Journal of Applied Analysis and Computation
Volume 12, Number 1, February 2022, 302–335
Website:http://www.jaac-online.com
DOI:10.11948/20210171

DYNAMICS OF A DELAYED PREDATOR-PREY MODEL WITH CONSTANT-YIELD PREY HARVESTING*

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Abstract In this paper, we study a delayed predator-prey model of Holling and Leslie type with constant-yield prey harvesting, in which two types of delays caused by maturation time of prey and the gestation time of predator are considered. We mainly investigate the local dynamics of the model with emphasis on the impact of delays. The stability of equilibrium and the existence conditions of Hopf bifurcation are discussed. First, based on the different values of delays, five cases of Hopf bifurcation are studied in detail. The critical values of Hopf bifurcation for each case are presented. In addition, we explore the properties of Hopf bifurcation. The direction of Hopf bifurcation and the stability of periodic solutions by using the normal form theory and central manifold theorem are determined. The qualitative analyses have demonstrated that the values of time delays can affect the stability of equilibrium and induce small amplitude period oscillations of population densities. Numerical simulations are carried out for illustrating the theoretical results. Meanwhile, we further investigate the effects of delay on the period of periodic solutions and the influence of the harvesting term on the stability of the equilibrium with time delays.

Keywords Predator-prey model, time delay, stability, Hopf bifurcation.

MSC(2010) 34K18, 34K20, 34K60, 34C60.

1. Introduction

The predator-prey model is a basic and important model to describe the relationship between the predator population and the prey population in natural phenomena [9]. Due to the complexity and diversity of the interaction between predator and their prey, the predator-prey models have been continuously investigated by many scholars in the last three decades [6,20,21,40,47]. Both the continuous-time predator-prey models described by ordinary differential equations and the discrete-time predator-prey models described by difference equations are often used to better understand and reveal the predation mechanisms by studying the dynamical properties of these models. The time-fractional derivative also has extensive applications in describing the predator-prey interaction [13,14]. The question of how the dynamical behaviors,

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*This work is supported by NSF of Shandong Province(No. ZR2018BF018), China Postdoctoral Science Foundation(No. 2019M652349) and the Youth Creative Team Sci-Tech Program of Shandong Universities(No. 2019KJI007).

such as stability, bifurcation, chaos phenomenon and spatiotemporal patterns, are influenced by the system parameters may naturally arise [26]. Boudjema et al. [2] investigated the Turing-Hopf bifurcation in a Gauss-type predator-prey model with cross-diffusion. Djilali [8] dealt with a predator-prey model with a spatial diffusion, a linear mortality and a herd behavior. The global stability of boundary equilibrium and the local stability of the positive equilibrium are studied. Recently, Djilali [10] also investigated the Hopf and Turing-Hopf bifurcations of a predator-prey model with the presence of herd behavior and spatial diffusion subject to the homogeneous Neumann boundary condition. Some results on spatiotemporal patterns and competition of predator-prey models can be found in [11, 12, 41].

In our daily life, there is an increasing exploitation of biological populations in predator-prey system due to the growing human needs for more food and resources [34]. The harvesting is a common technique used for the exploitation of biological and it leaves a strong influence on the dynamics of biological resources. About one-half of the endangered mammals and one-third of the endangered birds of the world are threatened by overharvesting [32]. In practice, the harvesting activity of human beings play a key role in the outcomes of population evolutions [33]. The exploitation of biological, the management of renewable resources, and the harvesting of species are commonly experienced by human in fishery, forestry and wildlife management [4, 21]. From the perspective of economic benefits and long term development, all human activities should be carried out under the premise of sustainable development of resources. Therefore, the scientific exploitation and management of resources is a topic of interest to biologists and economists [44]. The governments should make policies to avoid overexploitation to protect the sustainable development of species and to maximize profits of commercial harvesting [39]. For example, in China, starting from 2020, the fishing ban will be observed in 332 conservation areas in the Yangtze River basin, which will also be expanded to all natural waterways of the river and its major tributaries for 10 years starting from Jan 1, 2021. Xiao et al. [48] discussed a ratio-dependent predator-prey model when the prey is continuously being harvested at a nonzero constant rate, and observed numerous kinds of bifurcation as the values of parameters vary. Lv et al. [36] proposed and investigated a predator-prey model with selective nonlinear harvesting for the prey and predator. They founded that the existence of nonlinear harvesting makes the dynamics of the proposed model more complicated, including heteroclinic and homoclinic orbits, bistability, Bogdanov-Takens bifurcation, subcritical and supercritical Hopf bifurcation. Some other authors also studied the dynamics of predator-prey models with nonzero constant-yield harvesting [31, 35] and with constant proportion harvesting [27].

In [25], the authors have studied the following predator-prey system of Holling and Leslie type with constant-yield prey harvesting

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{mxy}{c+x} - H, \\ \dot{y} = qy\left(1 - \dfrac{py}{x}\right), \end{cases} \tag{1.1}$$

where x and y are densities of prey and predator at time t, respectively. r and K stand for the intrinsic growth rate and the environmental carrying capacity of the prey, respectively. The predator consumes the prey according to $\frac{mx}{c+x}$ which is the Holling type-II functional response with half-saturation constant c and grows logistically with intrinsic growth q and the environmental carrying capacity proportional to the prey density. The parameter p is the number of prey required to support one predator at equilibrium when y equals $\frac{x}{p}$ [17]. H is the constant-yield prey harvesting. All the parameters r, K, m, c, H, q, p are positive constants. In [17–19], the authors considered model (1.1) with no harvesting, i.e., $H = 0$. They investigated the relationship between local and global stability of the unique positive equilibrium, the conditions of multiple limit cycles and uniqueness of limit cycle. When $H > 0$, various kinds of bifurcations, such as Hopf bifurcation, repelling and attracting Bogdanov-Takens bifurcations of codimensions 2 and 3, were founded and investigated in [25]. The constant-yield prey harvesting H makes it more challenging and difficult to study the dynamics compared to the model with no harvesting since the equation for the interior equilibria is order three.

In addition to the constant-yield prey harvesting have a greater impact on the dynamics of model (1.1), in fact, time delay will also have a significant impact on the dynamics of model (1.1). Time delay has been proved to be an unavoidable factor in biological systems [3, 24, 30, 37, 45]. Any model of species dynamics without delays is an approximation at best [29]. Most natural and manmade processes [42], such as medicine, chemistry, physics, engineering, economics, involve time delays. There are two main reasons contributing to the delays in predator-prey models. One is the time of gestation, and the other one is the time of maturation [22]. Other factors that may cause time delays include the capturing, incubation, traveling time, digestion and energy conversion for predators, etc. [33]. Meanwhile, it reflects complexity in such models by showing stability transition phenomenon for equilibria, occurrence of Hopf-bifurcation, chaotic oscillations and extinction dynamics [38]. Djilali [7] dealt with the effect of the shape of herd behavior in a diffusive predator-prey model with time delay. The results shown that the delay can lead to the instability of interior equilibrium state and the existence of Hopf bifurcation. Huang et al. [23] investigated an issue of bifurcation control for a novel incommensurate fractional-order predator-prey system with time delay. It is shown that the time delay can heavily effect the dynamics of the model. Du et al. [15] considered the dynamics of a modified Leslie-Gower predator-prey system with two delays and diffusion and found that the two delays can induce complex dynamics near the double Hopf bifurcation point, including the existence of quasi-periodic solutions on a 2-torus, quasi-periodic solutions on a 3-torus, and strange attractors. For model (1.1), there are also time delays which are ignored by us, hence, some hidden dynamics have not been found yet.

Inspired by the work mentioned-above, the purpose of the present paper is further to study the dynamics of model (1.1) with the time delays. We incorporate two time delays into model (1.1) as follows

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{mx(t-\tau_1)y}{c + x(t-\tau_1)} - H, \\ \dot{y} = qy\left(1 - \dfrac{py(t-\tau_2)}{x}\right), \end{cases} \tag{1.2}$$

where τ_1 stands for the delay of maturation time of prey and τ_2 represents the delay of gestation time of predator. It is obvious that the time delays make model (1.1) more complicated. We will mainly discuss the effects of the time delays of model (1.2).

The present paper is organized as follows. In the present section, a predator-prey model with two time delays is formulated. In Sect. 2, the local stability of equilibria and the existence of Hopf bifurcation of model (1.2) are given by discussing the different cases of time delays. In Sect. 3, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are discussed by applying the normal form theory and the center manifold reduction for retarded functional differential equations (RFDEs) developed by Hassard et al. [16]. In Sect. 4, some numerical simulations are given for confirming the qualitative results. Meanwhile, we further investigate the effects of delay on the period of periodic solutions and the influence of the harvesting term on the stability of the equilibrium with time delays. A conclusion is given in Sect. 5.

2. Stability of equilibria and existence of Hopf bifurcation

For easy mathematical analysis one can reduce the number of parameters in system (1.2) by introducing new dimensionless variables. For simplicity we rescale (1.2) by using

$$\bar{t} = rt, \quad \bar{x} = \frac{x}{K}, \quad \bar{y} = \frac{my}{rK},$$

and drop the bars, then model (1.2) takes the form

$$\begin{cases} \dot{x} = x(1-x) - \dfrac{x(t-\tau_1)y}{a + x(t-\tau_1)} - h, \\ \dot{y} = y\left(\delta - \dfrac{\beta y(t-\tau_2)}{x}\right), \end{cases} \tag{2.1}$$

where $a = \frac{c}{K}$, $\delta = \frac{q}{r}$, $h = \frac{H}{rK}$ and $\beta = \frac{pq}{m}$ are positive constants. Let $f(x) = \frac{x}{a+x}$, $g(x) = \frac{1}{x}$ and $E(x^*, y^*)$ be a nontrivial equilibrium point of model (2.1). Then we transform the equilibrium to the origin by the following transformation

$$\bar{x}(t) = x(t) - x^*, \quad \bar{y}(t) = y(t) - y^*. \tag{2.2}$$

Taylor expanding $f(\bar{x} + x^*)$ and $g(\bar{x} + x^*)$ about $x = x^*$, we rewrite model (2.1) and again for simplicity denote $\bar{x}(t)$, $\bar{y}(t)$ as $x(t)$, $y(t)$, respectively, then

$$\begin{cases} \dot{x} = (1-2x^*)x - \dfrac{x^*}{a+x^*}y - x^2 - (y+y^*)\displaystyle\sum_{i=1}^{\infty}\frac{1}{i!}f^{(i)}(x^*)x^i(t-\tau_1), \\ \dot{y} = \delta y - \beta\left[yy(t-\tau_2) + y^*y(t-\tau_2) + y^*y\right]g(x^*) \\ \qquad - \beta\left[yy(t-\tau_2) + y^*(y + y(t-\tau_2)) + {y^*}^2\right]\displaystyle\sum_{i=1}^{\infty}\frac{1}{i!}g^{(i)}(x^*)x^i, \end{cases} \tag{2.3}$$

then model (2.3) have an equilibrium at the origin, where $f^{(i)}(x^*)$ and $g^{(i)}(x^*)$ denote the ith derivative of $f(x)$ and $g(x)$ evaluated at x^*, respectively. Model (2.3) linearized about the origin is given by

$$\begin{cases} \dot{x} = (1-2x^*)x - \dfrac{ay^*}{(a+x^*)^2}x(t-\tau_1) - \dfrac{x^*}{a+x^*}y, \\ \dot{y} = \delta y + \dfrac{\beta {y^*}^2}{{x^*}^2}x - \dfrac{\beta y^*}{x^*}\left(y(t-\tau_2) + y\right). \end{cases} \tag{2.4}$$

Defining a matrix A

$$A = \begin{pmatrix} 1 - 2x^* - \frac{ay^*}{(a+x^*)^2} e^{-\lambda\tau_1} & -\frac{x^*}{a+x^*} \\ \frac{\beta {y^*}^2}{{x^*}^2} & \delta - \frac{\beta y^*}{x^*} - \frac{\beta y^*}{x^*} e^{-\lambda\tau_2} \end{pmatrix}, \tag{2.5}$$

the characteristic equation of model (2.3) at the origin is given by

$$\lambda^2 + \kappa_1\lambda + \kappa_2\lambda e^{-\lambda\tau_1} + \kappa_3\lambda e^{-\lambda\tau_2} + \kappa_4 e^{-\lambda\tau_1} + \kappa_5 e^{-\lambda\tau_2} + \kappa_6 e^{-\lambda(\tau_1+\tau_2)} + \kappa_7 = 0, \tag{2.6}$$

where

$$\begin{aligned} &\kappa_1 = 2x^* - 1 - \delta + \frac{\beta y^*}{x^*},\ \kappa_2 = \frac{ay^*}{(a+x^*)^2},\ \kappa_3 = \frac{\beta y^*}{x^*},\ \kappa_4 = \frac{\beta a {y^*}^2}{x^*(a+x^*)^2} - \frac{\delta a y^*}{(a+x^*)^2}, \\ &\kappa_5 = 2\beta y^* - \frac{\beta y^*}{x^*},\ \kappa_6 = \frac{\beta a {y^*}^2}{x^*(a+x^*)^2},\ \kappa_7 = (1-2x^*)\delta + \beta y^* \left(2 - \frac{1}{x^*}\right) + \frac{\beta {y^*}^2}{x^*(a+x^*)}. \end{aligned} \tag{2.7}$$

According to the transformation (2.2), in order to obtain the stability of equilibria and existence of Hopf bifurcation of model (2.1), we only need to discuss the corresponding properties of model (2.3). Based on the different values of τ_1 and τ_2, we consider the following five cases.

Case 1. $\tau_1 = \tau_2 = 0$. Then the characteristic equation (2.6) becomes

$$\lambda^2 + (\kappa_1 + \kappa_2 + \kappa_3)\lambda + (\kappa_4 + \kappa_5 + \kappa_6 + \kappa_7) = 0. \tag{2.8}$$

Under the Hurwitz criterion, we can obtain the result as follows.

Lemma 2.1. *If $\kappa_i (i = 1, 2, \cdots, 7)$ which are defined by (2.7) satisfying the conditions*

$(\mathbf{H_1})$ $$\kappa_1 + \kappa_2 + \kappa_3 > 0,\ \kappa_4 + \kappa_5 + \kappa_6 + \kappa_7 > 0,$$

then all roots of (2.8) have negative real parts. Therefore, the equilibrium $(0,0)$ of model (2.3) is asymptotically stable. That is, the equilibrium $E(x^*, y^*)$ of model (2.1) is locally asymptotically stable.

Next, we will consider the effects of the positive time delay(s) on the stability of equilibrium $(0,0)$ of model (2.3). Since the roots of characteristic equation (2.6) depend continuously on time delay(s), a change of time delay(s) will lead to a change of the roots of characteristic equation (2.6). If there is a critical value of τ_1 or τ_2 which makes the roots of equation (2.6) have zero real parts, then the stability of the equilibrium $(0,0)$ of model (2.3) will switch at this critical value; that is, the stability of the positive equilibrium $E(x^*, y^*)$ of model (2.1) will change. In what follows, we will investigate the critical values of τ_1 and τ_2.

Case 2. $\tau_1 > 0, \tau_2 = 0$. Then characteristic equation (2.6) becomes

$$\lambda^2 + (\kappa_1 + \kappa_3)\lambda + \kappa_2\lambda e^{-\lambda\tau_1} + (\kappa_4 + \kappa_6) e^{-\lambda\tau_1} + (\kappa_5 + \kappa_7) = 0. \tag{2.9}$$

Thus $\lambda = \mathrm{i}\vartheta_1 (\vartheta_1 > 0)$ is a root of (2.9) if and only if ϑ_1 satisfies the following equation

$$\begin{aligned} &-\vartheta_1^2 + \mathrm{i}(\kappa_1 + \kappa_3)\vartheta_1 + \mathrm{i}\kappa_2\vartheta_1 [\cos(\vartheta_1\tau_1) - \mathrm{i}\sin(\vartheta_1\tau_1)] \\ &+ (\kappa_4 + \kappa_6)[\cos(\vartheta_1\tau_1) - \mathrm{i}\sin(\vartheta_1\tau_1)] + \kappa_5 + \kappa_7 = 0. \end{aligned} \tag{2.10}$$

Separating the real and imaginary parts of (2.10), we can get

$$\begin{cases}\kappa_2\vartheta_1\sin(\vartheta_1\tau_1)+(\kappa_4+\kappa_6)\cos(\vartheta_1\tau_1)=\vartheta_1^2-\kappa_5-\kappa_7,\\(\kappa_4+\kappa_6)\sin(\vartheta_1\tau_1)-\kappa_2\vartheta_1\cos(\vartheta_1\tau_1)=(\kappa_1+\kappa_3)\vartheta_1.\end{cases}$$

Then

$$\begin{cases}\sin(\vartheta_1\tau_1)=\dfrac{(\kappa_1+\kappa_3)(\kappa_4+\kappa_6)\vartheta_1+\kappa_2\vartheta_1(\vartheta_1^2-\kappa_5-\kappa_7)}{\kappa_2^2\vartheta_1^2+(\kappa_4+\kappa_6)^2},\\\cos(\vartheta_1\tau_1)=\dfrac{(\kappa_4+\kappa_6)\left(\vartheta_1^2-\kappa_5-\kappa_7\right)-\kappa_2(\kappa_1+\kappa_3)\vartheta_1^2}{\kappa_2^2\vartheta_1^2+(\kappa_4+\kappa_6)^2},\end{cases}\tag{2.11}$$

using $\sin^2(\vartheta_1\tau_1)+\cos^2(\vartheta_1\tau_1)=1$, we have

$$\vartheta_1^4+p_1\vartheta_1^2+p_2=0,\tag{2.12}$$

where

$$p_1=(\kappa_1+\kappa_3)^2-\kappa_2^2-2(\kappa_5+\kappa_7),\ p_2=(\kappa_5+\kappa_7)^2-(\kappa_4+\kappa_6)^2.$$

Setting $v_1=\vartheta_1^2$, (2.12) changes into

$$v_1^2+p_1v_1+p_2=0.\tag{2.13}$$

Lemma 2.2. *For (2.13), we have the following conclusions:*

(1) When $\Delta_1=p_1^2-4p_2<0$, (2.13) has no positive roots;

(2) When $\Delta_1=p_1^2-4p_2\geq 0$, (2.13) has positive root(s) if $p_2<0$ or $p_2\geq 0$ and $p_1<0$.

Suppose that (2.13) has two positive roots, denoted v_{1_1} and v_{1_2}, respectively. Thus (2.12) has corresponding two positive roots denoted $\vartheta_{1_1}=\sqrt{v_{1_1}}$ and $\vartheta_{1_2}=\sqrt{v_{1_2}}$, respectively. Then $\pm\mathrm{i}\vartheta_{1_n}(n=1,2)$ are two pairs of pure imaginary roots of (2.9). Substituting ϑ_{1_n} into (2.11), we have

$$\tau_{1_n}^{(j)}=\frac{1}{\vartheta_{1_n}}\left\{\arccos\left[\frac{(\kappa_4+\kappa_6)\left(\vartheta_{1_n}^2-\kappa_5-\kappa_7\right)-\kappa_2\left(\kappa_1+\kappa_3\right)\vartheta_{1_n}^2}{\kappa_2^2\vartheta_{1_n}^2+(\kappa_4+\kappa_6)^2}\right]+2j\pi\right\},$$

where $n=1,2;j=0,1,2,\cdots$. Furthermore, we define

$$\tau_{1_0}=\min\left\{\tau_{1_n}^{(j)}|n=1,2;j=0,1,2,\cdots\right\}.$$

Let $\lambda(\tau_1)=\alpha(\tau_1)+\mathrm{i}\vartheta_1(\tau_1)$ be a root of (2.9) with $\alpha(\tau_{1_n}^{(j)})=0$, $\vartheta_1(\tau_{1_n}^{(j)})=\vartheta_{1_n}$ near $\tau_1=\tau_{1_n}^{(j)}$. In what follows, we will investigate the transversality conditions.

Lemma 2.3. *Assume that the condition*

$(\mathbf{H_2})$ $$2\vartheta_{1_n}^2+p_1>0$$

is hold, then $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]_{\tau_1=\tau_{1_n}^{(j)}}>0(n=1,2;j=0,1,2,\cdots)$.

Proof. We computer the derivative of λ with respect to τ_1 in (2.9)

$$\left[\frac{\mathrm{d}\lambda(\tau_1)}{\mathrm{d}\tau_1}\right]^{-1}=\frac{(2\lambda+\kappa_1+\kappa_3)e^{\lambda\tau_1}}{\lambda(\kappa_2\lambda+\kappa_4+\kappa_6)}+\frac{\kappa_2}{\lambda(\kappa_2\lambda+\kappa_4+\kappa_6)}-\frac{\tau_1}{\lambda}. \tag{2.14}$$

Substitute $\tau_1=\tau_{1_n}^{(j)}$ into (2.14) and notice that $\lambda(\tau_{1_n}^{(j)})=\mathrm{i}\vartheta_{1_n}$, we have

$$\begin{aligned}
&\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]^{-1}_{\tau_1=\tau_{1_n}^{(j)}}\\
&=\mathrm{Re}\left[\frac{(2\lambda+\kappa_1+\kappa_3)e^{\lambda\tau_1}}{\lambda(\kappa_2\lambda+\kappa_4+\kappa_6)}\right]_{\tau_1=\tau_{1_n}^{(j)}}+\mathrm{Re}\left[\frac{\kappa_2}{\lambda(\kappa_2\lambda+\kappa_4+\kappa_6)}\right]_{\tau_1=\tau_{1_n}^{(j)}}\\
&=\frac{1}{M_1}\left\{(\kappa_4+\kappa_6)\vartheta_{1_n}\left[2\vartheta_{1_n}\cos(\vartheta_{1_n}\tau_1)+(\kappa_1+\kappa_3)\sin(\vartheta_{1_n}\tau_1)\right]\right.\\
&\quad\left.+\,\kappa_2\vartheta_{1_n}^2\left[2\vartheta_{1_n}\sin(\vartheta_{1_n}\tau_1)-(\kappa_1+\kappa_3)\cos(\vartheta_{1_n}\tau_1)\right]-\kappa_2^2\vartheta_{1_n}^2\right\}\\
&=\frac{1}{M_1}\left\{2\vartheta_{1_n}^2\left[\kappa_2\vartheta_{1_n}\sin(\vartheta_{1_n}\tau_1)+(\kappa_4+\kappa_6)\cos(\vartheta_{1_n}\tau_1)\right]\right.\\
&\quad\left.+\,(\kappa_1+\kappa_3)\vartheta_{1_n}\left[(\kappa_4+\kappa_6)\sin(\vartheta_{1_n}\tau_1)-\kappa_2\vartheta_{1_n}\cos(\vartheta_{1_n}\tau_1)\right]-\kappa_2^2\vartheta_{1_n}^2\right\}\\
&=\frac{1}{M_1}\left[2\vartheta_{1_n}^2(\vartheta_{1_n}^2-\kappa_5-\kappa_7)+(\kappa_1+\kappa_3)\vartheta_{1_n}(\kappa_1+\kappa_3)\vartheta_{1_n}-\kappa_2^2\vartheta_{1_n}^2\right]\\
&=\frac{\vartheta_{1_n}^2}{M_1}\left(2\vartheta_{1_n}^2+p_1\right),
\end{aligned}$$

where $M_1=\vartheta_{1_n}^2\left[\kappa_2^2\vartheta_{1_n}^2+(\kappa_4+\kappa_6)^2\right]>0$. Hence, if $2\vartheta_{1_n}^2+p_1>0$, we have $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]^{-1}_{\tau_1=\tau_{1_n}^{(j)}}>0$, that is, $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]_{\tau_1=\tau_{1_n}^{(j)}}>0$ $(n=1,2;j=0,1,2,\cdots)$. □

Theorem 2.1. *If the parameters of model (2.1) satisfy the condition* ($\mathbf{H_1}$)*, then*

- *(i) When $\Delta_1=p_1^2-4p_2<0$, all the roots of (2.9) have negative real parts, then the equilibrium $E(x^*,y^*)$ of model (2.1) is asymptotically stable when $\tau_1\geq 0$.*
- *(ii) When $\Delta_1=p_1^2-4p_2\geq 0$, if $p_2<0$ or $p_2\geq 0$ and $p_1<0$, then the equilibrium $E(x^*,y^*)$ is asymptotically stable when $\tau_1\in[0,\tau_{1_0})$ and is unstable when $\tau_1\geq\tau_{1_0}$;*
- *(iii) If the above conditions (ii) and* ($\mathbf{H_2}$) *hold, model (2.1) will undergo a Hopf bifurcation at the equilibrium $E(x^*,y^*)$ when τ_1 crosses through each critical values $\tau_{1_n}^{(j)}(n=1,2;j=0,1,2,\cdots)$.*

Case 3. $\tau_1=0,\tau_2>0$. Then characteristic equation (2.6) becomes

$$\lambda^2+(\kappa_1+\kappa_2)\lambda+\kappa_3\lambda e^{-\lambda\tau_2}+(\kappa_5+\kappa_6)e^{-\lambda\tau_2}+(\kappa_4+\kappa_7)=0. \tag{2.15}$$

Thus $\lambda=\mathrm{i}\vartheta_2(\vartheta_2>0)$ is a root of (2.15) if and only if ϑ_2 satisfies the following equation

$$\begin{aligned}
&-\vartheta_2^2+\mathrm{i}(\kappa_1+\kappa_2)\vartheta_2+\mathrm{i}\kappa_3\vartheta_2\left[\cos(\vartheta_2\tau_2)-\mathrm{i}\sin(\vartheta_2\tau_2)\right]\\
&+(\kappa_5+\kappa_6)\left[\cos(\vartheta_2\tau_2)-\mathrm{i}\sin(\vartheta_2\tau_2)\right]+\kappa_4+\kappa_7=0.
\end{aligned} \tag{2.16}$$

Separating the real and imaginary part of (2.16), we have

$$\begin{cases}\kappa_3\vartheta_2\sin(\vartheta_2\tau_2)+(\kappa_5+\kappa_6)\cos(\vartheta_2\tau_2)=\vartheta_2^2-\kappa_4-\kappa_7,\\(\kappa_5+\kappa_6)\sin(\vartheta_2\tau_2)-\kappa_3\vartheta_2\cos(\vartheta_2\tau_2)=(\kappa_1+\kappa_2)\vartheta_2.\end{cases}$$

Then

$$\begin{cases}\sin(\vartheta_2\tau_2)=\dfrac{(\kappa_1+\kappa_2)(\kappa_5+\kappa_6)\vartheta_2+\kappa_3\vartheta_2(\vartheta_2^2-\kappa_4-\kappa_7)}{\kappa_3^2\vartheta_2^2+(\kappa_5+\kappa_6)^2},\\\cos(\vartheta_2\tau_2)=\dfrac{\left(\vartheta_2^2-\kappa_4-\kappa_7\right)(\kappa_5+\kappa_6)-\kappa_3(\kappa_1+\kappa_2)\vartheta_2^2}{\kappa_3^2\vartheta_2^2+(\kappa_5+\kappa_6)^2},\end{cases}\tag{2.17}$$

furthermore, we have

$$\vartheta_2^4+q_1\vartheta_2^2+q_2=0,\tag{2.18}$$

where

$$q_1=(\kappa_1+\kappa_2)^2-\kappa_3^2-2(\kappa_4+\kappa_7),\ q_2=(\kappa_4+\kappa_5+\kappa_6+\kappa_7)(\kappa_4-\kappa_5-\kappa_6+\kappa_7).$$

Setting $v_2=\vartheta_2^2$, (2.18) becomes

$$v_2^2+q_1v_2+q_2=0.\tag{2.19}$$

Lemma 2.4. *For (2.19), we have the following conclusions:*

(1) When $\Delta_2=q_1^2-4q_2<0$, *(2.19) has no positive roots;*

(2) When $\Delta_2=q_1^2-4q_2\geq 0$, *(2.19) has positive root(s) if* $q_2<0$ *or* $q_2\geq 0$ *and* $q_1<0$.

Assuming that (2.19) has two positive roots, denoted v_{2_1} and v_{2_2}, respectively. Then (2.18) has two positive roots denoted $\vartheta_{2_1}=\sqrt{v_{2_1}}$ and $\vartheta_{2_2}=\sqrt{v_{2_2}}$, respectively. Then $\pm\mathrm{i}\vartheta_{2_n}(n=1,2)$ are two pairs of pure imaginary roots of (2.15). Substituting ϑ_{2_n} into (2.17), we have

$$\tau_{2_n}^{(j)}=\frac{1}{\vartheta_{2_n}}\left\{\arccos\left[\frac{\left(\vartheta_{2_n}^2-\kappa_4-\kappa_7\right)(\kappa_5+\kappa_6)-\kappa_3\left(\kappa_1+\kappa_2\right)\vartheta_{1_n}^2}{\kappa_3^2\vartheta_{2_n}^2+(\kappa_5+\kappa_6)^2}\right]+2j\pi\right\},$$

where $n=1,2;j=0,1,2,\cdots$. Furthermore, we define

$$\tau_{2_0}=\min\left\{\tau_{2_n}^{(j)}\middle|n=1,2;j=0,1,2,\cdots\right\}.$$

Let $\lambda(\tau_2)=\alpha(\tau_2)+\mathrm{i}\vartheta_2(\tau_2)$ be a root of equation (2.15) with $\alpha(\tau_{2_n}^{(j)})=0,\vartheta_2(\tau_{2_n}^{(j)})=\vartheta_{2_n}$ near $\tau_2=\tau_{2_n}^{(j)}$. Next, we need to check transversality.

Lemma 2.5. *Assume that the condition*

$$(\mathbf{H_3})\qquad 2\vartheta_{2_n}^2+q_1>0$$

is hold, then we have $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_2))}{\mathrm{d}\tau_2}\right]_{\tau_2=\tau_{2_n}^{(j)}}>0(n=1,2;j=0,1,2,\cdots)$.

Proof. Similar to the **Case 2**, from (2.15), we have

$$\left[\frac{\mathrm{d}\lambda(\tau_2)}{\mathrm{d}\tau_2}\right]^{-1} = \frac{(2\lambda+\kappa_1+\kappa_2)e^{\lambda\tau_2}}{\lambda(\kappa_3\lambda+\kappa_5+\kappa_6)} + \frac{\kappa_3}{\lambda(\kappa_3\lambda+\kappa_5+\kappa_6)} - \frac{\tau_2}{\lambda}, \tag{2.20}$$

substituting $\lambda = \mathrm{i}\vartheta_{2_n}$ into (2.20)

$$\begin{aligned}
&\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_2))}{\mathrm{d}\tau_2}\right]^{-1}_{\tau_2=\tau_{2_n}^{(j)}} \\
&= \mathrm{Re}\left[\frac{(2\lambda+\kappa_1+\kappa_2)e^{\lambda\tau_2}}{\lambda(\kappa_3\lambda+\kappa_5+\kappa_6)}\right]_{\tau_2=\tau_{2_n}^{(j)}} + \mathrm{Re}\left[\frac{\kappa_3}{\lambda(\kappa_3\lambda+\kappa_5+\kappa_6)}\right]_{\tau_2=\tau_{2_n}^{(j)}} \\
&= \frac{1}{M_2}\left\{(\kappa_5+\kappa_6)\vartheta_{2_n}\left[2\vartheta_{2_n}\cos(\vartheta_{2_n}\tau_2)+(\kappa_1+\kappa_2)\sin(\vartheta_{2_n}\tau_2)\right]\right. \\
&\quad \left. +\ \kappa_3\vartheta_{2_n}^2\left[2\vartheta_{2_n}\sin(\vartheta_{2_n}\tau_2)-(\kappa_1+\kappa_2)\cos(\vartheta_{2_n}\tau_2)\right] - \kappa_3^2\vartheta_{2_n}^2\right\} \\
&= \frac{1}{M_2}\left\{2\vartheta_{2_n}^2\left[\kappa_3\vartheta_{2_n}\sin(\vartheta_{2_n}\tau_2)+(\kappa_5+\kappa_6)\cos(\vartheta_{2_n}\tau_2)\right]\right. \\
&\quad \left. +\ (\kappa_1+\kappa_2)\vartheta_{2_n}\left[(\kappa_5+\kappa_6)\sin(\vartheta_{2_n}\tau_2)-\kappa_3\vartheta_{2_n}\cos(\vartheta_{2_n}\tau_2)\right] - \kappa_3^2\vartheta_{2_n}^2\right\} \\
&= \frac{1}{M_2}\left[2\vartheta_{2_n}^2(\vartheta_{2_n}^2-\kappa_4-\kappa_7)+(\kappa_1+\kappa_2)\vartheta_{2_n}(\kappa_1+\kappa_2)\vartheta_{2_n}-\kappa_3^2\vartheta_{2_n}^2\right] \\
&= \frac{\vartheta_{2_n}^2}{M_2}(2\vartheta_{2_n}^2+q_1),
\end{aligned}$$

where $M_2 = \vartheta_{2_n}^2\left[\kappa_3^2\vartheta_{2_n}^2+(\kappa_5+\kappa_6)^2\right]$. Hence, if $2\vartheta_{2_n}^2+q_1>0$, we have $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_2))}{\mathrm{d}\tau_2}\right]^{-1}_{\tau_2=\tau_{2_n}^{(j)}} > 0$, that is, $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_2))}{\mathrm{d}\tau_2}\right]_{\tau_2=\tau_{2_n}^{(j)}} > 0$ $(n=1,2; j=0,1,2,\cdots)$.

□

Theorem 2.2. *If the parameters of model (2.1) satisfy the condition* ($\mathbf{H_1}$), *then*

(i) When $\Delta_2 = q_1^2-4q_2<0$, *all the roots of (2.15) have negative real parts, then the equilibrium* $E(x^*,y^*)$ *of model (2.1) is asymptotically stable when* $\tau_2\geq 0$;

(ii) When $\Delta_2 = q_1^2-4q_2\geq 0$, *if* $q_2<0$ *or* $q_2\geq 0$ *and* $q_1<0$, *all the roots of (2.15) have negative real parts when* $\tau_2\in[0,\tau_{2_0})$, *then the equilibrium* $E(x^*,y^*)$ *of model (2.1) is asymptotically stable when* $\tau_2\in[0,\tau_{2_0})$ *and is unstable with* $\tau_2\geq\tau_{2_0}$;

(iii) If the above conditions (ii) and ($\mathbf{H_3}$) *hold, model (2.1) will undergo a Hopf bifurcation at the equilibrium* $E(x^*,y^*)$ *when* τ_2 *crosses through each critical values* $\tau_{2_n}^{(j)}(n=1,2;j=0,1,2,\cdots)$.

Case 4. $\tau_1=\tau_2=\tau\neq 0$. The characteristic equation (2.6) becomes

$$\lambda^2+\kappa_1\lambda+(\kappa_2+\kappa_3)\lambda e^{-\lambda\tau}+(\kappa_4+\kappa_5)e^{-\lambda\tau}+\kappa_6 e^{-2\lambda\tau}+\kappa_7=0. \tag{2.21}$$

The equation (2.21) multiplying $e^{\lambda\tau}$ is given by

$$\left(\lambda^2+\kappa_1\lambda+\kappa_7\right)e^{\lambda\tau}+\kappa_6 e^{-\lambda\tau}+(\kappa_2+\kappa_3)\lambda+(\kappa_4+\kappa_5)=0. \tag{2.22}$$

Then $\lambda=\mathrm{i}\vartheta_3(\vartheta_3>0)$ is a root of (2.22) if and only if ϑ_3 satisfies the following equation

$$\begin{aligned}
&\left(-\vartheta_3^2+\mathrm{i}\kappa_1\vartheta_3+\kappa_7\right)\left[\cos(\vartheta_3\tau)+\mathrm{i}\sin(\vartheta_3\tau)\right] \\
&+\kappa_6\left[\cos(\vartheta_3\tau)-\mathrm{i}\sin(\vartheta_3\tau)\right]+\mathrm{i}\vartheta_3(\kappa_2+\kappa_3)+\kappa_4+\kappa_5=0.
\end{aligned} \tag{2.23}$$

Separating the real and imaginary parts of (2.23), we have

$$\begin{cases} -\kappa_1\vartheta_3\sin(\vartheta_3\tau)+\left(\kappa_6+\kappa_7-\vartheta_3^2\right)\cos(\vartheta_3\tau)=-(\kappa_4+\kappa_5),\\ \left(\kappa_7-\kappa_6-\vartheta_3^2\right)\sin(\vartheta_3\tau)+\kappa_1\vartheta_3\cos(\vartheta_3\tau)=-(\kappa_2+\kappa_3)\vartheta_3,\end{cases}$$

and

$$\begin{cases} \sin(\vartheta_3\tau)=\dfrac{\kappa_1\vartheta_3(\kappa_4+\kappa_5)-(\kappa_2+\kappa_3)\vartheta_3\left(\kappa_7+\kappa_6-\vartheta_3^2\right)}{\kappa_1^2\vartheta_3^2+\left(\kappa_7+\kappa_6-\vartheta_3^2\right)\left(\kappa_7-\kappa_6-\vartheta_3^2\right)},\\ \cos(\vartheta_3\tau)=-\dfrac{\kappa_1\vartheta_3^2(\kappa_2+\kappa_3)+(\kappa_4+\kappa_5)\left(\kappa_7-\kappa_6-\vartheta_3^2\right)}{\kappa_1^2\vartheta_3^2+\left(\kappa_7+\kappa_6-\vartheta_3^2\right)\left(\kappa_7-\kappa_6-\vartheta_3^2\right)}.\end{cases} \tag{2.24}$$

Then we can obtain

$$\vartheta_3^8+m_1\vartheta_3^6+m_2\vartheta_3^4+m_3\vartheta_3^2+m_4=0, \tag{2.25}$$

where

$$\begin{aligned} m_1&=2\kappa_1^2-4\kappa_7-(\kappa_2+\kappa_3)^2,\\ m_2&=(\kappa_2+\kappa_3)^2(2\kappa_6+2\kappa_7-\kappa_1^2)-(\kappa_4+\kappa_5)^2+\kappa_1^4-4\kappa_1^2\kappa_7+6\kappa_7^2-2\kappa_6^2,\\ m_3&=4\kappa_1\kappa_6(\kappa_2+\kappa_3)(\kappa_4+\kappa_5)-\left[\kappa_1^2+2(\kappa_6-\kappa_7)\right](\kappa_4+\kappa_5)^2\\ &\quad-(\kappa_2+\kappa_3)^2(\kappa_6+\kappa_7)^2+(2\kappa_1^2-4\kappa_7)\left(\kappa_7^2-\kappa_6^2\right),\\ m_4&=(\kappa_6-\kappa_7)^2(\kappa_7-\kappa_4+\kappa_6-\kappa_5)(\kappa_7+\kappa_4+\kappa_6+\kappa_5).\end{aligned}$$

Since (2.25) can be treated as a fourth order equation by the transformation $u=\vartheta_3^2$, equation (2.25) has at most four real roots $u_n(n=1,2,3,4)$ which implies that there exist at most four positive roots $\vartheta_{3_n}(n=1,2,3,4)$. Without loss of generality, we suppose that (2.25) has four positive roots, denoted $\vartheta_{3_n}(n=1,2,3,4)$. Then $\pm\mathrm{i}\vartheta_{3_n}(n=1,2,3,4)$ are four pairs of pure imaginary roots of (2.22). Substituting ϑ_{3_n} into (2.24), we have

$$\tau_{3_n}^{(j)}=\frac{1}{\vartheta_{3_n}}\left\{\arccos\left[-\frac{\kappa_1\vartheta_{3_n}^2(\kappa_2+\kappa_3)+(\kappa_4+\kappa_5)\left(\kappa_7-\kappa_6-\vartheta_{3_n}^2\right)}{\kappa_1^2\vartheta_{3_n}^2+\left(\kappa_7+\kappa_6-\vartheta_{3_n}^2\right)\left(\kappa_7-\kappa_6-\vartheta_{3_n}^2\right)}\right]+2j\pi\right\},$$

where $n=1,2,3,4;j=0,1,2,\cdots$. Furthermore, we define

$$\tau_{3_0}=\min\left\{\tau_{3_n}^{(j)}|n=1,2,3,4;j=0,1,2,\cdots\right\}.$$

Let $\lambda(\tau)=\alpha(\tau)+\mathrm{i}\vartheta_3(\tau)$ be a root of (2.22) with $\alpha(\tau_{3_n}^{(j)})=0,\vartheta_3(\tau_{3_n}^{(j)})=\vartheta_{3_n}$ near $\tau_1=\tau_2=\tau_{3_n}^{(j)}$. We next check the transversality conditions.

Taking the derivative of λ with respect to τ in (2.22), we have

$$\left[\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\right]^{-1}=-\frac{(2\lambda+\kappa_1)e^{\lambda\tau}+\kappa_2+\kappa_3}{\lambda\left[\left(\lambda^2+\kappa_1\lambda+\kappa_7\right)e^{\lambda\tau}-\kappa_6e^{-\lambda\tau}\right]}-\frac{\tau}{\lambda},$$

then

$$\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_{3_n}^{(j)}}=\mathrm{Re}\left[-\frac{(2\lambda+\kappa_1)e^{\lambda\tau}+\kappa_2+\kappa_3}{\lambda\left[\left(\lambda^2+\kappa_1\lambda+\kappa_7\right)e^{\lambda\tau}-\kappa_6e^{-\lambda\tau}\right]}\right]_{\tau=\tau_{3_n}^{(j)}}=\frac{N_1}{M_3},$$

where

$$\begin{aligned} N_1 =& \left[2\kappa_1\kappa_6\cos(\vartheta_{3_n}\tau) + (\kappa_2+\kappa_3)\left(\kappa_6+\kappa_7-\vartheta_{3_n}^2\right)\right]\sin(\vartheta_{3_n}\tau) \\ &+ \vartheta_{3_n}\left[4\kappa_6\cos^2(\vartheta_{3_n}\tau) + \kappa_1(\kappa_2+\kappa_3)\cos(\vartheta_{3_n}\tau) + \kappa_1^2 + 2\vartheta_{3_n}^2 - 2\kappa_6 - 2\kappa_7\right], \\ M_3 =& \vartheta_{3_n}\left[4\kappa_1\kappa_6\vartheta_{3_n}\cos(\vartheta_{3_n}\tau)\sin(\vartheta_{3_n}\tau) - 4\kappa_6\left(\kappa_7-\vartheta_{3_n}^2\right)\cos^2(\vartheta_{3_n}\tau) \right.\\ &\left. +\vartheta_{3_n}^4 + \left(\kappa_1^2-2\kappa_6-2\kappa_7\right)\vartheta_{3_n}^2 + (\kappa_6+\kappa_7)^2\right]. \end{aligned}$$

Thus if $\frac{N_1}{M_3} > 0$, then we have $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_{3_n}^{(j)}} > 0$, that is, $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}\right]_{\tau=\tau_{3_n}^{(j)}} > 0$.

Lemma 2.6. *Assume that the condition*

$$(\mathbf{H_4}) \qquad \frac{N}{M_3} > 0$$

is hold, then we have $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau))}{\mathrm{d}\tau}\right]_{\tau=\tau_{3_n}^{(j)}} > 0 (n = 1,2,3,4; j = 0,1,2,\cdots)$.

Theorem 2.3. *If the parameters of model (2.1) satisfy the condition* $(\mathbf{H_1})$*, then*

(i) If (2.25) has no positive roots, then the equilibrium $E(x^*, y^*)$ of model (2.1) is asymptotically stable when $\tau_1 = \tau_2 = \tau \geq 0$;

(ii) If (2.25) has at least one positive root and the condition $(\mathbf{H_4})$ hold, then the equilibrium $E(x^*, y^*)$ of model (2.1) is locally asymptotically stable when $\tau_1 = \tau_2 = \tau \in [0, \tau_{3_0})$ and is unstable when $\tau_1 = \tau_2 = \tau \geq \tau_{3_0}$. Furthermore, when $\tau_1 = \tau_2 = \tau$ cross through each critical values $\tau_{3_n}^{(j)} (n = 1,2,3,4; j = 0,1,2,\cdots)$, model (2.1) will undergo a Hopf bifurcation at the equilibrium $E(x^*, y^*)$.

Case 5. $\tau_1 > 0, \tau_2 > 0$ and $\tau_1 \neq \tau_2$.

For this case, we suppose characteristic equation (2.6) with τ_2 in its stable interval $[0, \tau_{2_0})$ and choose τ_1 as a free parameter. Hence, we assume that $\lambda = \mathrm{i}\vartheta_4 (\vartheta_4 > 0)$ is a root of the characteristic equation (2.6), then

$$\begin{aligned} &-\vartheta_4^2 + \mathrm{i}\kappa_1\vartheta_4 + \mathrm{i}\kappa_2\vartheta_4\left[\cos(\vartheta_4\tau_1) - \mathrm{i}\sin(\vartheta_4\tau_1)\right] + \mathrm{i}\kappa_3\vartheta_4\left[\cos(\vartheta_4\tau_2) - \mathrm{i}\sin(\vartheta_4\tau_2)\right] \\ &+ \kappa_4\left[\cos(\vartheta_4\tau_1) - \mathrm{i}\sin(\vartheta_4\tau_1)\right] + \kappa_5\left[\cos(\vartheta_4\tau_2) - \mathrm{i}\sin(\vartheta_4\tau_2)\right] \\ &+ \kappa_6\left[\cos\left(\vartheta_4(\tau_1+\tau_2)\right) - \mathrm{i}\sin\left(\vartheta_4(\tau_1+\tau_2)\right)\right] + \kappa_7 = 0. \end{aligned} \tag{2.26}$$

Similarly, we have

$$\begin{cases} \sin(\vartheta_4\tau_1) = \dfrac{b_2c_1 - b_1c_2}{b_1^2 + b_2^2}, \\ \cos(\vartheta_4\tau_1) = \dfrac{b_2c_2 + b_1c_1}{b_1^2 + b_2^2}, \end{cases} \tag{2.27}$$

where

$$\begin{aligned} &b_1 = \kappa_4 + \kappa_6\cos(\vartheta_4\tau_2),\ b_2 = \kappa_2\vartheta_4 - \kappa_6\sin(\vartheta_4\tau_2), \\ &c_1 = \vartheta_4^2 - \kappa_3\vartheta_4\sin(\vartheta_4\tau_2) - \kappa_5\cos(\vartheta_4\tau_2) - \kappa_7, \\ &c_2 = -\kappa_1\vartheta_4 - \kappa_3\vartheta_4\cos(\vartheta_4\tau_2) + \kappa_5\sin(\vartheta_4\tau_2). \end{aligned}$$

Moreover, we have

$$\vartheta_4^4 + n_1\vartheta_4^3 + n_2\vartheta_4^2 + n_3\vartheta_4 + n_4 = 0, \tag{2.28}$$

where

$$
\begin{aligned}
n_1 &= -2\kappa_3 \sin(\vartheta_4 \tau_2), \\
n_2 &= 2(\kappa_1\kappa_3 - \kappa_5)\cos(\vartheta_4\tau_2) + \kappa_1^2 - \kappa_2^2 + \kappa_3^2 - 2\kappa_7, \\
n_3 &= -2(\kappa_1\kappa_5 - \kappa_2\kappa_6 - \kappa_3\kappa_7)\sin(\vartheta_4\tau_2), \\
n_4 &= -2(\kappa_4\kappa_6 - \kappa_5\kappa_7)\cos(\vartheta_4\tau_2) - \kappa_4^2 + \kappa_5^2 - \kappa_6^2 + \kappa_7^2.
\end{aligned}
$$

Suppose that (2.28) has positive roots. Without loss of generality, we assume that it has four positive roots, denoted $\vartheta_{4_n} (n = 1, 2, 3, 4)$. Then $\pm \mathrm{i}\vartheta_{4_n} (n = 1, 2, 3, 4)$ are four pairs of pure imaginary roots of (2.26). Substituting ϑ_{4_n} into (2.27), we have

$$
\tau_{4_n}^{(j)} = \frac{1}{\vartheta_{4_n}}\left[\arccos\left(\frac{b_2c_2 + b_1c_1}{b_1^2 + b_2^2}\right) + 2j\pi\right], n = 1, 2, 3, 4; j = 0, 1, 2, \cdots .
$$

Furthermore, we define

$$
\tau_{4_0} = \min\left\{\tau_{4_n}^{(j)} \middle| n = 1, 2, 3, 4; j = 0, 1, 2, \cdots\right\} \tag{2.29}
$$

and ϑ_{4_0} is the critical value corresponding to τ_{4_0}.

Next, we also need to check the transversality. From (2.6), we have

$$
\begin{aligned}
&\left[\frac{\mathrm{d}\lambda(\tau_1)}{\mathrm{d}\tau_1}\right]^{-1} \\
=& \frac{-\kappa_6\tau_2 e^{-\lambda(\tau_1+\tau_2)} + \kappa_2 e^{-\lambda\tau_1} + (-\kappa_3\lambda\tau_2 - \kappa_5\tau_2 + \kappa_3)e^{-\lambda\tau_2} + \kappa_1 + 2\lambda}{\lambda\left(\kappa_6 e^{-\lambda(\tau_1+\tau_2)} + (\kappa_2\lambda + \kappa_4)e^{-\lambda\tau_1}\right)} - \frac{\tau_1}{\lambda}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]^{-1}_{\tau_1 = \tau_{4_n}^{(j)}} \\
=& \mathrm{Re}\left[\frac{-\kappa_6\tau_2 e^{-\lambda(\tau_1+\tau_2)} + \kappa_2 e^{-\lambda\tau_1} + (-\kappa_3\lambda\tau_2 - \kappa_5\tau_2 + \kappa_3)e^{-\lambda\tau_2} + \kappa_1 + 2\lambda}{\lambda\left(\kappa_6 e^{-\lambda(\tau_1+\tau_2)} + (\kappa_2\lambda + \kappa_4)e^{-\lambda\tau_1}\right)}\right]_{\tau_1 = \tau_{4_n}^{(j)}} \\
=& \mathrm{Re}\left[\frac{N_2}{M_4}\right]_{\tau_1 = \tau_{4_n}^{(j)}} = \frac{M_{4_R}N_{2_R} + M_{4_I}N_{2_I}}{M_{4_R}^2 + M_{4_I}^2},
\end{aligned}
$$

where

$$
\begin{aligned}
N_2 =& -\kappa_6\tau_2\cos[\vartheta_4(\tau_1+\tau_2)] + \kappa_2\cos(\vartheta_4\tau_1) + (\kappa_3 - \kappa_5\tau_2)\cos(\vartheta_4\tau_2) \\
&- \kappa_3\vartheta_4\tau_2\sin(\vartheta_4\tau_2) + \kappa_1 + \mathrm{i}\left\{\kappa_6\tau_2\sin[\vartheta_4(\tau_1+\tau_2)] - \kappa_2\sin(\vartheta_4\tau_1)\right. \\
&\left. - \kappa_3\vartheta_4\tau_2\cos(\vartheta_4\tau_2) - (\kappa_3 - \kappa_5\tau_2)\sin(\vartheta_4\tau_2) + 2\vartheta_4\right\}, \\
M_4 =& \kappa_6\vartheta_4\sin[\vartheta_4(\tau_1+\tau_2)] + \kappa_4\vartheta_4\sin(\vartheta_4\tau_1) - \kappa_2\vartheta_4^2\cos(\vartheta_4\tau_1) \\
&+ \mathrm{i}\left\{\kappa_6\vartheta_4\cos[\vartheta_4(\tau_1+\tau_2)] + \kappa_4\vartheta_4\cos(\vartheta_4\tau_1) + \kappa_2\vartheta_4^2\sin(\vartheta_4\tau_1)\right\}
\end{aligned}
$$

and N_{2_R}, N_{2_I}, M_{4_R}, M_{4_I} are stand for the real part of N_2, imaginary part of N_2, real part of M_4 and imaginary part of M_4, respectively. Thus, if $M_{4_R}N_{2_R} + M_{4_I}N_{2_I} > 0$, then $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]^{-1}_{\tau_1 = \tau_{4_n}^{(j)}} > 0$, that is, we have $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]_{\tau_1 = \tau_{4_n}^{(j)}} > 0 (n = 1, 2, 3, 4; j = 0, 1, 2, \cdots)$.

Lemma 2.7. *Assume that the condition*

$(\mathbf{H_5})$ $$M_{4_R}N_{2_R} + M_{4_I}N_{2_I} > 0$$

is hold, then we have $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda(\tau_1))}{\mathrm{d}\tau_1}\right]_{\tau_1=\tau_{4_n}^{(j)}} > 0 (n = 1, 2, 3, 4; j = 0, 1, 2, \cdots)$.

Theorem 2.4. *If the parameters of model (2.1) satisfy the condition* $(\mathbf{H_1})$ *and* $\tau_2 \in [0, \tau_{2_0})$, *then*

- *(i) If (2.28) has no positive roots, then the equilibrium* $E(x^*, y^*)$ *of model (2.1) is asymptotically stable when* $\tau_1 \geq 0$*;*
- *(ii) If (2.28) has at least one positive root and the condition* $(\mathbf{H_5})$ *hold, then the equilibrium* $E(x^*, y^*)$ *of model (2.1) is asymptotically stable when* $\tau_1 \in [0, \tau_{4_0})$ *and is unstable when* $\tau_1 \geq \tau_{4_0}$. *Furthermore, when* τ_1 *cross through critical values* $\tau_{4_n}^{(j)} (n = 1, 2, 3, 4; j = 0, 1, 2, \cdots)$, *model (2.1) will undergo a Hopf bifurcation at the equilibrium* $E(x^*, y^*)$.

3. Properties of Hopf Bifurcation

In this section, using the normal form theory and central manifold theorem for RFDEs [16], we will study the properties of Hopf bifurcation obtained by Theorem 2.4.

Throughout this section, we always assume that model (2.1) undergoes Hopf bifurcation at the equilibrium $E(x^*, y^*)$ when $\tau_1 = \tau_{4_0}$ and then $\pm \mathrm{i}\vartheta_{4_0}$ are the corresponding purely imaginary roots of the characteristic equation (2.6). We assume that $\tau_2^* < \tau_{4_0}$, where $\tau_2^* \in (0, \tau_{2_0})$. Meanwhile, let $\tau_1 = \tau_{4_0} + \mu$, where τ_{4_0} is defined by (2.29) and $\mu \in \mathbb{R}$. Then there is the Hopf bifurcation value of model (2.1) at $E(x^*, y^*)$ when $\mu = 0$. Since model (2.1) is equivalent to model (2.3), hence we mainly investigate model (2.3) in the following discussion. Letting $X_1(t) = \overline{x}(\tau_1 t), X_2(t) = \overline{y}(\tau_1 t)$ to normalize the delays, model (2.3) is transformed into a RFDEs in phase space $\mathbb{C} = \mathbb{C}([-1, 0], \mathbb{R}^2)$ as

$$\dot{X}(t) = L_\mu(X_t) + F(\mu, X_t), \tag{3.1}$$

where $X(t) = (X_1(t), X_2(t))^T \in \mathbb{R}^2$, $X_t = X_t(\theta) = X(t+\theta) = (X_1(t+\theta), X_2(t+\theta))^T \in \mathbb{C}$ and $L_\mu : \mathbb{C} \to \mathbb{R}^2, F : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^2$ are given respectively by

$$L_\mu(\phi) = (\tau_{4_0} + \mu)B \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_{4_0} + \mu)C \begin{pmatrix} \phi_1\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) \\ \phi_2\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) \end{pmatrix} + (\tau_{4_0} + \mu)D \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} \tag{3.2}$$

and

$$F(\mu, \phi) = (\tau_{4_0} + \mu) \times \begin{pmatrix} -\phi_1^2(0) - \frac{a}{(a+x^*)^2}\phi_1(-1)\phi_2(0) - (\phi_2(0) + y^*)\sum_{i=2}^{\infty}\frac{1}{i!}f^{(i)}(x^*)\phi_1^i(-1) \\ \left(\frac{\beta}{x^{*2}}\phi_1(0) - \frac{\beta}{x^*}\right)\phi_2(0)\phi_2\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) + \frac{\beta y^*}{x^{*2}}\phi_1(0)\left(\phi_2(0) + \phi_2\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\right) - \beta \\ \times\left(\phi_2(0)\phi_2\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) + y^*\left(\phi_2(0) + \phi_2\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\right) + y^{*2}\right)\sum_{i=2}^{\infty}\frac{1}{i!}g^{(i)}(x^*)\phi_1^i(0) \end{pmatrix}, \tag{3.3}$$

where $\phi = \phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in \mathbb{C}$, and

$$B = \begin{pmatrix} 1-2x^* & -\frac{x^*}{a+x^*} \\ \frac{\beta y^{*2}}{x^{*2}} & \delta - \frac{\beta y^*}{x^*} \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\beta y^*}{x^*} \end{pmatrix}, \; D = \begin{pmatrix} -\frac{ay^*}{(a+x^*)^2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

By the Riese representation theorem, there exist a 2×2 matrix function $\eta(\theta,\mu)$, $-1 \le \theta \le 0$, whose elements are of bounded variation such that

$$L_\mu(\phi) = \int_{-1}^{0} \mathrm{d}\eta(\theta,\mu)\phi(\theta) \;\; \text{for} \;\; \phi \in \mathbb{C}([-1,0],\mathbb{R}^2).$$

In fact, we can choose

$$\eta(\theta,\mu) = \begin{cases} (\tau_{4_0}+\mu)(B+C+D), & \theta = 0, \\ (\tau_{4_0}+\mu)(B+C), & \theta \in \left[-\frac{\tau_2^*}{\tau_{4_0}}, 0\right), \\ (\tau_{4_0}+\mu)D, & \theta \in \left(-1, -\frac{\tau_2^*}{\tau_{4_0}}\right), \\ 0, & \theta = -1. \end{cases} \tag{3.5}$$

For $\phi \in \mathbb{C}^1([-1,0],\mathbb{R}^2)$, defining

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(\xi,\mu)\phi(\xi) = L_\mu\phi, & \theta = 0 \end{cases} \tag{3.6}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ F(\mu,\theta), & \theta = 0, \end{cases} \tag{3.7}$$

since $\frac{\mathrm{d}X_t}{\mathrm{d}\theta} = \frac{\mathrm{d}X_t}{\mathrm{d}t}$, then model (3.1) is equivalent to

$$\dot{X}_t = A(\mu)X_t + R(\mu)X_t, \tag{3.8}$$

where $X_t(\theta) = X(t+\theta)$.

For $\psi \in \mathbb{C}^1\left([-1,0],(\mathbb{R}^2)^*\right)$, defining

$$A^*(\mu)\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta^T(t,\mu)\psi(-t), & s = 0 \end{cases} \tag{3.9}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\cdot\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}^T(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi, \tag{3.10}$$

where η^T denotes the transpose of η, $\eta(\theta) = \eta(\theta,0)$. Here, for a and b in C^n, $a\cdot b$ means $\sum_{i=1}^{n} a_i b_i$, where a_i and b_i are the components of the vectors a and b, respectively. Then $A(0)$ and $A^*(0)$ are adjoint operators. Furthermore, $\langle\psi, A\phi\rangle = \langle A^*\psi, \phi\rangle$. Note that the above scaling transformation the corresponding characteristic exponents and the associated frequencies are transformed into $\tau_1\lambda$ and $\tau_1\omega$, respectively. Hence, when $\mu = 0$, $\pm \mathrm{i}\vartheta_{4_0}\tau_{4_0}$ are the eigenvalues of $A(0)$. Therefore, they are also eigenvalues of $A^*(0)$. Let $q(\theta)$ be the eigenvector for $A(0)$ corresponding to $\mathrm{i}\vartheta_{4_0}\tau_{4_0}$ and $q^*(\theta)$ be the eigenvector for $A^*(0)$ corresponding to $-\mathrm{i}\vartheta_{4_0}\tau_{4_0}$. Then we have

$$A(0)q(\theta) = \mathrm{i}\vartheta_{4_0}\tau_{4_0}q(\theta), \tag{3.11}$$

$$A^*(0)q^*(s) = -\mathrm{i}\vartheta_{4_0}\tau_{4_0}q^*(s). \tag{3.12}$$

From (3.6), we can rewrite (3.11) as follows

$$\begin{cases} \dfrac{\mathrm{d}q(\theta)}{\mathrm{d}\theta} = \mathrm{i}\vartheta_{4_0}\tau_{4_0}q(\theta), & \theta \in [-1, 0), \\ L_0 q(0) = \mathrm{i}\vartheta_{4_0}\tau_{4_0}q(0), & \theta = 0. \end{cases} \tag{3.13}$$

Using (3.13), we have

$$q(\theta) = Ve^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}, \quad \theta \in [-1, 0], \tag{3.14}$$

where $V = (v_1, v_2)^T$ is an undetermined constant vector, and from (3.13), the constant vector V must satisfy

$$\left(B + Ce^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} + De^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}} - \mathrm{i}\vartheta_{4_0}I\right)V = 0,$$

where I denotes the 2×2 identity matrix. The above algebraic equation has an infinite number of solutions. Without loss of generality, setting $v_1 = 1$, we have

$$v_2 = \frac{(1 - 2x^* - \mathrm{i}\vartheta_{4_0})(a + x^*)^2 - ay^*e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}}{x^*(a + x^*)}.$$

Similarly, from (3.9), we rewrite (3.12) as follows

$$\begin{cases} \dfrac{\mathrm{d}q^*(s)}{\mathrm{d}s} = \mathrm{i}\vartheta_{4_0}\tau_{4_0}q^*(s), & s \in (0, 1], \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta^T(t, 0)\varphi(-t) & \\ = \tau_{4_0}B^T\varphi(0) + \tau_{4_0}C^T\varphi\left(\dfrac{\tau_2^*}{\tau_{4_0}}\right) + \tau_{4_0}D^T\varphi(1) = -\mathrm{i}\vartheta_{4_0}\tau_{4_0}q(0), & s = 0. \end{cases} \tag{3.15}$$

Using (3.15), we have

$$q^*(s) = PV^*e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}s}, \quad s \in [0, 1], \tag{3.16}$$

where P and $V^* = (v_1^*, v_2^*)^T$ are a constant and constant vector, respectively. From (3.16), the constant vector V^* satisfies

$$\left(B^T + C^Te^{\mathrm{i}\vartheta_{4_0}\tau_2^*} + D^Te^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}} + \mathrm{i}\vartheta_{4_0}I\right)V^* = 0,$$

where I denotes the 2×2 identity matrix, setting $v_1^*=1$, we have

$$v_2^*=\frac{x^{*2}(1-2x^*+\mathrm{i}\vartheta_{4_0})(a+x^*)^2-ax^{*2}y^*e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}}{-\beta y^{*2}(a+x^*)^2}.$$

Next, we will find a constant $\overline{P}$ to make $\langle q^*(s),q(\theta)\rangle=1$. By (3.9), we get

$$\begin{aligned}
\langle q^*(s),q(\theta)\rangle &= \overline{q}^*(0)\cdot q(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{q}^{*T}(\xi-\theta)\mathrm{d}\eta(\xi)q(\xi)\mathrm{d}\xi\\
&= \overline{P}(\overline{v}_1^*,\overline{v}_2^*)(v_1,v_2)^T\\
&\quad -\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{P}(\overline{v}_1^*,\overline{v}_2^*)e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}(\xi-\theta)}\mathrm{d}\eta(\theta)(v_1,v_2)^Te^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\xi}\mathrm{d}\xi\\
&= \overline{P}(\overline{v}_1^*,\overline{v}_2^*)(v_1,v_2)^T-\overline{q}^{*T}(0)\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\mathrm{d}\xi\mathrm{d}\eta(\theta)q(0)\\
&= \overline{P}(\overline{v}_1^*,\overline{v}_2^*)(v_1,v_2)^T-\overline{q}^{*T}(0)\int_{\theta=-1}^{0}\xi e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\Big|_{\xi=0}^{\theta}\mathrm{d}\eta(\theta)q(0)\\
&= \overline{P}(\overline{v}_1^*,\overline{v}_2^*)(v_1,v_2)^T-\overline{q}^{*T}(0)\int_{\theta=-1}^{0}\theta e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\mathrm{d}\eta(\theta)q(0)\\
&= \overline{P}(v_1\overline{v}_1^*+v_2\overline{v}_2^*)+\overline{q}^{*T}(0)\tau_2^*\begin{pmatrix}0 & 0\\ 0 & -\frac{\beta y^*}{x^*}\end{pmatrix}e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}q(0)\\
&\quad +\overline{q}^{*T}(0)\tau_{4_0}\begin{pmatrix}-\frac{ay^*}{(a+x^*)^2} & 0\\ 0 & 0\end{pmatrix}e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}q(0)\\
&= \overline{P}\left[(v_1\overline{v}_1^*+v_2\overline{v}_2^*)-\frac{2\tau_2^*e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\overline{v}_2^*\beta y^*v_2}{x^*}-\frac{\tau_{4_0}e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\overline{v}_1^*ay^*v_1}{(a+x^*)^2}\right],
\end{aligned}$$

thus, we can choose $\overline{P}$ as

$$\begin{aligned}
\overline{P}&=\frac{1}{(v_1\overline{v}_1^*+v_2\overline{v}_2^*)-\frac{\tau_2^*e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\overline{v}_2^*\beta y^*v_2}{x^*}-\frac{\tau_{4_0}e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\overline{v}_1^*ay^*v_1}{(a+x^*)^2}},\\
P&=\frac{1}{(\overline{v}_1v_1^*+\overline{v}_2v_2^*)-\frac{\tau_2^*e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}v_2^*\beta y^*\overline{v}_2}{x^*}-\frac{\tau_{4_0}e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}v_1^*ay^*\overline{v}_1}{(a+x^*)^2}},
\end{aligned}\tag{3.17}$$

which assures that $\langle q^*(s),q(\theta)\rangle=1$. Furthermore, $\langle q^*(s),\bar{q}(\theta)\rangle=0$.

Using the same notations as in Hassard et al. [16], we first compute the coordinates to describe the center manifold C_μ at $\mu=0$, i.e., C_0, where

$$C_\mu=\left\{X_t\in\mathbb{R}^2|X_t=W(t,\theta;\mu)+2\mathrm{Re}\{z(t)q(\theta;\mu)\},|z(t)|<\delta\right\},$$

for some sufficiently small δ, and $W(t,\theta;\mu)$, $z(t)$ will be defined below. Let X_t be the solution of (3.1) when $\mu=0$. We can define

$$z(t)=\langle q^*,X_t\rangle,\quad W(t,\theta)=X_t(\theta)-z(t)q(\theta)-\overline{z}(t)\overline{q}(\theta)=X_t(\theta)-2\mathrm{Re}\{z(t)q(\theta)\}.\tag{3.18}$$

On the center manifold C_0 we have $W(t,\theta)=W(z(t),\overline{z}(t),\theta)$, where

$$W(z(t),\overline{z}(t),\theta)=W_{20}(\theta)\frac{z^2(t)}{2}+W_{11}(\theta)z(t)\overline{z}(t)+W_{02}(\theta)\frac{\overline{z}^2(t)}{2}+W_{30}\frac{z^3(t)}{6}+\cdots, \tag{3.19}$$

where z and $\overline{z}$ are local coordinates for the center manifold C_0 in the directions q^* and $\overline{q}^*$. Note that W is real if X_t is real, we shall deal with real solutions only. For the solution $X_t\in C_0$ of (3.1), since $\mu=0$, we have

$$\begin{aligned}\dot{z}(t)&=\langle q^*,\dot{X}_t\rangle=\langle q^*,A(0)X_t+R(0)X_t\rangle=\langle q^*,A(0)X_t\rangle+\langle q^*,R(0)X_t\rangle\\&=\langle A^*(0)q^*,X_t\rangle+\overline{q}^*(0)\cdot F(0,X_t)\\&=\mathrm{i}\vartheta_{4_0}\tau_{4_0}z(t)+\overline{q}^*(0)\cdot F\left(0,W(z(t),\overline{z}(t),0)+2\mathrm{Re}\{z(t)q(0)\}\right).\end{aligned}$$

Denoting $F\left(0,W(z(t),\overline{z}(t),0)+2\mathrm{Re}\{z(t)q(0)\}\right)$ by $F_0(z(t),\overline{z}(t))$, we have

$$\dot{z}(t)=\mathrm{i}\vartheta_{4_0}\tau_{4_0}z(t)+\overline{q}^*(0)\cdot F_0(z(t),\overline{z}(t)). \tag{3.20}$$

Furthermore, we rewrite (3.20) as

$$\dot{z}(t)=\mathrm{i}\vartheta_{4_0}\tau_{4_0}z(t)+g(z(t),\overline{z}(t)), \tag{3.21}$$

where $g(z(t),\overline{z}(t))=\overline{q}^*(0)\cdot F_0(z(t),\overline{z}(t))$, and expand $g(z(t),\overline{z}(t))$ in the following form

$$g\left(z(t),\overline{z}(t)\right)=g_{20}\frac{z^2(t)}{2}+g_{11}z(t)\overline{z}(t)+g_{02}\frac{\overline{z}^2(t)}{2}+g_{21}\frac{z^2(t)\overline{z}(t)}{2}+\cdots, \tag{3.22}$$

then it follows from (3.18) that

$$\begin{aligned}X_t(\theta)=W(t,\theta)+2\mathrm{Re}\{z(t)q(t)\}&=W_{20}(\theta)\frac{z^2(t)}{2}+W_{11}(\theta)z(t)\overline{z}(t)+W_{02}(\theta)\frac{\overline{z}^2(t)}{2}\\&+(v_1,v_2,v_3)^Te^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}z(t)+(\overline{v}_1,\overline{v}_2,\overline{v}_3)^Te^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\overline{z}(t)+\cdots.\end{aligned} \tag{3.23}$$

Since

$$q(\theta)=Ve^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}=(v_1,1)^Te^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}.$$

From (3.3) and (3.22),

$$\begin{aligned}&g(z(t),\bar{z}(t))=\bar{q}^*(0)F_0(z(t),\bar{z}(t))\\&=\bar{q}^*(0)F_0(0,X_t)=\bar{q}^*(0)\tau_{4_0}\begin{pmatrix}A\\B\end{pmatrix}\\&=\overline{P}\tau_{4_0}\left\{-\bar{v}_1^*\left[\left(W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}+z(t)v_1+\bar{z}(t)\bar{v}_1\right)^2\right.\right.\\&\quad+\frac{a}{(a+x^*)^2}\left(W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}+z(t)v_1e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\right.\\&\quad\left.+\bar{z}(t)\bar{v}_1e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\right)\left(W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+z(t)v_2+\bar{z}(t)\bar{v}_2\right)\\&\quad+\left(W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+z(t)v_2+\bar{z}(t)\bar{v}_2+y^*\right)\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i=2}^{\infty} \frac{1}{i!} f^{(i)}(x^*) \bigg(W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + z(t) v_1 e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \\
&+\bar{z}(t)\bar{v}_1 e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \bigg)^i \Bigg] + \bar{v}_2^* \Bigg[\bigg(\frac{\beta}{x^{*2}} \bigg(W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + z(t) v_1 \\
&+\bar{z}(t)\bar{v}_1 \bigg) - \frac{\beta}{x^*} \bigg) \bigg(W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + z(t) v_2 + \bar{z}(t)\bar{v}_2 \bigg) \\
&\times \bigg(W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{z^2}{2} + W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) z\bar{z} + W_{02}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{\bar{z}^2}{2} + z(t) v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} \\
&+\bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*} \bigg) + \frac{\beta}{x^{*2}} \bigg(W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + z(t) v_2 \\
&+\bar{z}(t)\bar{v}_2 \bigg) \bigg(W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + z(t) v_2 + \bar{z}(t)\bar{v}_2 \bigg) \\
&+ \bigg(W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{z^2}{2} + W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) z\bar{z} + W_{02}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{\bar{z}^2}{2} + z(t) v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} \\
&+\bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*} \bigg) - \beta \bigg(\bigg(W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + z(t) v_2 + \bar{z}(t)\bar{v}_2 \bigg) \\
&\times \bigg(W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{z^2}{2} + W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) z\bar{z} + W_{02}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{\bar{z}^2}{2} + z(t) v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} \\
&+\bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*} \bigg) + \bigg(W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + z(t) v_2 + \bar{z}(t)\bar{v}_2 \\
&+ W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{z^2}{2} + W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) z\bar{z} + W_{02}^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right)\frac{\bar{z}^2}{2} + z(t) v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} \\
&+\bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*} \bigg) + {y^*}^2 \bigg) \\
&\times \sum_{i=2}^{\infty} \frac{1}{i!} g^{(i)}(x^*) \bigg(W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + z(t) v_1 + \bar{z}(t)\bar{v}_1 \bigg)^i \Bigg] \Bigg\}, \qquad (3.24)
\end{aligned}
$$

where

$$
\begin{aligned}
A =& -\left(W^{(1)}(0) + z(t) v_1 + \bar{z}(t)\bar{v}_1 \right)^2 - \frac{a}{(a+x^*)^2} \Big(W^{(1)}(-1) + z(t) v_1 e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \\
&+\bar{z}(t)\bar{v}_1 e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \Big) \left(W^{(2)}(0) + z(t) v_2 + \bar{z}(t)\bar{v}_2 \right) - \Big(W^{(2)}(0) + z(t) v_2 + \bar{z}(t)\bar{v}_2 \\
&+ y^* \Big) \times \sum_{i=2}^{\infty} \frac{1}{i!} f^{(i)}(x^*) \left(W^{(1)}(-1) + z(t) v_1 e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}} + \bar{z}(t)\bar{v}_1 e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \right)^i, \\
B =& \left(\frac{\beta}{x^{*2}} (W^{(1)}(0) + z(t) v_1 + \bar{z}(t)\bar{v}_1) - \frac{\beta}{x^*} \right) \left(W^{(2)}(0) + z(t) v_2 + \bar{z}(t)\bar{v}_2 \right) \\
&\times \left(W^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) + z(t) v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} + \bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*} \right) + \frac{\beta}{x^{*2}} \Big(W^{(1)}(0) + z(t) v_1 \\
&+\bar{z}(t)\bar{v}_1 \Big) \bigg(\left(W^{(2)}(0) + z(t) v_2 + \bar{z}(t)\bar{v}_2 \right) + \bigg(W^{(2)}\left(-\frac{\tau_2^*}{\tau_{4_0}}\right) + z(t) v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}
\end{aligned}
$$

$$+\bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}\Big)\Big) - \beta\bigg[\Big(W^{(2)}(0)+z(t)v_2+\bar{z}(t)\bar{v}_2\Big)\bigg(W^{(2)}\Big(-\frac{\tau_2^*}{\tau_{4_0}}\Big)$$
$$+z(t)v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+\bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}\bigg)+y^*\bigg(W^{(2)}(0)+z(t)v_2+\bar{z}(t)\bar{v}_2$$
$$+W^{(2)}\Big(-\frac{\tau_2^*}{\tau_{4_0}}\Big)+z(t)v_2 e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+\bar{z}(t)\bar{v}_2 e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}\bigg)+y^{*2}\bigg]$$
$$\times\sum_{i=2}^{\infty}\frac{1}{i!}g^{(i)}(x^*)\Big(W^{(1)}(0)+z(t)v_1+\bar{z}(t)\bar{v}_1\Big)^i.$$

Comparing with the coefficients of (3.22), we can find

$$\begin{aligned}
g_{20}=&2\overline{P}\tau_{4_0}\bigg\{-\bar{v}_1^*\bigg[v_1^2+\frac{a}{(a+x^*)^2}v_1v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}+\frac{1}{2}f''(x^*)y^*v_1^2e^{-2\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\bigg]\\
&-\frac{\beta^2v_2^2}{x^{*3}}e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+\frac{\beta v_1v_2}{x^{*2}}\Big(e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+1\Big)-\frac{1}{2}g''(x^*)\beta y^{*2}{v_1}^2\bigg\},\\
g_{11}=&\overline{P}\tau_{4_0}\bigg\{-\bar{v}_1^*\bigg[2v_1\bar{v}_1+\frac{a}{(a+x^*)^2}\left(\bar{v}_1v_2e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}+v_1\bar{v}_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\right)\\
&+y^*f''(x^*)v_1\bar{v}_1\bigg]-\frac{\beta^2}{x^{*3}}\Big(v_2\bar{v}_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_2\bar{v}_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\Big)+\frac{\beta}{x^{*2}}\Big(v_1\Big(\bar{v}_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}\\
&+\bar{v}_2\Big)+\bar{v}_1\Big(v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_2\Big)\Big)-g''(x^*)\beta y^{*2}v_1\bar{v}_1\bigg\},\\
g_{02}=&2\overline{P}\tau_{4_0}\bigg\{-\bar{v}_1^*\bigg[\bar{v}_1^2+\frac{a}{(a+x^*)^2}\bar{v}_1\bar{v}_2e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}+\frac{1}{2}f''(x^*)y^*\bar{v}_1^2e^{2\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\bigg]\\
&-\frac{\beta^2\bar{v}_2^2}{x^{*3}}e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+\frac{\beta\bar{v}_1\bar{v}_2}{x^{*2}}\Big(e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+1\Big)-\frac{1}{2}g''(x^*)\beta y^{*2}\bar{v}_1^2\bigg\},\\
g_{21}=&2\overline{P}\tau_{4_0}\bigg\{-\bar{v}_1^*\bigg[2W_{11}^{(1)}(0)v_1+W_{20}^{(1)}(0)\bar{v}_1+\frac{a}{(a+x^*)^2}\ \Big(\frac{1}{2}W_{20}^{(2)}(0)\bar{v}_1e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\\
&+\frac{1}{2}W_{20}^{(1)}(-1)\bar{v}_2+W_{11}^{(1)}(-1)v_2+W_{11}^{(2)}(0)v_1e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\Big)\\
&+f''(x^*)\Big(\frac{1}{2}v_1^2\bar{v}_2e^{-2\mathrm{i}\vartheta_{4_0}\tau_{4_0}}+v_1\bar{v}_1v_2+\frac{1}{2}y^*W_{20}^{(1)}(-1)\bar{v}_1e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\\
&+y^*W_{11}^{(1)}(-1)v_1e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\Big)+\frac{1}{2}f'''(x^*)y^*v_1^2\bar{v}_1e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\bigg]+\frac{\beta}{x^{*2}}\Big(\bar{v}_1v_2^2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\\
&-\frac{\beta}{x^*}W_{11}^{(2)}(0)v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_1v_2\bar{v}_2\ \Big(e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}\Big)-\frac{\beta}{x^*}v_2W_{11}^{(2)}\Big(-\frac{\tau_2^*}{\tau_{4_0}}\Big)\\
&-\frac{\beta}{2x^*}W_{20}^{(2)}(0)\bar{v}_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}-\frac{\beta}{2x^*}\bar{v}_2W_{20}^{(2)}\Big(-\frac{\tau_2^*}{\tau_{4_0}}\Big)\Big)+\frac{\beta}{x^{*2}}\Big(\frac{1}{2}\bar{v}_1W_{20}^{(2)}(0)\\
&+\frac{1}{2}W_{20}^{(1)}(0)e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+W_{11}^{(1)}(0)v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\\
&+v_1W_{11}^{(2)}\Big(-\frac{\tau_2^*}{\tau_{4_0}}\Big)+\frac{1}{2}\bar{v}_1W_{20}^{(2)}\Big(-\frac{\tau_2^*}{\tau_{4_0}}\Big)+\frac{1}{2}W_{20}^{(1)}(0)\bar{v}_2+v_1W_{11}^{(2)}(0)\\
&+v_2W_{11}^{(1)}(0)\Big)-\beta y^*g''(x^*)\ \Big(v_1\bar{v}_1v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_1\bar{v}_1v_2+\frac{1}{2}v_1^2\bar{v}_2
\end{aligned}$$

$$+\frac{1}{2}v_1^2\bar{v}_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}\Big)-\beta y^{*2}g''(x^*)\ \left(\frac{1}{2}W_{20}^{(1)}(0)\bar{v}_1+W_{11}^{(1)}(0)v_1\right)$$
$$\left.-\frac{1}{2}g'''(x^*)\beta y^{*2}v_1^2\bar{v}_1\right\}.$$

Since there are unknown terms in g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (3.8), (3.18) and (3.21), we have

$$\begin{aligned}\dot{W}&=\dot{X}_t-\dot{z}q-\dot{\bar{z}}\bar{q}\\&=\begin{cases}AW-2\mathrm{Re}\{\bar{q}^*(0)\cdot F_0(z,\bar{z})q(\theta)\}, & \theta\in[-1,0)\\ AW-2\mathrm{Re}\{\bar{q}^*(0)\cdot F_0(z,\bar{z})q(0)\}+F_0(z,\bar{z}), & \theta=0\end{cases}\\&\triangleq AW+H(z,\bar{z},\theta),\end{aligned}\tag{3.25}$$

where

$$H(z,\bar{z},\theta)=H_{20}\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots.\tag{3.26}$$

From (3.25) and (3.26) and the definition of W, we need to use (3.19) and (3.21) to replace W_z and $\dot{z}$ and their conjugates. Comparing the coefficients (3.25), we obtain

$$(2\mathrm{i}\vartheta_{4_0}\tau_{4_0}I-A)\,W_{20}(\theta)=H_{20}(\theta),\quad -AW_{11}(\theta)=H_{11}(\theta),\tag{3.27}$$

where I denotes the 2×2 identity matrix.

From (3.25), we know that for $\theta\in[-1,0)$,

$$H(z,\bar{z},\theta)=-\bar{q}^*(0)\cdot f_0(z,\bar{z})q(\theta)-q^*(0)\cdot\overline{f}_0(z,\bar{z})\bar{q}(\theta)=-g(z,\bar{z})q(\theta)-\bar{g}(z,\bar{z})\bar{q}(\theta).\tag{3.28}$$

Substituting (3.22) into (3.28), we have

$$\begin{aligned}H(z,\bar{z},\theta)&=[-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta)]\,\frac{z^2}{2}+[-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta)]\,z\bar{z}\\&\quad+[-g_{02}q(\theta)-\bar{g}_{20}\bar{q}(\theta)]\,\frac{\bar{z}^2}{2}+\cdots.\end{aligned}\tag{3.29}$$

Comparing the coefficients in (3.29) in with those in (3.26), we can obtain

$$H_{20}(\theta)=-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta),\quad H_{11}(\theta)=-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta).\tag{3.30}$$

From (3.27), (3.30) and the definition of A, we have

$$\dot{W}_{20}(\theta)=2\mathrm{i}\vartheta_{4_0}\tau_{4_0}W_{20}(\theta)+g_{20}q(\theta)+\bar{g}_{02}\bar{q}(\theta).\tag{3.31}$$

Substituting $q(\theta)=(v_1,v_2)^Te^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}$ into (3.31), we can get the solution of (3.31)

$$W_{20}(\theta)=\frac{\mathrm{i}g_{20}}{\vartheta_{4_0}\tau_{4_0}}q(0)e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}+\frac{\mathrm{i}\bar{g}_{02}}{3\vartheta_{4_0}\tau_{4_0}}\bar{q}(0)e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}+G_1e^{2\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta},\tag{3.32}$$

similarly

$$W_{11}(\theta)=-\frac{\mathrm{i}g_{11}}{\vartheta_{4_0}\tau_{4_0}}q(0)e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}+\frac{\mathrm{i}\bar{g}_{11}}{\vartheta_{4_0}\tau_{4_0}}\bar{q}(0)e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}+G_2,\tag{3.33}$$

where $G_1 = \left(G_1^{(1)}, G_1^{(2)}\right)^T$, $G_2 = \left(G_2^{(1)}, G_2^{(2)}\right)^T$ and $G_1, G_2 \in \mathbb{R}^2$, they are constant vectors.

Next we will seek the values of constant vectors G_1 and G_2 in (3.32) and (3.33), respectively. From (3.27) and the definition of A, we have

$$A(0)W_{20}(\theta) = \int_{-1}^{0} \mathrm{d}\eta(\theta)W_{20}(\theta) = 2\mathrm{i}\vartheta_{4_0}\tau_{4_0}W_{20}(0) - H_{20}(0) \tag{3.34}$$

and

$$A(0)W_{11}(\theta) = \int_{-1}^{0} \mathrm{d}\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.35}$$

where $\eta(\theta) = \eta(\theta, 0)$. And from (3.25), we have

$$\begin{aligned} H_{20}(0) = {} & -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + \tau_{4_0} \\ & \times \begin{pmatrix} -v_1^2 - \dfrac{a}{(a+x^*)^2}v_1v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}} - \dfrac{1}{2}f''(x^*)y^*v_1^2e^{-2\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \\ -\dfrac{\beta^2v_2^2}{x^{*3}}e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} + \dfrac{\beta v_1v_2}{x^{*2}}\left(e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+1\right) - \dfrac{1}{2}g''(x^*)\beta {y^*}^2{v_1}^2 \end{pmatrix}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} H_{11}(0) = {} & -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + \tau_{4_0} \\ & \times \begin{pmatrix} -2v_1\bar{v}_1 - \dfrac{a}{(a+x^*)^2}\left(\bar{v}_1v_2e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}} + v_1\bar{v}_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\right) - y^*f''(x^*)v_1\bar{v}_1 \\ -\dfrac{\beta^2}{x^{*3}}\left(v_2\bar{v}_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*} + v_2\bar{v}_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\right) + \dfrac{\beta}{x^{*2}}\left(v_1\left(\bar{v}_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*} + \bar{v}_2\right) + \bar{v}_1 \right. \\ \left. \times \left(v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} + v_2\right)\right) - g''(x^*)\beta {y^*}^2 v_1\bar{v}_1 \end{pmatrix}. \end{aligned} \tag{3.37}$$

Substituting (3.32) and (3.36) into (3.34), we have

$$\begin{aligned} & \left(2\mathrm{i}\vartheta_{4_0}\tau_{4_0}I - \int_{-1}^{0} e^{2\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\mathrm{d}\eta(\theta)\right)G_1 \\ & = 2\tau_{4_0}\begin{pmatrix} -v_1^2 - \dfrac{a}{(a+x^*)^2}v_1v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}} - \dfrac{1}{2}f''(x^*)y^*v_1^2e^{-2\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \\ -\dfrac{\beta^2v_2^2}{x^{*3}}e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*} + \dfrac{\beta v_1v_2}{x^{*2}}\left(e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+1\right) - \dfrac{1}{2}g''(x^*)\beta {y^*}^2{v_1}^2 \end{pmatrix}. \end{aligned} \tag{3.38}$$

From the definition of A, we can obtain

$$\int_{-1}^{0} e^{2\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\mathrm{d}\eta(\theta) = A(\mu)e^{2\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta} = L_\mu\left(e^{2\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\right),$$

when $\mu = 0$,

$$\int_{-1}^{0} e^{2\mathrm{i}\vartheta_{4_0}\tau_{4_0}\theta}\mathrm{d}\eta(\theta) = \tau_{4_0}\left(B + Ce^{-2\mathrm{i}\vartheta_{4_0}\tau_2^*} + De^{-2\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\right).$$

Therefore, when $\mu = 0$, we have

$$\begin{pmatrix} 1 - 2x^* - \frac{ay^*e^{-2\mathrm{i}\omega_0\tau_{4_0}}}{(a+x^*)^2} & -\frac{x^*}{a+x^*} \\ \frac{\beta {y^*}^2}{{x^*}^2} & \delta - \frac{\beta y^*}{x^*} - \frac{\beta y^* e^{-2\mathrm{i}\omega_0\tau_2^*}}{x^*} \end{pmatrix} G_1$$

$$=2\begin{pmatrix} -v_1^2-\dfrac{a}{(a+x^*)^2}v_1v_2e^{-\mathrm{i}\omega_0\tau_{4_0}}-\dfrac{1}{2}f''(x^*)y^*v_1^2e^{-2\mathrm{i}\omega_0\tau_{4_0}} \\ -\dfrac{\beta^2v_2^2}{x^{*3}}e^{-\mathrm{i}\omega_0\tau_2^*}+\dfrac{\beta v_1v_2}{x^{*2}}\left(e^{-\mathrm{i}\omega_0\tau_2^*}+1\right)-\dfrac{1}{2}g''(x^*)\beta {y^*}^2{v_1}^2 \end{pmatrix}, \tag{3.39}$$

hence,

$$\begin{aligned} G_1=&2\begin{pmatrix} 1-2x^*-\frac{ay^*e^{-2\mathrm{i}\vartheta_{4_0}\tau_{4_0}}}{(a+x^*)^2} & -\frac{x^*}{a+x^*} \\ \frac{\beta {y^*}^2}{{x^*}^2} & \delta-\frac{\beta y^*}{x^*}-\frac{\beta y^*e^{-2\mathrm{i}\vartheta_{4_0}\tau_2^*}}{x^*} \end{pmatrix}^{-1} \\ &\times\begin{pmatrix} -v_1^2-\dfrac{a}{(a+x^*)^2}v_1v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}-\dfrac{1}{2}f''(x^*)y^*v_1^2e^{-2\mathrm{i}\vartheta_{4_0}\tau_{4_0}} \\ -\dfrac{\beta^2v_2^2}{x^{*3}}e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+\dfrac{\beta v_1v_2}{x^{*2}}\left(e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+1\right)-\dfrac{1}{2}g''(x^*)\beta {y^*}^2{v_1}^2 \end{pmatrix}. \end{aligned} \tag{3.40}$$

Similarly, we have

$$\begin{aligned} &\int_{-1}^{0}\mathrm{d}\eta(\theta)G_2 \\ =&-\begin{pmatrix} -2v_1\bar v_1-\dfrac{a}{(a+x^*)^2}\left(\bar v_1v_2e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}+v_1\bar v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\right)-y^*f''(x^*)v_1\bar v_1 \\ -\dfrac{\beta^2}{x^{*3}}\left(v_2\bar v_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_2\bar v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\right)+\dfrac{\beta}{x^{*2}}\left(v_1\left(\bar v_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+\bar v_2\right)+\bar v_1\right. \\ \left.\times\left(v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_2\right)\right)-g''(x^*)\beta {y^*}^2v_1\bar v_1 \end{pmatrix}, \end{aligned} \tag{3.41}$$

hence,

$$\begin{aligned} G_2=&\begin{pmatrix} 1-2x^*-\frac{ay^*}{(a+x^*)^2} & -\frac{x^*}{a+x^*} \\ \frac{\beta {y^*}^2}{{x^*}^2} & \delta-\frac{2\beta y^*}{x^*} \end{pmatrix}^{-1} \\ &\times\begin{pmatrix} -2v_1\bar v_1-\dfrac{a}{(a+x^*)^2}\left(\bar v_1v_2e^{\mathrm{i}\vartheta_{4_0}\tau_{4_0}}+v_1\bar v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_{4_0}}\right)-y^*f''(x^*)v_1\bar v_1 \\ -\dfrac{\beta^2}{x^{*3}}\left(v_2\bar v_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_2\bar v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}\right)+\dfrac{\beta}{x^{*2}}\left(v_1\left(\bar v_2e^{\mathrm{i}\vartheta_{4_0}\tau_2^*}+\bar v_2\right)+\bar v_1\right. \\ \left.\times\left(v_2e^{-\mathrm{i}\vartheta_{4_0}\tau_2^*}+v_2\right)\right)-g''(x^*)\beta {y^*}^2v_1\bar v_1 \end{pmatrix}. \end{aligned} \tag{3.42}$$

Thus, we can get the following quantities:

$$\begin{aligned} C_1(0)&=\frac{\mathrm{i}}{2\vartheta_{4_0}\tau_{4_0}}\left(g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right)+\frac{g_{21}}{2}, \\ \mu_2&=-\frac{\mathrm{Re}\,(C_1(0))}{\mathrm{Re}\,(\lambda'(\tau_{4_0}))}, \\ \beta_2&=2\mathrm{Re}\,(C_1(0)), \\ T_2&=\frac{\mathrm{Im}(C_1(0))+\mu_2\mathrm{Im}\,(\lambda'(\tau_{4_0}))}{\vartheta_{4_0}\tau_{4_0}}, \end{aligned}$$

which determine the properties of Hopf bifurcation at the critical value $\tau_1=\tau_{4_0}$ and $\tau_2\in[0,\tau_{2_0})$ on the center manifold. We have the following theorem.

Theorem 3.1. *For model (2.1), the following results hold:*

(i) The sign of μ_2 can determine the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the periodic solutions exist for $\tau_1 > \tau_{4_0}$ ($\tau_1 < \tau_{4_0}$);

(ii) The sign of β determines the stability of the bifurcating periodic solutions: if $\beta_2 > 0$ ($\beta_2 < 0$), the periodic solutions are stable (unstable).

(iii) The sign of T_2 determines the period of the bifurcating periodic solutions: if $T_2 > 0$ ($T_2 < 0$), the period of the periodic solutions increase (decrease).

4. Numerical simulations

In this section, we will present some numerical simulation results of model (2.1) for different parameter values to support the previous analytical results.

First, we consider model (2.1) with $a = 0.7$, $h = \frac{3}{25}$, $\delta = \frac{3}{5}$, $\beta = 1$. Then it has two positive equilibria $E_1(x^*, y^*) = (0.51219177, 0.30731719)$, $E_2(x^*, y^*) = (0.16741488, 0.10044918)$ and two boundary equilibria $E_3(x^*, y^*) = (0.13944460, 0)$, $E_4(x^*, y^*) = (0.86055556, 0)$(see Fig. 1(a)). By using a simple calculation, for E_1, we have $\kappa_1 + \kappa_2 + \kappa_3 = 0.77079226 > 0$, $\kappa_4 + \kappa_5 + \kappa_6 + \kappa_7 = 0.25458599 > 0$ and $p_1^2 - 4p_2 = -0.07912947 < 0$, hence, according to Theorem 2.1(i), the equilibrium E_1 is asymptotically stable when $\tau_1 > 0$(see Fig. 2(a)-(b)). The other three equilibria do not satisfy the condition ($\mathbf{H_1}$). In fact, when $\tau_1 = \tau_2 = 0$, E_2 and E_4 are two saddle points and E_3 is a nodal source, hence, they are unstable.

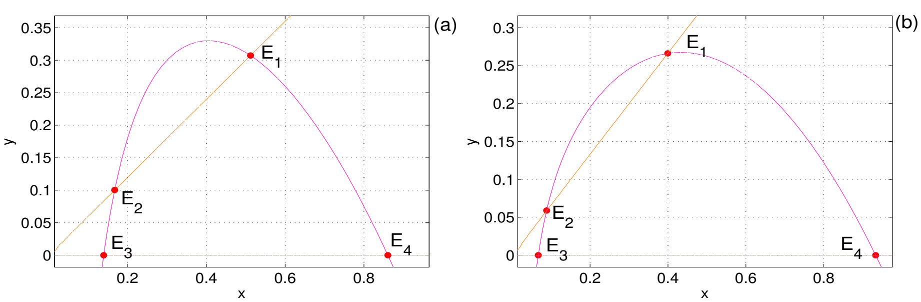

Figure 1. The equilibria of model (2.1). The magenta and yellow lines represent prey and predator nullclines, respectively. The points where the two nullclines cross are equilibria and there are four of these points for (a) $a = 0.7$, $h = \frac{3}{25}$, $\delta = \frac{3}{5}$, $\beta = 1$; (b) $a = 0.2$, $h = \frac{1}{16}$, $\delta = \frac{1}{5}$, $\beta = 0.3$.

Next, we set the following parameters $a = 0.2$, $h = \frac{1}{16}$, $\delta = \frac{1}{5}$, $\beta = 0.3$. Model (2.1) also has two positive equilibria $E_1(x^*, y^*) = (0.39929087, 0.26619585)$, $E_2(x^*, y^*) = (0.08835544, 0.05890365)$ and two boundary equilibria $E_3(x^*, y^*) = (0.06698700, 0)$, $E_4(x^*, y^*) = (0.93301270, 0)$(see Fig. 1(b)). For E_1, $\kappa_1 + \kappa_2 + \kappa_3 = 0.14682143 > 0$, $\kappa_4 + \kappa_5 + \kappa_6 + \kappa_7 = 0.07820114$, $\Delta_1 = 0.00826602 > 0$, $p_1 = -0.11907885 < 0$, $p_2 = 0.00147844 > 0$. Equation (2.13) has two positive roots $v_{1_1} = 0.10499826$ and $v_{1_2} = 0.01408058$. Hence, equation (2.12) has two positive roots $\vartheta_{1_1} = 0.32403435$ and $\vartheta_{1_2} = 0.11866163$. Furthermore, we have $\tau_{1_1}^{(0)} = 3.11596840$, $\tau_{1_2}^{(0)} = 21.92151453$ and $\tau_{1_0} = 3.11596840$, $2\vartheta_{1_1}^2 + p_1 = 0.090901768 > 0$, $2\vartheta_{1_2}^2 + p_1 = -0.09091768 < 0$. Therefore, according to Theorem 2.1(ii) and (iii),

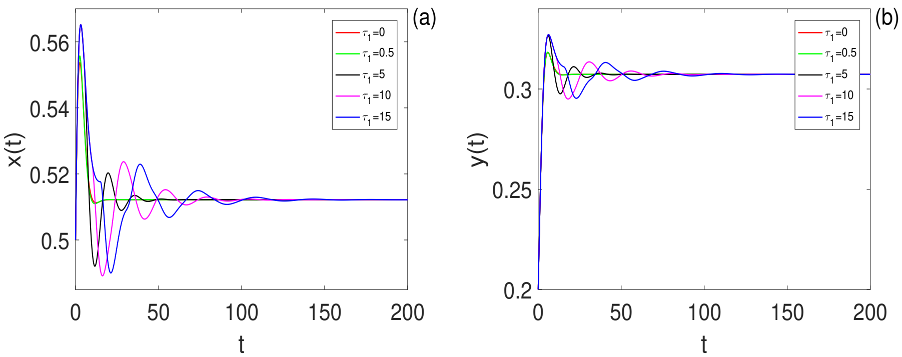

Figure 2. Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_2 = 0$ and $\tau_1 = 0, 0.5, 5, 10, 15$, respectively. The positive equilibrium $E_1(0.51219177, 0.30731719)$ is locally asymptotically stable. Here the initial value is $(0.5, 0.2)$.

the equilibrium E_1 is asymptotically stable when $\tau_1 \in [0, \tau_{1_0})$ (see Fig. 3(a)-(b)) and is unstable when $\tau_1 > \tau_{1_0}$(see Fig. 3(c)-(d)). Model (2.1) undergoes a Hopf bifurcation around E_1 when $\tau_1 > \tau_{1_0}$(see Fig. 3(c)-(d)).

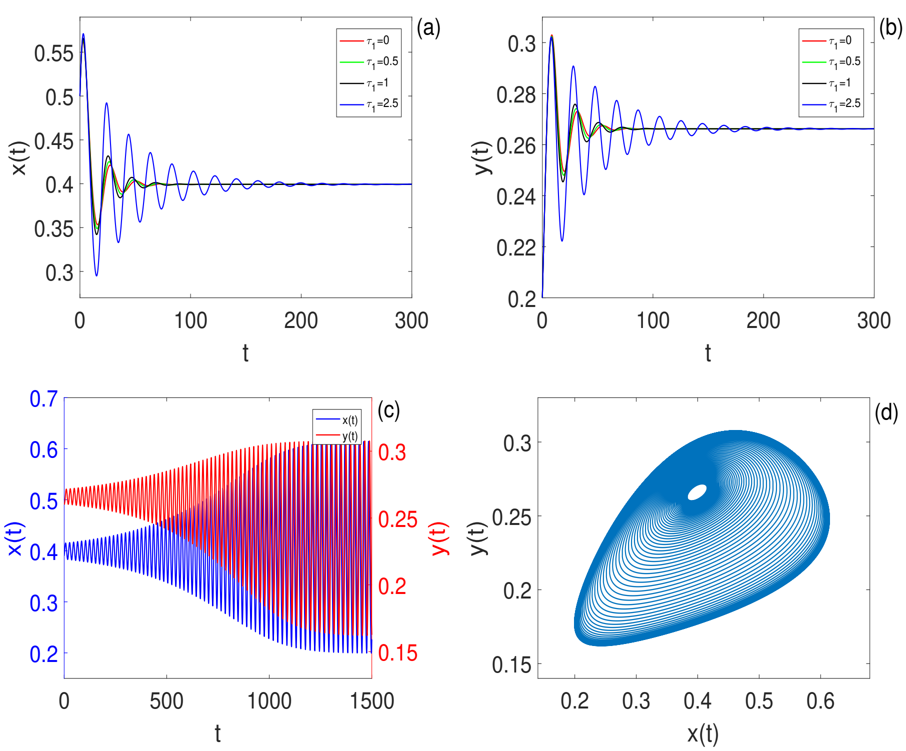

Figure 3. (a)-(b): Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_2 = 0$ and $\tau_1 = 0, 0.5, 1, 2.5 < \tau_{1_0} = 3.11596840$, respectively. The positive equilibrium $E_1(0.39929087, 0.26619585)$ is locally asymptotically. The initial value is $(0.5, 0.2)$. (c)-(d): Time series of $x(t)$, $y(t)$ and phase portrait of model (2.1) with $\tau_2 = 0$ and $\tau_1 = 3.2 > \tau_{1_0} = 3.11596840$. The positive equilibrium E_1 is unstable and the orbit from the initial value $(0.39, 0.26)$ located in a sufficiently small neighborhood of E_1 converges to a periodic solution.

For Theorem 2.2(i), we consider the following parameters $a = 0.1$, $h = \frac{1}{15}$, $\delta = \frac{3}{80}$, $\beta = 0.1$. There are two positive equilibria $E_1(x^*, y^*) = (0.56316894, 0.21118835)$, $E_2(x^*, y^*) = (0.09137792, 0.03426718)$ and two boundary equilibria $E_3(x^*, y^*) = (0.07182558, 0)$, $E_4(x^*, y^*) = (0.92817445, 0)$. For E_1, we can easily find that E_1 is asymptotically stable when $\tau_2 > 0$(see Fig. 4(a)-(b)). Again, we can find that E_2 and E_4 are two saddle points and E_3 is a nodal source, they are unstable.

For Theorem 2.2(ii) and (iii), we consider model (2.1) with parameters $a = 0.5$, $h = \frac{1}{80}$, $\delta = \frac{3}{5}$, $\beta = \frac{1}{2}$. There are two positive equilibria $E_1(x^*, y^*) = (0.42147248, 0.50576827)$, $E_2(x^*, y^*) = (0.01307041, 0.01568494)$ and two boundary equilibria $E_3(x^*, y^*) = (0.01266000, 0)$, $E_4(x^*, y^*) = (0.98734000, 0)$. By a simple calculation, only E_1 satisfies the conditions of the Theorem 2.2(ii) and $\tau_{2_0} = 1.76144387$. Therefore, E_1 is asymptotically stable when $\tau_2 \in [0, \tau_{2_0})$(see Fig. 5(a)-(b)) and is unstable when $\tau_2 > \tau_{2_0}$(Fig. 5(c)-(d)). Model (2.1) undergoes a Hopf bifurcation around E_1 when $\tau_2 > \tau_{2_0}$(Fig. 5(c)-(d)).

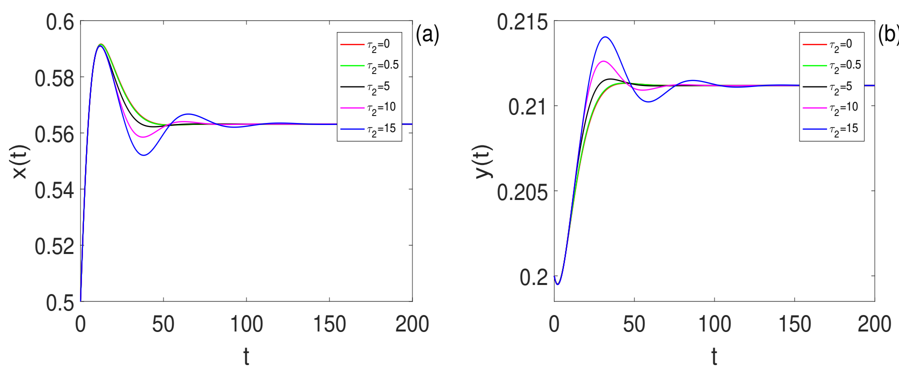

Figure 4. Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_1 = 0$ and $\tau_2 = 0, 0.5, 5, 10, 15$, respectively. The positive equilibrium $E_1(0.56316894, 0.21118835)$ is locally asymptotically stable. Here initial value is $(0.5, 0.2)$.

Now we will consider Theorem 2.3(i). Choosing the parameters as follows: $a = 0.5$, $h = \frac{3}{25}$, $\delta = \frac{1}{10}$, $\beta = \frac{3}{10}$, model (2.1) has two positive equilibria $E_1(x^*, y^*) = (0.62234655, 0.20744885)$, $E_2(x^*, y^*) = (0.15728432, 0.05242922)$ and two boundary equilibria $E_3(x^*, y^*) = (0.13944453, 0)$, $E_4(x^*, y^*) = (0.86055512, 0)$. For E_1, we have $\kappa_1 + \kappa_2 + \kappa_3 = 0.42703618 > 0$, $\kappa_4 + \kappa_5 + \kappa_6 + \kappa_7 = 0.05118711 > 0$ and (2.25) has no positive roots. Hence, E_1 is asymptotically stable when $\tau_1 = \tau_2 = \tau \geq 0$(see Fig. 6(a)-(b)).

For Theorem 2.3(ii), we consider the following parameters $a = 0.7$, $h = \frac{3}{80}$, $\delta = \frac{3}{5}$, $\beta = 1$. Model (2.1) has two positive equilibria $E_1(x^*, y^*) = (0.652995, 0.3917969)$, $E_2(x^*, y^*) = (0.04046421, 0.02428095)$ and two boundary equilibria $E_3(x^*, y^*) = (0.03902278, 0)$, $E_4(x^*, y^*) = (0.96097723, 0)$. Only E_1 satisfies the conditions of Theorem 2.3(ii) and $\tau_{3_0} = 2.12332287$. Hence, E_1 is asymptotically stable when $\tau_1 = \tau_2 = \tau \in [0, 2.12332287)$ (see Fig. 7(a)-(b)) and is unstable when $\tau_1 = \tau_2 = \tau > 2.12332287$(see Fig. 7(c)-(d)). Model (2.1) undergoes a Hopf bifurcation around E_1 when $\tau_1 = \tau_2 = \tau > 2.12332287$(see Fig. 7(c)-(d)).

For Theorem 2.4(i), we consider the following parameters: $a = 0.4$, $h = \frac{4}{30}$, $\delta = 0.2$, $\beta = 0.7$. Model (2.1) has two positive equilibria $E_1(x^*, y^*) = (0.60849, 0.17385)$, $E_2(x^*, y^*) = (0.18348630, 0.05242466)$ and two boundary equilibria $E_3(x^*, y^*) = (0.15843497, 0)$, $E_4(x^*, y^*) = (0.84156503, 0)$. For E_1, we have $\kappa_1 + \kappa_2 + \kappa_3 =$

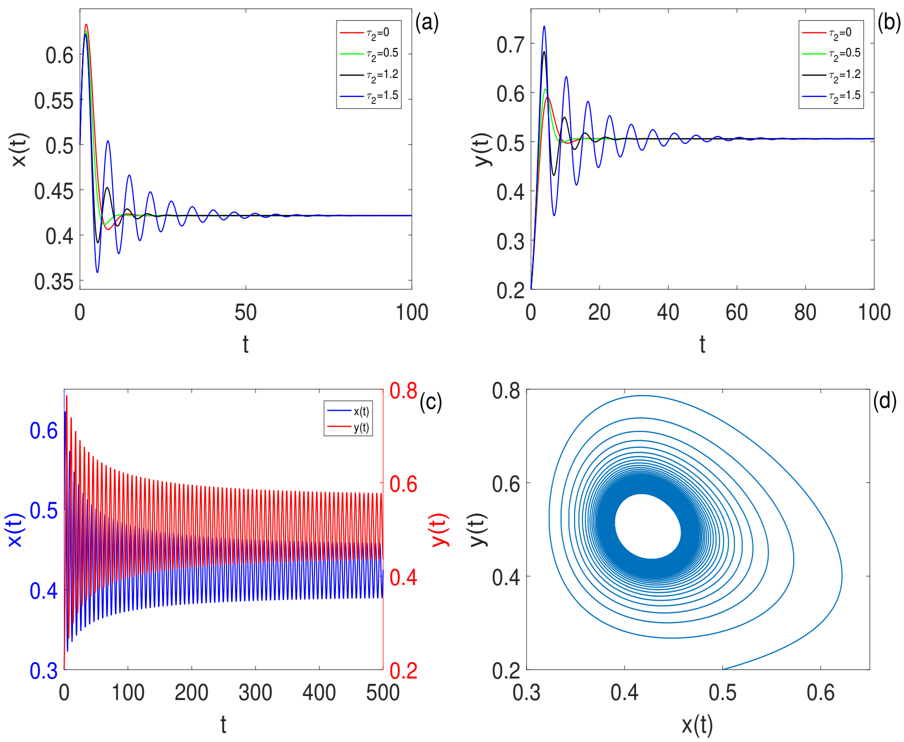

Figure 5. (a)-(b): Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_1 = 0$ and $\tau_2 = 0, 0.5, 1.2, 1.5 < \tau_{2_0} = 1.76144387$, respectively. The positive equilibrium $E_1(0.42147248, 0.50576827)$ is locally asymptotically. The initial value is $(0.5, 0.2)$. (c)-(d): Time series of $x(t)$, $y(t)$ and phase portrait of model (2.1) with $\tau_1 = 0$ and $\tau_2 = 1.77 > \tau_{2_0} = 1.76144387$. The positive equilibrium E_1 is unstable and the orbit from the initial value $(0.5, 0.2)$ located in a sufficiently small neighborhood of E_1 converges to a periodic solution.

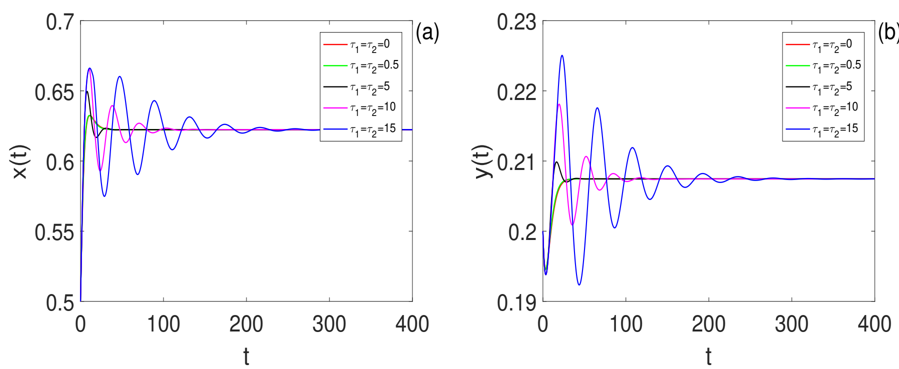

Figure 6. Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_1 = \tau_2 = 0, 0.5, 5, 10, 15$, respectively. The positive equilibrium $E_1(0.62234655, 0.20744885)$ is locally asymptotically stable. Here initial value is $(0.5, 0.2)$.

$0.48534963 > 0$, $\kappa_4 + \kappa_5 + \kappa_6 + \kappa_7 = 0.09154801 > 0$. If we choose $\tau_2 = 1 \in [0, \tau_{2_0})$, then (2.28) has no positive roots. According to Theorem 2.4(i), E_1 is asymptotically stable when $\tau_1 \geq 0$(see Fig. 8(a)-(b)).

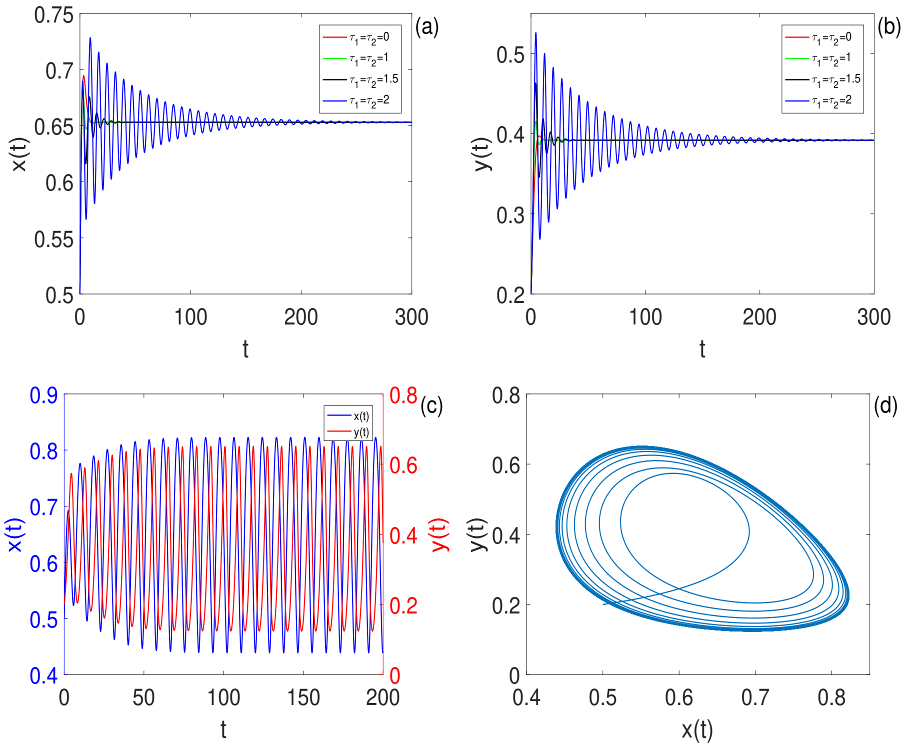

Figure 7. (a)-(b): Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_1 = \tau_2 = 0, 1, 1.5, 2 < \tau_{3_0} = 2.12332287$, respectively. The positive equilibrium $E_1(0.65299477, 0.39179686)$ is locally asymptotically. The initial value is $(0.5, 0.2)$. (c)-(d): Time series of $x(t)$, $y(t)$ and phase portrait of model (2.1) with $\tau_1 = \tau_2 = 2.35 > \tau_{3_0} = 2.12332287$. The positive equilibrium E_1 is unstable and the orbit from the initial value $(0.5, 0.2)$ located in a sufficiently small neighborhood of E_1 converges to a periodic solution.

For Theorem 2.4(ii), we consider the following parameters: $a = 0.5$, $h = \frac{1}{80}$, $\delta = \frac{3}{5}$, $\beta = \frac{1}{2}$. Model (2.1) has two positive equilibria $E_1(x^*, y^*) = (0.421472, 0.505768)$, $E_2(x^*, y^*) = (0.01307000, 0.015684000)$ and two boundary equilibria $E_3(x^*, y^*) = (0.01266000, 0)$, $E_4(x^*, y^*) = (0.98734000, 0)$. Only E_1 satisfies the conditions of Theorem 2.4(ii) and $\tau_{4_0} = 1.08245692$. According to Theorem 2.4(ii), choosing $\tau_2 = 1.5 \in [0, \tau_{2_0})$, E_1 is asymptotically stable when $\tau_1 \in [0, \tau_{4_0})$(see Fig. 9(a)-(b)) and is unstable when $\tau_1 \geq \tau_{4_0} = 1.08245692$(see Fig. 9(c)-(d)). Model (2.1) undergoes a Hopf bifurcation around E_1 when $\tau_1 \geq \tau_{4_0} = 1.08245692$(see Fig. 9(c)-(d)).

When $\tau_{4_0} = 1.08245692$, $\tau_2 = 1.5 \in [0, \tau_{2_0})$, we can obtain $C_1(0) = -5.97596954 - 8.25043868i$, $\mu_2 = 63.8610682$, $\beta_2 = -11.95193908$, $T_2 = 9.46870954$. Hence, according to the Theorem 3.1, we know that model (2.1) can undergo a supercritical Hopf bifurcation around equilibrium E_1. $\beta_2 < 0$ implies that the bifurcating periodic solution is asymptotically stable on the center manifold. $T_2 > 0$ means that the period of bifurcating periodic solutions are increasing with the increase of τ_1. The effect of τ_1 on the period T_2 with $\tau_2 = 1.5$ is shown in Fig. 10(a). We can find that the period of periodic solutions are increasing as the delay τ_1 increases. Some phase portraits of model (2.1) with $\tau_1 = 1.15, 1.3, 1.5, 1.8, 2 > \tau_{4_0} = 1.08245692$ and $\tau_2 = 1.5 \in [0, \tau_{2_0})$ are given in Fig. 10(b). To erase the transient behaviour, we just

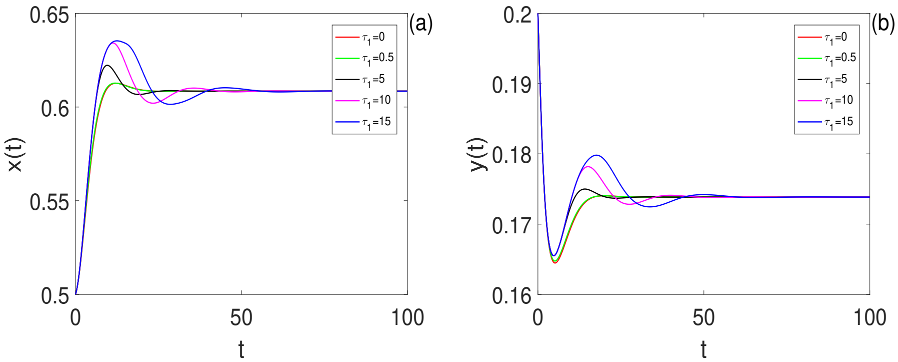

Figure 8. Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_2 = 1$ and $\tau_1 = 0, 0.5, 5, 10, 15$, respectively. The positive equilibrium $E_1(0.60848685, 0.17385342)$ is locally asymptotically stable. Here initial value is $(0.5, 0.2)$.

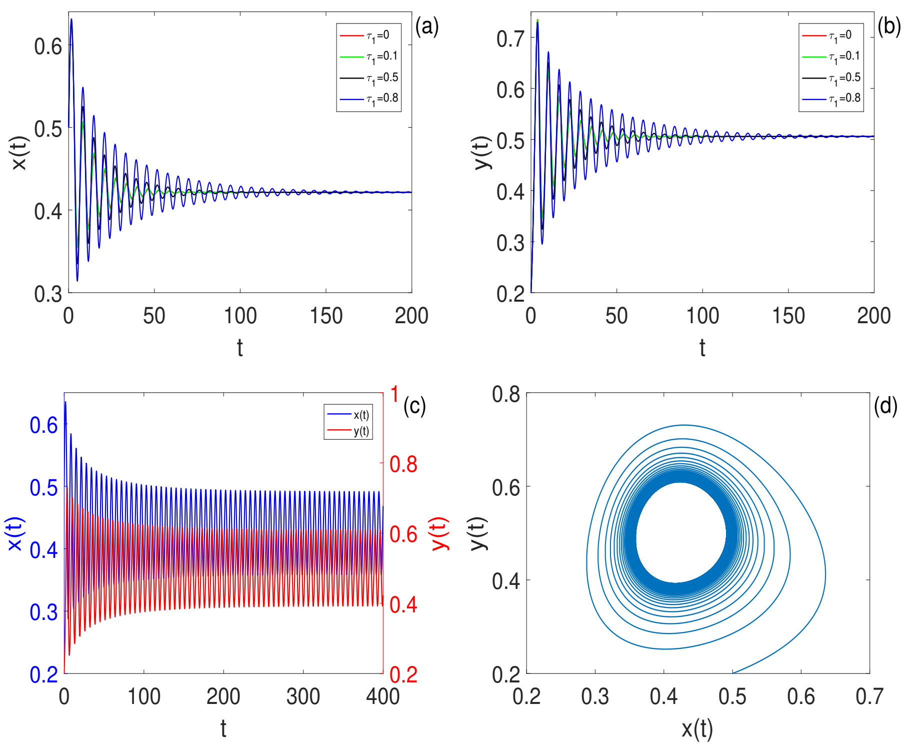

Figure 9. (a)-(b): Time series of $x(t)$, $y(t)$ of model (2.1) with $\tau_2 = 1.5$ and $\tau_1 = 0, 0.1, 0.5, 0.8 < \tau_{4_0} = 1.08245692$, respectively. The positive equilibrium $E_1(0.42147248, 0.50576827)$ is locally asymptotically. The initial value is $(0.5, 0.2)$. (c)-(d): Time series of $x(t)$, $y(t)$ and phase portrait of model (2.1) with $\tau_2 = 1.5$ and $\tau_1 = 1.15 > \tau_{4_0} = 1.08245692$. The positive equilibrium E_1 is unstable and the orbit from the initial value $(0.5, 0.2)$ located in a sufficiently small neighborhood of E_1 converges to a periodic solution.

keep the last 1000 points for each τ_1.

In the following, we will continue to use the numerical simulations to investigate the influence of constant-yield prey harvesting h on the system dynamics. In fact,

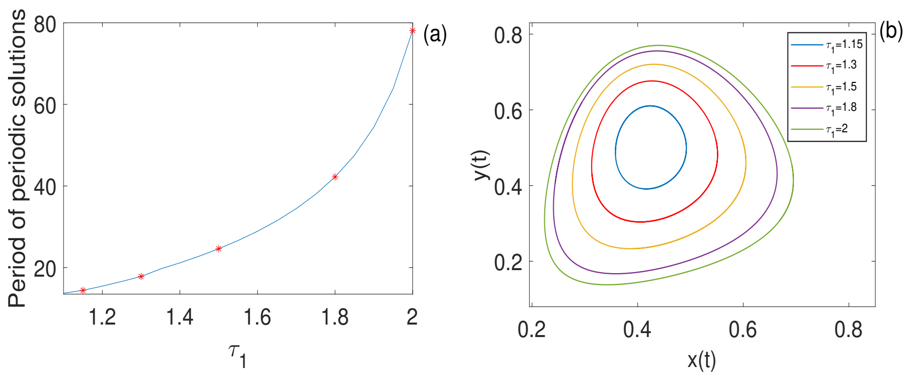

Figure 10. (a): The effect of τ_1 on the period T_2 with $\tau_2 = 1.5 \in [0, \tau_{2_0})$. The stars stand for $\tau_1 = 1.15, 1.3, 1.5, 1.8, 2 > \tau_{4_0} = 1.08245692$, respectively. (b): Phase portraits without showing the transient of model (2.1) with $\tau_2 = 1.5 \in [0, \tau_{2_0})$ and $\tau_1 = 1.15, 1.3, 1.5, 1.8, 2 > \tau_{4_0} = 1.08245692$, respectively. The parameters are the same as Fig. 9.

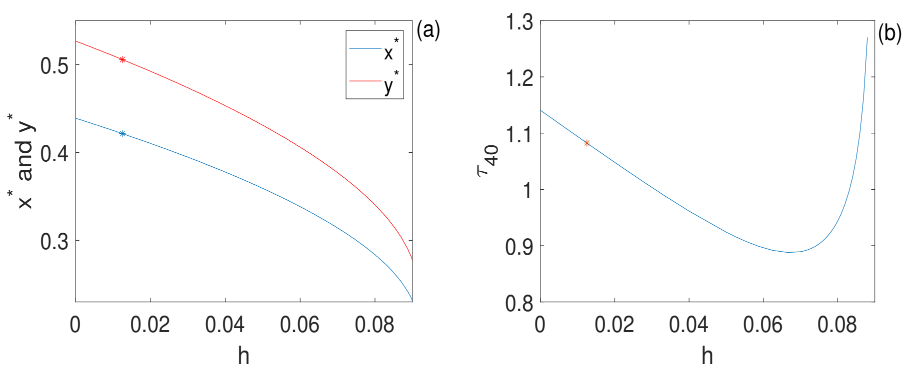

Figure 11. (a): The relationship between the location of the positive equilibrium $E_1(x^*, y^*)$ and h. The blue and red stars stand for $(\frac{1}{80}, 0.42147248)$ and $(\frac{1}{80}, 0.50576827)$, respectively. (b): The relationship between the critical value τ_{4_0} and h. The red star is $(\frac{1}{80}, 1.08245692)$.

the delays do not affect the number of equilibria for the model (2.1). The influence of the harvesting term h on the number of the equilibrium states has been studied in detail by the authors in the reference [25]. An obvious question is how does dynamics change as h is changed with time delays? Fig. 11(a) shows how the location of the positive equilibrium $E_1(x^*, y^*)$ changes when the harvesting term h is varied. We can find that the x^* and y^* of the equilibrium E_1 are monotonically decreasing, that is, the equilibrium E_1 will move to the lower left as h increases. Moreover, the equilibrium E_1 exists when h is a free parameter in the range $0 \le h \le 0.091449$ with $a = 0.5$, $\delta = \frac{3}{5}$ and $\beta = \frac{1}{2}$ fixed. Actually, this is also the range of harvesting h in which the positive equilibria exist. The blue and red stars in Fig. 11(a) stand for $(\frac{1}{80}, 0.42147248)$ and $(\frac{1}{80}, 0.50576827)$, respectively. That is when $h = \frac{1}{80}$, $E_1(x^*, y^*) = (0.42147248, 0.50576827)$ which is used to verify the Theorem 2.4(ii).

The relationship between the critical value τ_{4_0} and h is shown in Fig. 11(b). When $0 \le h \le h_c$, where $h_c \approx 0.066667$, the critical value τ_{4_0} is monotonically decreasing, whereas τ_{4_0} is monotonically increasing if $h_c \le h \le 0.091449$. That is, the critical value τ_{4_0} initially decreases as h increases and increases afterwards. The equilibrium $E_1(x^*, y^*)$ of model (2.1) is asymptotically stable where the parameters h and τ_{4_0} are below the curve in the (h, τ_{4_0})-plane and model (2.1) will under go a Hopf bifurcation at the equilibrium $E_1(x^*, y^*)$ where h and τ_{4_0} are above the curve. Therefore, the size of the critical value τ_{4_0} can be changed by adjusting the size of harvesting term h, and vice versa. The star in Fig. 11(b) is $(\frac{1}{80}, 1.08245692)$.

5. Conclusions

The biological resources are mostly harvested for achieving economic interest. However, the unreasonable exploitation of many natural and biological resources is a serious problem at present. Compared with the traditional ordinary differential equations without time delay, the delay differential equation can describe the laws of evolution of natural and objective things more accurately. In this paper, we mainly consider the effects of two time delays on a predator-prey model of Holling and Leslie type with constant-yield prey harvesting. By analyzed the corresponding characteristic equation of the model, we derive the stability of positive equilibria and the existence conditions of Hopf bifurcation. On the basis of the normal form theory and central manifold theorem, some explicit formulas are given for determining the direction of the Hopf bifurcation and the stability of bifurcated periodic solutions. And some numerical simulation results are carried out for illustrating these analyses.

In [25], the authors have discussed the effects of the harvesting term h and the conditions of Hopf bifurcation of model (2.1) in the absence of time delays. It is obvious for model (2.1) that the constant-yield harvesting h could lead some dangers in real-life harvesting such as there is no positive equilibria and either prey or predator will go to extinction for some values of the constant-yield harvesting h with some other parameters fixed(see Fig.11(a)). The period oscillations discussed by the authors of [25] arise from the Hopf bifurcation caused by the system parameters β and h. Actually, periodic oscillations are often observed in such laboratory experiments and attributed to time delayed responses [5]. In the present work, we give the periodic solutions caused by time delays. Based on the ecological meaning of these results, the presence of two time delays can make a perturbation of dynamics of predator and prey. Biologically, the bifurcation parameters τ_1 and τ_2 in the model (2.1) plays a major role of the measure of the biological maturation time of prey and the gestation time of predator, respectively. If the maturation time of prey τ_1 or(and) the gestation time of predator τ_2 is small enough, i.e., less than their corresponding critical values, the stability of the positive equilibrium of model (2.1) is unaffected. As the time delays τ_1 or(and) τ_2 increases, the stable positive equilibrium changed at the critical value $\tau_{k_0}(k = 1, 2, 3, 4)$ to a stable or unstable period solution. According to the relationship between the critical value τ_{4_0} and h, it could provide some suggestions and data supports for the government's policies, such as the period of fishing ban, the size of the harvesting. An understanding of the effects of different time delays in predation system can help us to explain population dynamics under different situations and monitor the sensitivity and recovery of predator populations [1].

As mentioned earlier, periodic oscillation behavior is a ubiquitous natural phenomenon with a wide range of applications, including neurological behavior, circadian rhythms, chemical reactions, and cell physiology [46]. It was observed in many naturally occurring nonconservative systems. The growth of some species does not agree with the increase described by logistic equation, such as fruit flies, flour beetles, and other organisms that have complex life cycles involving eggs, larvae, pupae, and adults. In these organisms, the predicted asymptotic approach to a steady carrying capacity was never observed–instead the populations exhibited large, persistent fluctuations after an initial period of logistic growth [43]. The possible causes of these fluctuations include age structure and time-delayed effects of overcrowding in the population. Some oscillations are intrinsic to the biological system and are not caused by external environmental changes while others might result from predator-prey interactions in field populations. For example, the Canada lynx eats snow-shoe hares and both species show dramatic cyclic oscillations in density with peaks every 9 to 10 years. This lynx-hare cycle has been interpreted as an example of an intrinsic predator-prey oscillation, but more recent experimental studies have suggested that both food shortage and predation are involved in generating cycles. Lynx depend on snow-shoe hares as primary prey, and are thus food-limited, whereas hares are affected by both food limitations and predators. The time delay inherent in the numerical response of lynx to hare numbers induces the density cycle of hares [28]. Therefore, the periodic oscillation can not only show the evolutionary changes of relationship between the predator and prey, but also act as a means of self-regulation of ecosystem.

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