{uuv}\\
&+ 3\delta_1^2 f_{uuw} + 3\delta_2^2\delta_1 f_{vvu} + 3\delta_2^2 f_{vvw} + 3\delta_1 f_{wwu} + 3\delta_2 f_{wwv} + 6\delta_1\delta_2 f_{uvw},\\
G_{2y} =& (X_{00} + h_{x*})(\delta_1 g_{uu} + \delta_2 g_{uv} + g_{uw}) + (Y_{00} + h_{y*})(\delta_1 g_{uv} + \delta_2 g_{vv} + g_{vw})\\
&+ (Z_{00} + h_{z*})(g_{ww} + \delta_1 g_{uw} + \delta_2 g_{vw}) + \delta_1^3 g_{uuu} + \delta_2^3 g_{vvv} + g_{www} + 3\delta_1^2\delta_2 g_{uuv}\\
&+ 3\delta_1^2 g_{uuw} + 3\delta_2^2\delta_1 g_{vvu} + 3\delta_2^2 g_{vvw} + 3\delta_1 g_{wwu} + 3\delta_2 g_{wwv} + 6\delta_1\delta_2 g_{uvw},\\
G_{2z} =& (X_{00} + h_{x*})(\delta_1 h_{uu} + \delta_2 h_{uv} + h_{uw}) + (Y_{00} + h_{y*})(\delta_1 h_{uv} + \delta_2 h_{vv} + h_{vw})\\
&+ (Z_{00} + h_{z*})(h_{ww} + \delta_1 h_{uw} + \delta_2 h_{vw}) + \delta_1^3 h_{uuu} + \delta_2^3 h_{vvv} + h_{www} + 3\delta_1^2\delta_2 h_{uuv}\\
&+ 3\delta_1^2 h_{uuw} + 3\delta_2^2\delta_1 h_{vvu} + 3\delta_2^2 h_{vvw} + 3\delta_1 h_{wwu} + 3\delta_2 h_{wwv} + 6\delta_1\delta_2 h_{uvw}.
\end{aligned}
$$

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