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BIFURCATIONS AND EXACT TRAVELING WAVE SOLUTIONS FOR THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION WITH WAVE OPERATOR*

Quting Chen[1], Yadong Shang[1,†] and Huafei Di[1]

Abstract In this paper, we investigate the dynamical bifurcations and exact traveling wave solutions for the generalized nonlinear Schrödinger equation with wave operator under different parametric conditions by means of the theory of singular system. We analyse the high order equilibrium point and give the phase portraits. We obtain many results under different values of the parameter p reflecting the strength of the nonlinearity in the model. For $p = 1$, we find explicit exact solutions of Jacobian elliptic functions type which is corresponding to the curves given by $H(\phi, y) = h$. According to the qualitative analysis of the phase portraits, we give the conclusions on the existence of solitary wave solutions and periodic wave solutions when $p \geq \frac{1}{2}$. In addition, we obtain the only explicit exact solitary wave solution corresponding to the curves given by $H(\phi, y) = 0$ for any p. Especially, we obtain some explicit exact double periodic solutions of elliptic functions type for $p = \frac{1}{2}$.

Keywords Generalized nonlinear Schrödinger equation, wave operator, bifurcations, solitary wave solution, periodic wave solution.

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1. Introduction

In 1980, Matsuuchi considered the nonlinear interaction between monochromatic waves and proposed one class of nonlinear Schrödinger equations with wave operator [20].

$$\begin{cases} i(A_t + c_g A_x) + \mathcal{D}_+ A = (q|A|^2 + r|B|^2)A, \\ i(B_t - c_g B_x) + \mathcal{D}_- B = -(q|B|^2 + r|A|^2)B, \end{cases} \tag{1.1}$$

where $A(x,t), B(x,t)$ are the complex amplitudes of the two waves and the second order operators $\mathcal{D}_\pm$ are given by

$$\mathcal{D}_\pm \equiv p_x \frac{\partial^2}{\partial x^2} \pm p_c \frac{\partial^2}{\partial x \partial t} + p_t \frac{\partial^2}{\partial t^2}. \tag{1.2}$$

[†]The corresponding author. Email address: gzydshang@126.com.(Y. Shang)
[1]School of Mathematics and Information Science, Guangzhou University, 510006, Guangzhou, Guangdong, China
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The coefficients p_x, p_c, and p_t in Eq. (1.2) are functions of the wave-number characterizing each wave system. They may satisfy the relation

$$p_x - c_g p_c + c_g^2 p_t = \frac{1}{2}\frac{dc_g}{dk}, \tag{1.3}$$

where k is the wave-number. The first equation of (1.1) without the coupling term can be regarded as the Schrödinger equation with wave operator

$$i(A_t + c_g A_x) + \mathcal{D}_+ A = q|A|^2 A, \tag{1.4}$$

Eq. (1.4) has a wide range of applications, such as nonrelativistic limit of the Klein-Gordon equation, the Langmuir wave envelope approximation in plasma and the modulated planar pulse approximation of the Sine-Gordon equation for light bullets. As we study the soliton in plasma physics, we can get the same type of equation, which is satisfied by the transverse velocity of high frequency electron. A lot of works have been done on qualitative research and numerical methods of nonlinear Schrödinger equations (systems) with wave operators. In 1983, Guo Boling [3] first studied the initial boundary value problems of a class of multidimensional nonlinear Schrödinger equations with wave operators. He proved the existence and uniqueness of the generalized and strong solution for the problem by means of Galerkin method, and discussed the regularity of the solution. He obtained the existence of the smooth solution under weaker assumption for one-dimensional case. In the same year, Guo and Liang [4] analyzed the initial boundary value problem of a class of Schrödinger equation with wave operator by using the difference method in another paper. They proved the convergence of the approximate solution and the stability of the difference scheme. In a subsequent paper, Guo [5] considered the blow up problem for the system of nonlinear Schrödinger equations with wave operator and the existence of soliton solutions for the system. He obtained the sufficient condition for the blow up problem and the existence of soliton solutions for this system under some assumptions. In 1989, Guo [6] studied the existence and nonexistence for the initial boundary value problem of one class of system of multidimensional nonlinear Schrödinger equations with wave operator and their soliton solutions.

In recent years, there have been many works on the numerical calculation and analysis of nonlinear Schrödinger equations with wave operators. The error estimates of local energy conservation law are given in the multisymplectic Fourier pseudospectral scheme [21]. Some new conservative finite difference schemes are presented for an initial-boundary value problem [23]. And a fully discrete scheme by discretizing the space with the local discontinuous Galerkin method and the time was put forward [7]. For further numerical analysis, please refer to [2, 8, 22, 24].

As we all know, exact solution can reflect more global properties than the approximate solution. Exact solution can also be used to judge the merits of numerical methods. In applications, exact solutions can be used to explain and predict the evolution of the physical state of the system. Exact solutions are of great significance both in mathematics and physics. However, for the nonlinear Schrödinger equation with wave operator, the results on exact solution are few.

In 2007, Li et al. [12, 13] developed the dynamic system approach of nonlinear evolution partial differential equation. One of the advantages is that it can handle the singular dynamical system. From then on, the exact solutions in different types of nonlinear partial equations were sought out [9–11, 14–16, 19]. The latest progress of this method can be known in references [17, 18]. The dynamic system method has been proved to be an effective way to deal with nonlinear evolution partial differential equations.

The main purpose of this paper is to investigate the exact solutions of the generalized nonlinear Schrödinger equation with wave operator. By applying to the theory of singular dynamic system, we consider the traveling wave solutions of generalized nonlinear Schrödinger equation with wave operator

$$Wu + i\alpha u_t + i\theta u_x + \delta^2 u + \beta |u|^{2p} u = 0, \tag{1.5}$$

where $Wu = u_{tt} - u_{xx} + \gamma u_{tx}$ is the wave operator, α, θ, δ, β, γ, and $p > 0$ are all given constants. $|u|^{2p}u$ is the nonlinear term, the parameter p reflects the strength of the nonlinearity. We investigate the dynamical bifurcations and exact traveling wave solutions for the generalized nonlinear Schrödinger equation with wave operator under different parametric conditions by means of the theory of singular system. We analyse the high order equilibrium point and give the phase portraits. We obtain the explicit exact solitary wave solutions and the double periodic wave solutions by means of Jacobian elliptic function for the generalized nonlinear Schrödinger equation with wave operator in various cases of parameters.

The paper is built up as follows. In section 2, we transform equation (1.5) into a plane dynamical system and consider the bifurcations of phase portraits of plane dynamical system. In section 3, for the case $p = 1$, corresponding to real curves defined by $H(\phi, y) = h$, we find all possible exact explicit solutions for equation (1.5). In section 4, for the case $p \geq \frac{1}{2}$, we obtain the existence of smooth solitary traveling wave and periodic traveling wave solutions of equation (1.5). Corresponding to real curves defined by $H(\phi, y) = 0$, we give an explicit expression for the only envelope solitary wave solution for any of $p > 0$. For $p = \frac{1}{2}$, we also obtain the explicit exact double periodic wave solutions by means of Jacobian elliptic function, corresponding to real curves defined by $H(\phi, y) = h$.

2. Bifurcations of generalized nonlinear Schrödinger equation with wave operator

To analyze the traveling wave solutions of generalized nonlinear Schrödinger equation with wave operator (1.5), we introduce a gauge transformation

$$u(x,t) = \varphi(x,t) \exp\left[i(kx + \omega t + \xi_0)\right], \tag{2.1}$$

where $\varphi(x,t)$ is a real-valued function, k, ω are two real constants to be determined, ξ_0 is an arbitrary constant. Substituting (2.1) into (1.5), yields

$$\begin{aligned} &\varphi_{tt} - \varphi_{xx} + \gamma\varphi_{tx} + \left(-\alpha\omega - k\theta - k\omega\gamma + \delta^2 + k^2 - \omega^2\right)\varphi \\ &+ \beta\varphi^{2p+1} + i\left[\left(\alpha + k\gamma + 2\omega\right)\varphi_t + \left(\theta - 2k + \omega\gamma\right)\varphi_x\right] = 0. \end{aligned} \tag{2.2}$$

Separating the real and imaginary parts, we get

$$\varphi_{tt} - \varphi_{xx} + \gamma\varphi_{tx} + \left(-\alpha\omega - k\theta - k\omega\gamma + \delta^2 + k^2 - \omega^2\right)\varphi + \beta\varphi^{2p+1} = 0, \tag{2.3}$$

$$\left(\alpha + k\gamma + 2\omega\right)\varphi_t + \left(\theta - 2k + \omega\gamma\right)\varphi_x = 0. \tag{2.4}$$

Solving Eq. (2.4), we have

$$\varphi(x,t)=\varphi(\xi)=\varphi\Big(\left(\alpha+k\gamma+2\omega\right)x-\left(\theta-2k+\omega\gamma\right)t+\xi_1\Big), \tag{2.5}$$

where ξ_1 is an arbitrary constant. Substituting (2.5) into (2.3), we get

$$\begin{aligned}&\big(-k\omega\gamma^3-\alpha\omega\gamma^2+k^2\gamma^2-k\theta\gamma^2-\omega^2\gamma^2-\alpha\theta\gamma-4k\omega\gamma\\&-\alpha^2-4\alpha\omega+4k^2-4k\theta-4\omega^2+\theta^2\big)\varphi''\\&+(-k\omega\gamma-\alpha\omega+\delta^2+k^2-k\theta-\omega^2)\varphi+\beta\varphi^{2p+1}=0.\end{aligned} \tag{2.6}$$

Setting $M=-k\omega\gamma^3-\alpha\omega\gamma^2+k^2\gamma^2-k\theta\gamma^2-\omega^2\gamma^2-\alpha\theta\gamma-4k\omega\gamma-\alpha^2-4\alpha\omega+4k^2-4k\theta-4\omega^2+\theta^2$, $N=-k\omega\gamma-\alpha\omega+\delta^2+k^2-k\theta-\omega^2$, then (2.6) becomes

$$M\varphi''(\xi)+N\varphi(\xi)+\beta\varphi^{2p+1}(\xi)=0. \tag{2.7}$$

For $p>0$, making transformation

$$\varphi(\xi)=(\phi(\xi))^{\frac{1}{2p}}, \tag{2.8}$$

we obtain

$$2Mp\phi\phi''+(1-2p)M\phi'^2+4p^2\phi^2(N+\beta\phi)=0, \tag{2.9}$$

which is equivalent to the system

$$\frac{d\phi}{d\xi}=y,\qquad \frac{dy}{d\xi}=\frac{(2p-1)My^2-4p^2\phi^2[N+\beta\phi]}{2Mp\phi}. \tag{2.10}$$

Apply $d\xi=2Mp\phi d\zeta$, the singular system (2.10) has the same invariant curve solutions as the associated regular system

$$\frac{d\phi}{d\zeta}=2Mp\phi y,\qquad \frac{dy}{d\zeta}=(2p-1)My^2-4p^2\phi^2[N+\beta\phi] \tag{2.11}$$

with the first integral

$$H(\phi,y)=\phi^{\frac{1}{p}-2}\left(\frac{4p^2\beta}{M(p+1)}\phi^3+\frac{4p^2N}{M}\phi^2+y^2\right)=h, \tag{2.12}$$

where h is an integral constant.

For $p\neq\frac{1}{2}$, $N\neq 0$, the origin $O(0,0)$ is a two-order equilibrium point of system (2.11). To consider the directions that the orbits of system (2.11) tend to the origin when $\zeta\to-\infty$ (or ∞), from [17], we have $G_1(\theta)=-\cos\theta(M\sin^2\theta+4p^2N\cos^2\theta)=0$. Therefore, there are six sectors lie the different types of orbits of system (2.11) when $MN<0$. When $MN>0$, there exist two areas (left phase plane and right phase plane) laying different orbits of system (2.11).

For $p\neq\frac{1}{2}$, $N=0$, the origin $O(0,0)$ is a three-order equilibrium point of system (2.11). Then $G_2(\theta)=-M\cos\theta\sin^2\theta=0$, the roots of $G_2(\theta)$ are 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π. Thus, there exist four areas laying different orbits of system (2.11).

Note that $f(\phi)=N+\beta\phi$, it is obvious that (2.11) has an equilibrium point $\left(-\frac{N}{\beta},0\right)$. Let $M(\phi,0)$ be the coefficient matrix of the linearized system of (2.11) at an equilibrium point and $J(\phi,0)=detM(\phi,0)$. We have

$$Trace(M(\phi,0))=0, \tag{2.13}$$

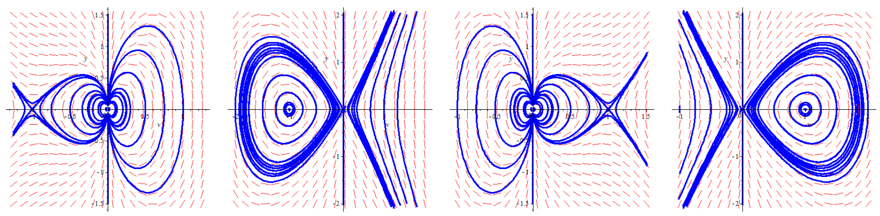
$$J(0,0) = \det(M(0,0)) = 0, \tag{2.14}$$

$$J(-\frac{N}{\beta}, 0) = -\frac{8Mp^3N^3}{\beta^2}. \tag{2.15}$$

We denote that

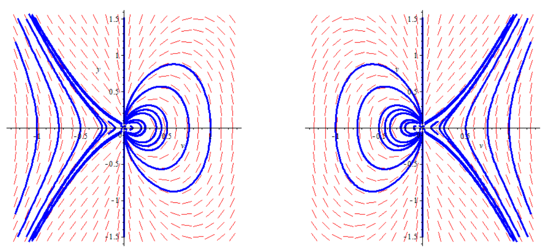
$$h_1 = H(-\frac{N}{\beta}, 0) = \frac{4Np^3}{M(p+1)} \left(-\frac{N}{\beta}\right)^{\frac{1}{p}}. \tag{2.16}$$

By using the above information, for $p > \frac{1}{2}$, depending on the change of parameters, we have the bifurcations of phase portraits of system (2.11) shown in Figure 1 and Figure 2.



(a) $\beta M > 0$, $MN > 0$ (b) $\beta M < 0$, $MN < 0$ (c) $\beta M < 0$, $MN > 0$ (d) $\beta M > 0$, $MN < 0$

Figure 1. The bifurcations of phase portraits of system (2.11) for $N \neq 0$



(a) $\beta M > 0$ (b) $\beta M < 0$

Figure 2. The bifurcations of phase portraits of system (2.11) for $N = 0$

3. The exact traveling wave solutions of equation (1.5) with $p = 1$

In this section, we investigate the exact solutions of equation (1.5) when $p = 1$ and $\beta > 0$. Since the transformation (2.8), we only consider the positive solutions of ϕ in the right phase plane. From (2.12), we have

$$\begin{aligned} y^2 =& \frac{2\beta}{M}\phi\left(-\phi^2 - \frac{2N}{\beta}\phi + \frac{Mh}{2\beta}\right), \quad \text{for } M > 0, \\ y^2 =& \frac{2\beta}{|M|}\phi\left(\phi^2 + \frac{2N}{\beta}\phi + \frac{|M|h}{2\beta}\right), \quad \text{for } M < 0. \end{aligned} \tag{3.1}$$

By using the first equation of system (2.10) and (3.1), we can get bounded exact solutions of equation (1.5) with $p = 1$.

Case I $M > 0$, $N > 0$(refer to Figure 1(a)).

For $h \in (0, \infty)$, the curves in the right phase plane defined by $H(\phi, y) = h$ are a family of periodic orbits of system (2.11) contact to the singular straight line at $O(0,0)$. Now $\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2\beta}{M}\phi\left(\phi - \phi_m\right)\left(\phi_M - \phi\right)$, where $(\phi_M, 0)$ and $(\phi_m, 0)$ are intersections of curves and ϕ-axis, $\phi_m = -\frac{N}{\beta} - \frac{\sqrt{4N^2+2Mh\beta}}{2\beta}$, $\phi_M = -\frac{N}{\beta} + \frac{\sqrt{4N^2+2Mh\beta}}{2\beta}$. Hence, for $\phi \in (0, \phi_M)$, we have the following family of double periodic wave solutions:

$$\phi(\xi) = \phi_M cn^2\left(\Omega_1\xi, K_1\right), \tag{3.2}$$

where $\Omega_1 = \left(\frac{\sqrt{4N^2+2Mh\beta}}{2M}\right)^{\frac{1}{2}}$, $K_1 = \sqrt{\frac{-2N+\sqrt{4N^2+2Mh\beta}}{2\sqrt{4N^2+2Mh\beta}}}$, $sn(\cdot, K)$, $cn(\cdot, K)$, $dn(\cdot, K)$, are Jacobian elliptic functions (refer to [1]).

From (2.1), (2.8), and (3.2), we obtain the exact envelope period wave solutions of equation (1.5) expressed by the Jacobian elliptic cosine function

$$u(x,t) = \pm\sqrt{\phi_M} cn\left(\Omega_1\xi, K_1\right)\exp\left(i\eta\right), \tag{3.3}$$

where $\xi = \left(\alpha + k\gamma + 2\omega\right)x - \left(\theta - 2k + \omega\gamma\right)t + \xi_1$, $\eta = kx + \omega t + \xi_0$, ξ_0 and ξ_1 are arbitrary constants.

Case II $M < 0$, $N < 0$(refer to Figure 1(c)).

(1) For $h \in (0, h_1)$, the curves in the right phase plane defined by $H(\phi, y) = h$ are close branches contact to the singular straight line $\phi = 0$ at $O(0,0)$. We have $\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2\beta}{|M|}\phi\left(\phi_m - \phi\right)\left(\phi_M - \phi\right)$. Here $\phi_m = -\frac{N}{\beta} - \frac{\sqrt{4N^2-2|M|h\beta}}{2\beta}$, $\phi_M = -\frac{N}{\beta} + \frac{\sqrt{4N^2-2|M|h\beta}}{2\beta}$. Thus, for $\phi \in (0, \phi_m)$, we have the following family of double periodic wave solutions:

$$\phi(\xi) = \phi_m sn^2\left(\Omega_2\xi, K_2\right), \tag{3.4}$$

where $\Omega_2 = \left(\frac{-2N+\sqrt{4N^2-2|M|h\beta}}{4|M|}\right)^{\frac{1}{2}}$, $K_2 = \sqrt{\frac{2N+\sqrt{4N^2-2|M|h\beta}}{2N-\sqrt{4N^2-2|M|h\beta}}}$.

Due to (2.1), (2.8), and (3.4), Eq. (1.5) admits the exact envelope period wave solutions expressed by Jacobian elliptic sine function

$$u(x,t) = \pm\sqrt{\phi_m} sn\left(\Omega_2\xi, K_2\right)\exp\left(i\eta\right), \tag{3.5}$$

where $\xi = \left(\alpha + k\gamma + 2\omega\right)x - \left(\theta - 2k + \omega\gamma\right)t + \xi_1$, $\eta = kx + \omega t + \xi_0$, ξ_0 and ξ_1 are arbitrary constants.

(2) For $h = h_1$, the curves in the right phase plane is a homoclinic orbit. We have that $\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2\beta}{|M|}\phi\left(-\frac{N}{\beta} - \phi\right)^2$. For $\phi \in \left(0, -\frac{N}{\beta}\right]$, we get the following exact solution of system (2.11):

$$\phi(\xi) = -\frac{N}{\beta}\left[1 - sech^2\left(\sqrt{-\frac{N}{2|M|}}\xi\right)\right]. \tag{3.6}$$

According to expressions (2.1), (2.8), and (3.6), we obtain the dark soliton solution of Eq.(1.5)

$$u(x,t) = \pm\sqrt{-\frac{N}{\beta}}\left[\tanh\left(\sqrt{-\frac{N}{2|M|}}\xi\right)\right]\exp\left(i\eta\right), \tag{3.7}$$

where $\xi = (\alpha + k\gamma + 2\omega)\, x - (\theta - 2k + \omega\gamma)\, t + \xi_1$, $\eta = kx + \omega t + \xi_0$, ξ_0 and ξ_1 are arbitrary constants.

Case III $M > 0$, $N < 0$(refer to Figure 1(d)).

(1) When $h \in (h_1, 0)$, the curves in the right phase plane defined by $H(\phi, y) = h$ are a family of close orbits of system (2.11), enclosing the equilibrium point $\left(-\frac{N}{\beta}, 0\right)$. In this case, we have $\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2\beta}{M}\phi\left(\phi - \phi_m\right)\left(\phi_M - \phi\right)$, where $\phi_m = -\frac{N}{\beta} - \frac{\sqrt{4N^2+2Mh\beta}}{2\beta}$, $\phi_M = -\frac{N}{\beta} + \frac{\sqrt{4N^2+2Mh\beta}}{2\beta}$. For $\phi \in [\phi_m, \phi_M]$, system (2.11) admits the following exact periodic wave solutions:

$$\phi(\xi) = \phi_M dn^2\left(\Omega_3\xi, K_3\right), \tag{3.8}$$

where $\Omega_3 = \left(\frac{-2N+\sqrt{4N^2+2Mh\beta}}{4M}\right)^{\frac{1}{2}}$, $K_3 = \sqrt{\frac{2\sqrt{4N^2+2Mh\beta}}{-2N+\sqrt{4N^2+2Mh\beta}}}$.

Expression (2.1), (2.8), and (3.8) implies that Eq.(1.5) has the exact envelope periodic wave solutions expressed by Jacobian elliptic function

$$u(x,t) = \pm\sqrt{\phi_M}dn\left(\Omega_3\xi, K_3\right)\exp\left(i\eta\right), \tag{3.9}$$

where $\xi = (\alpha + k\gamma + 2\omega)\, x - (\theta - 2k + \omega\gamma)\, t + \xi_1$, $\eta = kx + \omega t + \xi_0$, ξ_0 and ξ_1 are arbitrary constants.

(2) When $h = 0$, the curve in the right phase plane defined by $H(\phi, y) = h$ is a homoclinic orbit to the origin $O(0,0)$. From (3.1), we have that $\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2\beta}{M}\phi^2\left(-\frac{2N}{\beta} - \phi\right)$. For $\phi \in \left(0, -\frac{2N}{\beta}\right]$, it gives rise to the following solution of system (2.11):

$$\phi(\xi) = \left(-\frac{2N}{\beta}\right) sech^2\left(\sqrt{-\frac{N}{M}}\xi\right). \tag{3.10}$$

From (2.1), (2.8), and (3.10), we know that Eq. (1.5) admits the bright soliton solution

$$u(x,t) = \pm\sqrt{-\frac{2N}{\beta}} sech\left(\sqrt{-\frac{N}{M}}\xi\right)\exp\left(i\eta\right), \tag{3.11}$$

where $\xi = (\alpha + k\gamma + 2\omega)\, x - (\theta - 2k + \omega\gamma)\, t + \xi_1$, $\eta = kx + \omega t + \xi_0$, ξ_0 and ξ_1 are arbitrary constants.

(3) For $h \in (0, \infty)$, the curves in the right phase plane defined by $H(\phi, y) = h$ has a family of closed branch enclosing the homoclinic orbit defined by $H(\phi, y) = 0$. We have $\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2\beta}{M}\phi\left(\phi - \phi_m\right)\left(\phi_M - \phi\right)$, where $\phi_m = -\frac{N}{\beta} - \frac{\sqrt{4N^2+2Mh\beta}}{2\beta}$, $\phi_M = -\frac{N}{\beta} + \frac{\sqrt{4N^2+2Mh\beta}}{2\beta}$. For $\phi(\xi) \in (0, \phi_M]$, it gives rise to the same exact periodic solution family of system (2.11) as (3.2), and equation (1.5) has the same exact periodic wave solutions as (3.3).

Case IV $M>0$, $N=0$(refer to Figure 2(a)).

For $h \in (0,\infty)$, the curves in the right phase plane defined by $H(\phi,y)=h$ are a family of periodic orbits of system (2.11). Then $\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2\beta}{M}\phi\left(\phi-\phi_m\right)\left(\phi_M-\phi\right)$, where $\phi_m = -\sqrt{\frac{Mh}{2\beta}}$, $\phi_M = \sqrt{\frac{Mh}{2\beta}}$. Hence, for $\phi \in (0,\phi_M)$, we have the following family of periodic wave solutions:

$$\phi(\xi) = \phi_M cn^2\left(\Omega_4\xi, K_4\right), \tag{3.12}$$

where $\Omega_4 = \left(\frac{h\beta}{2M}\right)^{\frac{1}{4}}$, $K_4 = \sqrt{\frac{1}{2}}$.

Owing to (2.1), (2.8), and (3.12), we obtain the exact envelope periodic wave solutions of equation (1.5) expressed by Jacobian elliptic cosine function

$$u(x,t) = \pm\sqrt{\phi_M} cn\left(\Omega_4\xi, K_4\right)\exp\left(i\eta\right), \tag{3.13}$$

where $\xi = (\alpha + k\gamma + 2\omega)\,x - (\theta - 2k + \omega\gamma)\,t + \xi_1$, $\eta = kx + \omega t + \xi_0$, ξ_0 and ξ_1 are arbitrary constants.

To sum up, we have the following main conclusion:

Theorem 3.1. *For $p=1$, under different parameter conditions, corresponding to the bounded real curves defined by $H(\phi,y)=h$, equation* (1.5) *has the exact explicit solutions given by* (3.3)*,* (3.5)*,* (3.7)*,* (3.9)*,* (3.11)*,* (3.13)*.*

4. The existence of smooth traveling wave solutions when $p \geq \frac{1}{2}$, $\beta > 0$

In section 3, we give a various of different explicit smooth wave solutions of Eq.(1.5) when $p=1$ and $\beta>0$. But for any p, it's not easily to find the explicit exact solutions from (2.12). In this section, we discuss the existence of the bounded solutions of equation (1.5) by singular dynamical system approach [12, 13]. As the parameters varying, we can also obtain the explicit expressions of solitary wave and periodic wave in special cases. We draw conclusions as follow.

Theorem 4.1. *For $p \geq \frac{1}{2}$, $\beta > 0$, equation* (1.5) *has the solitary wave solutions, under one of the following conditions*

(1) $M<0$, $N<0$, curves correspond to $H(\phi,y)=h_1$ in (2.12)*(refer to Figure* 1*(c)).*

(2) $M>0$, $N<0$, curves correspond to $H(\phi,y)=0$ in (2.12)*(refer to Figure* 1*(d)).*

Especially, in the case (2) of Theorem 4.1, we obtain the unique explicit envelope solitary wave solution of equation (1.5)

$$u(x,t) = \left(-\frac{N(p+1)}{\beta}\right)^{\frac{1}{2p}} sech^{\frac{1}{p}}\left(\sqrt{-\frac{N}{M}}p\xi\right)\exp\left(i\eta\right), \tag{4.1}$$

where $\xi = (\alpha + k\gamma + 2\omega)\,x - (\theta - 2k + \omega\gamma)\,t + \xi_1$, $\eta = kx + \omega t + \xi_0$, ξ_0 and ξ_1 are arbitrary constants.

Theorem 4.2. *For $p \geq \frac{1}{2}$, $\beta > 0$, equation* (1.5) *has the bounded periodic wave solutions, under one of the following conditions*

(1) $M>0$, $N>0$, *curves correspond to* $H(\phi,y)=h$, $h\in(0,\infty)$ *in* (2.12)*(refer to Figure* 1*(a)).*

(2) $M<0$, $N<0$, *curves correspond to* $H(\phi,y)=h$, $h\in(0,h_1)$ *in* (2.12)*(refer to Figure* 1*(c)).*

(3) $M>0$, $N<0$, *curves correspond to* $H(\phi,y)=h$, $h\in(h_1,0)\cup(0,\infty)$ *in* (2.12)*(refer to Figure* 1*(d)).*

(4) $M>0$, $N=0$, *curves correspond to* $H(\phi,y)=h$, $h\in(0,\infty)$ *in* (2.12)*(refer to Figure* 2*(a)).*

In special case of $p=\frac{1}{2}$, we can obtain the explicit exact traveling wave solutions of Eq.(1.5). For $M>0$, $N>0$, $h\in(0,h_1)$ (or $M>0$, $N<0$, $h\in(h_1,0)$), Eq.(1.5) has the envelope double periodic wave solutions expressed by Jacobian elliptic function

$$u(x,t)=\left\{r_2+(r_3-r_2)\,cn^2\left(\sqrt{\frac{\beta(r_3-r_1)}{6M}}\xi,\sqrt{\frac{r_3-r_2}{r_3-r_1}}\right)\right\}\exp\left(i\eta\right), \tag{4.2}$$

where $\xi=(\alpha+k\gamma+2\omega)\,x-(\theta-2k+\omega\gamma)\,t+\xi_1$, $\eta=kx+\omega t+\xi_0$, ξ_0 and ξ_1 are arbitrary constants. It corresponds to Theorem 4.2 (1) and (3).

For $M<0$, $N<0$, $h\in(0,h_1)$, we obtain the envelope double periodic wave solutions expressed by Jacobian elliptic function

$$u(x,t)=\left\{r_1+(r_2-r_1)\,sn^2\left(\sqrt{-\frac{\beta(r_3-r_1)}{6M}}\xi,\sqrt{\frac{r_2-r_1}{r_3-r_1}}\right)\right\}\exp\left(i\eta\right), \tag{4.3}$$

where $\xi=(\alpha+k\gamma+2\omega)\,x-(\theta-2k+\omega\gamma)\,t+\xi_1$, $\eta=kx+\omega t+\xi_0$, ξ_0 and ξ_1 are arbitrary constants. It corresponds to Theorem 4.2 (2).

For $M>0$, $N>0$, $h\in(h_1,\infty)$ (or $M>0$, $N<0$, $h\in(-\infty,h_1)$), we obtain the envelope double periodic wave solutions expressed by Jacobian elliptic function

$$u(x,t)=\frac{\left(r_1+\sqrt{\Delta}\right)cn(\Omega_5,K_5)-\left(r_1-\sqrt{\Delta}\right)}{cn(\Omega_5,K_5)-1}\exp\left(i\eta\right), \tag{4.4}$$

where $\Delta=(r_1-b_1)^2+a_1^3$, $\Omega_5=\left(\frac{2\beta\sqrt{\Delta}}{3M}\right)^{\frac{1}{2}}$, $K_5=\sqrt{\frac{\sqrt{\Delta}+r_1-b_1}{2\sqrt{\Delta}}}$, $\xi=(\alpha+k\gamma+2\omega)\,x-(\theta-2k+\omega\gamma)\,t+\xi_1$, $\eta=kx+\omega t+\xi_0$, ξ_0 and ξ_1 are arbitrary constants. It corresponds to Theorem 4.2 (1) and (3).

The above r_1, r_2, r_3 are the real roots of $-\frac{2\beta}{3M}\phi^3-\frac{N}{M}\phi^2+h=0$, $r_1<r_2<r_3$. b_1+ia_1 and b_1-ia_1 are the conjugate complex roots of $-\frac{2\beta}{3M}\phi^3-\frac{N}{M}\phi^2+h=0$.

For $p\geq\frac{1}{2}$, $\beta<0$, we can draw similar conclusions as above.

5. Conclusion

In this paper, we consider the generalized nonlinear Schrödinger equation with wave operator. This partial differential equation can be transformed into a two-dimensional integrable system with a singular line. Then we analyze the dynamics and bifurcation behavior of traveling wave solutions of system by using the theory of singular systems. Particularly, we analyze the cases of high order equilibrium point. Moreover, we find the solitary wave and periodic wave solutions for the classical nonlinear Schrödinger equation with wave operator. Finally, we obtain the existence of different type traveling wave solutions and give the explicit expressions of exact solutions in the special cases by applying singular dynamical system theory. All above results indicate that the theory of singular systems is very helpful for studying the traveling wave solution of nonlinear differential equation.

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