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GLOBAL WEAK SOLUTION TO COMPRESSIBLE NAVIER-STIKES-LANDAU-LIFSHITZ-MAXWELL EQUATIONS FOR QUANTUM FLUIDS IN DIMENSION THREE

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Abstract This paper is concerned with viscous quantum Navier-Stokes-Landau-Lifshitz-Maxwell equations in dimension three. We use Faedo-Galerkin method to prove the local existence of weak solution, then combine the a priori estimates to obtain the global existence of solution.

Keywords Viscous quantum Navier-Stokes-Landau-Lifshitz-Maxwell system, weak solution, global solution, Faedo-Galerkin method.

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1. Introduction and main results

In this paper, we study the viscous quantum Navier-Stokes-Landau-Lifshitz-Maxwell system on $\Omega \times (0, T)$

$$\partial_t \rho + \mathrm{div}(\rho u) = \nu_1 \triangle \rho, \tag{1.1}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \frac{\nabla P}{m} = -\frac{\rho e}{m}(E + u \times H) + \frac{\mu \hbar^2}{2m^2}\rho \nabla\Big(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\Big) + \nu_2 \triangle(\rho u)$$

$$-\frac{\rho u}{\tau} - \lambda \nabla \cdot \Big(\nabla d \odot \nabla d - \frac{|\nabla d|^2}{2} I\Big), \tag{1.2}$$

$$d_t + u \cdot \nabla d + \alpha_1 d \times (d \times (\triangle d + H)) = \alpha_2 d \times (\triangle d + H), \tag{1.3}$$

$$E_t - \nabla \times H = e\rho u, \tag{1.4}$$

$$H_t + \nabla \times E = -\lambda m (d_t + u \cdot \nabla d), \tag{1.5}$$

$$\nabla \cdot H = 0, \tag{1.6}$$

$$|d(x, t)| = 1 \tag{1.7}$$

with initial data

$$\rho|_{t=0} = \rho_0(x),\ u|_{t=0} = u_0(x),\ E|_{t=0} = E_0(x),\ d|_{t=0} = d_0(x),\ H|_{t=0} = H_0(x) \tag{1.8}$$

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which satisfy that

$$\begin{aligned} &\rho_0(x) > 0, \quad x \in \Omega, \\ &|d_0(x)| = 1, \ d_0(x) \in H^2(\Omega), \ \inf_x d_0^2 > 0, \\ &E_0(x), H_0(x) \in L^2(\Omega). \end{aligned} \tag{1.9}$$

And the domain Ω we consider belongs to $\mathbb{R}^3$ and is $2D$-periodic. We also assume that $\rho_0(x), u_0(x), d_0(x), E_0(x), H_0(x)$ are $2D$-periodic, $D > 0$ is a constant. Here we consider isentropic case $P = \rho^\gamma (\gamma > 3)$ and P denotes the pressure. The unknown ρ represents the mass density, $u(x,t) : \Omega \times (0,T)$ represents the velocity field of the flow. E and H represent the electric field and the magnetic field respectively. $d(x,t) : \Omega \times (0,T) \to S^2$ is a unit vector that represents the macroscopic molecular orientation of the liquid crystal material. The physic constants $m, e, \hbar$ are positive and represent the mass, the charge of the particle and Planck constant respectively. ν_1, ν_2 and μ are positive viscosity constants. λ represents the competition between kinetic energy and potential energy, τ denotes the relaxation time of electron. $\alpha_1 \geqslant 0$ is Gilbert damping coefficient and α_2 is a positive constant. $u \otimes u$ is the matrix with components $u_i u_j$, $\nabla d \odot \nabla d$ denotes the 3×3 matrix with components $\nabla_i d \cdot \nabla_j d$ for $1 \leqslant i, j \leqslant 3$, " $\times$ " denotes the vector outer product.

Notice that if $d = E = H = 0$, system (1.1)-(1.6) is called quantum hydrodynamic model(QHD). Jüngle [13] obtained the existence of global-in-time solutions to the multidimensional equations (1.1)-(1.2) with a strictly positive particle density. Quantum hydrodynamic models are used to describe superfluids [14], quantum semiconductors Loffredo etc [2] and so on. We can also refer to [5,6] for more details. There are many studies for the QHD system, one can see [4, 15].

If the system $\mu = \nu = 0$, d is a constant vector, it becomes the Navier-Stokes-Landau-Lifshitz-Maxwell(NSLLM) system, we can see [7,8] and their references for more details about the Landau-Lifshitz equations.

The existence of global-in-time solutions to the two-dimensional equations (1.1)-(1.7) has been shown in Guo etc [10]. To our knowledge there are no results for the three dimensional situation. We will give such a result in this paper. Inspired by Jüngel [13] and Guo etc [10], the key is to deal with the maganetization field in the momentum equation.

Theorem 1.1 (Global existence)**.** *For any $T > 0$, $P(\rho) = A\rho^\gamma (\gamma > 3)$. Under the condition of (1.8) and that $E(\rho_0, u_0, d_0, E_0, H_0)$ is finite, where $E(\rho, u, d, E, H)$ will be defined in (4.2). There exists a weak solution (ρ, u, d, E, H) to (1.1)-(1.7) with the regularity*

$$\sqrt{\rho} \in L^\infty([0,T]; H^1(\Omega)) \cap L^2([0,T]; H^2(\Omega)), \quad \rho \geqslant 0, \tag{1.10}$$

$$\rho \in H^1([0,T]; L^2(\Omega)) \cap L^\infty([0,T]; L^\gamma(\Omega)) \cap L^2([0,T]; W^{1,3}(\Omega)), \tag{1.11}$$

$$\sqrt{\rho} u \in L^\infty([0,T]; L^2(\Omega)), \quad \rho u \in L^2([0,T]; W^{1,\frac{3}{2}}(\Omega)), \tag{1.12}$$

$$\sqrt{\rho} \nabla u \in L^2([0,T]; L^2(\Omega)), \tag{1.13}$$

$$E \in L^\infty([0,T]; L^2(\Omega)), \quad H \in L^\infty([0,T]; L^2(\Omega)), \tag{1.14}$$

$$d \in L^2([0,T]; H^2(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)), \tag{1.15}$$

$$\nabla d \in L^4([0,T]; L^4(\Omega)), \tag{1.16}$$

satisfying (1.1) pointwise and for all smooth functions satisfying $\phi(\cdot,T)=0$,

$$\begin{aligned}
&-\int_\Omega \rho_0^2 u_0\phi(\cdot,0)dx\\
=&\int_0^T\int_\Omega \big(\rho^2 u\cdot\phi_t-\rho^2 div(u)u\cdot\phi-\nu_2(\rho u\otimes\nabla\rho):\nabla\phi\\
&+\rho u\otimes\rho u:\nabla\phi+\frac{\gamma}{\gamma+1}\rho^{\gamma+1}div\phi+\frac{\rho e}{m}(E+u\times H)\cdot\rho\phi\\
&-\frac{\mu\hbar^2}{2m^2}\triangle\sqrt{\rho}(2\sqrt{\rho}\nabla\rho\cdot\phi+\rho^{\frac{3}{2}}div\phi)-\nu_2\nabla(\rho u):(\rho\nabla\phi+2\nabla\rho\otimes\phi)\\
&+\lambda(\nabla d\odot\nabla d-\frac{|\nabla d|^2}{2}I)\cdot\nabla(\rho\phi)\big)dxdt,
\end{aligned}\tag{1.17}$$

$$\begin{aligned}
-\int_\Omega d_0\rho\phi(\cdot,0)dx=\int_0^T\int_\Omega&\big(d\rho\phi_t+\rho u\cdot\nabla d\cdot\phi+\alpha_1 d\times(d\times(\triangle d+H))\cdot\rho\phi\\
&-\alpha_2 d\times(\triangle d+H)\cdot\rho\phi\big)dxdt,
\end{aligned}\tag{1.18}$$

$$-\int_\Omega E_0\phi(\cdot,0)dx=\int_0^T\int_\Omega\big(E\phi_t-H\cdot(\nabla\times\phi)-e\rho u\cdot\phi\big)dxdt,\tag{1.19}$$

$$\begin{aligned}
-\int_\Omega(H_0+\lambda m d_0)\phi(\cdot,0)dx=\int_0^T\int_\Omega&\big((H+\lambda md)\cdot\phi_t\\
&+E\cdot(\nabla\times\phi)+\lambda m(u\cdot\nabla d)\cdot\phi\big)dxdt.
\end{aligned}\tag{1.20}$$

Similar to Jüngel [13], to deal with the lack of compactness, we need to get the estimates of u. We first add the right hand side of (1.2) a viscosity term $\delta\triangle u-\delta u$:

$$\partial_t\rho+\mathrm{div}(\rho u)=\nu_1\triangle\rho,\tag{1.21}$$

$$\begin{aligned}
&\partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)+\frac{\nabla P}{m}=-\frac{\rho e}{m}(E+u\times H)+\frac{\mu\hbar^2}{2m^2}\rho\nabla(\frac{\triangle\rho}{\sqrt{\rho}})+\nu_2\triangle(\rho u)\\
&-\frac{\rho u}{\tau}-\lambda\nabla\cdot(\nabla d\odot\nabla d-\frac{|\nabla d|^2}{2}I)+\delta\triangle u-\delta u,
\end{aligned}\tag{1.22}$$

$$d_t+u\cdot\nabla d+\alpha_1 d\times(d\times(\triangle d+H))=\alpha_2 d\times(\triangle d+H),\tag{1.23}$$

$$E_t-\nabla\times H=e\rho u,\tag{1.24}$$

$$H_t+\nabla\times E=-\lambda m(d_t+u\cdot\nabla d),\tag{1.25}$$

$$\nabla\cdot H=0,\tag{1.26}$$

$$|d(x,t)|=1.\tag{1.27}$$

Then we will let $\delta\to 0$. Finally, we obtain the desired weak solution to the original system (1.1)-(1.7).

This paper is organised as following. In section 2, we denote some preliminaries for this paper. Then we show the local existence solution to (1.1)-(1.7) in section 3. In section 4, we prove the global existence solution to (1.21)-(1.27). After some a priori estimates in section 5, we obtain the solution to (1.1)-(1.7) letting $n\to 0$ and $\delta\to 0$ respectively.

2. Preliminaries

C is a constant and may assume different values in different formulates.

The product $A : B$ means summation over both indices of matrices A and B.

$L^p([0,T], L^q(\Omega))$ is the space whose element is the p-integrable respect to time variable and q-integrable respect to space variable function.

Denote $H^m(\Omega), m = 1, 2, \cdots$ being the Sobolev space of complex-valued functions with the norm

$$\|u\|_{H^m} = \Big(\int_\Omega \sum_{|\alpha|\leqslant m} |D^\alpha u|^2 dx\Big)^{\frac{1}{2}},$$

$(H^m)^*$ is the dual space of H^m.

" $\hookrightarrow\hookrightarrow$ " denotes compact imbedding, " $\hookrightarrow$ " denotes continuous imbedding.

Lemma 2.1 (The Gagliardo-Nirenberg inequality, [16]). *Assume that* $u \in L^q(\Omega)$, $D^m u \in L^r(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, $1 \leqslant q, r \leqslant \infty$, $0 \leqslant j \leqslant m$. *Let* p *and* α *satisfy*

$$\frac{1}{p} = \frac{j}{n} + \alpha\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-\alpha)\frac{1}{q}; \quad \frac{j}{m} \leqslant \alpha \leqslant 1.$$

Then

$$\|D^j u\|_p \leqslant C(p, m, j, q, r)\|D^m u\|_r^\alpha \|u\|_q^{1-\alpha} \tag{2.1}$$

where $C(p, m, j, q, r)$ *is a positive constant.*

Lemma 2.2 (The Gronwall's inequality, [1]). *Let* c *be a constsnt, and* $b(t), u(t)$ *be nonnegative continuous functions in the interval* $[0, T]$ *satisfying*

$$u(t) \leqslant c + \int_0^t b(\tau)u(\tau)d\tau, \quad t \in [0, T].$$

Then $u(t)$ *satisfies the estimate*

$$u(t) \leqslant c\ exp\Big(\int_0^t b(\tau)d\tau\Big), \quad t \in [0, T]. \tag{2.2}$$

Lemma 2.3 (Aubin-Lions Lemma, [17]). *Assume* $X \hookrightarrow\hookrightarrow E \hookrightarrow Y$ *are Banach spaces. Then the following imbeddings are compact, if* $1 < q \leqslant \infty$ *and* $1 \leq p < q$

$$L^q(0, T; E) \cap L^1(0, T; X) \cap \{\varphi : \frac{\partial \varphi}{\partial t} \in L^1(0, T; Y)\} \hookrightarrow\hookrightarrow L^p(0, T; E). \tag{2.3}$$

3. Local Existence of Solution

In this section we will show the local existence of solution to the viscosity system (1.21)-(1.27) by Faedo-Galerkin method. Let $T > 0$, and w_j be an orthonormal basis of $L^2(\Omega)$ which is also an orthogonal basis of $H^1(\Omega)$, with periodicity $w_n(x - De_i) = w_n(x + De_i)(i = 1, 2, 3)$. Consider the space $X_n = \text{span}\{w_1, \cdots, w_n\}, n \in \mathbb{N}$.

Denote the approximate solution of the problem (1.21)-(1.27) as following form

$$u_m^\delta(x,t) = \sum_{s=1}^m \alpha_{sm}(t)w_s(x), \qquad d_m^\delta(x,t) = \sum_{s=1}^m \beta_{sm}(t)w_s(x),$$

$$E_m^\delta(x,t) = \sum_{s=1}^m \gamma_{sm}(t)w_s(x), \qquad H_m^\delta(x,t) = \sum_{s=1}^m \zeta_{sm}(t)w_s(x),$$

where $\alpha_{sm}(t), \beta_{sm}(t), \gamma_{sm}(t), \zeta_{sm}(t)(t \in \mathbb{R}^+)(s = 1, 2, \cdots, m; m = 1, 2, \cdots)$ are 3-dimensional vector valued functions.

We introduce the operator $S_1 : C^0([0,T]; X_n) \to C^0([0,T]; C^3(\Omega))$ by $S_1(u) = \rho$. Since the equation for ρ is linear, S_1 is Lipshitz continuous:

$$\|S_1(u_1) - S_1(u_2)\|_{C^0([0,T];C^k(\Omega))} \leqslant C(n,k)\|u_1 - u_2\|_{C^0([0,T];L^2(\Omega))}. \tag{3.1}$$

Next we wish to solve (1.21)-(1.27) on the space X_n. For $S_1(u) = \rho$, we are looking for functions $(u_n^\delta, d_n^\delta, E_n^\delta, H_n^\delta) \in (C^0([0,T]; X_n)^4$ such that

$$\begin{aligned}
-\int_\Omega \rho_0 u_0 \phi(\cdot,0)dx = \int_0^T \int_\Omega \big(&\rho u_n^\delta \cdot \phi_t + \rho(u \otimes u_n^\delta) : \nabla\phi + \frac{P(\rho)\mathrm{div}(\phi)}{m} \\
&- \frac{\rho e}{m}(E + u_n^\delta \times H)\cdot\phi - \frac{\mu\hbar^2}{2m^2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\mathrm{div}(\rho\phi) - \nu_2\nabla(\rho u_n^\delta) : \nabla\phi \\
&- \frac{\rho u_n^\delta}{\tau}\phi + \lambda(\nabla d \odot \nabla d - \frac{|\nabla d|^2}{2}I)\cdot\nabla\phi \\
&- \delta(\nabla u_n^\delta : \nabla\phi + u_n^\delta \cdot \phi)\big)dxdt,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
-\int_\Omega d_0\phi(\cdot,0)dx = \int_0^T \int_\Omega \big(&d_n^\delta\phi_t + u_n^\delta \cdot \nabla d_n^\delta \cdot \phi + \alpha_1 d_n^\delta \times (d_n^\delta \times (\triangle d_n^\delta + H_n^\delta))\cdot\phi \\
&- \alpha_2 d_n^\delta \times (\triangle d_n^\delta + H_n^\delta)\cdot\rho\phi\big)dxdt,
\end{aligned} \tag{3.3}$$

$$-\int_\Omega E_0\phi(\cdot,0)dx = \int_0^T \int_\Omega \big(E_n^\delta\phi_t - H_n^\delta\cdot(\nabla\times\phi) - e\rho u_n^\delta\cdot\phi\big)dxdt, \tag{3.4}$$

$$\begin{aligned}
-\int_\Omega (H_0 + \lambda m d_0)\phi(\cdot,0)dx = \int_0^T \int_\Omega \big(&(H_n^\delta + \lambda m d_n^\delta)\cdot\phi_t + E_n^\delta\cdot(\nabla\times\phi) \\
&+ \lambda_m(u_n^\delta\cdot\nabla d_n^\delta)\cdot\phi\big)dxdt
\end{aligned} \tag{3.5}$$

for all $\phi \in (C^1([0,T]; X_n)$ such that $\phi(\cdot,T) = 0$. We will apply Banach fixed point theorem to prove the local-in-time existence of solution, so we add the regularization term $\delta(\triangle u_n^\delta - u_n^\delta)$. The regularization yields the H^1 regularity of u_n^δ needed to conclude the global existence of solution.

For some functions $\alpha_{sm}(t)$, and the norm of u in $C^0([0,T]; X_n)$ can be formulated as

$$\|u\|_{C^0([0,T];X_n)} = \max_{t\in[0,T]} \sum_{m=1}^n \sum_{s=1}^m |\alpha_{sm}(t)|.$$

Then u belongs to $C^0([0,T]; C^k(\Omega))$ for any $k \in \mathbb{N}$, and there exists a constant C depending on k such that

$$\|u\|_{C^0([0,T];C^k(\Omega))} \leqslant C\|u\|_{C^0([0,T];L^2(\Omega))}. \tag{3.6}$$

The approximate system is defined as follows. Let $\rho \in C^1([0,T]; C^3(\Omega))$ be the classical solution to

$$\partial_t\rho + \mathrm{div}(\rho u) = \nu\triangle\rho, \quad \rho|_{t=0} = \rho_0(x). \tag{3.7}$$

The maximum principle provides the lower and upper bounds (Jiang etc [12])

$$\inf_{x\in\Omega}\rho_0(x)\exp(-\int_0^t \|\mathrm{div}u\|_{L^\infty(\Omega)}ds) \leqslant \rho(x) \leqslant \sup_{x\in\Omega}\rho_0(x)\exp(\int_0^t \|\mathrm{div}u\|_{L^\infty(\Omega)}ds).$$

Since $\rho_0(x) > \bar{\rho} > 0$, $\rho(x)$ is strictly positive. In view of (3.6), for $\|u\|_{C^0([0,T];L^2(\Omega))} \leqslant C$, there exist constants $\rho_1(C)$ and $\rho_2(C)$ such that

$$0 < \rho_1(C) \leqslant \rho(x,t) \leqslant \rho_2(C).$$

To solve (3.2)-(3.5), we follow Feireisl [1] and introduce the following family of operators, given a function $\varrho \in L^1(\Omega)$ with $\varrho \geqslant \bar{\varrho} > 0$:

$$M[\varrho] : X_n \to X_n^*, \quad < M[\varrho]u, w >= \int_\Omega \varrho u \cdot w, \quad u, w \in X_n.$$

These operators are symmetric and positive definite with the smallest eigenvalue

$$\int_{\|u\|_{L^2(\Omega)}=1} < M[\varrho]u, u >= \int_{\|u\|_{L^2(\Omega)}=1} \varrho |u|^2 dx \geqslant \inf_{x\in\Omega} \varrho(x) \geqslant \bar{\varrho}.$$

Hence since X_n is finite-dimensional, the operators are invertible with

$$\|M^{-1}[\varrho]\|_{L(X_n^*, X_n)} \leqslant \rho^{-1},$$

where $L(X_n^*, X_n)$ is the set of bounded linear mappings from X_n^* to X_n. Moreover (see Feireisl [1]), M^{-1} is Lipschitz continuous in the sense

$$\|M^{-1}[\varrho_1] - M^{-1}[\varrho_2]\|_{L(X_n^*, X_n)} \leqslant C(n, \varrho)\|\varrho_1 - \varrho_2\|_{L^1(\Omega)} \tag{3.8}$$

for all ϱ_1, $\varrho_2 \in L^1(\Omega)$ such that ϱ_1, $\varrho_2 \geqslant \bar{\rho} > 0$.

Now the integral equation (3.2) can be rephrased as an ordinary differential equation on the finite-dimensional space X_n,

$$\frac{d}{dt}(M[\rho(t)]u_n^\delta(t)) = N[u, d, H, E, u_n^\delta(t)], \quad M[\rho_0]u_n^\delta(0) = M[\rho_0]u_0, \tag{3.9}$$

when $\rho = S_1(u)$

$$\begin{aligned}
&< N(u, d, E, H, u_n^\delta), \phi > \\
= &\int_0^T \int_\Omega \Big(\rho(u \otimes u_n^\delta) : \nabla\phi + \frac{P(\rho)\mathrm{div}(\phi)}{m} \\
&- \frac{\rho e}{m}(E + u_n^\delta \times H) \cdot \phi - \frac{\mu\hbar^2}{2m^2} \frac{\triangle\sqrt{\rho}}{\sqrt{\rho}} \mathrm{div}(\rho\phi) - \nu_2 \nabla(\rho u_n^\delta) : \nabla\phi \\
&- \frac{\rho u_n^\delta}{\tau}\phi + \lambda(\nabla d \odot \nabla d - \frac{|\nabla d|^2}{2} I) \cdot \nabla\phi - \delta(\nabla u_n^\delta : \nabla\phi + u_n^\delta \cdot \phi)\Big) dxdt.
\end{aligned}$$

For operator $N(u, d, E, H, \cdot) : X_n \to X_n^*$ is continuous in time. Standard theory for systems of ordinary differential equations then provides the existence of a unique classical solution to (3.9), that is, for a given $u \in C^0([0,T]; X_n)$, there exists a unique solution $u_n \in C^1([0,T]; X_n)$ to (3.2).

Integrating (3.9) over $(0,t)$ yields the following nonlinear equation:

$$u_n^\delta(t) = M^{-1} S_1(u_n^\delta)(t)(M[\rho_0]u_0 + \int_0^t N[u_n^\delta, u_n^\delta(s)]ds). \tag{3.10}$$

Since the operators S_1 and M are Lipschitz type, (3.10) can be solved by evoking the fixed point theorem of Banach on a short time interval $[0, T']$, where $T' \leqslant T$, in the space $C^0([0,T]; X_n)$. In fact, we have even $u_n^\delta \in C^1([0,T']; X_n)$. Then we can solve system (3.3)-(3.5). Thus, there exists a unique local-in-time solution $(\rho_n^\delta, u_n^\delta, d_n^\delta, E_n^\delta, H_n^\delta)$ to (1.21)-(1.27).

4. Global Existence solution to (1.21)-(1.27)

In this section, we will give the a-priori estimates. Using these estimates, we can show that the local-in-time solution $(\rho_n^\delta, u_n^\delta, d_n^\delta, E_n^\delta, H_n^\delta)$ which are proved in Section 3 can be extended globally. For simplicity, we omit the subscript n and superscript δ in this section.

Theorem 4.1. *Assume the conditions of Theorem 1.1 to be hold. Then we have the following energy equality:*

$$\begin{aligned}&\frac{d}{dt}E(\rho,u,d,E,H)+\int_\Omega\big(\frac{\nu_1}{m}\tilde{H}''(\rho)|\nabla\rho|^2+\frac{\mu\nu_1\hbar^2}{4m^2}\rho|\nabla^2 log\rho|^2+\nu_2\rho|\nabla u|^2\\&\frac{1}{\tau}\rho|u|^2+\lambda\alpha_1|d\times(\triangle d+H)|^2+\delta|\nabla u|^2+\delta|u|^2\big)dx=0,\end{aligned}\tag{4.1}$$

where

$$E(\rho,u,d,E,H)=\int_\Omega\big(\tilde{H}(\rho)+\frac{\mu\hbar^2}{2m^2}|\nabla\sqrt{\rho}|^2+\frac{1}{2}\rho|u|^2+\frac{\lambda}{2}|\nabla d|^2+\frac{1}{2m}|E|^2+\frac{1}{2m}|H|^2\big)dx,\tag{4.2}$$

here, $\tilde{H}(\rho)=\frac{A\rho^\gamma}{\gamma-1}$ *for* $\gamma>3$.

Proof. Multiplying (1.21) by $\frac{1}{m}\tilde{H}'(\rho)-\frac{|u|^2}{2}-\frac{\mu\hbar^2}{2m^2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}$, and integrating by parts in Ω, we have

$$\begin{aligned}&\frac{1}{m}\frac{d}{dt}\int_\Omega\tilde{H}(\rho)dx-\int_\Omega\rho_t\frac{|u|^2}{2}dx+\frac{\mu\hbar^2}{2m^2}\frac{d}{dt}\int_\Omega|\nabla\sqrt{\rho}|^2dx\\&+\frac{1}{m}\int_\Omega\mathrm{div}(\rho u)\tilde{H}'(\rho)dx-\int_\Omega\mathrm{div}(\rho u)\frac{|u|^2}{2}dx-\int_\Omega\mathrm{div}(\rho u)\frac{\mu\hbar^2}{2m^2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}dx\\=&-\frac{\nu_1}{m}\int_\Omega\tilde{H}''(\rho)|\nabla\rho|^2dx+\nu_1\int_\Omega\nabla\rho:\nabla u:udx\\&-\int_\Omega\big(\frac{\mu\nu_1\hbar^2}{4m^2}\rho|\nabla^2\log\rho|^2+\delta|\nabla u|^2+\delta|u|^2\big)dx.\end{aligned}\tag{4.3}$$

Here we have used

$$\begin{aligned}&\int_\Omega\partial_t\rho\tilde{H}'(\rho)dx=\frac{d}{dt}\int_\Omega\tilde{H}(\rho)dx,\\&-\int_\Omega\partial_t\rho\frac{\mu\hbar^2}{2m^2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}dx=-\frac{\mu\hbar^2}{m^2}\int_\Omega\partial_t\sqrt{\rho}\triangle\sqrt{\rho}dx=\frac{\mu\hbar^2}{2m^2}\partial_t\int_\Omega|\nabla\sqrt{\rho}|^2dx,\\&\int_\Omega\nu_1\triangle\rho\frac{1}{m}\tilde{H}'(\rho)dx=-\frac{\nu_1}{m}\int_\Omega\nabla\rho\nabla\tilde{H}'(\rho)dx=-\frac{\nu_1}{m}\int_\Omega\tilde{H}''(\rho)|\nabla\rho|^2dx,\\&-\int_\Omega\nu_1\triangle\rho\frac{|u|^2}{2}dx=\nu_1\int_\Omega\nabla\rho:\nabla u:udx,\\&\int_\Omega\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\triangle\rho dx=-\int_\Omega\rho\nabla\log\rho\nabla\big(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\big)dx=\frac{1}{2}\int_\Omega\rho|\nabla^2\log\rho|^2dx.\end{aligned}$$

Then multiplying (1.22) by u, and integrating both sides of it by parts respectively in Ω, we have

$$\begin{aligned}
&\int_\Omega [\partial_t\rho|u|^2+\rho\partial_t(\frac{|u|^2}{2})+\frac{1}{2}\nabla\cdot(\rho u)|u|^2]dx-\frac{1}{m}\int_\Omega \tilde{H}'(\rho)\mathrm{div}(\rho u)dx\\
=&-\int_\Omega \frac{\rho e}{m}E\cdot udx-\int_\Omega \frac{\mu\hbar^2}{2m^2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\mathrm{div}(\rho u)dx-\int_\Omega \nu_2\nabla\rho:\nabla u:udx\\
&-\int_\Omega \nu_2\rho|\nabla u|^2dx-\int_\Omega \frac{1}{\tau}\rho|u|^2dx+\lambda\int_\Omega \nabla d\odot\nabla d:\nabla udx-\int_\Omega \frac{|\nabla d|^2}{2}\nabla udx.
\end{aligned} \tag{4.4}$$

Here, we have used

$$\begin{aligned}
&\int_\Omega (\partial_t(\rho u)\cdot u+\mathrm{div}(\rho u\otimes u)\cdot u)dx\\
=&\int_\Omega [\partial_t\rho|u|^2+\rho\partial_t(\frac{|u|^2}{2})+\nabla\cdot(\rho u)|u|^2+\rho u\nabla(\frac{|u|^2}{2})]dx\\
=&\int_\Omega [\partial_t\rho|u|^2+\rho\partial_t(\frac{|u|^2}{2})+\frac{1}{2}\nabla\cdot(\rho u)|u|^2]dx,\\
&\int_\Omega \frac{\nabla P}{m}\cdot udx=\frac{1}{m}\int_\Omega \nabla\tilde{H}'(\rho)\rho udx=-\frac{1}{m}\int_\Omega \tilde{H}'(\rho)\mathrm{div}(\rho u)dx,\\
&-\int_\Omega \frac{\rho e}{m}u\times H\cdot udx=0,\\
&\int_\Omega \frac{\mu\hbar^2}{2m^2}\rho\nabla(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}})\cdot udx=-\int_\Omega \frac{\mu\hbar^2}{2m^2}(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}})\mathrm{div}(\rho u)dx,\\
&\int_\Omega \nu_2\triangle(\rho u)\cdot udx=-\int_\Omega \nu_2\nabla(\rho u)\cdot\nabla udx=-\int_\Omega \nu_2\nabla\rho:\nabla u:udx-\int_\Omega \nu_2\rho|\nabla u|^2dx,\\
&-\lambda\int_\Omega \nabla\cdot(\nabla d\odot\nabla d-\frac{|\nabla d|^2}{2}I)\cdot udx=\lambda\int_\Omega \nabla d\odot\nabla d:\nabla udx-\lambda\int_\Omega \frac{|\nabla d|^2}{2}\nabla udx.
\end{aligned}$$

Multiplying (1.23) by $\triangle d+H$, integrating both sides by parts respectively in Ω, we get

$$\begin{aligned}
&-\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla d|^2dx-\int_\Omega \nabla d\odot\nabla d:\nabla udx+\int_\Omega \frac{|\nabla d|^2}{2}\nabla udx+\int_\Omega d_tHdx\\
&+\int_\Omega u\cdot\nabla d\cdot Hdx-\alpha_1\int_\Omega |d\times(\triangle d+H)|^2dx=0,
\end{aligned} \tag{4.5}$$

here we use the following computation:

$$\begin{aligned}
&\int_\Omega d_t\cdot\triangle ddx=-\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla d|^2dx,\\
&\int_\Omega (u\cdot\nabla d)\cdot\triangle ddx=\int_\Omega d_{jj}u^id_idx=\int_\Omega [(d_ju^id_i)_j-d_id_ju^i_j-u^i(\frac{|\nabla d|^2}{2})_i]dx\\
&\qquad=-\int_\Omega [d_id_ju^i_j+u^i_i(\frac{|\nabla d|^2}{2})]dx\\
&\qquad=-\int_\Omega \nabla d\odot\nabla d:\nabla udx+\int_\Omega \frac{|\nabla d|^2}{2}\nabla udx,
\end{aligned}$$

$$\alpha_1 \int_\Omega d\times(d\times(\triangle d+H))\cdot(\triangle d+H)dx=-\alpha_1\int_\Omega |d\times(\triangle d+H)|^2dx,$$
$$\alpha_2 \int_\Omega d\times(\triangle d+H)\cdot(\triangle d+H)dx=0.$$

Multiplying (1.24) by $\frac{E}{m}$, integrating by parts in Ω, we have

$$\frac{1}{2m}\frac{d}{dt}\int_\Omega |E|^2dx-\int_\Omega \nabla\times H\cdot\frac{E}{m}dx=\int_\Omega e\rho u\cdot\frac{E}{m}dx. \tag{4.6}$$

Multiplying (1.25) by $\frac{H}{m}$, integrating by parts in Ω, we have

$$\frac{1}{2m}\frac{d}{dt}\int_\Omega |H|^2dx-\int_\Omega \nabla\times E\cdot\frac{H}{m}dx=-\int_\Omega \beta d_t\cdot\frac{H}{m}dx-\int_\Omega \beta u\cdot\nabla d\cdot\frac{H}{m}dx. \tag{4.7}$$

Notice the fact that

$$\int_\Omega \nabla\times E\cdot\frac{H}{m}dx=\int_\Omega \nabla\times H\cdot\frac{E}{m}dx.$$

From (4.3)-(4.7) we can get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega(\tilde{H}(\rho)+\frac{\mu\hbar^2}{2m^2}|\nabla\sqrt{\rho}|^2+\frac{1}{2}\rho|u|^2+\frac{\lambda}{2}|\nabla d|^2+\frac{1}{2m}|E|^2+\frac{1}{2m}|H|^2)dx\\&+\frac{\nu_1}{m}\int_\Omega \tilde{H}''(\rho)|\nabla\rho|^2dx+\nu_2\int_\Omega \rho|\nabla u|^2dx+\int_\Omega\frac{\mu\nu_1\hbar^2}{4m^2}\rho|\nabla^2\log\rho|^2dx\\&+\int_\Omega(\frac{1}{\tau}\rho|u|^2+\lambda\alpha_1|d\times(\triangle d+H)|^2+\delta|\nabla u|^2+\delta|u|^2)dx=0.\end{aligned} \tag{4.8}$$

□

Combining Theorem 4.1 with Gronwall's inequality, we can get the following estimates:

5. A priori estimates.

Lemma 5.1.

$$\|\sqrt{\rho}\|_{L^\infty([0,T];H^1(\Omega))}\leqslant C, \tag{5.1}$$
$$\|\rho\|_{L^\infty([0,T];L^\gamma(\Omega))}\leqslant C, \tag{5.2}$$
$$\|\sqrt{\rho}u\|_{L^\infty([0,T];L^2(\Omega))}\leqslant C, \tag{5.3}$$
$$\|\sqrt{\rho}\nabla u\|_{L^2([0,T];L^2(\Omega))}\leqslant C, \tag{5.4}$$
$$\|H\|_{L^\infty([0,T];L^2(\Omega))}+\|E\|_{L^\infty([0,T];L^2(\Omega))}\leqslant C, \tag{5.5}$$
$$\|\nabla d\|_{L^\infty([0,T];L^2(\Omega))}\leqslant C, \tag{5.6}$$
$$\|d\times(\triangle d+H)\|_{L^2([0,T];L^2(\Omega))}\leqslant C, \tag{5.7}$$
$$\|\sqrt{\rho}\nabla^2 log\rho\|_{L^2([0,T];L^2(\Omega))}\leqslant C, \tag{5.8}$$
$$\delta\|u\|_{L^2([0,T];H^1(\Omega))}\leqslant C. \tag{5.9}$$

The energy equality (4.1) and Lemma 5.1 allow us to achieve some estimates.

Lemma 5.2. *The following uniform estimate holds for constant $C > 0$ which is independent of n and δ:*

$$\|\sqrt{\rho}\|_{L^2([0,T];H^2(\Omega))} + \|\sqrt[4]{\rho}\|_{L^4([0,T];W^{1,4}(\Omega))} \leqslant C. \tag{5.10}$$

Proof. The lemma follows from the energy estimate in Theorem 4.1. The inequality

$$\int_\Omega \rho|\nabla^2 \log\rho|^2 dx \geqslant \kappa_2 \int_\Omega |\nabla^2\sqrt{\rho}|^2 dx, \tag{5.11}$$

with κ_2, was shown in Jüngel [13], and the inequality

$$\int_\Omega \rho|\nabla^2 \log\rho|^2 dx \geqslant \kappa \int_\Omega |\nabla\sqrt[4]{\rho}|^4 dx, \quad \kappa > 0$$

was proved in Jüngel [13]. □

We are able to deduce more regularity from the H^2 bound for $\sqrt{\rho}$.

Lemma 5.3. *The following uniform estimates hold for some constants $C > 0$ that are independent of n and δ:*

$$\|\rho u\|_{L^2([0,T];W^{1,\frac{3}{2}}(\Omega))} \leqslant C, \tag{5.12}$$

$$\|\rho\|_{L^2([0,T];W^{2,p}(\Omega))} \leqslant C, \tag{5.13}$$

$$\|\rho\|_{L^{\frac{4\gamma}{3}+1}([0,T];L^{\frac{4\gamma}{3}+1}(\Omega))} \leqslant C, \tag{5.14}$$

where $p = 2\gamma/(\gamma+1)$.

Proof. Since the space $H^2(\Omega)$ embeds continuously into $L^\infty(\Omega)$, $\sqrt{\rho}$ is bounded in $L^2([0,T];L^\infty(\Omega))$. Thus, in view of (5.3), $\rho u = \sqrt{\rho}\sqrt{\rho}u$ is uniformly bounded in $L^2([0,T];L^2(\Omega))$. By (5.1) and (5.10), $\nabla\sqrt{\rho}$ is bounded in $L^2([0,T];L^6(\Omega))$ and $\sqrt{\rho}$ is bounded in $L^\infty([0,T];L^6(\Omega))$. This, together with (5.4), implies that

$$\nabla(\rho u) = 2\nabla\sqrt{\rho}\otimes(\sqrt{\rho}u) + \sqrt{\rho}\nabla u\sqrt{\rho}$$

is uniformly bounded in $L^2([0,T];L^{3/2}(\Omega))$, proving the first claim.

For the second claim, we observe first that, by the Gagliardo-Nirenberg inequality, with $p = 2\gamma/(\gamma+1)$ and $\theta = 1/2$,

$$\begin{aligned}\|\nabla\sqrt{\rho}\|^4_{L^4([0,T];L^{2p}(\Omega))} &\leqslant C\int_0^T \|\sqrt{\rho}\|^{4\theta}_{H^2(\Omega)}\|\sqrt{\rho}\|^{4(1-\theta)}_{L^{2\gamma}(\Omega)} dt \\ &\leqslant C\|\sqrt{\rho}\|^{4(1-\theta)}_{L^\infty([0,T];L^{2\gamma}(\Omega))}\int_0^T \|\sqrt{\rho}\|^2_{H^2(\Omega)} dt \leqslant C.\end{aligned}$$

Then, $\sqrt{\rho}$ is bounded in $L^4(0,T;W^{1,2p}(\Omega))$. Notice that $\gamma > 3$, so $2p > 3$ gives a uniform bound for $\sqrt{\rho}$ in $L^4(0,T;L^\infty(\Omega))$. The estimate on $\nabla\sqrt{\rho}$ in $L^4(0,T;L^{2p}(\Omega))$ shows that

$$\nabla^2\rho = 2(\sqrt{\rho}\nabla^2\sqrt{\rho} + \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho})$$

is bounded in $L^2(0,T;L^p(\Omega))$, which proves the second claim.

Finally, the Gagliardo-Nirenberg inequality, with $\theta = 3/(4\gamma+3)$ and $q = 2(4\gamma+3)/3$,

$$\|\sqrt{\rho}\|^q_{L^q([0,T];L^q(\Omega))} \leqslant C\int_0^T \|\sqrt{\rho}\|^{q\theta}_{H^2(\Omega)}\|\sqrt{\rho}\|^{q(1-\theta)}_{L^{2\gamma}(\Omega)} dt$$

$$\leqslant C\|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))}^{q(1-\theta)}\int_0^T\|\sqrt{\rho}\|_{H^2(\Omega)}^2dt\leqslant C$$

shows that ρ is bounded in $L^{\frac{q}{2}}([0,T];L^{\frac{q}{2}}(\Omega))$. This finishes the proof. □

Lemma 5.4. *The following uniform estimates hold for $s>5/2$:*

$$\|\partial_t\rho\|_{L^2([0,T];L^{3/2}(\Omega))}\leqslant C, \tag{5.15}$$

$$\|\partial_t(\rho u)\|_{L^{4/3}([0,T];(H^s(\Omega))^*)}\leqslant C. \tag{5.16}$$

Further,

$$\|\partial_t\sqrt{\rho}\|_{L^2([0,T];(H^1(\Omega))^*)}\leqslant C. \tag{5.17}$$

Proof. By (5.12), (5.13), we find that $\partial_t\rho=-\mathrm{div}(\rho u)+\nu\triangle\rho$ is uniformly bounded in $L^2([0,T];\,L^{3/2}(\Omega))$, achieving the first claim.

The sequence $(\rho u\otimes u)$ is bounded in $L^\infty([0,T];L^1(\Omega))$; hence, $\mathrm{div}(\rho u\otimes u)$ is bounded in $L^\infty([0,T];(W^{1,\infty}(\Omega))^*)$ and, because of the continuous embedding of $H^s(\Omega)$ into $W^{1,\infty}(\Omega)$ for $s>5/2$, also in $L^\infty([0,T];(H^s(\Omega))^*)$. The estimate

$$\begin{aligned}&\int_0^T\int_\Omega\rho\nabla(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}})\cdot\phi dxdt\\ =&-\int_0^T\int_\Omega\triangle\sqrt{\rho}(2\nabla\sqrt{\rho}\cdot\phi+\sqrt{\rho}\mathrm{div}\phi)dxdt\\ \leqslant&\|\triangle\sqrt{\rho}\|_{L^2([0,T];L^2(\Omega))}(2\|\sqrt{\rho}\|_{L^4([0,T];W^{1,3}(\Omega))}\|\phi\|_{L^4([0,T];L^6(\Omega))}\\ &+\|\sqrt{\rho}\|_{L^\infty([0,T];L^6(\Omega))}\|\phi\|_{L^2([0,T];W^{1,3}(\Omega))})\\ \leqslant&C\|\phi\|_{L^4([0,T];W^{1,3}(\Omega))},\end{aligned}$$

for all $\phi\in L^4([0,T];W^{1,3}(\Omega))$ proves that $\rho\triangle\sqrt{\rho}/\sqrt{\rho}$ is uniformly bounded in $L^{4/3}([0,T];(W^{1,3}(\Omega))^*)\hookrightarrow L^{4/3}([0,T];(H^s(\Omega))^*)$. By virtue of $(\rho^\gamma\ )$ is bounded in $L^{4/3}([0,T];L^{4/3}(\Omega))\hookrightarrow L^{4/3}([0,T];\,(H^s(\Omega))^*)$. Moreover, by (5.12) $\triangle(\rho u)$ is uniformly bounded in $L^2([0,T];(W^{1,3}(\Omega))^*)$, and $(\delta\triangle u)$ is bounded in $L^2([0,T];(H^1(\Omega))^*)$. Therefore, using Lemma 5.1 and Lemma 5.3 we have

$$\begin{aligned}(\rho u)_t=&-\mathrm{div}(\rho u\otimes u)-\frac{\nabla P(\rho)}{m}-\frac{\rho e}{m}(E+u\times H)+\frac{\mu\hbar^2}{2m^2}\nabla(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}})\\ &+\nu_2\triangle(\rho u)-\frac{\rho u}{\tau}-\lambda\nabla\cdot(\nabla d\odot\nabla d-\frac{|\nabla d|^2}{2}I)\end{aligned}$$

is uniformly bounded in $L^{4/3}([0,T];(H^s(\Omega))^*)$.

Dividing the mass equation by $\sqrt{\rho}$ gives

$$\begin{aligned}\partial_t\sqrt{\rho}&=-\nabla\sqrt{\rho}\cdot u-\frac{1}{2}\sqrt{\rho}\mathrm{div}u+\nu_1(\triangle\sqrt{\rho}+4|\nabla\sqrt[4]{\rho}|^2)\\ &=-\mathrm{div}(\sqrt{\rho}u)+\frac{1}{2}\sqrt{\rho}\mathrm{div}u+\nu_1(\triangle\sqrt{\rho}+4|\nabla\sqrt[4]{\rho}|^2).\end{aligned}$$

The first term on the right-hand side is bounded in $L^2([0,T];(H^1(\Omega))^*)$ by (5.3), (5.4). By (5.3), (5.4) and (5.9), the remaining terms are uniformly bounded in $L^2([0,T];L^2(\Omega))$. The proof is completed. □

Lemma 5.5. *The following uniform estimates hold*

$$\|\triangle d\|_{L^2([0,T];L^2(\Omega))} \leqslant C, \tag{5.18}$$

$$\|\partial_t d\|_{L^2([0,T];H^1(\Omega))\cap L^\infty([0,T];L^2(\Omega))} \leqslant C. \tag{5.19}$$

Proof. On one hand, Gagliardo-Nirenberg inequality and elliptic estimates yield that

$$\|\nabla d\|^4_{L^4(\Omega)} \leqslant C(\Omega)(\|\nabla^2 d\|^2_{L^2(\Omega)}\|d\|^2_{L^\infty(\Omega)} + \|d\|^4_{L^4(\Omega)})$$

and

$$\|\nabla^2 d\|_{L^2(\Omega)} \leqslant C(\Omega)(\varepsilon\|\triangle d\|_{L^2(\Omega)} + \frac{1}{\varepsilon}\|\nabla d\|_{L^2(\Omega)}) \text{ for any } d\in H^1(\Omega).$$

Whence,

$$\begin{aligned}\|\nabla d\|^4_{L^4(\Omega)} &\leqslant C(\Omega)\varepsilon\|\triangle d\|^2_{L^2(\Omega)} + \frac{C(\Omega)}{\varepsilon}(\|d\|^2_{L^\infty(\Omega)}\|\nabla d\|^2_{L^2(\Omega)} + \|d\|^4_{L^4(\Omega)})\\ &\leqslant C(\Omega)\varepsilon\|\triangle d\|^2_{L^2(\Omega)} + \frac{C(\Omega)}{\varepsilon}(\|\nabla d\|^2_{L^2(\Omega)} + \|d\|^2_{L^2(\Omega)}).\end{aligned}$$

On the other hand, since

$$\begin{aligned}\int_0^T\int_\Omega |\triangle d|^2 dxdt &\leqslant \int_0^T\int_\Omega \big(|d\times\triangle d + H|^2 - 2(H\cdot\triangle d)\\ &\quad + 2(|\nabla d|^2 d)^2 + 2(d\cdot\triangle d)(d\cdot H) - |d\times H|^2\big)dxdt\\ &\leqslant C\varepsilon\int_0^T\int_\Omega |\triangle d|^2 dxdt + \frac{C}{\varepsilon}\int_0^T \|\nabla d\|^4_{L^4(\Omega)}dt + C.\end{aligned}$$

Combining above estimates, we have

$$\int_0^T\int_\Omega |\triangle d|^2 dxdt \leqslant C\varepsilon\int_0^T \|\triangle d\|^2_{L^2(\Omega)} + \frac{C}{\varepsilon}\int_0^T (\|\nabla d\|^2_{L^2(\Omega)} + \|d\|^2_{L^2(\Omega)})dt.$$

So, $\triangle d \in L^2([0,T];L^2(\Omega))$.

Multiplying (1.3) by d_t and integrating by parts, we have

$$\begin{aligned}&\int_\Omega |d_t|^2 dx\\ &= -\int_\Omega \{u\cdot\nabla d\cdot d_t + \alpha_1 d\times(d\times(\triangle d + H))\cdot d_t - \alpha_2 d\times(\triangle d + H)\cdot d_t\}dx\\ &= \int_\Omega \{u\cdot\nabla d_t\cdot d + \alpha_1(\triangle d + H)\cdot d_t + \alpha_2(d\times\nabla d_t)\cdot\nabla d + \alpha_2(d\times H)\cdot d_t\}dx\\ &\leqslant \|u\|_{L^2(\Omega)}\|\nabla d\|_{L^2(\Omega)} + \alpha_1\|\triangle d\|_{L^2(\Omega)}\|d_t\|_{L^2(\Omega)}\\ &\quad + (\alpha_1+\alpha_2)\|H\|_{L^2(\Omega)}\|d_t\|_{L^2(\Omega)} + \alpha_2\|\nabla d\|_{L^2(\Omega)}\|d_t\|_{L^2(\Omega)} - \alpha_1\frac{d}{dt}\int_\Omega |\nabla d_t|^2 dx\\ &\leqslant \frac{1}{2}\|d_t\|^2_{L^2(\Omega)} + C\|\triangle d\|^2_{L^2(\Omega)} + C\|H\|^2_{L^2(\Omega)} + C\|\nabla d\|^2_{L^2(\Omega)}\\ &\quad + C\|u\|^2_{L^2(\Omega)} - \alpha_1\frac{d}{dt}\int_\Omega |\nabla d_t|^2 dx.\end{aligned}$$

Here C denotes different constants independent of n. Then integrating by parts respect to t in $[0,T]$, then the Lemma 5.1 and Lemma 5.3 finish the proof. □

6. Proof of Theorem 1.1

6.1. The limit $n \to \infty$.

We perform first the limit $n \to \infty$, $\delta > 0$ being fixed. The limit $\delta \to 0$ is carried out in section 6.2. We consider both limits separately since the weak formulation (1.17)-(1.20) for the continuous viscous quantum Euler model is different from its approximation (3.7), (3.2)-(3.5).

We conclude from the Aubin lemma, taking into account the regularity (5.13) and (5.15) for ρ_n , the regularity (5.10) and (5.17) for $\sqrt{\rho_n}$, and the regularity (5.12) and (5.16) for $\rho_n u_n$, that there exist subsequences of $\rho_n, \sqrt{\rho_n}$, and $(\rho_n u_n)$, which are not relabeled, such that, for some functions ρ and J, as $n \to \infty$,

$$\begin{aligned}
&\rho_n \to \rho \ \text{ strongly in } \ L^2([0,T]; L^\infty(\Omega)),\\
&\sqrt{\rho_n} \rightharpoonup \sqrt{\rho} \ \text{ weakly in } \ L^2([0,T]; H^2(\Omega)),\\
&\sqrt{\rho_n} \to \sqrt{\rho} \ \text{ strongly in } \ L^2([0,T]; H^1(\Omega)),\\
&\rho_n u_n \to J \ \text{ strongly in } \ L^2([0,T]; L^2(\Omega)).
\end{aligned}$$

Here we have used that the embeddings $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)(p > 3/2)$, and $W^{1,3/2}(\Omega) \hookrightarrow L^2(\Omega)$ are compact. The estimate (5.9) on u_n yields that as $n \to \infty$,

$$u_n \rightharpoonup u \ \text{ weakly in } \ L^2([0,T]; H^1(\Omega)).$$

Then, since $(\rho_n u_n)$ converges weakly to ρu in $L^1([0,T]; L^6(\Omega))$, we infer that $J = \rho u$. We are now in the position to let $n \to \infty$ in the approximate system (3.2)-(3.5) with $\rho = \rho_n$, $u = u_n$, $d = d_n$, $E = E_n$. It is clear to have that ρ solves

$$\partial_t \rho + \operatorname{div}(\rho u) = \nu_1 \triangle \rho.$$

Next we consider the weak formulation (3.2) term by term. The strong convergence of $(\rho_n u_n)$ in $L^2([0,T]; L^2(\Omega))$ and the weak convergence of ρ_n in $L^2([0,T]; L^6(\Omega))$ leads to

$$\rho_n u_n \otimes u_n \rightharpoonup \rho u \otimes u \ \text{ weakly in } \ L^1([0,T]; L^{3/2}(\Omega)).$$

Furthermore, in view of (5.12) (up to a subsequence),

$$\nabla(\rho_n u_n) \rightharpoonup \nabla(\rho u) \ \text{ weakly in } \ L^2([0,T]; L^{3/2}(\Omega)).$$

The $L^\infty([0,T]; L^\gamma(\Omega))$ bound for ρ_n shows that $\rho_n^\gamma \rightharpoonup y$ weakly $*$ in $L^\infty([0,T]; L^1(\Omega))$ for some function y and, since $\rho_n^\gamma \to \rho^\gamma$, a.e., $y = \rho^\gamma$. Finally, the above convergence results show that the limit $n \to \infty$ of

$$\int_\Omega \operatorname{div}(\rho_n \phi) \frac{\triangle\sqrt{\rho_n}}{\sqrt{\rho_n}} dx = \int_\Omega \triangle\sqrt{\rho_n}(2\nabla\sqrt{\rho_n} \cdot \phi + \sqrt{\rho_n}\operatorname{div}\phi) dx$$

equals, for sufficiently smooth test functions,

$$\int_\Omega \triangle\sqrt{\rho}(2\nabla\sqrt{\rho} \cdot \phi + \sqrt{\rho}\operatorname{div}\phi) dx.$$

From Lemma 5.5 we know that $\|\nabla d\|_{L^2([0,T];H^2(\Omega)) \cap L^\infty([0,T];L^2(\Omega))}$ is bounded, then

$$\nabla d_n \odot \nabla d_n \rightharpoonup \nabla d \odot \nabla d \ \text{ weakly in } \ L^2([0,T]; L^2(\Omega)),$$

$$\frac{|\nabla d_n|^2}{2}I \rightharpoonup \frac{|\nabla d|^2}{2}I \ \text{ weakly in } \ L^2([0,T];L^2(\Omega)),$$
$$\nabla d_n \odot \nabla d_n \to \nabla d \odot \nabla d \ \text{ strongly in } \ L^1([0,T];L^1(\Omega)),$$
$$\frac{|\nabla d_n|^2}{2}I \to \frac{|\nabla d|^2}{2}I \ \text{ strongly in } \ L^1([0,T];L^1(\Omega)).$$

Since $E, H \in L^\infty([0,T];L^2(\Omega))$, we have

$$\frac{\rho_n e}{m}E_n \rightharpoonup \frac{\rho e}{m}E \ \text{ weakly in } \ L^1([0,T];L^1(\Omega)),$$
$$\frac{\rho_n e}{m}u_n \times H_n \rightharpoonup \frac{\rho e}{m}u \times H \ \text{ weakly in } \ L^1([0,T];L^1(\Omega)).$$

Thus we have shown that (ρ, u, d, E, H) solves $\partial_t\rho + \mathrm{div}(\rho u) = \nu_1 \triangle \rho$ pointwise and for all test function such that the integrals are defined,

$$\begin{aligned}
-\int_\Omega \rho_0 u_0 \phi(\cdot,0)dx = \int_0^T \int_\Omega \Big(&\rho u \cdot \phi_t + \rho(u \otimes u) : \nabla\phi + P(\rho)\mathrm{div}(\phi) \\
&- \frac{\rho e}{m}(E + u \times H)\cdot\phi - \frac{\mu\hbar^2}{2m^2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\mathrm{div}(\rho\phi) - \nu_2\nabla(\rho u) : \nabla\phi \\
&- \frac{\rho u}{\tau}\phi + \lambda(\nabla d \odot \nabla d - \frac{|\nabla d|^2}{2}I)\cdot\nabla\phi \\
&- \delta(\nabla u : \nabla\phi + u \cdot \phi)\Big)dxdt.
\end{aligned} \tag{6.1}$$

Then, we consider the weak formulation (3.3) term by term. By (5.6) (5.9) and Lemma 5.5 we obtain

$$u_n \nabla d_n \rightharpoonup u \nabla d \ \text{ weakly in } \ L^2([0,T];L^2(\Omega)),$$

(5.6) and (5.7) imply that

$$d_n \times (d_n \times (\triangle d_n + H_n)) \rightharpoonup d \times (d \times (\triangle d + H)) \ \text{ weakly in } \ L^1([0,T];L^1(\Omega)),$$

(5.2) and (5.7) imply that

$$d_n \times (\triangle d_n + H_n) \rightharpoonup d \times (\triangle d + H) \ \text{ weakly in } \ L^1([0,T];L^1(\Omega)).$$

Then the limit of d_n satisfy

$$\begin{aligned}
-\int_\Omega d_0\phi(\cdot,0)dx = \int_0^T \int_\Omega \Big(&d\phi_t + u\cdot\nabla d \cdot \phi + \alpha_1 d \times (d \times (\triangle d + H))\cdot\phi \\
&- \alpha_2 d \times (\triangle d + H)\cdot\rho\phi\Big)dxdt.
\end{aligned} \tag{6.2}$$

Analogously, using the a priori estimates we can show that as $n \to \infty$, the limit of (E_n, H_n) satisfy

$$-\int_\Omega E_0\phi(\cdot,0)dx = \int_0^T \int_\Omega \Big(E\phi_t - H\cdot(\nabla\times\phi) - e\rho u\cdot\phi\Big)dxdt, \tag{6.3}$$

$$\begin{aligned}
-\int_\Omega (H_0 + \lambda m d_0)\phi(\cdot,0)dx = \int_0^T \int_\Omega \Big(&(H + \lambda m d)\cdot\phi_t + E\cdot(\nabla\times\phi) \\
&+ \lambda m(u\cdot\nabla d)\cdot\phi\Big)dxdt.
\end{aligned} \tag{6.4}$$

6.2. The limit $\delta \to 0$.

Let $(\rho^\delta, u^\delta, d^\delta, E^\delta, H^\delta)$ be a solution to (3.2)-(3.5) with the regularity proved in the previous. By employing the test function $\rho^\delta \phi$ in (3.2), we obtain,

$$\begin{aligned}
&-\int_\Omega \rho_0^2 u_0 \phi(\cdot,0)dx \\
=&\int_0^T\int_\Omega [(\rho^\delta)^2 u^\delta \cdot \phi_t - (\rho^\delta)^2 \mathrm{div}(u^\delta) u^\delta \cdot \phi - \nu_2(\rho^\delta u^\delta \otimes \nabla\rho^\delta) : \nabla\phi \\
&+ \rho^\delta u^\delta \otimes \rho^\delta u^\delta : \nabla\phi + \frac{\gamma}{\gamma+1}\frac{(\rho^\delta)^{\gamma+1}}{m}\mathrm{div}\phi + \frac{(\rho^\delta)^2 e}{m}(E^\delta + u^\delta \times H^\delta)\cdot\phi \\
&- \frac{\mu\hbar^2}{2m^2}\triangle\sqrt{\rho^\delta}(2\sqrt{\rho^\delta}\nabla\rho^\delta\cdot\phi + (\rho^\delta)^{3/2}\mathrm{div}\phi) \\
&- \nu_2\nabla(\rho^\delta u^\delta) : (\rho^\delta\nabla\phi + 2\nabla\rho^\delta\otimes\phi) + \lambda(\nabla d^\delta \odot \nabla d^\delta - \frac{|\nabla d^\delta|^2}{2}I)\cdot\nabla(\rho^\delta\phi) \\
&- \delta\nabla u^\delta : (\rho^\delta\nabla\phi + \nabla\rho^\delta\otimes\phi) - \delta\rho^\delta u^\delta\cdot\phi]dxdt.
\end{aligned} \tag{6.5}$$

By Aubin-Lions lemma and the regularity results, for some functions ρ and J, we have that as $\delta \to 0$,

$$\rho^\delta \to \rho \text{ strongly in } L^2([0,T]; W^{1,p}(\Omega)),\quad 3 < p < 6\gamma/(\gamma+3), \tag{6.6}$$

$$\rho^\delta u^\delta \to J \text{ strongly in } L^2([0,T]; L^q(\Omega)),\quad 1 \leqslant q < 3, \tag{6.7}$$

$$\sqrt{\rho^\delta} \to \sqrt{\rho} \text{ strongly in } L^\infty([0,T]; L^r(\Omega)),\quad 1 \leqslant r < 6. \tag{6.8}$$

Estimate (5.3) (5.4) and Fatou's lemma yield

$$\int_\Omega \liminf_{\delta\to 0}\ \inf_\delta \frac{|\rho^\delta u^\delta|^2}{\rho^\delta} < \infty.$$

This implies that $J = 0$ in $\rho = 0$. Then, when we define the limit velocity $u := J/\rho$ in $\{\rho \neq 0\}$ and $u := 0$ in $\rho = 0$, thus $J = \rho u$. By (5.3) (5.4) there exists a subsequence such that

$$\sqrt{\rho^\delta}u^\delta \rightharpoonup g \text{ weakly } * \text{ in } L^\infty([0,T]; L^2(\Omega)), \tag{6.9}$$

for some function g. Hence, since $\sqrt{\rho^\delta}$ converges strongly to $\sqrt{\rho}$ in $L^2([0,T]; L^\infty(\Omega))$, we infer that $\rho^\delta u^\delta = \sqrt{\rho^\delta}\sqrt{\rho^\delta}u^\delta$ converges weakly to $\sqrt{\rho}g$ in $L^2([0,T]; L^2(\Omega))$ and $\sqrt{\rho}g = \rho u = J$. In particular, $g = J/\sqrt{\rho}$ in $\{\rho \neq 0\}$.

Now we are able to pass the limit $\delta \to 0$ in the weak formulation (6.5) term by term. The strong convergences (6.6) and (6.7) imply that

$$(\rho^\delta)^2 u^\delta \to \rho^2 u \text{ strongly in } L^1([0,T]; L^q(\Omega)),\ 1 \le q < 3,$$

$$\rho^\delta u^\delta \otimes \nabla\rho^\delta \to \rho u \otimes \nabla\rho \text{ strongly in } L^1([0,T]; L^{3/2}(\Omega)).$$

The strong convergence of $\rho^\delta u^\delta$ yields

$$\rho^\delta u^\delta \otimes \rho^\delta u^\delta \to \rho u \otimes \rho u \text{ strongly in } L^1([0,T]; L^{q/2}(\Omega)),\ 1 \le q < 3.$$

Furthermore, we have

$$\nabla\rho^\delta \to \nabla\rho \text{ strongly in } L^2([0,T]; L^p(\Omega)),\ p > 3,$$

$$\sqrt{\rho^\delta} \to \sqrt{\rho} \text{ strongly in } L^\infty([0,T];L^r(\Omega)) \text{ with } r = 2p/(p-2),$$
$$\triangle\sqrt{\rho^\delta} \rightharpoonup \triangle\sqrt{\rho} \text{ weakly in } L^2([0,T];L^2(\Omega)).$$

Notice that $r < 6$ since $p > 3$, which implies that

$$\triangle\sqrt{\rho^\delta}\sqrt{\rho^\delta}\nabla\rho^\delta \rightharpoonup \triangle\sqrt{\rho}\sqrt{\rho}\nabla\rho \text{ weakly in } L^1([0,T];L^1(\Omega)).$$

Since $\nabla(\rho^\delta u^\delta)$ converges weakly in $L^2([0,T];L^{3/2}(\Omega))$ (see (5.12)) and $\nabla\rho^\delta$ converges strongly in $L^2([0,T];L^3(\Omega))$ (see (6.6)), we obtain

$$\nabla(\rho^\delta u^\delta)\cdot\nabla\rho^\delta \rightharpoonup \nabla(\rho u)\cdot\nabla\rho \text{ weakly in } L^1([0,T];L^1(\Omega)).$$

The a.e. convergence of ρ^δ and the $L^{4\gamma/3+1}([0,T];L^{4\gamma/3+1}(\Omega))$ bound on ρ^δ (see (5.14)), together with the fact that $4\gamma/3+1 > \gamma+1$, proves that

$$(\rho^\delta)^{\gamma+1} \to \rho^{\gamma+1} \text{ strongly in } L^1([0,T];L^1(\Omega)).$$

Using the estimate (5.9) for $\sqrt{\delta}u^\delta$, we obtain

$$\begin{aligned}
&\delta\int_\Omega \nabla u^\delta : (\rho^\delta\nabla\phi + \nabla\rho^\delta\otimes\phi)dx\\
\leqslant&\sqrt{\delta}\|\sqrt{\delta}\nabla u^\delta\|_{L^2([0,T];L^2(\Omega))}(\|\rho^\delta\|_{L^2([0,T];L^\infty(\Omega))}\|\phi\|_{L^\infty([0,T];H^1(\Omega))}\\
&+\|\rho^\delta\|_{L^2([0,T];W^{1,3}(\Omega))}\|\phi\|_{L^\infty([0,T];L^6(\Omega))}) \to 0, \quad \text{as } \delta\to 0,\\
&\delta\int_\Omega \rho^\delta u^\delta\cdot\phi dx \leqslant \delta\|\rho^\delta u^\delta\|_{L^2([0,T];L^3(\Omega))}\|\phi\|_{L^2([0,T];L^{3/2}(\Omega))} \to 0, \quad \text{as } \delta\to 0.
\end{aligned}$$

It remains to show the convergence of $(\rho^\delta)^2\mathrm{div}(u^\delta)u^\delta$. We proceed similarly as in Guo etc [11] and introduce the functions $G_\alpha \in C^\infty([0,\infty))$, $\alpha > 0$, satisfying $G_\alpha(x) = 1$ for $x \geqslant 2\alpha$, $G_\alpha(x) = 0$ for $x \leqslant \alpha$, and $0 \leqslant G_\alpha(x) \leqslant 1$. Then we estimate the low-density part of $(\rho^\delta)^2\mathrm{div}(u^\delta)u^\delta$ by

$$\begin{aligned}
&\|(1-G_\alpha(\rho^\delta))(\rho^\delta)^2\mathrm{div}(u^\delta)u^\delta\|_{L^1([0,T];L^1(\Omega))}\\
\leqslant&\|(1-G_\alpha)\sqrt{\rho^\delta}\|_{L^\infty([0,T];L^\infty(\Omega))}\|\sqrt{\rho^\delta}\mathrm{div}(u^\delta)u^\delta\|_{L^2([0,T];L^2(\Omega))}\|\rho^\delta u^\delta\|_{L^2([0,T];L^2(\Omega))}\\
\leqslant&C\|(1-G_\alpha)\sqrt{\rho^\delta}\|_{L^\infty([0,T];L^\infty(\Omega))} \leqslant C\sqrt{\alpha},
\end{aligned} \tag{6.10}$$

where $C > 0$ is independent of α. We write

$$G_\alpha(\rho^\delta)\rho^\delta\mathrm{div}u^\delta = \mathrm{div}(G_\alpha(\rho^\delta)\rho^\delta u^\delta) - \rho^\delta u^\delta\otimes\nabla\rho^\delta(G'_\alpha(\rho^\delta) + \frac{G_\alpha(\rho^\delta)}{\rho^\delta}). \tag{6.11}$$

As $\delta\to 0$, the first term on the right-hand side converges strongly to $\mathrm{div}(G_\alpha(\rho)\rho u)$ in $L^1([0,T];(H^1(\Omega))^*)$ since $G_\alpha(\rho^\delta)$ converges strongly to $G_\alpha(\rho)$ in $L^p([0,T];L^p(\Omega))$ for any $p < \infty$ and $\rho^\delta u^\delta$ converges strongly to ρu in $L^2([0,T];L^q(\Omega))$ for any $q < 3$. In view of (6.8) and (6.9), we infer the weak* convergence $\rho^\delta u^\delta \rightharpoonup \sqrt{\rho}g = \rho u$ in $L^\infty([0,T];L^{2r/(r+2)}(\Omega))$ for all $r < 6$. Thus, by (6.6),

$$\rho^\delta u^\delta\otimes\nabla\rho^\delta \rightharpoonup \rho u\otimes\nabla\rho \text{ weakly in } L^2([0,T];L^\theta(\Omega))$$

where $\theta = 2pr/(2p+2r+pr)$. It is possible to choose $3 < p6\gamma/(\gamma+3)$ and $r < 6$ such that $\theta > 1$. Then, together with strong convergence of $G'_\alpha(\rho^\delta) + G_\alpha(\rho^\delta)/\rho^\delta$

to $G'_\alpha(\rho)+G_\alpha(\rho)/\rho$ in $L^p([0,T];L^p(\Omega))$ for any $p<\infty$, the limit $\delta\to 0$ in (6.11) yields the identity

$$G_\alpha(\rho)\rho\mathrm{div}u=\mathrm{div}(G_\alpha(\rho)\rho u)-\rho u\otimes\nabla\rho(G'_\alpha(\rho)+\frac{G_\alpha(\rho)}{\rho}).$$

in $L^1([0,T];(H^2(\Omega))^*)$. Since $G_\alpha(\rho^\delta)\rho^\delta\mathrm{div}(u^\delta)$ is bounded in $L^2([0,T];L^2(\Omega))$, we conclude that

$$G_\alpha(\rho^\delta)\rho^\delta\mathrm{div}(u^\delta)\rightharpoonup G_\alpha(\rho)\rho\mathrm{div}u\ \text{ weakly in }\ L^2([0,T];L^2(\Omega)).$$

Moreover, in view of the strong convergence of $\rho^\delta u^\delta$ to ρu in $L^2([0,T];L^q(\Omega))$ for all $q<3$, we infer that

$$G_\alpha(\rho^\delta)\rho^\delta\mathrm{div}(u^\delta)\rho^\delta u^\delta\rightharpoonup G_\alpha(\rho)\rho^2\mathrm{div}(u)u\ \text{ weakly in }\ L^1([0,T];L^{q/2}(\Omega)).$$

We write, for $\phi\in L^\infty([0,T];L^\infty(\Omega))$,

$$\begin{aligned}&\int_\Omega((\rho^\delta)^2\mathrm{div}(u^\delta)u^\delta-\rho^2\mathrm{div}(u)u)\cdot\phi dx\\ =&\int_\Omega(G_\alpha(\rho^\delta)(\rho^\delta)^2\mathrm{div}(u^\delta)u^\delta-G_\alpha(\rho)\rho^2\mathrm{div}(u)u)\cdot\phi dx\\ &+\int_\Omega(G_\alpha(\rho)-G_\alpha(\rho^\delta))\rho^2\mathrm{div}(u)u\cdot\phi dx\\ &+\int_\Omega(1-G_\alpha(\rho^\delta))((\rho^\delta)^2\mathrm{div}(u^\delta)u^\delta-\rho^2\mathrm{div}(u)u)\cdot\phi dx.\end{aligned}\tag{6.12}$$

For fixed $\alpha>0$, the first integral converges to zero as $\delta\to 0$. Furthermore, the last integral can be estimated by $C\sqrt{\alpha}$ uniformly in δ. For the second term, we recall that $G_\alpha(\rho^\delta)\to G_\alpha(\rho)$ strongly in $L^p([0,T];L^p(\Omega))$ for all $p<\infty$. Furthermore, by the Gagliardo-Nirenberg inequality, the bounds of $\rho u\in L^2([0,T];W^{1,3/2}(\Omega))$ and $L^\infty([0,T];L^{3/2}(\Omega))$ imply that $\rho u\in L^{5/2}([0,T];L^{5/2}(\Omega))$. Thus, since $\sqrt{\rho}\mathrm{div}u\in L^2([0,T];L^2(\Omega))$ and $\sqrt{\rho}\in L^q([0,T];L^q(\Omega))$ with $q=8\gamma/3+2$,

$$\rho^2\mathrm{div}(u)u=\sqrt{\rho}(\sqrt{\rho}\mathrm{div}u)\rho u\in L^r([0,T];L^r(\Omega)),\quad r=\frac{18\gamma+21}{20\gamma+15}>1.$$

So the second integral converges to zero as $\delta\to 0$. Thus, in the limit $\delta\to 0$, (6.12) can be made arbitrarily small, and hence,

$$(\rho^\delta)^2\mathrm{div}(u^\delta)u^\delta\rightharpoonup\rho^2\mathrm{div}(u)u\ \text{ weakly in }\ L^1([0,T];L^1(\Omega)).$$

Here we will omit the rest term convergence about d,E,H, you can refer to Guo etc [11].

We have proved that (ρ,u,d,E,H) solves (1.21)-(1.27) for smooth initial data. Let (ρ_0,u_0,d_0,E_0,H_0) be some finite-energy initial data, i.e., $\rho_0\geqslant 0$, $E(\rho_0,u_0,d_0,E_0,H_0)<\infty$, and let $(\rho_0^\delta,u_0^\delta,d_0^\delta,E_0^\delta,H_0^\delta)$ be smooth approximations satisfying $\rho_0^\delta\geqslant\delta>0$ in Ω and $\sqrt{\rho_0^\delta}\to\sqrt{\rho_0}$ strongly in $H^1(\Omega)$ and $\sqrt{\rho_0^\delta}u_0^\delta\to\sqrt{\rho_0}u_0$ strongly in $L^{3/2}(\Omega)$. From the above estimates, there exists a weak solution $(\rho^\delta,u^\delta,d^\delta,E^\delta,H^\delta)$ to (1.21)-(1.27) with initial data $(\rho_0^\delta,u_0^\delta,d_0^\delta,E_0^\delta,H_0^\delta)$ satisfying all the above bounds. Since $(\rho^\delta,\rho^\delta u^\delta)$ converges strongly to $(\rho,\rho u)$ as $\delta\to 0$, and there exist uniform bounds for ρ^δ in $H^1([0,T];L^{3/2}(\Omega))$ and for $\rho^\delta u^\delta$ in $W^{1,4/3}([0,T];(H^s(\Omega))^*)$. Thus, up to subsequences, as $\delta \to 0$,

$$\rho_0^\delta = \rho^\delta(\cdot,0) \rightharpoonup \rho(\cdot,0) \quad \text{weakly in} \quad L^{3/2}(\Omega),$$
$$\rho_0^\delta u_0^\delta = \rho^\delta u^\delta(\cdot,0) \rightharpoonup (\rho u)(\cdot,0) \quad \text{weakly in} \quad (H^s(\Omega))^*.$$

This shows that $\rho(\cdot,0) = \rho_0$ and $\rho u(\cdot,0) = \rho_0 u_0$ in the sense of distributions. We conclude the proof of Theorem 1.1. □

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