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ASYMPTOTICS OF THE SOLUTION TO A PIECEWISE-SMOOTH QUASILINEAR SECOND-ORDER DIFFERENTIAL EQUATION*

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Abstract We investigate a singularly perturbed boundary value problem for a piecewise-smooth second-order quasilinear differential equation in the case when the discontinuous curve which separates the domain is monotone. Applying the boundary layer function method, the asymptotic expansion of a solution with internal layer appearing in the neighborhoods of some point on the monotone curve and the point itself is constructed. For sufficiently small parameter values, using the matching method, the existence of a smooth solution with an internal transition layer in the neighborhood of a point of the monotone curve is proved. A simple example is given to show the effectiveness of our method.

Keywords Quasilinear differential equation, internal layer, asymptotic method, piecewise-smooth dynamical system

MSC(2010) 35B25, 35B40, 35B65, 35G30.

1. Introduction

When studying the problems like boundary layer in fluid mechanics and turning point in quantum mechanics, etc, scholars often encounter the phenomenons that variables in the models change quickly in a narrow domain on account of small parameters [12, 25, 34]. To describe these phenomenons, mathematical models of differential equations whose highest derivatives are multiplied by small parameters are established [1, 5, 17, 18, 24, 33]. Generally speaking, the solutions of singularly perturbed problems change radically on the boundary of domain or some point in the interior of region, which are called boundary layer of internal layer [29, 30, 32].

In recent years, piecewise-smooth dynamical systems have been received much attention due to their wide use as mathematical models in monetary policy, epidemics, neural network [4, 8–11, 13]. In particular, this type of problems with a small parameter has appealed to many mathematicians [3, 6, 7]. As seen in the city development models [14], some model parameters are discontinuous because of the

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barriers in media, which leads to the appearance of internal layer in the neighborhoods of some point on the given discontinuous curve. This kind of solution is called **contrast (spatial) structure solution**. In particular, boundary value problems with discontinuous right-hand sides are significant to apply in physical and biological fields [3, 6, 7]. Since the dynamical behavior of solution to this problem on the left and right side of discontinuous curve is totally different, it is necessary to determine the existence of smooth solution and the transition point where solution crosses the discontinuous curve. So far, problems in the case that the function on the right of equation is discontinuous on a vertical discontinuity line have been studied in some papers [15, 21–23].

In this paper, a stationary reaction-advection-diffusion equation with discontinuous advective and reactive terms is considered. We shall extend and generalize the basic ideas in the case of problems with discontinuous time variables [23] to the case of equations whose state variables are discontinuous. The essential characteristic is that the discontinuity curve is general. In this case, the biggest challenge is that both abscissas and ordinates of the transition point is unknown, the original method used in [23] and inclusion theory mentioned in [6,8] failed. Therefore, a new method should be constructed to solve piecewise-smooth dynamical systems. First of all, the asymptotic expansion of transition point and the solution is constructed by a new method based on boundary layer function technique [27, 28]. Then applying the matching asymptotic expansion method in contrast structure theory [26, 29], the existence of a smooth solution is proved. Moreover, the results we obtain can be used when developing models of problems with discontinuous characteristic on account of the fact that media is not even. And our results provides an efficient numerical algorithm for some models with discontinuous coefficients [19, 31].

The rest of the paper is organized as follows. In Sect. 1, the model problem is stated and the needed assumptions are given to make the solution pass through the discontinuous curve smoothly. In Sect. 2, we propose an algorithm for constructing an asymptotic approximation of the transition point on the discontinuous curve and the solution with an internal layer near the transition point. By using the matching asymptotic expansion method, the existence of contrast structure solution and uniformly validity of formal asymptotic solution are proved in Sect. 3. Sect. 4 consists of a simulation study, whose numerical result shows that our asymptotic solution is accurate.

1.1. Model Problem

We will consider the singularly perturbed boundary value problem

$$\begin{cases} \mu \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = A(y,x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + B(y,x), \quad (y,x) \in D, \\ y(0,\mu) = y^0, \quad y(1,\mu) = y^1, \end{cases} \tag{1.1}$$

where $D = \{(y,x) | \ -l \le y \le l, \, 0 < x < 1\}$, $\mu > 0$ is a small parameter, and y is an unknown scalar function.

The problem (1.1) will be considered under all the following assumptions.

Assumption 1.1. *The functions $A(y,x)$ and $B(y,x)$ appearing in the equation of* *problem (1.1) are piecewise-smooth on the domain D, namely,*

$$A(y,x)=\begin{cases}A^{(-)}(y,x), & (y,x)\in D^{(-)},\\ A^{(+)}(y,x), & (y,x)\in D^{(+)},\end{cases}\qquad B(y,x)=\begin{cases}B^{(-)}(y,x), & (y,x)\in D^{(-)},\\ B^{(+)}(y,x), & (y,x)\in D^{(+)},\end{cases}$$

where

$$\begin{aligned}D^{(-)}&=\{(y,x)|\ d(x)<y\le l,\ 0\le x\le 1\},\\ D^{(+)}&=\{(y,x)|\ -l\le y\le d(x),\ 0\le x\le 1\}.\end{aligned}$$

Here functions $A^{(\mp)}(y,x)$ and $B^{(\mp)}(y,x)$ are sufficiently smooth on the sets $\bar{D}^{(\mp)}$ and admit the inequalities

$$\begin{aligned}\lim_{y\to d(x)^+}A^{(-)}(d(x),x)&\neq\lim_{y\to d(x)^-}A^{(+)}(d(x),x), \qquad 0\le x\le 1,\\ \lim_{y\to d(x)^+}B^{(-)}(d(x),x)&\neq\lim_{y\to d(x)^-}B^{(+)}(d(x),x), \qquad 0\le x\le 1,\end{aligned}$$

As is shown in the Figure 1, functions $A^{(\mp)}(y,x)$ and $B^{(\mp)}(y,x)$ are discontinuous on the curve function $d(x)$ which separates the domain D and is sufficiently smooth and monotonely nonincreasing in the interval $[0,1]$. Moreover, the curve $y=d(x)$ crosses the boundaries of the rectangular region at $x=t_0$ and $x=t_1$ $(0\le t_0,t_1\le 1)$.

Assumption 1.2. *(i) Assume that on the subset $D^{(-)}$, the Cauchy problem*

$$A^{(-)}(y,x)\frac{\mathrm{d}y}{\mathrm{d}x}+B^{(-)}(y,x)=0,\quad y(0)=y^0$$

has an infinitely differentiable solution $y=\varphi^{(-)}(x)$, and we assume that

$$A^{(-)}(\varphi^{(-)}(x),x)>0,\quad 0\le x\le 1;$$

(ii) Assume that on the subset $D^{(+)}$, the Cauchy problem

$$A^{(+)}(y,x)\frac{\mathrm{d}y}{\mathrm{d}x}+B^{(+)}(y,x)=0,\quad y(1)=y^1$$

has an infinitely differentiable solution $y=\varphi^{(+)}(x)$, and we assume that

$$A^{(+)}(\varphi^{(+)}(x),x)<0,\quad 0\le x\le 1.$$

As shown in Figure 1, two curves $y=\varphi^{(-)}(x)$ and $y=\varphi^{(+)}(x)$ intersect the boundary curve $y=d(x)$ at two points Q and P, whose abscissas are denoted by q and p respectively, in the xy-plane. If $p<q$, then, by virtue of Assumptions 1.1-1.2, problem (1.1) may have a solution with a sharp transition layer in the neighborhood of $x=x^*(0<x^*<1)$. By a similar analysis, we find that problem (1.1) is unsolvable if $p\ge q$. The transition point x^* where the solution passes through the monotone curve remains undetermined. Note that the case when the function $d(x)$ is monotonely nondecreasing in the interval $[0,1]$ can be considered accordingly.

1.2. Attached system

Consider the attached system [27]

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = \tilde{z}, \quad \frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau} = A(\tilde{y}, x^*)\tilde{z}, \quad \tau \in R, \tag{1.2}$$

where $\tau = (x - x^*)/\mu$. According to Assumption 1.2, in the phase plane $(\tilde{y}, \tilde{z})$, there exist separatrices

$$\tilde{z}(\tilde{y}) = \int_{\varphi^{(\mp)}(x^*)}^{\tilde{y}} A^{(\mp)}(s, x^*)\,\mathrm{d}s \tag{1.3}$$

passing equilibrium points $(\varphi^{(\mp)}(x^*), 0)$ as $\tau \to \mp\infty$.

In the course of constructing the leading term in the asymptotic representation of the internal layer, it is necessary to consider the solvability of the following boundary value problem

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = \tilde{z}, \quad \dfrac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau} = A(\tilde{y}, x^*)\tilde{z}, \quad \tau \in R, \\ \tilde{y}(0) = d(x^*), \quad \tilde{y}(\mp\infty) = \varphi^{(\mp)}(x^*), \quad \tilde{z}(\mp\infty) = 0, \end{cases} \tag{1.4}$$

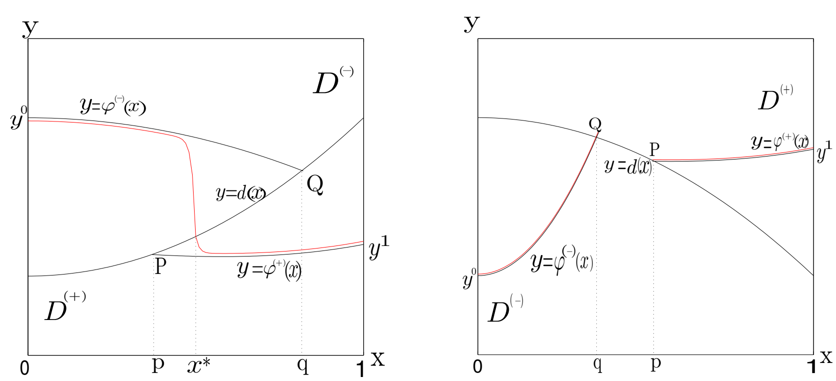
where $d(x^*) \in [\varphi^{(-)}(x^*), \varphi^{(+)}(x^*)]$, and x^* shall be determined in the course of proving the existence of a smooth solution to problem (1.1).

Thus, one can obtain the following sufficient condition needed to guarantee that problem (1.4) is solvable.

Assumption 1.3. *Suppose that the following inequalities are satisfied:*

$$\int_{\varphi^{(-)}(x^*)}^{\tilde{y}} A^{(-)}(s, x^*)\,\mathrm{d}s < 0, \quad \varphi^{(+)}(x^*) \le \tilde{y} < \varphi^{(-)}(x^*), \quad x^* \in [p, q],$$

$$\int_{\varphi^{(+)}(x^*)}^{\tilde{y}} A^{(+)}(s, x^*)\,\mathrm{d}s < 0, \quad \varphi^{(+)}(x^*) < \tilde{y} \le \varphi^{(-)}(x^*), \quad x^* \in [p, q].$$



(a) $p < q$ (b) $p \ge q$

Figure 1. The illustration of solution to problem (1.1).

Figure 1 shows that it is necessary to determine the transition point x^* where the solution of problem (1.1) passes the monotone curve. To this end, for any $x \in [p, q]$, we introduce the function

$$I(x) = \int_{\varphi^{(-)}(x)}^{d(x)} A^{(-)}(y, x)\,\mathrm{d}y - \int_{\varphi^{(+)}(x)}^{d(x)} A^{(+)}(y, x)\,\mathrm{d}y. \tag{1.5}$$

By the phase plane analysis method, we give the following sufficient condition.

Assumption 1.4. *Assume that the equation $I(x) = 0$ has a solution $x = x_0$, $x_0 \in [p, q]$, and we suppose that $I'(x_0) \neq 0$.*

2. Construction of a Formal Asymptotic Solution

The asymptotic approximation of problem (1.1) shall be constructed by boundary layer function method [27]. Since there is an internal transition layer in the neighborhood of monotone curve $y = d(x)$, problem (1.1) can be attributed to two classical singularly perturbed boundary value problem that are considered on both sides of $y = d(x)$. Taking account of the character of differential equations of problems (1.1), we introduce new variables

$$z^{(-)}(x, \mu) = \frac{\mathrm{d}y^{(-)}}{\mathrm{d}x}(x, \mu), \quad z^{(+)}(x, \mu) = \frac{\mathrm{d}y^{(+)}}{\mathrm{d}x}(x, \mu).$$

It is easy to see that $y^{(-)}(x^*, \mu) = y^{(+)}(x^*, \mu) = d(x^*)$, which means that solutions of two problems are sewed at the transition point x^*. In order to obtain a smooth solution of the original problem (1.1), especially at the transition point $x = x^*$, it is necessary to satisfy the following smoothness condition

$$z^{(-)}(x^*, \mu) = z^{(+)}(x^*, \mu) = z(\mu), \tag{2.1}$$

where x^* and $z(\mu)$ shall be determined in the course of constructing the asymptotics of solution to problem (1.1).

As shown in Figure 1, these two problems are equivalent to the systems of first-order differential equations:

$$\begin{cases} \dfrac{\mathrm{d}y^{(-)}}{\mathrm{d}x} = z^{(-)}, \quad \mu\dfrac{\mathrm{d}z^{(-)}}{\mathrm{d}x} = A^{(-)}z^{(-)} + B^{(-)}(y^{(-)}, x), \quad 0 < x < x^*, \\ y^{(-)}(0, \mu) = y^0, \quad y^{(-)}(x^*, \mu) = d(x^*), \quad z^{(-)}(x^*, \mu) = z(\mu) \end{cases} \tag{2.2}$$

and

$$\begin{cases} \dfrac{\mathrm{d}y^{(+)}}{\mathrm{d}x} = z^{(+)}, \quad \mu\dfrac{\mathrm{d}z^{(+)}}{\mathrm{d}x} = A^{(+)}z^{(+)} + B^{(+)}(y^{(+)}, x), \quad x^* < x < 1, \\ y^{(+)}(x^*, \mu) = d(x^*), \quad y^{(+)}(1, \mu) = y^1, \quad z^{(+)}(x^*, \mu) = z(\mu). \end{cases} \tag{2.3}$$

Applying boundary layer function method, the solutions to problems (2.2), (2.3) shall be constructed in the form of a sum of two terms:

$$\begin{cases} y^{(\mp)}(x, \mu) = \bar{y}^{(\mp)}(x, \mu) + Q^{(\mp)}y(\tau, \mu), \\ z^{(\mp)}(x, \mu) = \bar{z}^{(\mp)}(x, \mu) + Q^{(\mp)}z(\tau, \mu), \end{cases} \tag{2.4}$$

where
$$\tau = \frac{x - x^*}{\mu},$$
here $\bar{y}^{(\mp)}(x,\mu)$, $\bar{z}^{(\mp)}(x,\mu)$ are the regular parts of asymptotic approximation to the solutions $y^{(\mp)}(x,\mu)$, $z^{(\mp)}(x,\mu)$, and $Q^{(\mp)}y(\tau,\mu)$, $Q^{(\mp)}z(\tau,\mu)$ are the internal transition layer parts of asymptotic expansion of the solution in the neighborhood of monotone curve $y = d(x)$.

Each part of the formula (2.4) can be written as a power series of μ:

$$\bar{y}^{(\mp)}(x,\mu) = \bar{y}_0^{(\mp)}(x) + \mu\bar{y}_1^{(\mp)}(x) + \cdots + \mu^k\bar{y}_k^{(\mp)}(x) + \cdots, \tag{2.5}$$
$$\bar{z}^{(\mp)}(x,\mu) = \bar{z}_0^{(\mp)}(x) + \mu\bar{z}_1^{(\mp)}(x) + \cdots + \mu^k\bar{z}_k^{(\mp)}(x) + \cdots; \tag{2.6}$$
$$Q^{(\mp)}y(\tau,\mu) = Q_0^{(\mp)}y(\tau) + \mu Q_1^{(\mp)}y(\tau) + \cdots + \mu^k Q_k^{(\mp)}y(\tau) + \cdots, \tag{2.7}$$
$$Q^{(\mp)}z(\tau,\mu) = \mu^{-1}Q_{-1}^{(\mp)}z(\tau) + Q_0^{(\mp)}z(\tau) + \cdots + \mu^k Q_k^{(\mp)}z(\tau) + \cdots. \tag{2.8}$$

By separation of fast and slow variables (τ, x), we can obtain the problems for determining the regular terms $\bar{y}_k^{(\mp)}(x)$, $\bar{z}_k^{(\mp)}(x)$ of asymptotics of solutions to problems (2.2), (2.3) from the following expressions

$$\begin{cases} \dfrac{\mathrm{d}\bar{y}^{(\mp)}}{\mathrm{d}x} = \bar{z}^{(\mp)}, \quad \mu\dfrac{\mathrm{d}\bar{z}^{(\mp)}}{\mathrm{d}x} = A^{(\mp)}(y^{(\mp)},x)\bar{z}^{(\mp)} + B^{(\mp)}(y^{(\mp)},x), \\ \bar{y}^{(-)}(0,\mu) = y^0, \quad \bar{y}^{(+)}(1,\mu) = y^1, \end{cases} \tag{2.9}$$

and problems for the terms of internal layer parts $Q^{(\mp)}y(\tau)$, $Q^{(\mp)}z(\tau)$ are as follows:

$$\begin{cases} \dfrac{\mathrm{d}Q^{(\mp)}y}{\mathrm{d}\tau} = \mu Q^{(\mp)}z, \\ \dfrac{\mathrm{d}Q^{(\mp)}z}{\mathrm{d}\tau} = A^{(\mp)}(\bar{y}^{(\mp)}(x^*+\mu\tau) + Q^{(\mp)}y, x^*+\mu\tau)Q^{(\mp)}z \\ \qquad +A^{(\mp)}(\bar{y}^{(\mp)}(x^*+\mu\tau) + Q^{(\mp)}y, x^*+\mu\tau)\bar{z}^{(\mp)}(x^*+\mu\tau) \\ \qquad +B^{(\mp)}(\bar{y}^{(\mp)}(x^*+\mu\tau) + Q^{(\mp)}y, x^*+\mu\tau) \\ \qquad -A^{(\mp)}(\bar{y}^{(\mp)}(x^*+\mu\tau), x^*+\mu\tau)\bar{z}^{(\mp)}(x^*+\mu\tau) \\ \qquad -B^{(\mp)}(\bar{y}^{(\mp)}(x^*+\mu\tau), x^*+\mu\tau), \\ Q^{(\mp)}y(0,\mu) = d(x^*) - \bar{y}^{(\mp)}(x^*,\mu), \quad Q^{(\mp)}y(\mp\infty,\mu) = 0, \\ Q^{(\mp)}z(0,\mu) = z(\mu) - \bar{z}^{(\mp)}(x^*,\mu), \quad Q^{(\mp)}z(\mp\infty,\mu) = 0. \end{cases} \tag{2.10}$$

We substitute the formulas (2.5)-(2.8) into corresponding expressions (2.9), (2.10), then equate terms of the same powers of μ on both sides of equalities, and subsequently obtain Cauchy problems shown in Assumption 1.2 for determining $\bar{y}_0^{(\mp)}(x)$. Thus, we have
$$\bar{y}_0^{(\mp)}(x) = \varphi^{(\mp)}(x), \quad \bar{z}_0^{(\mp)}(x) = \varphi^{(\mp)\prime}(x).$$

For $\bar{y}_k^{(\mp)}$, $k \geq 1$, under Assumption 1.2, one can obtain the linear Cauchy problems whose solutions are represented in explicit forms:

$$\bar{y}_k^{(-)}(x) = \int_0^x \exp\left(-\int_s^x \frac{M^{(-)}(\xi)}{A^{(-)}(\varphi^{(-)}(\xi),\xi)}\mathrm{d}\xi\right)\frac{N_k^{(-)}(s)}{A^{(-)}(\varphi^{(-)}(\xi),\xi)}\mathrm{d}s, \tag{2.11}$$

$$\bar{y}_k^{(+)}(x) = \int_1^x \exp\left(-\int_s^x \frac{M^{(+)}(\xi)}{A^{(+)}(\varphi^{(+)}(\xi),\xi)}\mathrm{d}\xi\right)\frac{N_k^{(+)}(s)}{A^{(+)}(\varphi^{(+)}(\xi),\xi)}\mathrm{d}s, \tag{2.12}$$

where

$$M^{(\mp)}(x) = A_y^{(\mp)}(\varphi^{(\mp)}(x),x)\varphi^{(\mp)'}(x) + B_y^{(\mp)}(\varphi^{(\mp)}(x),x),$$

and $N_k^{(\mp)}(x)$ are known functions depending on $\bar{y}_j^{(\mp)}(x)$, $\bar{z}_j^{(\mp)}(x), j < k$. In particular, $N_1^{(\mp)}(x) = 0$. Obviously speaking, $\bar{z}_k^{(\mp)}$, $k \geq 1$ are defined as follows:

$$\bar{z}_k^{(\mp)} = \frac{-M^{(\mp)}(x)}{A^{(\mp)}(\varphi^{(\mp)}(x),x)}\bar{y}_k^{(\mp)} + \frac{N_k^{(\mp)}(x)}{A^{(\mp)}(\varphi^{(\mp)}(x),x)}.$$

After that, we write the problems for defining the leading terms of internal layer functions $Q_0^{(\mp)}y(\tau)$, $Q_{-1}^{(\mp)}z(\tau)$

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(\mp)}y}{\mathrm{d}\tau} = Q_{-1}^{(\mp)}z, \\ \dfrac{\mathrm{d}Q_{-1}^{(\mp)}z}{\mathrm{d}\tau} = A^{(\mp)}(\varphi^{(\mp)}(x^*) + Q_0^{(\mp)}y, x^*)Q_{-1}^{(\mp)}z, \\ Q_0^{(\mp)}y(0) = d(x^*) - \varphi^{(\mp)}(x^*), \quad Q_0^{(\mp)}y(\mp\infty) = 0, \\ Q_{-1}^{(\mp)}z(\mp\infty) = 0. \end{cases} \tag{2.13}$$

To solve this problem, we introduce the variable

$$\tilde{y}_0 = \varphi^{(\mp)}(x^*) + Q_0^{(\mp)}y, \quad \tilde{z} = Q_{-1}^{(\mp)}z. \tag{2.14}$$

By means of changes of variables (2.14), problem (2.13) are equivalent to the system (1.4). By virtue of Assumption 1.3 and discussion of attached system (1.2) in Section 2, there exists a solution $(Q_0^{(\mp)}y(\tau), Q_{-1}^{(\mp)}z)$ to problem (2.13). And one has the following exponential estimate

$$|Q_0^{(\mp)}y(\tau)| \leq C\mathrm{e}^{-\kappa|\tau|}, \quad |Q_{-1}^{(\mp)}z(\tau)| \leq C\mathrm{e}^{-\kappa|\tau|}, \tag{2.15}$$

where $C > 0$, $\kappa > 0$. From (1.3), we have

$$Q_{-1}^{(\mp)}z = \int_{\varphi^{(\mp)}(x_0)}^{\tilde{y}_0} A^{(\mp)}(y, x_0)\,\mathrm{d}y. \tag{2.16}$$

Taking account of the above formula, $Q_0^{(\mp)}y$ are determined from the first equation and boundary value conditions for $Q_0^{(\mp)}y$ of problem (2.13).

Generally speaking, $Q_k^{(\mp)}y(\tau)$, $Q_{k-1}^{(\mp)}z(\tau)$, $k \geq 1$ can be determined in a similar way, thus, for sake of simplicity, here we consider how to search for $Q_1^{(\mp)}y(\tau)$, $Q_0^{(\mp)}z(\tau)$, which are defined from the following linear differential systems

$$\begin{cases} \dfrac{\mathrm{d}Q_1^{(\mp)}y}{\mathrm{d}\tau} = Q_0^{(\mp)}z, \\ \dfrac{\mathrm{d}Q_0^{(\mp)}z}{\mathrm{d}\tau} = \tilde{A}^{(\mp)}(\tau)Q_0^{(\mp)}z + \tilde{A}_y^{(\mp)}(\tau)Q_{-1}^{(\mp)}zQ_1^{(\mp)}y + G_0^{(\mp)}(\tau), \\ Q_1^{(\mp)}y(0) = -\bar{y}_1^{(\mp)}(x^*), \quad Q_1^{(\mp)}y(\mp\infty) = 0, \\ Q_0^{(\mp)}z(\mp\infty) = 0, \end{cases} \tag{2.17}$$

where

$$
\begin{aligned}
G_0^{(\mp)}(\tau) = {} & \left\{ \tilde{A}_y^{(\mp)}(\tau) \left[\bar{y}_0^{(\mp)'}(x^*) + \bar{y}_1^{(\mp)}(x^*) \right] + \tilde{A}_x^{(\mp)}(\tau)\tau \right\} Q_{-1}^{(\mp)} z \\
& + \tilde{A}^{(\mp)}(\tau) z_0^{(\mp)}(x^*) + \tilde{B}^{(\mp)}(\tau), \\
\tilde{A}^{(\mp)}(\tau) := {} & A^{(\mp)}(\varphi^{(\mp)}(x^*) + Q_0^{(\mp)} y, x^*), \quad \tilde{A}_y^{(\mp)}(\tau) := A_y^{(\mp)}(\varphi^{(\mp)}(x^*) + Q_0^{(\mp)} y, x^*), \\
\tilde{A}_x^{(\mp)}(\tau) := {} & A_x^{(\mp)}(\varphi^{(\mp)}(x^*) + Q_0^{(\mp)} y, x^*), \quad \tilde{B}^{(\mp)}(\tau) := B^{(\mp)}(\varphi^{(\mp)}(x^*) + Q_0^{(\mp)} y, x^*).
\end{aligned}
$$

Using the equality

$$
\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\tilde{A}^{(\mp)}(\tau) Q_1^{(\mp)} y \right) = \tilde{A}^{(\mp)}(\tau) Q_0^{(\mp)} z + \tilde{A}_y^{(\mp)}(\tau) Q_{-1}^{(\mp)} z Q_1^{(\mp)} y,
$$

and integrating both sides of the first two equations of (2.17) with regard to boundary value conditions subsequently, one can obtain the representations for $Q_0^{(\mp)} z(\tau)$ and $Q_1^{(\mp)} y(\tau)$

$$
\begin{cases}
Q_0^{(\mp)} z(\tau) = \tilde{A}^{(\mp)}(\tau) Q_1^{(\mp)} y(\tau) + \displaystyle\int_{\mp\infty}^{\tau} G_0^{(\mp)}(s)\,\mathrm{d}s, \\
Q_1^{(\mp)} y(\tau) = Q_1^{(\mp)} y(0) \exp\left(\displaystyle\int_0^{\tau} \tilde{A}^{(\mp)}(s)\,\mathrm{d}s \right) \\
\qquad\qquad + \displaystyle\int_0^{\tau} \exp\left(\int_s^{\tau} \tilde{A}^{(\mp)}(q)\mathrm{d}q \right) \mathrm{d}s \int_{\mp\infty}^{s} G_0^{(\mp)}(q)\mathrm{d}q.
\end{cases}
\tag{2.18}
$$

Since $G_0^{(\mp)}(\tau)$ are going down exponentially, it is easy to see that $Q_1^{(\mp)} y(\tau)$, $Q_0^{(\mp)} z(\tau)$ are also decreasing exponentially.

3. Existence of a smooth solution to the original problem (1.1)

It is noted that two classical boundary value problems (2.2), (2.3) have smooth asymptotic solutions on the left and right side of monotone curve $y = d(x)$. However, one problem is that if there exists a point x^* where asymptotic approximation to problems (2.2), (2.3) can be sewed. And the other problem is whether the composite solution obtained in Sect.2 is smooth at the transition point x^* or not.

According to boundary layer function method [27], problems (2.2), (2.3) have solutions $y^{(\mp)}(x,\mu)$, $z^{(\mp)}(x,\mu)$, whose asymptotic representations are as follows:

$$
y^{(\mp)}(x,\mu) = Y_n^{(\mp)}(x,\mu) + O(\mu^{n+1}), \quad z^{(\mp)}(x,\mu) = Z_{n-1}^{(\mp)}(x,\mu) + O(\mu^n), \tag{3.1}
$$

where

$$
\begin{cases}
Y_n^{(\mp)}(x,\mu) = \sum\limits_{k=0}^{n} \mu^k \left(\bar{y}_k^{(\mp)}(x) + Q_k^{(\mp)} y(\tau) \right), \\
Z_{n-1}^{(\mp)}(x,\mu) = \mu^{-1} Q_{-1}^{(\mp)} z(\tau) + \sum\limits_{k=0}^{n-1} \mu^k \left(\bar{z}_k^{(\mp)}(x) + Q_k^{(\mp)} z(\tau) \right).
\end{cases}
\tag{3.2}
$$

We introduce the notation

$$
I(x^*,\mu) = \mu \left(z_\mu^{(-)}(x^*) - z_\mu^{(+)}(x^*) \right).
$$

It follows from (3.2), (2.6) and (2.8) that

$$\begin{aligned}I(x^*,\mu) =&Q_{-1}^{(-)}z(0,x^*) - Q_{-1}^{(+)}z(0,x^*)\\&+\mu\left(z_0^{(-)}(0,x^*) + Q_0^{(-)}z(0,x^*) - z_0^{(+)}(x^*) + Q_0^{(-)}z(x^*)\right)\\&+\cdots + O(\mu^{n+1}).\end{aligned}$$

We reconsider two classical boundary value problems (2.2), (2.3) by modifying x^* in the form

$$x^* = x_\delta := x_0 + \mu x_1 + \cdots + \mu^{n+1}(x_{n+1} + \delta), \tag{3.3}$$

where δ is a parameter. And we write the function $I(x^*,\mu)$ in the form

$$I(x^*,\mu) = I_0(x_0) + \mu I_1(x_1) + \cdots + \mu^{n+1} I_{n+1}(x_{n+1} + \delta, \mu),$$

where

$$\begin{aligned}I_0(x_0) &= \int_{\varphi^{(-)}(x_0)}^{d(x_0)} A^{(-)}(y,x_0)\,\mathrm{d}y - \int_{\varphi^{(+)}(x_0)}^{d(x_0)} A^{(+)}(y,x_0)\,\mathrm{d}y,\\ I_k(x_k) &= I'(x_0)x_k + H_k, \quad k = 1,\cdots,n,\end{aligned}$$

here H_k are known functions depending on the coefficients x_j, $j < k$, and

$$\begin{aligned}I'(x_0) =&\int_{\varphi^{(-)}(x_0)}^{d(x_0)} A_x^{(-)}(y,x_0)\,\mathrm{d}y - \int_{\varphi^{(+)}(x_0)}^{d(x_0)} A_x^{(+)}(y,x_0)\,\mathrm{d}y\\&+A^{(-)}(d(x_0),x_0)d'(x_0) - A^{(+)}(d(x_0),x_0)d'(x_0)\\&+A^{(+)}(\varphi^{(-)}(x_0),x_0)\varphi^{(+)'}(x_0) - A^{(-)}(\varphi^{(-)}(x_0),x_0)\varphi^{(-)'}(x_0),\end{aligned}$$

and finally

$$I_{n+1}(\delta,\mu) = I'(x_0)x_{n+1} + I'(x_0)\delta + H_{n+1} + O(\mu),$$

here H_n is a known quantity independent of μ or δ. In particular,

$$\begin{aligned}H_1 =&-A^{(-)}(\varphi^{(-)}(x_0),x_0)\bar{y}_1^{(-)}(x_0) + A^{(+)}(\varphi^{(+)}(x_0),x_0)\bar{y}_1^{(+)}(x_0)\\&-\bar{z}_0^{(+)}(x_0) + \bar{z}_0^{(-)}(x_0) - \int_{+\infty}^0 \tilde{B}^{(+)}(s)\mathrm{d}s + \int_{-\infty}^0 \tilde{B}^{(-)}(s)\mathrm{d}s\\&-\int_{+\infty}^0 \tilde{A}_x^{(+)}(s)sQ_{-1}^{(+)}z(s)\mathrm{d}s + \int_{-\infty}^0 \tilde{A}_x^{(-)}(s)sQ_{-1}^{(-)}z(s)\mathrm{d}s\\&-\int_{+\infty}^0 \tilde{A}_y^{(+)}s\bar{y}_0^{(+)'}(x_0)Q_{-1}^{(+)}z(s)\mathrm{d}s + \int_{-\infty}^0 \tilde{A}_y^{(-)}s\bar{y}_0^{(-)'}(x_0)Q_{-1}^{(-)}z(s)\mathrm{d}s\\&-z_0^{(+)}(x_0)\int_{+\infty}^0 \tilde{A}^{(+)}(s)\mathrm{d}s + z_0^{(-)}(x_0)\int_{-\infty}^0 \tilde{A}^{(-)}(s)\mathrm{d}s.\end{aligned}$$

By Assumption 1.4, x_0 can be obtained from $I_0(x_0) = 0$. And the coefficients x_k, $k = 1,\cdots,n$ can be determined from the linear equations $I_k(x_k) = 0$. And finally, for a sufficiently small μ, since $I'(x_0) \neq 0$, there exists $\delta = \bar{\delta}$, such that $I_n(\bar{\delta},\mu) = 0$. Therefore, we conclude that the asymptotic solution $y(x,\mu)$ is smooth at the point $x = x^*$. Thus, the problem (1.1) has a smooth solution

$$y(x,\mu) = \begin{cases} y^{(-)}(x,\mu,\bar{\delta}(\mu)), & 0 \le x \le x_{\bar{\delta}},\\ y^{(+)}(x,\mu,\bar{\delta}(\mu)), & x_{\bar{\delta}} \le x \le 1,\end{cases}$$

which appears internal layers in the neighborhood of discontinuous curve $y = d(x)$. In particular, the existence of transition point $x = x_{\bar{\delta}}$ is proved.

Replacing $x_{\bar{\delta}}$ by $\bar{x}$, the error estimate of the first equation of (3.1) remains constant. So main result in this paper can be summarized as the following theorem.

Theorem 3.1. *If Assumptions 1.1-1.4 are satisfied, then for sufficiently small parameter $\mu > 0$, problem (1.1) has a smooth solution $y(x,\mu)$, whose asymptotic expression can be represented as*

$$y(x,\mu) = \begin{cases} \sum_{k=0}^{n} \mu^k \left(\bar{y}_k^{(-)}(x) + Q_k^{(-)} y(\tau) \right) + O(\mu^{n+1}), & 0 \le x < \bar{x}, \\ \sum_{k=0}^{n} \mu^k \left(\bar{y}_k^{(+)}(x) + Q_k^{(+)} y(\tau) \right) + O(\mu^{n+1}), & \bar{x} \le x \le 1, \end{cases} \tag{3.4}$$

where $\bar{x} = x_0 + \mu x_1 + \cdots + \mu^{n+1} x_{n+1}$, $\tau = (x - \bar{x})/\mu$.

4. Numerical example

We consider the boundary value problem

$$\begin{cases} \begin{cases} \mu y'' = (1-x)y' - 4x(1-x), & (y,x) \in D^{(-)}, \\ \mu y'' = (x-2)y' + 6x(x-2), & (y,x) \in D^{(+)}, \end{cases} \\ y(0) = 0, \quad y(1) = -3, \end{cases} \tag{4.1}$$

where $d(x) = 2x^2 + 3x - 2$.

It is easy to verify that Assumption 1.1 in Theorem 3.1 is fulfilled.

The Cauchy problems for defining $\varphi^{(\mp)}(x)$ are as follows

$$y' - 4x = 0, \quad y(0) = 0; \qquad y' + 6x = 0, \quad y(1) = -3,$$

whose solutions are

$$\varphi^{(-)}(x) = 2x^2, \quad (y,x) \in D^{(-)}; \qquad \varphi^{(+)}(x) = -3x^2, \quad (y,x) \in D^{(+)}.$$

Thus, Assumption 1.2 is satisfied.

From $\bar{y}_0^{(\mp)}(x) = \varphi^{(\mp)}(x)$, $\bar{z}_0^{(\mp)}(x) = \bar{y}_0^{(\mp)'}(x)$, we have

$$\bar{y}_0^{(-)}(x) = 2x^2, \quad \bar{y}_0^{(+)}(x) = -3x^2; \quad \bar{z}_0^{(-)}(x) = 4x, \quad \bar{z}_0^{(+)}(x) = -6x.$$

It follows from (2.11), (2.12) that

$$\bar{y}_1^{(\mp)}(x) = 0.$$

The function (1.5) is rewritten as

$$I(x) = \int_{2x^2}^{2x^2+3x-2} (1-x)\,\mathrm{d}y - \int_{-3x^2}^{2x^2+3x-2} (x-2)\,\mathrm{d}y,$$

and the equation $I(x) = 0$ has a solution $x_0 = 0.435$. Simple computation shows that $I'(x_0) = 12.2515 \neq 0$. And since

$$\int_{\varphi^{(-)}(x_0)}^{\tilde{y}} (1-x_0)\,\mathrm{d}s < 0, \quad \varphi^{(+)}(x_0) \le \tilde{y} < \varphi^{(-)}(x_0), \quad x_0 \in [0.4, 2/3],$$

$$\int_{\varphi^{(+)}(x_0)}^{\tilde{y}} (2 - x_0)\,\mathrm{d}s < 0, \quad \varphi^{(+)}(x_0) < \tilde{y} \le \varphi^{(-)}(x_0), \quad x_0 \in [0.4, 2/3],$$

Assumptions 1.3-1.4 can be satisfied.

Taking $x_0 = 0.435$ into account, the Cauchy problems for defining $Q_0^{(\mp)}y$ are as follows

$$\frac{\mathrm{d}Q_0^{(-)}y}{\mathrm{d}\tau} = 0.565\,Q_0^{(-)}y, \quad Q_0^{(-)}y(0) = -0.6951, \quad \tau \le 0;$$

$$\frac{\mathrm{d}Q_0^{(+)}y}{\mathrm{d}\tau} = -1.565\,Q_0^{(+)}y, \quad Q_0^{(+)}y(0) = 0.2511, \quad \tau \ge 0,$$

whose solutions are in the form

$$Q_0^{(-)}y(\tau) = -0.6951\mathrm{e}^{0.565\tau}, \quad Q_0^{(+)}y(\tau) = 0.2511\mathrm{e}^{-1.565\tau}.$$

From $Q_{-1}^{(\mp)}z = \mathrm{d}Q_0^{(\mp)}y/\mathrm{d}\tau$, we have

$$Q_{-1}^{(-)}z(\tau) = -0.3927\mathrm{e}^{0.565\tau}, \quad Q_{-1}^{(+)}z(\tau) = -0.3930\mathrm{e}^{-1.565\tau}.$$

It follows from the equation $I'(x_0)x_1 + H_1 = 0$ that $x_1 = -0.4686$.

By Theorem 3.1, the problem (4.1) has a smooth solution $y(x,\mu)$, which takes the form

$$y(x,\mu) = \begin{cases} 2x^2 - 0.6951\mathrm{e}^{0.565\tau} + O(\mu), & 0 \le x < \bar{x}, \\ -3x^2 + 0.2511\mathrm{e}^{-1.565\tau} + O(\mu), & \bar{x} \le x \le 1, \end{cases}$$

where $\bar{x} = 0.435 - 0.4686\mu$, $\tau = (x - \bar{x})/\mu$.

This problem has not an analytical solution. To describe the behavior of the exact solution, an asymptotic solution is obtained by our method. And the expression of smooth solution $y(x,\mu)$ shows that the obtained asymptotic approximation is close to the exact solution as μ takes sufficiently small value. In order to verify our result, a reliable numerical solution is compared with the asymptotic representation $Y_0(x,\mu)$. As is shown in Figure 2, our asymptotic solution is accurate and the internal layer is easy to see in the neighborhood of monotone curve $y = 2x^2 + 3x - 2$.

5. Conclusion

In this paper, a stationary problem for a piecewise-smooth reactive-advection-diffusion differential equation is studied. By using contrast structure theory, a smooth solution with an internal layer in the neighborhood of a point on the discontinuous curve is obtained. This work can be seen as a further development for the results in [21, 23]. Furthermore, our results can be generalized to equation systems and also be applied to provide an efficient numerical algorithm for similar problems in [19, 31].

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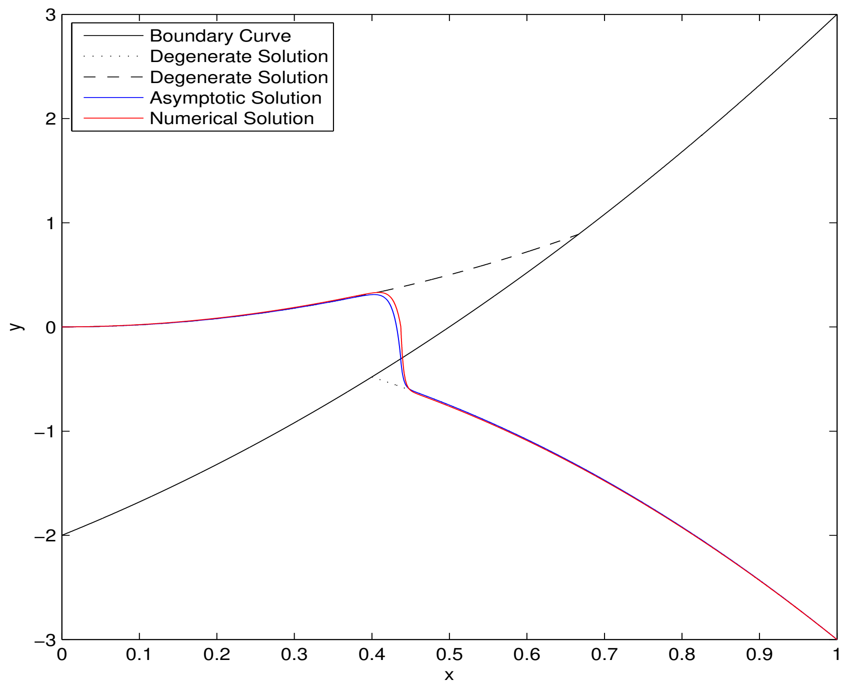

Figure 2. Numerical solution of problem (4.1) and its zero-order asymptotic approximation($\mu = 0.001$).

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