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SENSITIVITY ANALYSIS OF PESTICIDE DOSE ON PREDATOR-PREY SYSTEM WITH A PREY REFUGE*

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Abstract A type of predator-prey model with refuge and nonlinear pulse feedback control is established. First, we construct the Poincaré map of the model and analyse its main properties. Then based on the Poincaré map, we explore the existence, uniqueness and global stability of the order-1 periodic solution, and also the existence of order-k($k \geq 2$) periodic solutions of the system. These theoretical analysis gives the relationship between pesticide dosage and spraying cycle and economic threshold, and the relationship between the pesticide dose D and threshold ET. The results show that choosing the appropriate spraying period and finding the corresponding pesticide dose under the certain economic threshold can control the number of pests and the healthy growth of crops could be controlled. Moreover, we study the influence of refuge on population density. The results show that with the increase of refuge intensity, the population of predators decreases or even becomes extinct. The parameter sensitivity analysis shows that the change of control parameters and pesticide dosage is very sensitive to the critical condition of the stability of the boundary periodic solution.

Keywords Impulsive semi-dynamic system, refuge, Poincaré map, periodic solutions.

MSC(2010) 34C25, 34C60, 92B05.

1. Introduction

Pest management has always been an important subject of research, because once pest management is not timely or the control measures are inappropriate, it will bring immeasurable losses to agriculture and forestry [14, 44]. Generally, there are three commonly used methods to control pests:

(a) **chemical control** that is to control pests by spraying insecticides. Although this method is simple and effective, long-term use will cause damage to the soil environment and increase the resistance of pests etc. [42].

(b) **biological control** that is to control pests by releasing natural enemies or viruses of the pests. However, this method is only suitable for situations

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where there is a small amount of pests [3,4].

(c) When the amount of pests is large, people often use the **integrated pest management (IPM)** that is a combination of the chemical control and biological control. Evidences have shown it is more effective than the classic methods [22,23,26].

Since the pests will be reduced instantaneously after applying comprehensive pest management, the aforementioned management strategies can be mathematically modelled by systems of impulse differential equations. Impulsive differential equations are widely used in biology, agriculture and medicine, which often used to describe transient phenomena in nature [9, 11, 16, 19, 27, 29, 34, 35, 41]. Depending when "control" is applied, people proposed two types of impulsive differential equations (IDEs): time impulse IDE, which is used to describe the periodic instantaneous change of the population; and state impulse IDE, which is often used to study the instantaneous change of the population when the pest number or density reaches a certain threshold [14,17,22,23,26,31,36–39].

To investigate models with a state impulse feedback control and IPM, the qualitative theory for impulsive semi-dynamical system has been developed [1,27–29]. For example, Wei and Chen [32] established and proved the existence and stability of the order 1 periodic solution. Tian et al. [10] studied the Holling II functional response predator-prey system with state-dependent impulse control by defining the successor function of the semi-continuous dynamical system, and extended a new theoretical proof to obtain similar conclusions. Huang et al. [7] proposed a state pulse mathematical model for pulse injection of insulin for type 1 and type 2 diabetes, and proved the existence and stability of the first-order periodic solution. Sun and his collaborators [24] proposed a chemostat model of maintaining the biomass concentration below a certain threshold, and considered the existence of order 1 and 2 periodic solutions. As shown in the aforementioned references, the existence and local stability of the order 1 periodic solution, and even the existence of the order 2 periodic solution can be established by using the successor function and Poincaré map due to the use of the linear impulse control or feedback control [27,28]. However, there are scenarios where models have more complicated dynamics that need nonlinear control strategies, for example in the model to be formulated in this study where a prey refuge will be considered in the prey-predator model. It is known that the refuge plays an important role in affecting the dynamic behavior of predator-prey population [2,15,30]. Prey can use a shelter to prevent predation by natural enemies. In recent years, many scholars have studied the role of refuge on populations [13,21]. A Holling II prey-predator model with refuge was studied in [43], where authors demonstrated the existence and stability of the equilibrium point of the model, and also the existence of Hopf bifurcation and limit cycle. Bhaskar, in [2] studied a diffusion predator-prey model with a prey refuge and a type III response function, and considered the effects of the one-dimensional and two-dimensional diffusion rates of the model system on the population. Ji et al. [8] proposed a Holling II predator-prey model with a constant prey refuge and a constant ratio of prey harvest.

In addition to the nonlinear impulse control strategy, the sensitivity analysis of pesticide application intensity to pests and natural enemies requires further research. To this end, we will develop a nonlinear impulse control model with prey refuge to further study its dynamic properties by Poincaré map. The parameter sensitivity analysis of the model include (a) the sensitivity of parameters to the stability threshold of the boundary period solution, (b) the relationship between the pest control period T, the threshold ET and the sprayed pesticide dose D, and (c) the changing law of predator and prey population density as the proportion of prey refuge changes. We outline our study as follows. In Section 2, we will formulate a predator-prey model based on pulse feedback control. We define the Poincaré map of the model and analyse its properties, such as the monotonicity, continuous and discontinuous intervals, differentiability, fixed points and asymptote of the Poincaré map in Section 3. Then it is followed by Section 4 where we discuss the existence, uniqueness and global stability of the order-1 periodic solution of the model, and the conditions for the existence of the order-$k(k \geq 2)$ periodic solution. In Sections 5.1 and 5.2, we investigate the impact of the prey refuge and numerically demonstrate our results. We finally conclude our study in Section 6.

2. Model Formulation

In this section, based on the model formulated by [15], we will propose our model under impulse control. Let $x(t)$ and $y(t)$ represent the densities of the prey and the predator population at time t, respectively, $g(x)$ the growth rate, $\phi(x)$ the functional response. Then the general predator-prey model takes the following form as in [15]

$$\begin{cases} x' = xg(x) - y\phi(x), \\ y' = (p\phi(x) - c)y. \end{cases} \tag{2.1}$$

If further assume that the growth rate of the prey population takes the logistic form, and the predator has a Holling II saturation functional response, system (2.1) becomes

$$\begin{cases} x' = rx(1 - \dfrac{x}{K}) - \dfrac{qxy}{x+a}, \\ y' = b(\dfrac{px}{x+a} - c)y, \end{cases} \tag{2.2}$$

where r is the internal growth rate of the prey population, K is the environmental capacity, $\frac{qxy}{x+a}$ denotes the Holling II functional response, p is the conversion rate from prey to predator growth, c is the death rate of predators. All parameters r, K, q, a, p, c are positive constants. If we further consider the refuge in prey that is the prey is protected by the refuge, a predator-prey system with refuge effect takes the following form

$$\begin{cases} x' = rx(1 - \dfrac{x}{K}) - \dfrac{q(x - x_r)y}{(x - x_r) + a}, \\ y' = b(\dfrac{p(x - x_r)}{(x - x_r) + a} - c)y, \end{cases} \tag{2.3}$$

where x_r denote the density of prey in the refuge. Generally, the refuge effect of the prey population may be mathematically modelled by one of the following forms:

(a) The density of prey in the refuge is proportional to the existing density, that is, $x_r = \beta x, (0 \leq \beta \leq 1)$;

(b) The density of prey in the refuge is constant, that is $x_r = \gamma$.

In this study, we consider the first case. Then system (2.3) becomes

$$\begin{cases} x' = (r(1-\dfrac{x}{K}) - \dfrac{q(1-\beta)y}{(1-\beta)x+a})x, \\ y' = b(\dfrac{p(1-\beta)x}{(1-\beta)x+a} - c)y. \end{cases} \tag{2.4}$$

Let $x \to Kx, y \to \frac{rK}{qy}, t \to \frac{r}{x+A}\tau$, and denote $m = \frac{a}{1-\beta}, A = \frac{m}{K}, B = \frac{bp}{r}, C = \frac{c}{p}$. System (2.4) becomes

$$\begin{cases} x\prime = ((1-x)(x+A) - y)x, \\ y\prime = B(x - C(x+A))y, \end{cases} \tag{2.5}$$

which has 3 nonnegative equilibrium points, among them there are two boundary equilibria $O(0,0)$ and $P_1(1,0)$, and a unique interior equilibrium $Q_e(x^*,y^*)$ that is positive provided $1-C-AC>0$ with

$$x^* = \frac{AC}{1-C}, y^* = \frac{A(1-C-AC)}{(1-C)^2}.$$

Reference [15] indicated that when Q_e exists, a simple analysis shows that $O(0,0)$ and $P_1(1,0)$ are saddle points, and Q_e is a weak focus. In what follows, unless specified otherwise, we discuss the interior equilibrium Q_e only.

It is known that if pests are not controlled they may cause immeasurable damage to the biological population. Therefore, we propose to implement comprehensive control strategy for model (2.5), and only when the density of pests population reaches a critical density called economic threshold (ET), control is implemented: the density of the prey population decreases by spraying chemicals and the density of the predator population increase by releasing of natural enemies [25]. More precisely, we will implement the nonlinear pulse control technique developed in [20, 25] where authors showed that limited resources could affect pest control, and they also investigated that the nonlinear pulsed control may induce periodic solutions with any period.

Based on the above discussion, we can have a schematic diagram as shown in Figure 1. From the schematic diagram a prey-predator model concerning refuge and the effects of pesticide doses takes the following form:

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= ((1-x)(x+A)-y)x \\ \frac{dy(t)}{dt} &= B(x-C(x+A))y \end{aligned}\right\} x(t) < ET, \\ \left.\begin{aligned} x(t^+) &= H_1(D)x(t) \\ y(t^+) &= H_2(D)y(t) + \tau \end{aligned}\right\} x(t) = ET. \end{cases} \tag{2.6}$$

Here $H_1(D)$ and $H_2(D)$ denote survival fractions of prey and predator population, respectively, after applying dose D of pesticide and $0 \le H_1(D), H_2(D) \le 1$. A common assumption is that insecticides not only kill prey but also predators, but with different kill rates:

$$H_1(D) = e^{-k_1 D},\ H_2(D) = e^{-k_2 D},\ 0 \le k_1, k_2 \le 1.$$

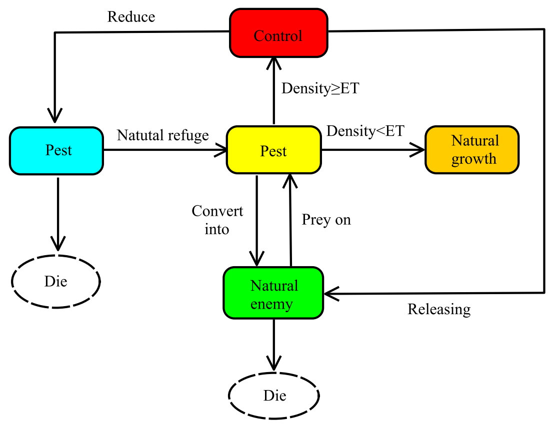

Figure 1. Schematic diagram of the mathematical model.

$\tau \geq 0$ is the fixed number of natural enemies released at time t. We let $x(t^+) = x^+, y(t^+) = y^+$ denote the density of prey and natural enemies after implementing comprehensive control strategy at time t. In addition, $x(0^+)$ and $y(0^+)$ represent the initial density of prey and natural enemies, respectively. Without loss of generality, in this study, we assume that the initial density of the prey population is always less than ET, that is $x(0^+) = x_0^+ < ET, y(0^+) = y_0^+ > 0$ and $ET < K$.

To investigate the dynamics of (2.6), we will first construct the corresponding Poincaré map, using the pulse points on the phase set and then study the accurate domain of the Poincaré map. From biological view, we restrict study to region $R_+^2 = \{(x, y) : x \geq 0, y \geq 0\}$ in the following section.

3. Poincaré Map and Its Properties

3.1. The domain of the Poincaré map

First, it is easy to see that model (2.5) have two isoclines, which are denoted by

$$L_1 : x = \frac{AC}{1-C}; \qquad L_2 : y = (1-x)(1+x).$$

Then for the convenience of discussion, we define two lines as follow:

$$L_3 = \{(x, y) : x = e^{-k_1 D} ET, y \geq 0\}; \qquad L_4 = \{(x, y) : x = ET, y \geq 0\}.$$

Since $0 < ET < K$, substituting $x = ET$ into line L_2, one yields the intersection point of line L_2 and line L_4 ,denote the only intersect point by $Q(ET, y_{ET})$. Then denote the unique intersect point of L_2 and L_3 by $H(e^{-k_1 D} ET, y_H)$, where

$$y_{ET} = -ET^2 + (1-A)ET + A, \quad y_H = -(e^{-k_1 D} ET)^2 + (1-A)e^{-k_1 D} ET + A.$$

In order to define the pulse semi-dynamic system of system (2.6), the exact pulse set and phase set should be specified. Now based on the relative positions between the threshold ET and the equilibrium point $Q_e(x^*, y^*)$, we have the following two cases:

Case I: $ET < x^*$

In this case, L_3 and L_4 are located on the left side of the equilibrium point Q_e. Any solution starting from the line L_3 will definitely reach line L_4 after a finite time by the vector fields of the model (2.6). Thus, the trajectory initiating from the intersection of two lines L_2 and L_3, $A^+(e^{-k_1 D}ET, y_H) \in L_3$ must intersect L_4 at a point that is denoted by $A_{11}(ET, y_{11})$ (see Fig. 2(a)). Then the impulsive set M_1 for system (2.6) can be defined as follows:

$$M_1 = \{(x,y) \in R^2 \,|x = ET, \quad 0 \le y \le y_{11}\},$$

which is a closed subset of R^2, see Fig. 2(a). Furthermore, define the continuous function I as follow:

$$I : (ET, y) \in M_1 \to (x^+, y^+) = (e^{-k_1 D}ET, e^{-k_2 D}y + \tau) \in R^2.$$

So the corresponding phase set can be defined by

$$N_1 = I(M_1) = \left\{ (x^+, y^+) \middle| \, x^+ = e^{-k_1 D}ET, \tau \le y^+ \le e^{-k_2 D}y_{11} + \tau \right\}.$$

Case II: $ET > x^*$

When $ET > x^*$, L_4 is on the right side of equilibrium Q_e, and L_3 could be on the left or right side equilibrium Q_e. Since the vector fields of the system (2.6), any solution starting from point $(e^{-k_1 D}ET, y_0^+)$ with $y_0^+ > 0$ may experience infinitely many pulses or will be free from impulsive effects which depends on the initial conditions. Therefore, according to different situations, the exact domain of impulsive sets and phase sets of the system (2.6) will be different under case II.

For example, if trajectory starting from point A^+ intersects L_4 at point $A_{22}(ET, y_{22})$ (see Fig. 2(b)), then any solution initiating from $(e^{-k_1 D}ET, y_0^+)$ with $y_0^+ > 0$ may experience infinitely many pulses. Thus the impulse set M_2 for system (2.6) can be determined as follows:

$$M_2 = \{(x,y) \in R^2 \,|x = ET, \quad 0 \le y \le y_{22}\}.$$

So the corresponding phase set can be defined by

$$N_2 = I(M_2) = \left\{ (x^+, y^+) \middle| \, x^+ = e^{-k_1 D}ET, \tau \le y^+ \le e^{-k_2 D}y_{22} + \tau \right\}.$$

If there exists a trajectory Γ_s which tangents to the line L_4 at point $S(ET, y_s)$ and intersects the line L_3 at two points, denoted by $P_1 = (e^{-k_1 D}ET, y_{P_1})$ and $P_2 = (e^{-k_1 D}ET, y_{P_2})$ with $y_{P_1} > y_{P_2}$ (see Fig. 2(c),1(d)). Moreover, any solution initiating from point (x_0^+, y_0^+) with $y_{P_2} < y_0^+ < y_{P_1}$ will be free from impulsive effects. Thus, We can infer the pulse set as follow:

$$M_3 = \{(x,y) \in R^2 \,|x = ET, \quad 0 \le y \le y_s\}.$$

So the corresponding phase set can be defined as

$$N_3 = I(M_3) = \left\{ (x^+, y^+) \middle| \, x^+ = e^{-k_1 D}ET, \tau \le y^+ \le e^{-k_2 D}y_s + \tau \right\}.$$

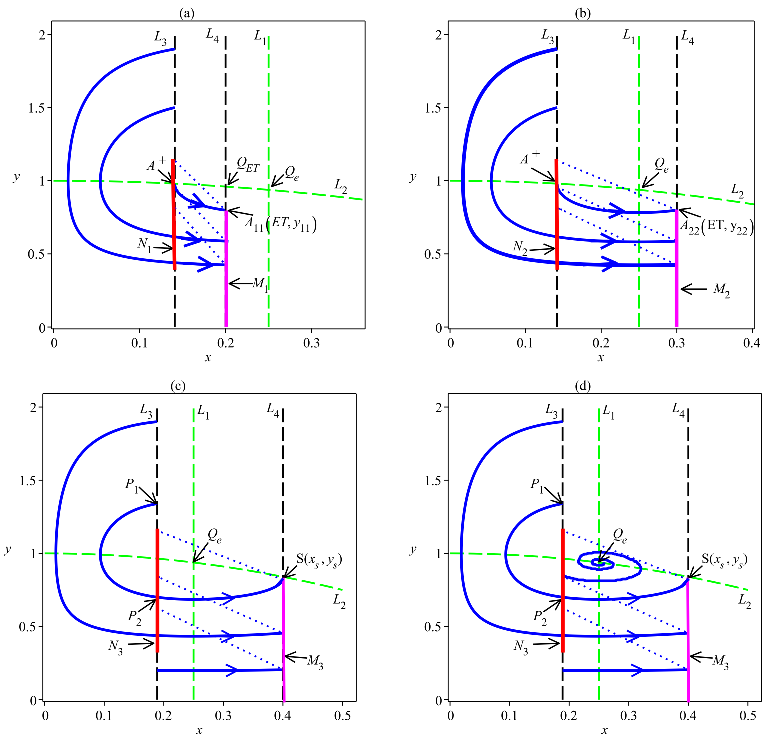

Figure 2. The domains of the phase and pulse sets in three cases, $A = 1$, $B = 0.8$, $C = 0.2$. (a)$k_1 = 0.7, k_2 = 0.3, D = 0.5, ET = 0.2$, $\tau = 0.45$.(b)$k_1 = 1.5, k_2 = 0.3, D = 0.5, ET = 0.3$, $\tau = 0.45$.(c)$k_1 = 1.5, k_2 = 0.3, D = 0.5, ET = 0.4$, $\tau = 0.45$.(d)$k_1 = 1.5, k_2 = 0.3, D = 0.5, ET = 0.4$, $\tau = 0.45$.

3.2. The construction of Poincaré map

Based on the discussion in preceding section, we construct the Poincaré map, where the phase sets are part of L_3.

For case I, the trajectory from any one point $H_k^+(e^{-k_1 D}ET, y_k^+) \in L_3$ on the phase set must intersect with the straight line L_4 at the point $\mathrm{H}_{k+1}(ET, y_{k+1})$. Then, the value of y_k^+ is only determined by y_{k+1} through Cauchy-Lipschitz theorem , which will be denoted by $y_{k+1} \overset{def}{=} g(y_k^+)$ for the convenience of the following discussion. Point H_{k+1} on the impulse set L_4 maps to point $\mathrm{H}_{k+1}^+(e^{-k_1 D}ET, y_{k+1}^+)$ on L_3 after a pulse. Therefore, based on the impulse function $y(t^+) = e^{-k_2 D}g(t) + \tau$ of system (2.6), $y_{k+1}^+ = e^{-k_2 D}y_{k+1} + \tau$, and point H_{k+1}^+ will be the initial point of the next impulse function on the phase set.

Using the same fashion, we can define a similar Poincaré map for case II. Now we can express the Poincaré map of system (2.6) as:

$$y_{i+1}^+ = e^{-k_2 D}g(y_i^+) + \tau = \varphi(y_i^+). \tag{3.1}$$

Next, we infer the expression of Poincaré map function φ and discuss its properties. Let

$$\begin{cases} P(x(t), y(t)) = ((1-x)(x+A) - y)x, \\ Q(x(t), y(t)) = B(x - C(x+A))y. \end{cases} \tag{3.2}$$

System (2.6) can be rewritten as a scalar differential equation on the phase set:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{Q}{P} = \omega(x,y), \\ y(e^{-k_1 D}ET) = y_0^+, \end{cases} \tag{3.3}$$

which is restricted on region

$$\Omega_1 = \{(x,y)|x>0, y>0, y<(1-x)(x+A)\}.$$

Thus $\omega(x,y)$ is differentiable in the region Ω_1. Let $x_0^+ = e^{-k_1 D}ET$ and $y_0^+ = S$, where $S \in N$, $S < y_H$, $(x_0^+, y_0^+) \in \Omega_1$. Then we can get

$$y(x) = y(x; e^{-k_1 D}ET, S) \stackrel{def}{=} y(x,S), \quad e^{-k_1 D}ET \le x \le ET, \tag{3.4}$$

and from (3.3), we get

$$y(x,S) = S + \int_{e^{-k_1 D}ET}^{x} \omega(S, y(s,S))\, ds. \tag{3.5}$$

Based on equations (3.1) and (3.3), the definition of Poincaré map φ as follow:

$$\varphi(S) = e^{-k_2 D}ETy(ET,S) + \tau. \tag{3.6}$$

Next, we discuss the properties of the Poincaré map φ.

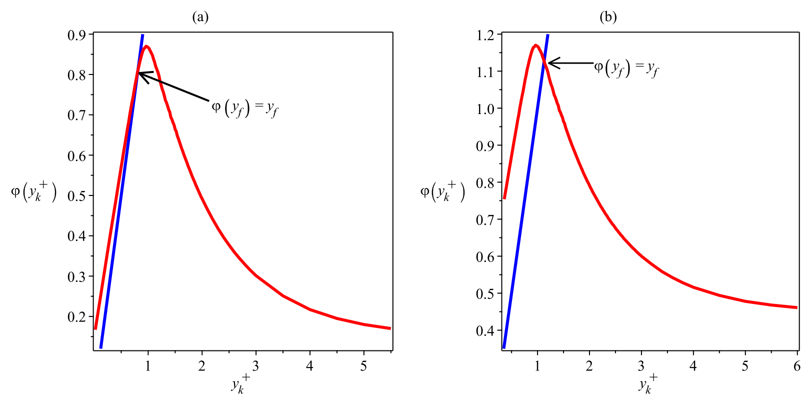

Figure 3. The Poincaré map φ related to the impulsive point series y_k^+ in case I. The parameters fixed as $A=1$, $B=0.8$, $C=0.2$, $k_1=0.7$, $k_2=0.3$, $D=0.5$, $ET=0.25$. (a)$\tau=0.15$; (b)$\tau=0.45$.

3.3. The main properties of the Poincaré map

According to the mathematical model and expression of Poincaré map φ, we can address some properties about Poincaré map φ.

Theorem 3.1. *For case I, that is, when $ET < x^*$, the Poincaré map φ of system* (2.6) *satisfies the following properties, which are illustrated by Fig.3:*

i) The domain of φ is $[0,+\infty)$ and the range is $[\tau, \varphi(y_H)] = [\tau, e^{-k_2 D}y(e^{-k_1 D}ET, y_H) + \tau]$. In addition φ monotonically increases on $[0, y_H]$ and monotonically decreases on $[y_H, +\infty)$.

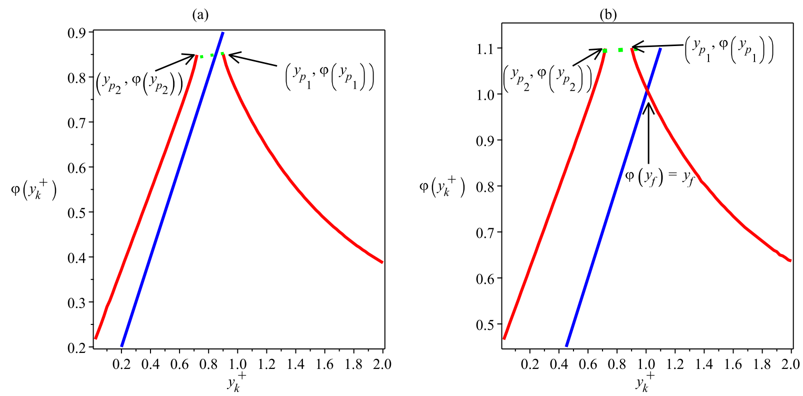

Figure 4. The Poincaré map φ related to the impulsive point series y_k^+ in case II. The parameters fixed as:$A = 0.8,\ B = 0.8,\ C = 0.2,\ k_1 = 0.7,\ k_2 = 0.3,\ D = 0.5,\ ET = 0.25$. (a) $\tau = 0.2$; (b)$\tau = 0.45$.

ii) *φ is continuously differentiable.*

iii) *φ is concave on $[0, y_H)$.*

iv) *φ has a unique fixed point y_f. Furthermore, when $\tau > 0$ and $\varphi(y_H) < y_H, y_f \in (0, y_H)$; while $\varphi(y_H) > y_H$ we have $y_f \in (y_H, +\infty)$.*

v) *φ has a horizontal asymptote $y = \tau$ as $y_k^+ \to +\infty$.*

Proof. (i) According to the vector field of system (2.6), the domain of φ is defined on the interval $[0, \infty)$. Given two points $A_i^+(e^{-k_1 D}ET, y_i^+), A_j^+(e^{-k_1 D}ET, y_j^+)$ with $y_i^+, y_j^+ \in [0, y_H], y_i^+ < y_j^+$. According to the Cauchy-Lipschitz theorem, the trajectories staring from two points A_i^+ and point A_j^+ will meet line L_4 at points $A_{i+1}(ET, y_{i+1})$, $A_{j+1}(ET, y_{j+1})$, where $y_{i+1} < y_{j+1}$. After one time impulsive effect reach the phase set, one yields $\varphi(y_i^+) = e^{-k_2 D}y_{i+1} + \tau < e^{-k_2 D}y_{j+1} + \tau = \varphi(y_j^+)$.

Similarly, for any $y_i^+, y_j^+ \in [y_H, +\infty)$ with $y_i^+ < y_j^+$, the trajectories staring from points A_i^+ and A_j^+ will first meet L_3 at points $(e^{-k_1 D}ET, y_i')$ and $(e^{-k_1 D}ET, y_j')$ with $y_i' > y_j'$ and then reach L_4 at points (ET, y_{i+1}) and (ET, y_{j+1}), where $y_{i+1} > y_{j+1}$. After impulsive effect, get an inequality relationship $\varphi(y_i^+) = e^{-k_2 D}y_{i+1} + \tau > e^{-k_2 D}y_{j+1} + \tau = \varphi(y_j^+)$.

Therefore, the Poincaré map φ is increasing on $[0, y_H]$ and decreasing on $[y_H, +\infty)$, and the range of φ is $[\tau, \varphi(y_H)] = [\tau, e^{-k_2 D}y(e^{-k_1 D}ET, y_H) + \tau]$.

(ii) We can get that both $P(x, y)$ and $Q(x, y)$ functions are continuously differentiable in the first quadrant in accordance with system (2.3). So by Cauchy theorem and Lipschitz theorem with parameters, the φ is a continuously differentiable function can be proved.

(iii) Form (3.5), we get

$$\frac{\partial \omega}{\partial y} = \frac{Bx[x - C(x + A)][(1 - x)(x + A)]}{[((1 - x)(x + A) - y)x]^2}, \tag{3.7}$$

$$\frac{\partial^2 \omega}{\partial^2 y} = \frac{2Bx^2[x - C(x + A)][(1 - x)(x + A)]}{[(1 - x)(x + A)x]^3}. \tag{3.8}$$

It is obvious that $[(1-x)(x+A)-y]x > 0$ and $x - C(x+A) < 0$ when $0 < y < y_{\theta ET}$ with $x < \frac{A}{1-C}$, which indicate that the $\partial\omega/\partial y < 0$ holds, $\partial^2\omega/\partial^2 y < 0$ also true for all $0 < y < y_H$.

Form the Cauchy-Lipschitz theorem with parameters has

$$\frac{\partial y(x,S)}{\partial S} = exp[\int_{e^{-k_1 D}ET}^{x} \frac{\partial}{\partial y}(\frac{Q(z,y(z,S))}{P(z,y(z,S))})dz] > 0, \tag{3.9}$$

and

$$\frac{\partial^2 y(x,S)}{\partial S^2} = \frac{\partial y(x,S)}{\partial S} \cdot \int_{e^{-k_1 D}ET}^{x} \frac{\partial^2}{\partial y^2}(\frac{Q(z,y(z,S))}{P(z,y(z,S))})\frac{\partial y(x,S)}{\partial S}dz. \tag{3.10}$$

Based on the above analysis, the inequality $\partial^2 y(x,S)/\partial S^2 < 0$ holds true. Therefore, φ is monotonically increasing and concave for $y < y_H$.

(iv) From (i), the φ monotonically increasing on $[0, y_H]$ and monotonically decreasing on $[y_H, +\infty)$. Therefore, we discuss the existence of fixed point about φ in two cases.

(a) When $\varphi(y_H) < y_H$, since $\varphi(0) = \tau > 0$, there is at least one number y_f such that $\varphi(y_f) = y_f$ on $[0, y_H)$; Notice that φ is monotonically decrease on $[y_H, +\infty)$ and $\varphi(y_H) < y_H$. We obtain φ has no fixed point on $[y_H, +\infty)$. Thus, according to the property (ii) we can conclude that when $\varphi(y_H) < y_H$ the function φ has at least one fixed point.

(b) When $\varphi(y_H) \geq y_H$, let $\varphi(y_A) > y_A$ and $\varphi(y_A) = y_A^+$, that is $y_A^+ > y_A$. Because of φ monotonically decrease on $[y_H, +\infty)$, We have $\varphi(y_A^+) < \varphi(y_A) = y_A^+$. According to the above proof and the property (ii), φ has at least one point y_f on $[y_H, +\infty)$ such that $\varphi(y_f) = y_f$. In addition, φ there is no fixed point on $[0, y_H)$ because of $\varphi(0) = \tau > 0$. Therefore, the φ has at least one fixed point when $\varphi(y_H) \geq y_H$.

In terms of the above discussion and the concavity property of φ, φ always has the only fixed point y_f.

(v) Denote the closure of the Ω_1 as

$$\bar{\Omega}_1 = \{(x,y)|x > 0, y > 0, y \leq (1-A)(x+A)\}. \tag{3.11}$$

Denote

$$L = y - (1-x)(x+A).$$

If the vector field will eventually reach the boundary $\bar{\Omega}_1$ and following formula holds, the $\bar{\Omega}_1$ is an invariant set of system (2.6).

$$[(P(x,y), Q(x,y)) \cdot (2x + A - 1, 1)]_{L=0} \leq 0,$$

here $\cdot$ is the scalar product of two vectors, carry out calculations, then

$$\begin{aligned} V(x)|_{L=0} &\doteq (x(1-x)(x+A) - y) \cdot (2x + A - 1) + B(x - C(x+A))y \\ &= B(x - C(x+A))y < 0. \end{aligned}$$

Since φ increasing on $[0, y_H]$ and decreasing on $[y_H, +\infty)$, $\varphi(y_0^+)$ is bounded for any $y_0^+ \in [0, y_H]$ and $\varphi([y_H, +\infty)) \subset \varphi([0, y_{ET}])$. Furthermore, any point on the phase set we always have $\frac{dy}{dx} < 0$, which leads to $\lim_{y_k^+ \to \infty} g(y_k^+) = 0$. Therefore, we get

$$\lim_{y_k^+ \to \infty} \varphi(y_k^+) = \lim_{y_k^+ \to \infty} e^{-k_2 D} g(y_k^+) + \tau = \tau.$$

□

Theorem 3.2. *For case II, that is, when $ET \geq x^*$, the trajectory Γ_S intersects the line L_4 at point $S(x_s, y_s)$ and the line L_3 at two points $P_1(e^{-k_1 D}ET, y_{P_1})$ and $P_2(e^{-k_1 D}ET, y_{P_2})$ respectively with $y_{P_1} > y_{P_2}$, we know the Poincaré map φ has the following properties as illustrated in Figure 4:*

i) The range of φ are $[0, y_{P_2}] \cup [y_{P_1}, +\infty)$. In addition φ monotonically increases on $[0, y_{P_2}]$, monotonically decreases on $[y_{P_1}, +\infty)$.

ii) φ is continuously differentiable on $[0, y_{P_2}] \cup [y_{P_1}, +\infty)$.

iii) When $\varphi(y_{P_1}) \geq y_{P_1}$, there exists a point $y_f^* \in [y_{P_1}, +\infty)$ satisfies $\varphi(y_f^*) = y_f^*$; when $\varphi(y_{P_1}) < y_{P_1}$, there is no fixed point $y_f^* \in (0, y_{P_1}]$ satisfies $\varphi(y_f^*) = y_f^*$.

Proof. (i) Firstly, we discuss the range of φ. Because $Q_e(x^*, y^*)$ is balance point, when $ET \geq x^*$, the trajectory Γ_S intersects the line L_4 at point $S(x_s, y_s)$ and intersect the line L_3 at two points $P_1(e^{-k_1 D}ET, y_{P_1})$ and $P_2(e^{-k_1 D}ET, y_{P_2})$ with $y_{P_1} > y_{P_2}$. We take an arbitrary point, say $A_n^+(x_n^+, y_n^+)$ on line L_3. Then, if $y_n^+ \in (0, y_{P_2}] \cup [y_{P_1}, +\infty)$, the point A_n^+ will reach point $A_{n+1}(ET, y_{n+1})$ on the impulse sets through impulsive effect; if $y_n^+ \in (y_{P_2}, y_{P_1})$, then trajectory starting from point A_n^+ have no intersect point with line $x = ET$. Therefore, the range of φ are $[0, y_{P_2}] \cup [y_{P_1}, +\infty)$.

Secondly, for any solution $y_{n_1}^+, y_{n_2}^+ \in (0, y_{p_2}]$ with $y_{n_1}^+ < y_{n_2}^+$, we can get

$$g(x, y_{n_1}^+) < g(x, y_{n_2}^+), \quad e^{-k_1 D}ET \leq x \leq ET,$$

according to the uniqueness of the solution of the differential equation. Through the definition of Poincaré map, we get $\varphi(y_{n_1}^+) < \varphi(y_{n_2}^+)$. So the φ is increasing on $[0, y_{P_2}]$.

When $y_{n_1}^+, y_{n_2}^+ \in [y_{P_1}, +\infty)$ with $y_{n_1}^+ < y_{n_2}^+$, the trajectory staring from $A_{n_1}^+(e^{-k_1 D} ET, y_{n_1}^+)$ and $A_{n_2}^+(e^{-k_1 D}ET, y_{n_2}^+)$ will meet phase set at points $A_{n_{11}}^+(e^{-k_1 D}ET, y_{n_{11}}^+)$ and $A_{n_{21}}^+(e^{-k_1 D}ET, y_{n_{21}}^+)$ with $y_{n_{11}}^+ > y_{n_{21}}^+$. According to the uniqueness of the solution of the differential equation, we can see $\varphi(y_{n_1}^+) > \varphi(y_{n_2}^+)$. So the φ is decreasing on $[y_{P_1}, +\infty)$.

(ii) According to formula (11), we known $\omega(x, y)$ is continuously differentiable in the first quadrant, and φ is continuously differentiable on $[0, y_{P_2}] \cup [y_{P_2}, +\infty)$ through the continuously differentiable theorem of differential equations.

(iii) According to the positional relationship between Poincaré $\varphi(y_{P_1})$ and y_{P_1}, consider the following two cases:

(a)When $\varphi(y_{P_1}) \geq y_{P_1}$, we suppose $\varphi(y_{P_1}) = y_P \geq y_{P_1}$. We known φ is decreasing on $[y_{P_1}, +\infty)$, then $\varphi(y_P) \leq \varphi(y_{P_1}) = y_P$, that is $\varphi(y_P) \leq y_P$. Thus, there is a point $y_f^* \in [y_{P_1}, +\infty)$ satisfies $\varphi(y_f^*) = y_f^*$ on the basis of the continuous differentiability of closed interval. (b)When $\varphi(y_{P_1}) < y_{P_1}$, there is no fixed point $y_f^* \in (0, y_{P_1}]$ satisfies $\varphi(y_f^*) = y_f^*$. For any points $y_n^+ \in (0, y_{P_1}]$, the trajectories starting from point $A_n^+(e^{-k_1 D}ET, y_n^+)$ first meet the pulse set M_3 at point

$A_{n+1}(ET, y_{n+1})$ and then the point $A_{n+1}(ET, y_{n+1})$ will pulse to the phase set N_3 at point $A^+_{n+1}(e^{-k_1 D}ET, y^+_{n+1})$, which easy to get $y_{n+1} < y^+_n < y^+_{n+1}$. On account of $y_{n+1} \neq y^+_{n+1}$, there is no fixed point $y^*_f \in (0, y_{P_1}]$ satisfies to $\varphi(y^*_f) = y^*_f$. $\square$

4. Order-k Periodic Solutions

4.1. Boundary periodic solutions of system (2.6)

Assume that the predator population tends to extinction and also stop releasing predators. Then the system (2.6) becomes to the following system:

$$\begin{cases} \dfrac{dx(t)}{dt} = rx(t)(1 - \dfrac{x(t)}{K}), \qquad x(t) < ET, \\ x(t^+) = e^{-k_1 D}x(t), \qquad x(t) = ET. \end{cases} \tag{4.1}$$

In this scenario, we shall show that the system (2.6) has a boundary periodic solution and investigate its stability. Solving equation (4.1) with initial value $x(0^+) = e^{-k_1 D}ET$ yields

$$x^T(t) = \frac{e^{-k_1 D}ETK\exp(rt)}{K - e^{-k_1 D}ET + e^{-k_1 D}ET\exp(rt)}. \tag{4.2}$$

Then any solution of (2.6) from the initial point $(x^T(t), 0)$ will intersect with the line L_4 over time. Solving

$$ET = \frac{e^{-k_1 D}ETK\exp(rt)}{K - e^{-k_1 D}ET + e^{-k_1 D}ET\exp(rt)},$$

for t and D:

$$t = \frac{1}{r}\ln\frac{K - e^{-k_1 D}ET}{e^{-k_1 D}(K - ET)} \overset{def}{=} T, \quad D = \frac{1}{k_1}\ln\frac{ET + e^{rT}(K - ET)}{K},$$

here T denotes the period of the boundary periodic solution, D denotes the insecticide dose required to control the pest population density under ET, so the boundary periodic solution of system (2.6) with a period of T becomes:

$$\begin{cases} x^T(t) = \dfrac{e^{-k_1 D}ETK\exp(r(t-(k-1)T))}{K - e^{-k_1 D}ET + e^{-k_1 D}ET\exp(r(t-(k-1)T))}, \\ y^T(t) = 0. \end{cases} \tag{4.3}$$

Theorem 4.1. *If $R_0 < 1$, then boundary periodic solution $\left(x^T(t), 0\right)$ of system (2.6) asymptotically stable, where the expression of R_0 is*

$$R_0 = \left|\frac{e^{-k_1 D}e^{-k_2 D}(K - e^{-k_1 D}ET)}{K - ET}\exp\left(\int_0^T\left[r - bc - \frac{2rET}{K} + \frac{bp(1-\beta)ET}{(1-\beta)ET + a}\right]dt\right)\right|. \tag{4.4}$$

Proof. By simple calculations, we get

$$P(x,y) = rx(1 - \frac{x}{K}) - \frac{q(1-\beta)xy}{(1-\beta)x + a}, \quad Q(x,y) = \frac{bp(1-\beta)xy}{(1-\beta)x + a} - bcy,$$

$$\alpha(x,y) = x(e^{-k_1 D} - 1), \quad \beta(x,y) = y(e^{-k_2 D} - 1) + \tau, \quad \phi(x,y) = x - ET,$$
$$(x^T(T), y^T(T)) = (ET, 0), \quad (x^T(T^+), y^T(T^+)) = (e^{-k_1 D} ET, 0).$$

Thus, we have

$$\begin{aligned}
&\frac{\partial P}{\partial x} = r - \frac{2rx}{K} - \frac{aq(1-\beta)y}{[(1-\beta)x + a]^2}, \quad \frac{\partial Q}{\partial y} = \frac{bp(1-\beta)x}{(1-\beta)x + a} - bc,\\
&\frac{\partial \alpha}{\partial x} = e^{-k_1 D} - 1, \quad \frac{\partial \beta}{\partial y} = e^{-k_2 D} - 1,\\
&\frac{\partial \phi}{\partial x} = 1, \quad \frac{\partial \alpha}{\partial y} = \frac{\partial \beta}{\partial x} = \frac{\partial \phi}{\partial y} = 0.
\end{aligned}$$

So

$$\begin{aligned}
\Delta_1 &= \frac{P_+(\frac{\partial \beta}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial \beta}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}) + Q_+(\frac{\partial \alpha}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y})}{P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y}}\\
&= \frac{e^{-k_1 D} e^{-k_2 D}(K - e^{-k_1 D} ET)}{K - ET}.
\end{aligned}$$

Noticing

$$\begin{aligned}
&\exp\left(\int_0^T \left[\frac{\partial P}{\partial x}\left(x^T(t), y^T(t)\right) + \frac{\partial Q}{\partial y}\left(x^T(t), y^T(t)\right)\right] dt\right)\\
&= \exp\left(\int_0^T \left[r - bc - \frac{2rx^T(t)}{K} + \frac{bp(1-\beta)x^T(t)}{(1-\beta)x^T(t) + a}\right] dt\right)
\end{aligned}$$

we have

$$\begin{aligned}
\mu_1 &= \Delta_1 \exp\left(\int_0^T \left[\frac{\partial P}{\partial x}\left(x^T(t), y^T(t)\right) + \frac{\partial Q}{\partial y}\left(x^T(t), y^T(t)\right)\right] dt\right)\\
&= \frac{e^{-k_1 D} e^{-k_2 D}(K - e^{-k_1 D} ET)}{K - ET}\\
&\quad \times \exp\left(\int_0^T \left[r - bc - \frac{2rx^T(t)}{K} + \frac{bp(1-\beta)x^T(t)}{(1-\beta)x^T(t) + a}\right] dt\right) < 1
\end{aligned}$$

under condition (4.4). Hence by [40, Lemma 2.2] the periodic solution $(x^T(t), 0)$ is asymptotically stable. □

4.2. Existence and stability of order-k($k \geq 1$) periodic solutions at $\tau > 0$

From the definition of the Poincaré map, we know that the existence of the periodic solution of the system (2.6) same as the existence of the fixed point of the corresponding Poincaré map. The periodic solution of Poincaré map φ when $\tau > 0$ are discussed in this section. From Theorems 3.1 and 3.2 and references [5, 6, 18, 33], system (2.6) at least exist one order-1 periodic solution, whose stability can be determined by the following theorem.

Theorem 4.2. *The order-1 periodic solution $(\xi(t), \eta(t))$ is orbitally asymptotically stable if and only if*

$$\left|\frac{e^{-k_2 D} e^{-k_1 D}(r(1-\frac{e^{-k_1 D}ET}{K}) - \frac{q(1-\beta)(e^{-k_2 D}\eta_0+\tau)}{a+(1-\beta)e^{-k_1 D}ET})}{r(1-\frac{ET}{K}) - \frac{q(1-\beta)\eta_0}{a+(1-\beta)ET}} exp(\int_0^T U(t)dt)\right| < 1, \quad (4.5)$$

where

$$U(t) = r - bc - \frac{2r\xi(t)}{K} + \frac{bp(1-\beta)\xi(t)}{(1-\beta)\xi(t)+a}.$$

Proof. We omit the proof details here since it similar to the proof of Theorem 4.1, except here we

$$\begin{aligned}\Delta_1 =& \frac{P_+(\frac{\partial\beta}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\beta}{\partial x}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial x}) + Q_+(\frac{\partial\alpha}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial\alpha}{\partial y}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y})}{P\frac{\partial\phi}{\partial x} + Q\frac{\partial\phi}{\partial y}} \\ =& \frac{e^{-k_2 D} e^{-k_1 D}(r(1-\frac{e^{-k_1 D}ET}{K}) - \frac{q(1-\beta)(e^{-k_2 D}\eta_0+\tau)}{a+(1-\beta)e^{-k_1 D}ET})}{r(1-\frac{ET}{K}) - \frac{q(1-\beta)\eta_0}{a+(1-\beta)ET}}, \\ &\exp\left(\int_0^T \left[\frac{\partial P}{\partial x}\left(x^T(t), y^T(t)\right) + \frac{\partial Q}{\partial y}\left(x^T(t), y^T(t)\right)\right]dt\right) \\ =& \exp\left(\int_0^T \left[r - bc - \frac{2r\xi(t)}{K} + \frac{bp(1-\beta)\xi(t)}{(1-\beta)\xi(t)+a}\right]dt\right)\end{aligned}$$

and the Floquet multiplier

$$\begin{aligned}\mu_1 =& \Delta_1 \exp\left(\int_0^T \left[\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))\right]dt\right) \\ =& \frac{e^{-k_2 D} e^{-k_1 D}(r(1-\frac{e^{-k_1 D}ET}{K}) - \frac{q(1-\beta)(e^{-k_2 D}\eta_0+\tau)}{a+(1-\beta)e^{-k_1 D}ET})}{r(1-\frac{ET}{K}) - \frac{q(1-\beta)\eta_0}{a+(1-\beta)ET}} \\ &\exp\left(\int_0^T \left[r - bc - \frac{2r\xi(t)}{K} + \frac{bp(1-\beta)\xi(t)}{(1-\beta)\xi(t)+a}\right]dt\right).\end{aligned}$$

□

Theorem 4.3. *When $ET < x^*, \varphi(y_H) < y_H$, the system* (2.6) *admits a unique order-1 periodic solution, which is globally asymptotically stable.*

Proof. If $\varphi(y_H) < y_H$, Poincaré map φ has a unique fixed point y_f on $(0, y_H)$ form the property (iv) of Theorem 3.1,which implies the uniqueness of the order-1 solution. Next we prove the stability of the solution.

For trajectory starting from any point $(e^{-k_1 D}ET, y_0^+)$, $y_0^+ \in [0, y_f)$, that is $0 \leq y_0^+ < y_f$. Because φ is monotonically increase in $[0, y_f)$, then after a pulse effect, it is obtained $y_0^+ < \varphi(y_0^+) < \varphi(y_f) = y_f$. After another pulse effect, then $y_0^+ < \varphi(y_0^+) < \varphi^2(y_0^+) < \varphi^2(y_f) = y_f$, once again, $\varphi(y_0^+) < \varphi^2(y_0^+) < \varphi^3(y_0^+) < y_f$. Sequentially after n times of pulse effect, $\varphi(y_0^+) < \varphi^2(y_0^+) < \cdots < \varphi^n(y_0^+) <$

y_f. When $n \to +\infty$, there is $\varphi(y_0^+) < \varphi^2(y_0^+) < \cdots < \varphi^n(y_0^+) < \cdots < y_f$. Because the sequence $\{\varphi^n(y_0^+)\}$ is monotonically increasing and bounded, we can only get $\lim_{n\to+\infty}\varphi^n(y_0^+)$ existence, can't claim that $\lim_{n\to+\infty}\varphi^n(y_0^+) = y_f$. Next we will prove $\lim_{n\to+\infty}\varphi^n(y_0^+) = y_f$, which can be proven by contradiction as follow. Set $\lim_{n\to+\infty}\varphi^n(y_0^+) = y^*$, that is, we will prove $y^* = y_f$. Otherwise,we set$\lim_{n\to+\infty}\varphi^n(y_0^+) = y^*$,$y^* < y_f$. Then we have $y^* < \varphi(y^*) < \varphi(y_f) = y_f$. After n times pulse effect, $y^* < \varphi(y^*) < \varphi^2(y^*) < \cdots < \varphi^n(y^*) < y_f$. This is contradict to the fact that $\lim_{n\to+\infty}\varphi^n(y_0^+) = y^*$. So we obtain $y^* = y_f$ and $\lim_{n\to+\infty}\varphi^n(y_0^+) = y^*$. Then, by [23, Section 6] we know the solution is stable.

Similarly we can prove the case of $y_0^+ > y_f$. This completes the proof. □

Next we show the existence of order-2 periodic solution and its stability by using the definitions in [23, Section 6].

Theorem 4.4. *When $ET < x^*, \varphi(y_H) > y_H$, $\varphi^2(y_H) \geq y_H$, system* (2.6) *either has a stable order-1 periodic solution or a stable order-2 periodic solution.*

Proof. From Theorem 3.1, if φ does not has fixed points on $[0, y_H]$, then will exist a positive constant i such that $y_i^+ \leq y_H$ and $y_{i+1}^+ \geq y_H$. Noticing $y_{i+1}^+ = \varphi\left(y_i^+\right) \leq \varphi\left(y_H\right)$, we get $y_{i+1}^+ \in [y_H, \varphi\left(y_H\right)]$. Since φ is decreasing on $[y_H, +\infty)$ for any $y_1^+ > y_H$, $y_2^+ = \varphi\left(y_1^+\right) \leq \varphi\left(y_H\right)$ through one Poincaré map. And then $y_{i+1}^+ \in [y_H, \varphi\left(y_H\right)]$ holds for all $i \geq 2$.

Furthermore, φ is monotonically decreasing on $[y_H, \varphi(y_H))$, and φ^2 is monotonically increasing on $[y_H, \varphi(y_H))$ so that

$$\varphi\left([y_H, \varphi\left(y_H\right)]\right) = \left[\varphi^2\left(y_H\right), \varphi\left(y_H\right)\right] \subset \left[y_H, \varphi\left(y_H\right)\right].$$

Next, we discuss the existence of the periodic solution. If there is $y_0^+ \in [y_H, \varphi\left(y_H\right)]$, the conclusion is clearly established. Thus, without loss of generality, we assume that for any $y_0^+ \in [y_H, \varphi\left(y_H\right)]$ we have $y_1^+ = \varphi\left(y_0^+\right) \neq y_0^+$ and $y_2^+ = \varphi^2\left(y_0^+\right) \neq y_0^+$. Then, we have the following cases.

(I) If $y_H \leq y_2^+ < y_0^+ < y_1^+ \leq \varphi\left(y_H\right)$, then $y_3^+ = \varphi\left(y_2^+\right) > \varphi\left(y_0^+\right) = y_1^+$, $y_4^+ = \varphi\left(y_3^+\right) < \varphi\left(y_1^+\right) = y_2^+$. It can be obtained by mathematical induction that

$$\begin{aligned} y_H &\leq \cdots < y_{2n+2}^+ < y_{2n}^+ < \cdots < y_2^+ < y_0^+ \\ &< y_1^+ < \cdots < y_{2n-1}^+ < y_{2n+1}^+ < \cdots \leq \varphi\left(y_H\right). \end{aligned}$$

(II) If $y_H \leq y_0^+ < y_2^+ < y_1^+ \leq \varphi\left(y_H\right)$, then $y_1^+ = \varphi\left(y_0^+\right) > \varphi\left(y_2^+\right) = y_3^+ > y_2^+ = \varphi\left(y_1^+\right)$, $y_2^+ = \varphi\left(y_1^+\right) < \varphi\left(y_3^+\right) = y_4^+ < y_3^+ = \varphi\left(y_2^+\right) < y_1^+$, i.e. it can be obtained by mathematical induction that

$$\begin{aligned} y_H &\leq y_0^+ < y_2^+ < \cdots < y_{2n}^+ < y_{2n+2}^+ \\ &< \cdots < y_{2n+1}^+ < y_{2n-1}^+ < \cdots < y_1^+ \leq \varphi\left(y_H\right). \end{aligned}$$

(III) $y_H \leq y_1^+ < y_2^+ < y_0^+ \leq \varphi\left(y_H\right)$. In the same way as case II, we can obtain

$$\begin{aligned} y_H &\leq y_1^+ < \cdots < y_{2n-1}^+ < y_{2n+1}^+ \\ &< \cdots < y_{2n+2}^+ < y_{2n}^+ < \cdots < y_2^+ < y_0^+ \leq \varphi\left(y_H\right). \end{aligned}$$

(IV) $y_H \leq y_1^+ < y_0^+ < y_2^+ \leq \varphi\left(y_H\right)$. By using the same method as case I, we can obtain

$$y_H \leq \cdots < y_{2n+1}^+ < y_{2n-1}^+ < \cdots < y_1^+ < y_0^+$$

$$< y_2^+ < \cdots < y_{2n}^+ < y_{2n+2}^+ < \cdots \leq \varphi(y_H).$$

On the hand, for case (II), $\varphi^{2n}\left(y_0^+\right) = y_{2n}^+$ monotonically increasing, however, $\varphi^{2n+1}\left(v_0^+\right) = y_{2n+1}^+$ monotonically decreasing; on the other hand, for case (III), $\varphi^{2n}\left(y_0^+\right) = y_{2n}^+$ is monotonically decreasing and $\varphi^{2n+1}\left(y_0^+\right) = y_{2n+1}^+$ is monotonically increasing. It is obvious that for case (II) and case (III) there either exists a unique fixed point y_f such that

$$\lim_{n\to\infty} y_{2n}^+ = \lim_{n\to\infty} y_{2n+1}^+ = y_f, \quad y_f \in [y_H, \varphi(y_H)],$$

or exists two distinct values y_{f_1} and y_{f_2} and $y_{f_1} \neq y_{f_2}$ such that

$$\lim_{n\to\infty} y_{2n}^+ = y_{f_1}, \quad y_{f_1} \in [y_H, \varphi(y_H)],$$
$$\lim_{n\to\infty} y_{2n+1}^+ = y_{f_2}, \; y_{f_2} \in [y_H, \varphi(y_H)].$$

While for cases (I) and (IV), only the later case can be true. □

Theorem 4.5. *The system* (2.6) *admits an order-3 periodic solution if* $\varphi(y_H) > y_H$, $\varphi^2(y_H) < y_m^+$, *where*

$$y_m^+ = \min\{y^+ : \varphi(y^+) = y_H\}.$$

Proof. If $\varphi(y_H) > y_H$, from Theorem 3.1, we know that there is a point y_f in $(y_H, \varphi(y_H))$,

$$\varphi(y_f) = y_f, \quad y_f \in (y_H, \varphi(y_H))$$

because the Poincaré map φ is continuous on closed intervals $[0, y_f]$, and

$$\varphi(0) = \tau, \quad \varphi(y_f) = y_f,$$

according to the intermediate value theorem, there exists $y_m^+ \in (0, y_f)$, and $\varphi(y_m^+) = y_H$.

Furthermore

$$\varphi^3(y_m^+) = \varphi^2(y_H) < y_m^+.$$

Thus,

$$\varphi^3(y_m^+) - y_m^+ < 0$$

and

$$\varphi^3(0) = \varphi^2(\tau) = \tau > 0.$$

Owing to the existence theorem of zero point on closed intervals, at least one value of $\tilde{y}^+$ make

$$\varphi^3(\tilde{y}^+) = \tilde{y}^+,$$

which implying that the system (2.6) has an order-3 periodic solution. □

Remark 4.1. When $\varphi(y_H) > y_H$, $\varphi^{k-1}(y_H) < y_m^+$, using the same fashion, we can prove that system (2.6) admits an order-k periodic solution.

5. Numerical Simulation and Discussion

5.1. The impact of the prey refuge on population density

In order to better understand the impact of prey refuge on population density in the system (2.4), we discuss the impact of the prey refuge on prey population density and predator population density. Reference [15] indicated that the system (2.4) has a globally asymptotically stable equilibrium point $P_e(x_e, y_e)$ with

$$x_e = \frac{ca}{(bp-c)(1-\beta)}, \qquad y_e = \frac{rabp}{qK}\left(\frac{K(bp-c)(1-\beta)-ca}{((bp-c)(1-\beta))^2}\right).$$

In order to explore the impact of the prey refuge on population density, the relationship between x_e and y_e relative to β needs to be explored. It is observed that x_e and y_e are continuous functions about β, $\beta \in (0, 1-\frac{ac}{K(bp-c)}]$. We compute the derivative along x_e and y_e with respect to β, that is

$$\frac{dx_e}{d\beta} = \frac{ca}{(bp-c)(1-\beta)^2} > 0, \qquad \frac{dy_e}{d\beta} = \frac{rabp}{qK}\left(\frac{K(bp-c)(1-\beta)-2ac}{(bp-c)^2(1-\beta)^3}\right).$$

When $bp-c>0$, we know that $\frac{dx_e}{d\beta} > 0$, suggesting x_e increases with β, i.e., increasing the amount of refuges β can increase prey density. For $\frac{dy_e}{d\beta}$, after a simple calculation, we can know that:

1) when $0 < \beta \leq 1-\frac{2ac}{K(bp-c)}$, then $\frac{dy_e}{d\beta} > 0$, which can know that y_e is a strictly increasing function on $\beta \in [0, 1-\frac{2ac}{K(bp-c)})$;

2) when $1-\frac{2ac}{K(bp-c)} < \beta < 1-\frac{ac}{K(bp-c)}$, then y_e is a strictly decreasing function on $\beta \in (1-\frac{2ac}{K(bp-c)}, 1-\frac{ac}{K(bp-c)})$;

3) when $\beta = 1-\frac{2ac}{K(bp-c)}$,the predator species reaches its upper bound, and when $\beta = 1-\frac{ac}{K(bp-c)}$, the predator species goes to extinction.

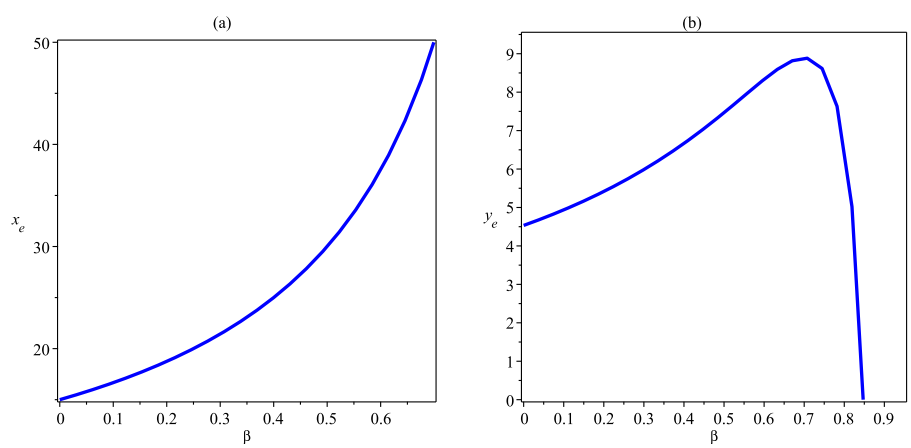

Figure 5. The density of prey x_e and predator y_e in system (2.4)as the refuge β changes. Here $r = 0.2$, $a = 1$, $b = 0.8$, $p = 0.6$, $q = 0.6$, and $c = 0.45$.

Fig. 5 demonstrates how prey refuge affects the density of predator and prey population. On one hand, increasing the amount of refuge (i.e.,as β increase) can increase prey density (Fig. 5(a)); on the other hand, the population of predators first increases and then decreases, and there is an upper bound at $\beta = 1 - \frac{2ac}{K(bp-c)}$ (Fig. 5(b)). It also indicates that the refuge is important in the continued survival and extinction of the predator population. When $\beta \geq 1 - \frac{2ac}{K(bp-c)}$, overmuch prey hide in the refuge, the predator population cannot get enough food, which leads to the extinction of the population; when $\beta < 1 - \frac{2ac}{K(bp-c)}$, two species can coexist well.

5.2. Numerical simulation

In this section we numerically illustrate our findings in preceding sections.

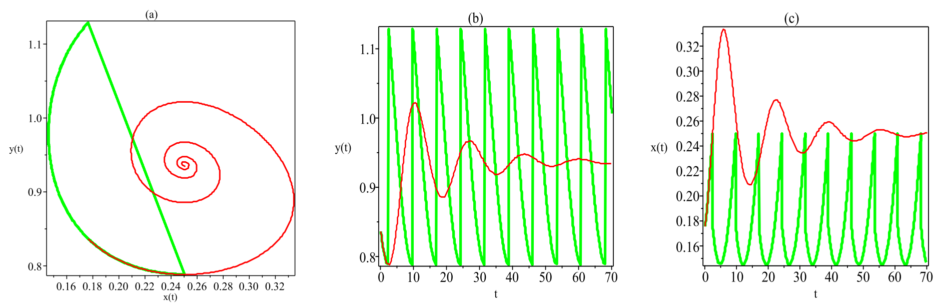

Figure 6. State pulse feedback control strategy of system (2.6), $A = 1$, $B = 0.8$, $C = 0.2$, $ET = 0.24$, $k_1 = 0.7$, $k_2 = 0.3$, $D = 0.5$, $\tau = 0.45$.

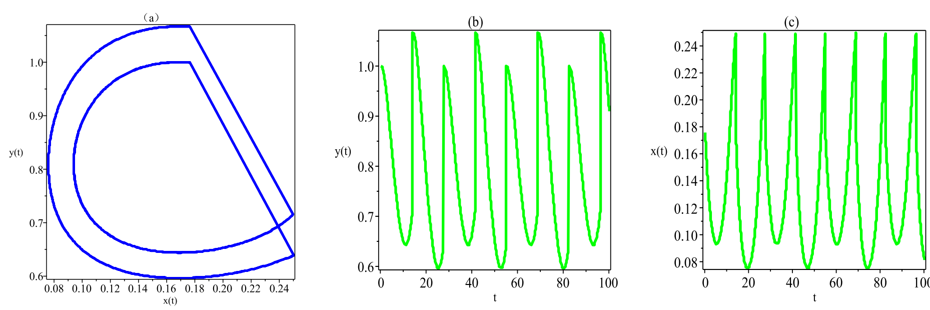

Figure 7. The order-2 periodic solution of system (2.6). $A = 0.8$, $B = 0.98$, $C = 0.174$, $ET = 0.25$, $k_1 = 0.7$, $k_2 = 0.3$, $D = 0.5$, $\tau = 0.45$.

In Fig. 6 and Fig. 7, the red line represents the system trajectory without pulses, and the green line represents the system trajectory with pulses, which suggests that the existence of the periodic solution due to the control strategy: order-1 periodic solution (Fig. 6) and order-2 periodic solution (Fig. 7) of the Poincaré map. The green line is the motion trajectory of the order-1 periodic solution in Fig. 6(a) , and the blue line is the motion trajectory of the order-2 periodic solution in Fig. 7(a).

Fig.6(b) and Fig.6(c) represent the change of the densities of predator and prey with time t under the condition of order-1 periodic solution in Fig.6 (a) respectively.

The density changes of predator and prey with time t are shown in Fig.7(b) and Fig.7(c) respectively under the condition of order-2 periodic solution in Fig.7(a).

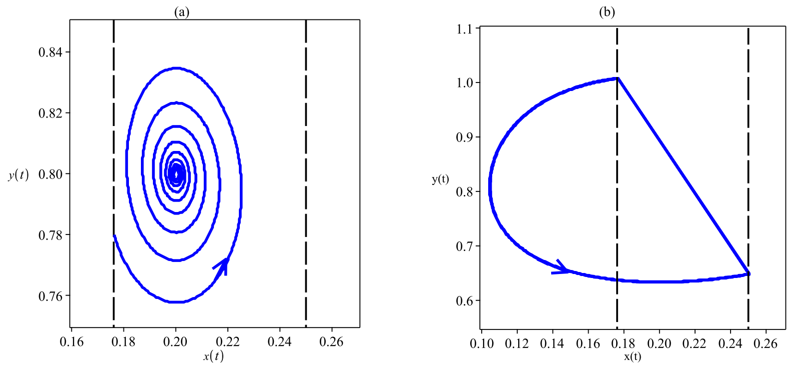

Figure 8. Periodic solutions in the case of Fig.4a and Fig.4b, respectively. $A = 0.8$, $B = 0.8$, $C = 0.2$, $k_1 = 0.7$, $k_2 = 0.3$, $D = 0.5$, $ET = 0.25$. (a) $\tau = 0.2$; (b)$\tau = 0.45$.

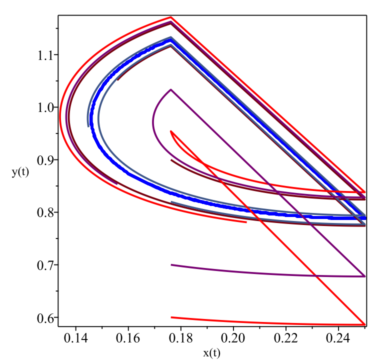

Figure 9. The path curve of system (2.6) starting from different initial points. $A = 1$, $B = 0.8$, $C = 0.2$, $ET = 0.25$, $k_1 = 0.7$, $k_2 = 0.3$, $D = 0.5$, $\tau = 0.45$.

Fig. 8(a) and fig. 8(b) show the existence of the order-1 periodic solution under the conditions of Fig. 4(a) and Fig. 4(b) respectively. Fig. 8(a) shows that there is no order-1 periodic solution, and fig. 8(b) proves that there is order-1 periodic solution. From Fig. 9, we can observed that all trajectories converge to the same order-1 periodic solution, although the trajectories start at different initial points, which proves that the order-1 periodic solution of the system (2.6) is globally asymptotically stable.

According to Section 4, if enemy goes extinct, that is $y(t) = 0$, and pest arrive at the economic threshold, that is $x = ET$, after simple calculation, we can see two expressions about the boundary periodic solution and insecticide dosage. Then we discuss the key parameters that affect the pesticide dose D. By giving appropriate parameters in the numerical simulation, we know that when the threshold ET decreases, the pesticide dose D must increase (Fig. 10(a)). Moreover, when the T of chemical control increases, so the dose D must increase (Fig. 10(b)). In the biological sense, the time interval for chemical control period T is increasing, then

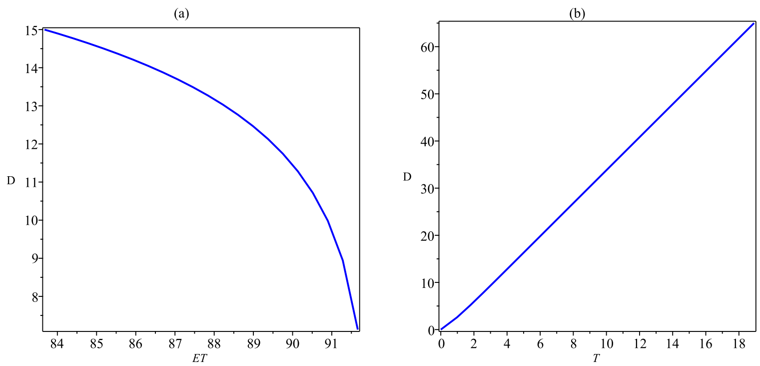

Figure 10. The effects of key parameters on the dose D of chemical control, and the parameters are fixed as:(a)$r = 0.4,\ k_1 = 0.4,\ T = 6,\ K = 92$; (b)$r = 1.4,\ k_1 = 0.4,\ ET = 35,\ K = 92$.

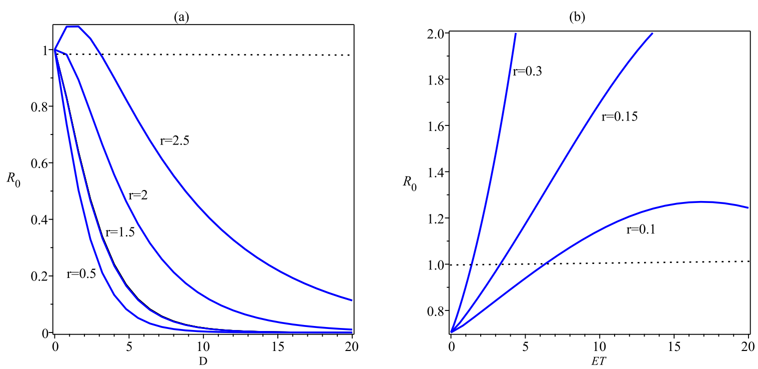

Figure 11. The effects of control parameters on the threshold condition R_0. (a)$k_1 = 0.5$, $k_2 = 0.2$, $K = 92$, $b = 0.3$, $p = 0.2$, $\beta = 0.2$, $a = 0.3$:(a)$ET = 35$ and $c = 0.6$; (b)$D = 0.5$ and $c = 0.3$.

pests live in an environment with pesticide concentrations that are not harmful to themselves, which will lead to the rapid mass reproduction of pest populations, so we must increase the dosage of spraying pesticides. Therefore, in actual agricultural production, the required pesticide dose D should consider both threshold ET and control period T.

Through threshold condition (4.4), we want to know whether a single chemical control can make the boundary periodic solution stable. $R_0 < 1$ denote single chemical control of dose D can the conclusion holds. But when $R_0 \geq 1$, the conclusion does not hold. Therefore, We need to know that how the dose D and the threshold ET affect the threshold conditions of R_0. To explore these, we conducted numerical simulations in Fig. 11. Fig. 11(a) shows that a large dose of D can control the pest density under ET. In fig. 11(b), we get $R_0 < 1$ with a relative small ET, while when ET reachs a certain leave, then $R_0 \geq 1$. It reveals that a smaller value of ET is more helpful for pest control for a fixed parameter r . In addition, when $R_0 \geq 1$, the boundary periodic solution is unstable, which indicates that chemical control alone can not control pests well under the threshold. Moreover, repeated use of pesticides will lead to pest resistance to pesticides. Therefore, releasing natural enemies as a biological control method combined with chemical control can make pest control measures more reasonable and effective.

6. Conclusion

In this study we proposed and studied a Holling II prey-predator system incorporating both prey refuge and effect of pesticide dose. First, we constructed a Poincaré map of system (2.6), and discussed the main properties, which are monotonicity, continuous and discontinuous interval, differentiability, fixed point and asymptote. Furthermore, the existence and global asymptotic stability of the order-1 periodic solution of the global dynamics about system have been further studied, and the existence of the $k(k \geq 2)$ periodic solution was proved. Secondly, we perform a parameter sensitivity analysis on the stability of the boundary periodic solution. The results indicate that dose D is inversely proportional to the threshold ET, but proportional to the chemical control time T. Therefore, the threshold ET and period T should be considered when spraying the pesticide dose D. At the same time, we gave sensitivity analysis to the parameters in R_0 threshold condition. Analysis shows that when the parameter r is larger, for the purpose of the stability of the system, that is, $R_0 < 1$, the pesticide dose D needs to be increased accordingly; its corresponding economic threshold ET is smaller, so as to keep $R_0 < 1$. In addition, the relationship between shelter intensity β and population was also studied. The results show that the density of prey increases with the increase of shelter strength β, but β increases to a certain extent, the predator population will eventually become extinct.

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