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SOLITON, BREATHER AND LUMP MOLECULES IN THE (2+1)-DIMENSIONAL B-TYPE KADOMTSEV-PETVIASHVILI-KORTEWEG DE-VRIES EQUATION*

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Abstract The N-soliton solutions of the (2+1)-dimensional B-type Kadomtsev-Petviashvili-Korteweg de-Vries (BKP-KdV) equation are constructed through the Hirota bilinear method. By inserting the velocity resonance conditions into the soliton solutions, soliton molecule, breather molecule, breather-soliton molecule, lump-soliton molecule and lump-breather molecule are presented. The interaction of two soliton molecules, and the interaction between one soliton molecule and breather are discussed and shown to be the elastic collisions.

Keywords (2+1)-dimensional BKP-KdV equation, breather-soliton molecule, lump-soliton molecule, lump-breather molecule, elastic interaction.

MSC(2010) 35Q51, 35Q53, 35C08.

1. Introduction

The (2+1)-dimensional nonlinear B-type Kadomtsev-Petviashvili (BKP) equation is given by $(u \equiv u(x, y, t))$ [22]

$$u_t+u_{xxxxx}-5(u_{xxy}+\int u_{yy}\mathrm{d}x)+15(u_xu_{xx}+uu_{xxx}-uu_y-u_x\int u_y\mathrm{d}x)+45u^2u_x=0, \tag{1.1}$$

which can be used to describe weakly dispersive waves propagating in the quasi media and fluid mechanics. Wazwaz used the simplified form of the Hirota method to establish multiple soliton solutions for this equation [24]. By virtue of the binary Bell polynomial and the bilinear form, the N-soliton solutions, periodic wave solutions and breather wave solutions were constructed in [5]. The bilinear form,

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Bäcklund transformation, Lax pair and multi-soliton solutions were constructed as well as the propagation and interaction of the solitons were illustrated graphically in [10]. The general lump waves, lumpoff and special rogue wave solutions were derived in [20]. By means of the homoclinic breather limit method, its rogue waves and homoclinic breather waves were investigated in [4]. Soliton molecules and some novel interaction solutions were generated from the N-soliton solution by using a new velocity resonance condition [29].

Further, one integrable nonlinear equation combining with another by using the Galilei transformation will give rise to unexpected results. Hirota studied the resonance of solitons in one-dimensional space theoretically for the Sawaka-Kotera-Korteweg-de Vries (SK-KdV) equation with a nonvanishing boundary condition [7]. Wazwaz applied this technique to the generalized fifth-order Caudrey-Dodd-Gibbon (CDG) and its Lax equations, and derived the multiple-soliton solutions for the extended KdV-CDG and KdV-Lax equations [25]. Based on the bilinear differential operator extension method, a combined model of the generalized bilinear Kadomtsev-Petviashvili (KP) and Boussinesq equations in terms of the function f was proposed and the lump solutions were constructed [11]. By introducing the velocity resonant mechanism, Lou established some new types of solutions in the Sharma-Tasso-Olver-Burgers (STOB) equation, including the soliton (kink) molecules, half periodic kink molecules and breathing soliton molecules [30]. Inspired by the results of the Galilei transformation, by taking the transformation $u \to u + c, u \to u(x, y + at, t)$ and $y + at \to y$ successively on the following BKP equation

$$u_t+b[u_{xxxxx}-5(u_{xxy}+\int u_{yy}\mathrm{d}x)+15(u_xu_{xx}+uu_{xxx}-uu_y-u_x\int u_y\mathrm{d}x)+45u^2u_x]=0, \tag{1.2}$$

the so-called BKP-KdV system can be derived

$$\begin{aligned} &u_t + a(u_{xxx} + 6uu_x) + b[u_{xxxxx} - 5(u_{xxy} + \int u_{yy}\mathrm{d}x) \\ &+ 15(u_xu_{xx} + uu_{xxx} - uu_y - u_x \int u_y\mathrm{d}x) + 45u^2u_x] = 0, \end{aligned} \tag{1.3}$$

where the constants a, b and c are arbitrary and $c = \frac{a}{15b}$.

For an integrable nonlinear system, to derive some molecules is one of hot topics recently. From experiment observation in the dispersion-managed optical fibers and numerical prediction in the Bose-Einstein condensates [6, 12, 13, 23], the soliton molecules-the bound states of solitons, play an important role theoretically in the field of the integrable system. It was found that the velocity resonance which is a new possible mechanism, was introduced to form soliton molecules and asymmetric solitons for three (1+1)-dimensional fluid models [14]. For the combination of the KP3 and the KP4 (cKP3-4) equations, the soliton molecules and the missing D'Alembert type solutions were found in [15]. The soliton molecules, breather molecules and breather-soliton molecules were presented for a (2+1)-dimensional fifth-order KdV equation [31]. Soliton molecules and novel smooth positons for the complex modified Korteweg-de Vries (mKdV) equation and the nonlinear Schrödinger (NLS) equation were obtained based on Darboux transformation (DT) [26, 33]. For the integrable higher-order NLS equation, Xu proved that the interactions among the dark soliton molecules are elastic [28]. The interactions of soliton molecules were proved to be nonelastic in two nonlocal Alice-Bob Sawada-Kotera (ABSK) systems [34]. Soliton molecules and hybrid/mixed solutions in (1+1)/(2+1)-dimensions, such as the fifth-order KdV equation, the variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation, the bidirectional Sawada-Kotera (bSK) equation, the extended modified Korteweg-de Vries (mKdV) system and the modified Korteweg-de Vries-sine-Gordon (mKdV-sG) equation were also investigated [2,9,21,32,35].

The present paper is organized as follows. In section 2, the N-soliton solutions of the (2+1)-dimensional BKP-KdV equation (1.3) are firstly constructed through its Hirota bilinear form. Then one soliton molecule, the breather and the lump solutions are obtained successively for two-soliton solution. Interaction of one soliton molecule and a line soliton, and soliton-breather molecule and lump-soliton molecule are derived from three-soliton solution. Further, the elastic interaction between two soliton molecules, one soliton molecule and the breather, the breather and a lump soliton as well as the soliton molecule consisting of one soliton molecule and the breather, the breather and a lump are presented from four-soliton solution. Figures are depicted to demonstrate these dynamics features. In the last section, a brief summary is given for this paper.

2. Soliton, breather and lump molecules in the (2+1) -dimensional BKP-KdV equation

To study the soliton molecules and their related interaction solutions for the (2+1)-dimensional BKP-KdV equation (1.3), we introduce the following transformation

$$u_y = v_x, \quad u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy}, \tag{2.1}$$

where v and f are functions of variables x, y and t. Then the bilinear form of Eq. (1.3) can be obtained

$$[D_xD_t + aD_x^4 + b(D_x^6 - 5D_x^3D_y - 5D_y^2)]f \cdot f = 0, \tag{2.2}$$

where the Hirota D-operator is defined by [8]

$$\begin{aligned}&D_x^mD_y^nD_t^l(f\cdot g)\\ =&(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'})^n(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^l f(x,y,t)\cdot g(x',y',t')|_{x'=x,y'=y,t'=t}.\end{aligned} \tag{2.3}$$

Identically, the bilinear form (2.2) can be expressed by

$$\begin{aligned}&ff_{xt} - f_xf_t + a(ff_{xxxx} - 4f_xf_{xxx} + 3f_{xx}^2) + b[ff_{xxxxxx} - 6f_xf_{xxxxx} + 15f_{xx}f_{xxxx}\\ &- 10f_{xxx}^2 - 5(ff_{xxxy} - f_{xxx}f_y - 3f_xf_{xxy} + 3f_{xx}f_{xy} + ff_{yy} - f_y^2)] = 0.\end{aligned} \tag{2.4}$$

Hence, the N-soliton solutions which include the mixed solutions consisting of soliton molecules, breathers and lumps of the (2+1)-dimensional BKP-KdV equation(1.3) can be given by [2,32,35]

$$f = \sum_{\tau=0,1}\exp(\sum_{i<j}^{N}\tau_i\tau_jA_{ij} + \sum_{i=1}^{N}\tau_i\xi_i), \tag{2.5}$$

where $\tau = (\tau_1, \tau_2, \cdots, \tau_N)$, $\tau = 0, 1$ means that each τ_i takes 0 or 1. Here, the relations of these parameters are

$$\xi_i = k_i x + p_i y + q_i t + \eta_i,\ q_i = -\frac{ak_i^4 + b(k_i^6 - 5k_i^3 p_i - 5p_i^2)}{k_i} \quad (i = 1, 2, \cdots, N), \tag{2.6}$$

$$\begin{aligned} e^{A_{ij}} &= a_{ij} \\ &= -\frac{a(k_i - k_j)^4 + b[(k_i - k_j)^6 - 5(k_i - k_j)^3(p_i - p_j) - 5(p_i - p_j)^2] + (k_i - k_j)(q_i - q_j)}{a(k_i + k_j)^4 + b[(k_i + k_j)^6 - 5(k_i + k_j)^3(p_i + p_j) - 5(p_i + p_j)^2] + (k_i + k_j)(q_i + q_j)} \\ &(i < j, \quad i, j = 1, 2, \cdots, N). \end{aligned} \tag{2.7}$$

In addition to the multiple line soliton solutions, Eqs. (2.1) and (2.5) possess many kinds of resonant excitations [1,16]. In particular, one can derive the breather solution through the restricted conditions $k_i = \overline{k_j}$, $p_i = \overline{p_j}$, $\eta_i = \overline{\eta_j}$ $(i \neq j)$ and further the rational lump solution via the limit process $k_i \to 0$, $p_i \to 0$ and $\eta_i \to 0$.

2.1. Soliton molecule, breather and lump solutions

For the two-soliton solution of Eq.(1.3), the auxiliary function can be expressed by

$$\begin{aligned} &u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy}, \quad f = 1 + e^{\xi_1} + e^{\xi_2} + a_{12} e^{\xi_1 + \xi_2}, \\ &\xi_i = k_i x + p_i y + q_i t + \eta_i, \quad (i = 1, 2), \end{aligned} \tag{2.8}$$

where the phase shift a_{12} is

$$a_{12} = -\frac{a(k_1 - k_2)^4 + b[(k_1 - k_2)^6 v - 5(k_1 - v k_2)^3(p_1 - p_2) - 5(p_1 - p_2)^2] + (k_1 - k_2)(q_1 - q_2)}{a(k_1 + k_2)^4 + b[(k_1 + k_2)^6 - 5(k_1 + k_2)^3(p_1 + p_2) - 5(p_1 + p_2)^2] + (k_1 + k_2)(q_1 + q_2)}, \tag{2.9}$$

with the parameters

$$q_i = -\frac{ak_i^4 + b(k_i^6 - 5k_i^3 p_i - 5p_i^2)}{k_i}, \quad (i = 1, 2). \tag{2.10}$$

Then we list three typical cases.

(1) The soliton molecule. To generate the nonsingular resonant excitation solution, the following velocity resonance condition need to be satisfied

$$\frac{k_1}{k_2} = \frac{p_1}{p_2} = \frac{q_1}{q_2}, \quad k_1 \neq |k_2|, \quad p_1 \neq |p_2|. \tag{2.11}$$

From Eq.(2.9) with (2.10), the specific parmeters' relations in the soliton molecule are derived

$$\begin{aligned} &\frac{k_1}{p_1} = \frac{k_2}{p_2} = \frac{5b}{a + b(k_1^2 + k_2^2)}, \quad \frac{k_1}{q_1} = \frac{k_2}{q_2} = \frac{5b}{(a + bk_1^2)^2 + bk_2^2(2a + 7bk_1^2 + bk_2^2)}, \\ &a_{12} = \frac{(k_1 - k_2)^2(2k_1^2 - 5k_1 k_2 + 2k_2^2)}{(k_1 + k_2)^2(2k_1^2 + 5k_1 k_2 + 2k_2^2)}. \end{aligned} \tag{2.12}$$

By taking the parameters as

$$a = b = k_1 = 1, \quad k_2 = \frac{2}{5}, \quad \eta_1 = 0, \quad \eta_2 = 5, \tag{2.13}$$

the others are calculated as follows:

$$p_1 = \frac{54}{125}, \quad p_2 = \frac{108}{625}, \quad q_1 = \frac{3416}{3125}, \quad q_2 = \frac{6832}{15625}, \quad a_{12} = \frac{2}{147}. \tag{2.14}$$

Then the soliton molecule structure for the (2+1)-dimensional BKP-KdV equation (1.3) is constructed, in which two-line solitons possess identical velocities, but have different heights and widths as shown in Fig. 1(a).

(2) The breather solution. The breather solution is usually expressed by the hyperbolic and periodic functions, so two-line soliton solution should satisfy the conjugate relation $\xi_1 = \overline{\xi_2}$ [3, 27]. To this end, we take the parameters as the following form

$$k_1 = \overline{k_2} = k + \kappa\mathrm{i}, \quad p_1 = \overline{p_2} = \rho + \varrho\mathrm{i}, \quad \eta_1 = \overline{\eta_2} = \zeta + \varsigma\mathrm{i}, \tag{2.15}$$

(i is an imaginary unit). Hence, the other terms can be rewritten as

$$\begin{aligned} q_1 &= -\frac{a(k+\kappa\mathrm{i})^4 + b[(k+\kappa\mathrm{i})^6 - 5(k+\kappa\mathrm{i})^3(\rho+\varrho\mathrm{i}) - 5(\rho+\varrho\mathrm{i})^2]}{k+\kappa\mathrm{i}} \equiv Q_1 + Q_2\mathrm{i}, \\ q_2 &= \overline{q_1} = -\frac{a(k-\kappa\mathrm{i})^4 + b[(k-\kappa\mathrm{i})^6 - 5(k-\kappa\mathrm{i})^3(\rho-\varrho\mathrm{i}) - 5(\rho-\varrho\mathrm{i})^2]}{k-\kappa\mathrm{i}} \\ &\equiv Q_1 - Q_2\mathrm{i}, \end{aligned} \tag{2.16}$$

$$a_{12} = \frac{-4a\kappa^4 + b(16\kappa^6 + 20\kappa^3\varrho - 5\varrho^2) + Q_2\kappa}{4ak^4 + b(16k^6 - 20k^3\rho - 5\rho^2) + Q_1 k}, \tag{2.17}$$

which lead to the following form of the auxiliary function

$$f = 1 + 2\cos(\phi)e^{\varphi} + a_{12}e^{2\varphi}, \tag{2.18}$$

with

$$\phi = \kappa x + \varrho y + Q_2 t + \varsigma, \quad \varphi = kx + \rho y + Q_1 t + \zeta. \tag{2.19}$$

Figure 1(b) exhibits the breather structure for the solution u of Eq. (1.3) with the parameters

$$k = \frac{3}{10}, \quad \kappa = -\frac{2}{5}, \quad \rho = \frac{1}{5}, \quad \varrho = \frac{2}{5}, \quad a = b = 1, \quad \zeta = \varsigma = 0. \tag{2.20}$$

This kind of special solution contains trigonometric and exponential functions, which describes the property of the breather structure. Figure 1(b) shows that the periodicity moves along the propagation direction while the localization comes along the straight line $\frac{3x}{2} + y = 0$ $(t = 0)$.

(3) The lump soliton. The lump soliton is localized in all directions in the space and its solution can be expressed by the rational function [17–19]. Based on the long-wave limit idea of generating the lump solution, one can consider the following parameters' conditions:

$$k_i = \lambda_i\epsilon_i, \quad p_i = l_i\lambda_i\epsilon_i, \quad \eta_i = \lambda_{i0}\epsilon_i, \quad \epsilon_i \to 0 \quad (i = 1, 2). \tag{2.21}$$

Then the auxiliary function f is given by

$$\begin{aligned} &f = \theta_1\theta_2 + B_{12}, \quad B_{12} = -\frac{6\lambda_1\lambda_2[2a - 5b(l_1 + l_2)]}{5b(l_1 - l_2)^2}, \\ &\theta_i = \lambda_i(x + l_i y + 5bl_i^2 t) + \lambda_{i0} \quad (i = 1, 2). \end{aligned} \tag{2.22}$$

By taking the complex parameters

$$l_1 = \overline{l_2} = \frac{a_2 + a_6\mathrm{i}}{a_1 + a_5\mathrm{i}}, \quad \lambda_1 = \overline{\lambda_2} = a_1 + a_5\mathrm{i}, \quad \lambda_{10} = \overline{\lambda_{20}} = a_4 + a_8\mathrm{i}, \tag{2.23}$$

the quadratic function is confirmed from Eq.(2.22)

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9, \tag{2.24}$$

with

$$\begin{aligned} a_3 &= \frac{5b(a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)}{a_1^2 + a_5^2}, \quad a_7 = \frac{5b(2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)}{a_1^2 + a_5^2}, \\ a_9 &= \frac{3(a_1^2 + a_5^2)^2[a(a_1^2 + a_5^2) - 5b(a_1a_2 + a_5a_6)]}{5b(a_1a_6 - a_2a_5)^2}. \end{aligned} \tag{2.25}$$

This means that the moving route and velocity of the lump are

$$x = \frac{5b(a_2^2 + a_6^2)t}{a_1^2 + a_5^2} + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \quad y = -\frac{10b(a_1a_2 + a_5a_6)t}{a_1^2 + a_5^2} - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \tag{2.26}$$

$$v_x = \frac{5b(a_2^2 + a_6^2)}{a_1^2 + a_5^2}, \quad v_y = -\frac{10b(a_1a_2 + a_5a_6)}{a_1^2 + a_5^2}. \tag{2.27}$$

By choosing the parameters

$$a_1 = -2, \quad a_2 = a_5 = a = b = 1, \quad a_4 = a_8 = 0, \quad a_6 = -4, \tag{2.28}$$

the rational lump solution u of Eq. (1.3) through (2.8), (2.24) and (2.25) is expressed by

$$u = \frac{28(-175x^2 + 420xy + 91y^2 + 910xt - 9324yt + 48209t^2 + 375)}{(35x^2 - 84xy + 119y^2 - 182xt - 1428yt + 10115t^2 + 75)^2}, \tag{2.29}$$

which is shown in Fig. 1(c). The maximum amplitude, the moving velocity and the moving path of this lump are $\frac{28}{15}$, $\sqrt{433}$ and $y = \frac{12}{17}x$, respectively. Figure 1(c) displays the three-dimensional structure of this lump with $t = 0$.

2.2. Soliton molecule-soliton interaction, soliton-breather molecule and soliton-lump molecule

When $N = 3$ in (2.5), the auxiliary function f is taken as

$$\begin{aligned} &f = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3} + a_{23}e^{\xi_2+\xi_3} + a_{12}a_{13}a_{23}e^{\xi_1+\xi_2+\xi_3}, \\ &\xi_i = k_ix + p_iy + q_it + \eta_i \quad (i = 1, 2, 3), \end{aligned} \tag{2.30}$$

with the phase shifts

$$\begin{aligned} &a_{ij} = -\frac{a(k_i-k_j)^4+b[(k_i-k_j)^6-5(k_i-k_j)^3(p_i-p_j)-5(p_i-p_j)^2]+(k_i-k_j)(q_i-q_j)}{a(k_i+k_j)^4+b[(k_i+k_j)^6-5(k_i+k_j)^3(p_i+p_j)-5(p_i+p_j)^2]+(k_i+k_j)(q_i+q_j)}, \\ &(i < j, \quad i, j = 1, 2, 3), \end{aligned} \tag{2.31}$$

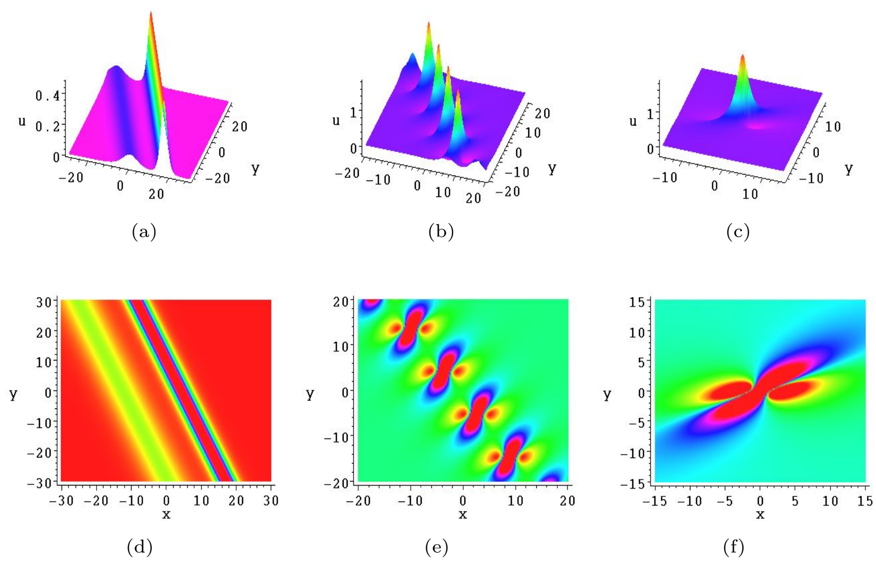


Figure 1. (a) The soliton molecule with the parameters (2.12), (2.13) and (2.14) at $t = 0$. (b) The breather with the parameters (2.16), (2.17) and (2.20) at $t = 0$. (c) The lump structure given by (2.29) at $t = 0$. (d)-(f) are the corresponding density figures (a)-(c), respectively.

and the parameters

$$q_i = -\frac{ak_i^4 + b(k_i^6 - 5k_i^3 p_i - 5p_i^2)}{k_i}, \quad (i = 1, 2, 3). \tag{2.32}$$

Three-soliton solution can be obtained by substituting Eq.(2.30) with Eqs.(2.31) and (2.32) into the transformation (2.1) for the (2+1)-dimensional BKP-KdV equation (1.3). In the following, we present three typical cases.

(1) Soliton molecule-soliton interaction. The interaction of one soliton molecule and a line soliton can be produced from three-soliton solution. For this situation, k_i, p_i and q_i $(i = 1, 2)$ satisfy the velocity resonant condition (2.11), but k_3, p_3 and q_3 should not obey this ratio. For example, when taking the related parameters

$$a = b = k_1 = p_3 = 1, \quad k_2 = \frac{2}{5}, \quad k_3 = -\frac{3}{5}, \quad \eta_1 = \eta_3 = 0, \quad \eta_2 = 5, \tag{2.33}$$

the others are calculated as

$$\begin{aligned} &p_1 = \frac{54}{125}, \quad p_2 = \frac{108}{625}, \quad q_1 = \frac{3416}{3125}, \quad q_2 = \frac{6832}{15625}, \quad q_3 = -\frac{58496}{9375}, \\ &a_{12} = \frac{2}{147}, \quad a_{13} = \frac{2113249}{597199}, \quad a_{23} = \frac{1100569}{664249}. \end{aligned} \tag{2.34}$$

Figure 2 depicts this interaction between one soliton molecule and a line soliton under the condition (2.33) and (2.34) at the different time.

(2) Soliton-breather molecule. This type of molecule contains a line soliton and a breather. For this purpose, the module resonant condition is defined as $k_1 =$

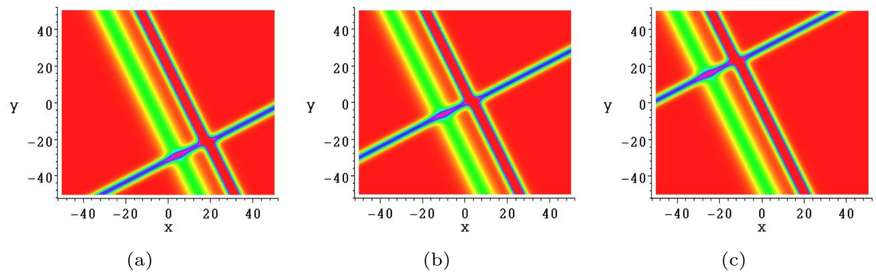


Figure 2. Interaction of one soliton molecule and a line soliton for the (2+1)-dimensional BKP-KdV equation (1.3) with the parameters (2.33) and (2.34) at (a) $t=-5$, (b) $t=0$ and (c) $t=5$, respectively.

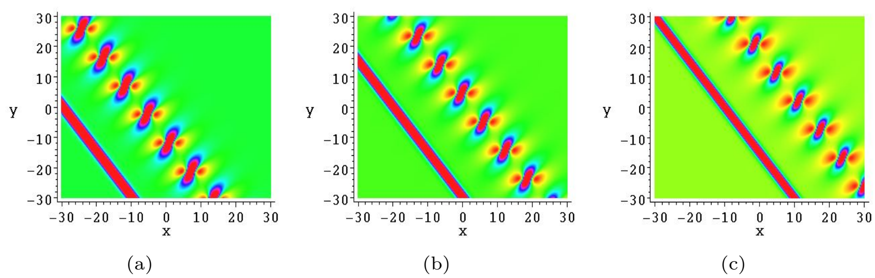


Figure 3. The soliton-breather molecule for the (2+1)-dimensional BKP-KdV equation (1.3) with the parameters (2.36) and (2.37) at (a) $t=-2$, (b) $t=0$ and (c) $t=2$, respectively.

$\overline{k_2}$, $p_1=\overline{p_2}$, $q_1=\overline{q_2}$ and $\eta_1=\overline{\eta_2}$ in Eqs.(2.15) and (2.16), while the parameters a_{13} and a_{23} can be obtained from Eq. (2.31) and need to satisfy $a_{13}=\overline{a_{23}}\equiv m_1+m_2\mathrm{i}$, then the auxiliary function f of Eq. (2.18) is derived

$$f=1+2\cos(\phi)e^{\varphi}+a_{12}e^{2\varphi}+2[m_1\cos(\phi)-m_2\sin(\phi)]e^{\varphi+\xi_3}+a_{12}(m_1^2+m_2^2)e^{2\varphi+\xi_3}+e^{\xi_3}, \tag{2.35}$$

where a_{12}, ϕ and φ satisfy Eqs. (2.17), (2.19) and $\xi_3=k_3x+p_3y+q_3t+\eta_3$.

Figure 3 shows one molecule structure which exhibits a line soliton and a breather at the different time with the velocity resonance conditions

$$\frac{k_3}{k}=\frac{p_3}{\rho}=\frac{q_3}{Q_1}, \quad k_3\neq|k|, \quad p_3\neq|\rho|, \tag{2.36}$$

where the parameters are taken as

$$a=b=1, \quad k=\frac{3}{10}, \quad \kappa=-\frac{2}{5}, \quad \rho=\frac{1}{5}, \quad \varrho=\frac{2}{5}, \quad \zeta=\varsigma=0, \quad \eta_3=40. \tag{2.37}$$

(3) Soliton-lump molecule. From the three-soliton solution, we can construct a molecule consisting of a line soliton and a lump through the long-wave limit idea. When taking

$$k_i=\lambda_i\epsilon_i, \quad p_i=l_i\lambda_i\epsilon_i, \quad \eta_i=\lambda_{i0}\epsilon_i, \quad \epsilon_i\to 0 \ \ (i=1,2,3), \tag{2.38}$$

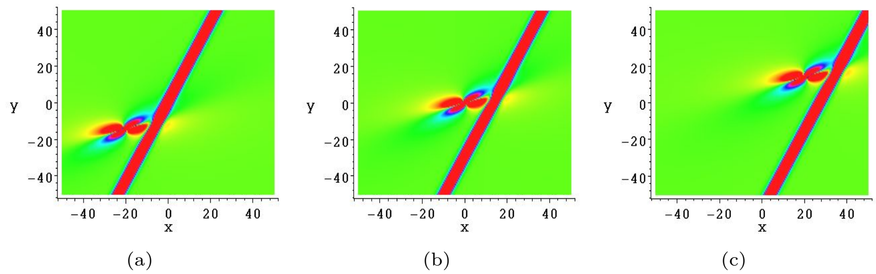


Figure 4. The soliton-lump molecule in the (2+1)-dimensional BKP-KdV equation (1.3) with the parameters (2.28) and (2.41) at (a) $t = -1.2$, (b) $t = 0$ and (c) $t = 1.2$, respectively.

the transformation function f of Eq. (2.30) is deduced

$$f = (\theta_1\theta_2 + B_{12})(1 + e^{\xi_3}) + (C_{23}\theta_1 + C_{13}\theta_2 + C_{13}C_{23})e^{\xi_3}, \tag{2.39}$$

with

$$\begin{aligned} C_{13} &= -\frac{12a\lambda_1 k_3^3 + b[30\lambda_1 k_3^2(k_3^3 - p_3) - 30\lambda_1 l_1 k_3^3]}{3ak_3^4 + b[5k_3^2 l_1^2 - 5k_3 l_1(k_3^3 + 2p_3) + 5(k_3^3 - p_3)^2]}, \\ C_{23} &= -\frac{12a\lambda_2 k_3^3 + b[30\lambda_2 k_3^2(k_3^3 - p_3) - 30\lambda_2 l_2 k_3^3]}{3ak_3^4 + b[5k_3^2 l_2^2 - 5k_3 l_2(k_3^3 + 2p_3) + 5(k_3^3 - p_3)^2]}. \end{aligned} \tag{2.40}$$

Figure 4 shows one molecule consisting of a line soliton and a lump at the different time with the parameters (2.28) and

$$k_3 = \frac{3}{2}, \quad p_3 = -\frac{279 - 3\sqrt{5549}}{80}, \quad q_3 = \frac{327 - 9\sqrt{5549}}{20}, \quad \eta_3 = -20. \tag{2.41}$$

The corresponding moving velocity of this lump soliton is $\sqrt{433}$.

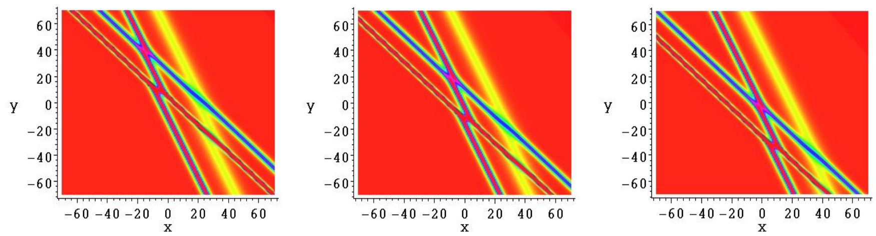
2.3. The interaction among soliton molecule, breather and lump

The auxiliary function f of the four-soliton solution can be expressed as

$$\begin{aligned} f(x,y,t) =& 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_4} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3} + a_{14}e^{\xi_1+\xi_4} + a_{23}e^{\xi_2+\xi_3} \\ &+ a_{24}e^{\xi_2+\xi_4} + a_{34}e^{\xi_3+\xi_4} + a_{12}a_{13}a_{23}e^{\xi_1+\xi_2+\xi_3} + a_{23}a_{24}a_{34}e^{\xi_2+\xi_3+\xi_4} \\ &+ a_{13}a_{14}a_{34}e^{\xi_1+\xi_3+\xi_4} a_{12}a_{14}a_{24}e^{\xi_1+\xi_2+\xi_4} \\ &+ a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}e^{\xi_1+\xi_2+\xi_3+\xi_4}, \end{aligned} \tag{2.42}$$

with the phase shifts $a_{ij}(i < j, \ \ i, j = 1, 2, 3, 4)$ and the parameters $q_i(i = 1, 2, 3, 4)$ satisfying Eqs. (2.6) and (2.7). The four-soliton solution can be derived through substituting Eq.(2.42) with Eqs.(2.6) and (2.7) into the transformation (2.1). Then we list three typical cases.

(1) The interaction of two soliton molecules. When the parameters k_i, p_i and q_i $(i = 1, 2)$ satisfy the velocity resonant condition (2.11), while k_j, p_j and q_j $(j = 3, 4)$ also obey this ratio: $\frac{k_3}{k_4} = \frac{p_3}{p_4} = \frac{q_3}{q_4}$, $k_3 \neq |k_4|$, $p_3 \neq |p_4|$, the structure of



(a) (b) (c)

Figure 5. The elastic interaction between two soliton molecules in the (2+1)-dimensional BKP-KdV equation (1.3) with the parameters (2.43) at (a) $t = -2$, (b) $t = 0$ and (c) $t = 2$, respectively.

two soliton molecules is generated from the four-soliton solution. Figure 5 shows this structure in the (2+1)-dimensional BKP-KdV equation (1.3) with the parameters

$$a = b = k_1 = 1, \quad k_2 = -\frac{2}{5}, \quad k_3 = -\frac{5}{6}, \quad k_4 = \frac{17}{10} \quad \eta_1 = \eta_3 = 0, \quad \eta_2 = 2, \quad \eta_4 = 10. \tag{2.43}$$

In this case, we have $\frac{k_1}{k_3} = -\frac{6}{5}$, $\frac{p_1}{p_3} = -\frac{5832}{10315}$ and $\frac{q_1}{q_3} = -\frac{3794688}{17965715}$. This means that their ratios are different and the collision of two soliton molecules occurs. From Figure 5, one can see that their interactions among separated solitons are elastic, that is, the heights and velocities of these wave peaks remain unchanged except for the phase shifting after collisions.

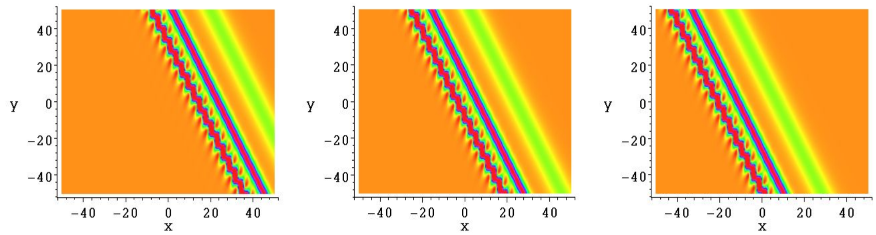
(2) The interaction between one soliton molecule and one breather. For this situation, the parameters k_i, p_i and q_i $(i = 1, 2)$ should satisfy the velocity resonant conditions (2.11) and (2.12), while the complex conjugate relations $\xi_3 = \overline{\xi_4}$, i.e. $k_3 = \overline{k_4} = k + \kappa\mathrm{i}$, $p_3 = \overline{p_4} = \rho + \varrho\mathrm{i}$, $\eta_3 = \overline{\eta_4} = \zeta + \varsigma\mathrm{i}$, and the ratio $\frac{k}{k_1} = \frac{\rho}{p_1} = \frac{Q_1}{q_1}$, $q_3 = \overline{q_4} \equiv Q_1 + Q_2\mathrm{i}$ need to be hold just as Eqs. (2.15)-(2.19). Figure 6 shows the molecule consisting of one soliton molecule and one breather in the (2+1)-dimensional BKP-KdV equation (1.3) where the parameters are chosen as

$$a = b = k_1 = 1, \ k_2 = \frac{2}{5}, \ k = \frac{13}{20}, \ \kappa = \frac{6}{5}, \ \eta_1 = \zeta = \varsigma = 0, \ \eta_2 = -3, \tag{2.44}$$

and others are given by

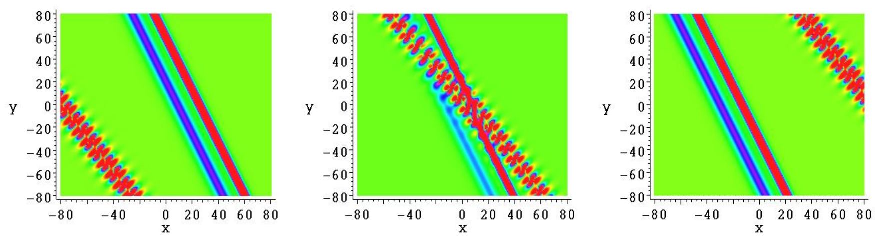
$$p_1 = \frac{54}{125}, \ p_2 = \frac{108}{625}, \ q_1 = \frac{3416}{3125}, \ q_2 = \frac{6832}{15625}, \ \rho = \frac{351}{1250}, \ \varrho = -\frac{8583}{5000} + \frac{3\sqrt{11301203}}{8000}. \tag{2.45}$$

In fact, there usually exists the elastic collision between soliton molecule and breather. For instance, we choose the parameters $\kappa = -\frac{2}{5}$, $\eta_2 = 5$ in Eq.(2.44) with $\rho = \frac{1}{5}$, $\varrho = \frac{2}{5}$, then the ratio is $\frac{k}{k_1} = \frac{3}{10} \neq \frac{\rho}{p_1} = \frac{25}{54}$. The elastic interaction structure of one soliton molecule and one breather in the (2+1)-dimensional BKP-KdV equation (1.3) can be generated from the four-soliton solution as shown in Figure 7.



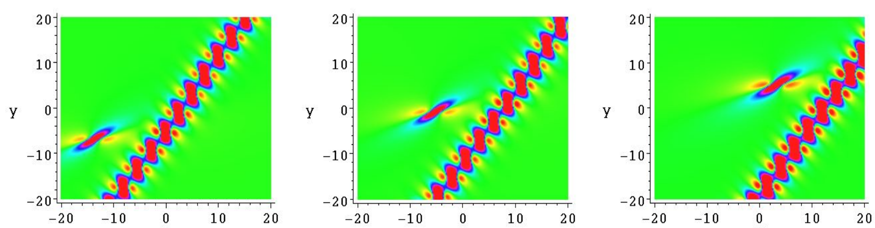
(a) (b) (c)

Figure 6. The molecule consisting of one soliton molecule and one breather in the (2+1)-dimensional BKP-KdV equation (1.3) with the parameter (2.44) and (2.45) at (a) $t = -16$, (b) $t = 0$ and (c) $t = 16$, respectively.



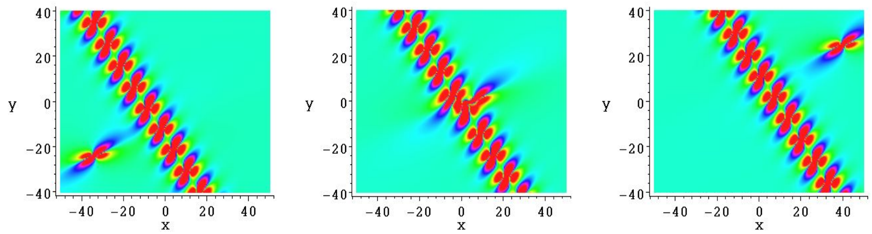
(a) (b) (c)

Figure 7. The elastic interaction between one soliton molecule and one breather in the (2+1)-dimensional BKP-KdV equation (1.3) with the parameter (2.44) and (2.45), but $\kappa = -\frac{2}{5}$, $\eta_2 = 5$, $\rho = \frac{1}{5}$, $\varrho = \frac{2}{5}$ at (a) $t = -16$, (b) $t = 0$ and (c) $t = 16$, respectively.



(a) (b) (c)

Figure 8. The molecule consisting of one breather and one lump in the (2+1)-dimensional BKP-KdV equation (1.3) with the parameter (2.46), (2.47) and (2.48) at (a) $t = -\frac{1}{2}$, (b) $t = 0$ and (c) $t = \frac{1}{2}$, respectively.



(a) (b) (c)

Figure 9. The elastic interaction between one breather and one lump in the (2+1)-dimensional BKP-KdV equation (1.3) with the parameters (2.20) and (2.28) at (a) $t = -2$, (b) $t = 0$ and (c) $t = 2$, respectively.

(3) The lump-breather molecule. Based on Eqs. (2.21), (2.23) and the complex conjugate relation $\xi_3 = \overline{\xi_4}$, i.e. $k_3 = \overline{k_4} = k+\kappa\mathrm{i}$, $p_3 = \overline{p_4} = \rho+\varrho\mathrm{i}$, $\eta_3 = \overline{\eta_4} = \zeta+\varsigma\mathrm{i}$, the transformation function of Eq.(2.42) is simplified as

$$\begin{aligned} f(x,y,t)=&(\theta_1\theta_2 + B_{12})(1 + e^{\xi_3} + e^{\xi_4}) + a_{34}[(C_{23} + C_{24})\theta_1 + (C_{13} + C_{14})\theta_2 \\ &+\theta_1\theta_2+(C_{13}+C_{14})(C_{23}+C_{24})+B_{12}]e^{\xi_3+\xi_4}+(C_{23}\theta_1+C_{13}\theta_2+C_{13}C_{23})e^{\xi_3} \\ &+ (C_{24}\theta_1 + C_{14}\theta_2 + C_{14}C_{24})e^{\xi_4}, \end{aligned} \tag{2.46}$$

with $q_4 = \overline{q_3} \equiv Q_1 - Q_2\mathrm{i}$, $C_{14} = \overline{C_{23}}$ and $C_{24} = \overline{C_{13}}$. Figure 8 displays the evolution of a molecule consisting of one breather and one lump in the (2+1)-dimensional BKP-KdV equation (1.3) where f is given by Eq.(2.46) and the parameters are taken as

$$k = -\frac{4}{5},\ \kappa = 1,\ \rho = \frac{1}{2},\ \varrho = -\frac{453}{200} + \frac{\sqrt{249103905}}{5000},\ \zeta = 5,\ \varsigma = 0. \tag{2.47}$$

Besides, the relation

$$\frac{5bk(a_2^2 + a_6^2)}{a_1^2 + a_5^2} - \frac{10b\rho(a_1a_2 + a_5a_6)}{a_1^2 + a_5^2} + Q_1 = 0, \tag{2.48}$$

should obey for the soliton molecule structure.

It is noted that when we take the parameters just as Eqs.(2.20) and (2.28), Eq.(2.48) is not equal to zero and then the elastic interaction between one breather and one lump appears as shown in Figure 9.

3. Summary

In conclusion, starting from the N-soliton solution and the velocity resonant condition, we present the soliton molecules and some related interaction solutions for the (2+1)-dimensional BKP-KdV equation (1.3). From two-soliton solution, one soliton molecule, one breather and one lump soliton are constructed successively (Figure 1). From three-soliton solution, the interaction solution of one soliton molecule and one line soliton, the molecule consisting of one line soliton and one breather/lump soliton are established, respectively (Figures 2-4). Further, the elastic interaction of two soliton molecules (Figure 5), the molecule consisting of one soliton molecule and the breather (Figure 6), the lump-breather molecule (Figure 8) are presented from four-soliton solution according to the velocity resonance, the module resonance and the long-wave limit mechanisms. Abundant and vivid figures for these molecule and their interaction are illustrated to demonstrate these dynamics features.

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