Journal of Applied Analysis and Computation
Volume 12, Number 1, February 2022, 347–360
Website:http://www.jaac-online.com
DOI:10.11948/20210218

EXISTENCE AND MULTIPLICITY OF SIGN-CHANGING SOLUTIONS FOR THE DISCRETE PERIODIC PROBLEMS WITH MINKOWSKI-CURVATURE OPERATOR*

Zhongzi Zhao[1,†], Ruyun Ma[1,2] and Yan Zhu[2]

Abstract We are concerned with the discrete periodic problems with Minkowski-curvature operator

$$\begin{cases} -\nabla(\dfrac{\Delta u(t)}{\sqrt{1-(\Delta u(t))^2}}) = \lambda g(t,u(t)), \quad t\in\mathbb{T}, \\ u(0)=u(T), u(1)=u(T+1), \end{cases} \qquad (P)$$

where $T>2$ is an integer, $\mathbb{T}:=\{1,2,\cdots,T\}$, $\hat{\mathbb{T}}=\{0,1,\cdots,T,T+1\}$, $\mathbb{Z}:=\{\cdots,-1,0,1,\cdots\}$, $g:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is continuous, $g_0:=\lim\limits_{s\to0}\frac{g(t,s)}{s}=+\infty$ uniformly for $t\in\hat{\mathbb{T}}$, g is T-periodic respect to t, $\lambda\in(0,\infty)$ is a parameter. We show that (P) has multiple odd sign-changing solutions and multiple even sign-changing solutions when $g(t,\cdot)$ is odd. The proof of our main result is based upon bifurcation techniques.

Keywords Minkowski-curvature operator, discrete periodic problem, odd solutions, even solutions.

MSC(2010) 39A22, 39A23, 39A27, 39A28.

1. Introduction

Let $T>2$ be an integer, $\mathbb{T}:=[1,T]_{\mathbb{Z}}=\{1,2,\cdots,T\}$, $\hat{\mathbb{T}}:=[0,T+1]_{\mathbb{Z}}=\{0,1,\cdots,T,T+1\}$, $\mathbb{Z}:=\{\cdots,-1,0,1,\cdots\}$.

Define Δ and ∇ by

$$\Delta u(t)=u(t+1)-u(t), \quad \nabla u(t)=u(t)-u(t-1),$$

respectively. In this paper, we are concerned with the discrete periodic problems with Minkowski-curvature operator

$$\begin{cases} -\nabla(\dfrac{\Delta u(t)}{\sqrt{1-(\Delta u(t))^2}}) = \lambda g(t,u(t)), \quad t\in\mathbb{T}, \\ u(0)=u(T), u(1)=u(T+1), \end{cases} \qquad (1.1)$$

† The corresponding author. Email: 15193193403@163.com(Z. Zhao)
1 School of Mathematics and Statistics, Xidian University, Xi'an 710071, China
2 Department of Mathematics, Northwest Normal University, Lanzhou, 730070, China
* The authors were supported by National Natural Science Foundation of China (No. 12061064).

where $g:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is continuous, g is T-periodic respect to t, $\lambda\in(0,\infty)$ is a parameter. Assume that $g(t,\cdot)$ is odd and $g_0:=\lim\limits_{s\to 0}\frac{g(t,s)}{s}=+\infty$ uniformly for $t\in\hat{\mathbb{T}}$. A *solution* of (1.1) is a function $u:\hat{\mathbb{T}}\to\mathbb{R}$ such that $\max\limits_{t\in\hat{\mathbb{T}}}|\Delta u(t)|<1$ and u satisfies (1.1).

The equation in (1.1) is the one-dimensional discrete version of the following

$$-\mathrm{div}\Big(\frac{\mathrm{grad}u}{\sqrt{1-|\mathrm{grad}u|^2}}\Big)=\lambda g(x,u)\quad \text{in } \Omega, \tag{1.2}$$

where $u\mapsto -\mathrm{div}(\frac{\mathrm{grad}u}{\sqrt{1-|\mathrm{grad}u|^2}})$ in turn is usually meant as a mean-curvature operator in Lorentz-Minkowski spaces, Ω is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\Omega$ of class C^2.

The existence and multiplicity of solutions for the prescribed mean curvature equations with Dirichlet boundary conditions and the discrete analogue have been widely discussed by various methods, see [9,10,12,15,25] and the references therein.

In [14] some general solvability results for (1.2) with Dirichlet boundary conditions were proved under the assumption that the function g is globally bounded. Yet, as all spacelike solutions of the Dirichlet problem are uniformly bounded by the quantity $\frac{1}{2}d(\Omega)$, with $d(\Omega)$ the diameter of Ω, one can always reduce to that situation by truncation, see Coelho et al. [9,10].

Nevertheless it should be observed that the situation differs substantially for periodic solutions of (1.2). In the continuous case, existence and multiplicity of periodic solutions for the equations with mean-curvature operator have been studied extensively, see Benevieri, Marcos do Ó and de Medeiros [3], Bereanu, Jebelean and Mawhin [4], Bereanu and Mawhin [5], Bereanu and Zamora [7], Bosecaggin and Feltrin [8].

However, there are many differences, even essential differences, between the difference equation and the corresponding differential equation. From the definition of Strogatz [28], chaos sensitivity depends on initial conditions. That is shown that nearby trajectories diverge exponentially. Continuous systems in a 2-dimensional phase space cannot experience such a divergence, hence chaotic behaviors can only be observed in deterministic continuous systems with a phase space of dimension 3, at least. On the other hand, in a discrete map it is well known that chaos occurs also in one-dimension. Therefore, discrete chaotic systems exhibit chaos whatever their dimension.

In addition, unlike the continuous case, the discrete linear eigenvalue problem

$$\begin{cases} -\nabla(\Delta u(t))=\lambda u(t), & t\in\mathbb{T},\\ u(0)=u(T), \quad u(1)=u(T+1) \end{cases} \tag{1.3}$$

has only a finite number of eigenvalues.

Let

$$N:=\begin{cases} \dfrac{T}{2}, & \text{as } T \text{ is even},\\ \dfrac{T-1}{2}, & \text{as } T \text{ is odd}. \end{cases}$$

From [19] and [20], the eigenvalues of (1.3) are the following

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_N.$$

The eigenspace corresponding to λ_0 is span$\{1\}$, the eigenspace corresponding to λ_j, $j \in [1, N-1]_{\mathbb{Z}}$ is

$$M_j = \text{span}\{\cos\frac{2j\pi t}{T}, \sin\frac{2j\pi t}{T}\}, \quad t \in \mathbb{T},$$

and the eigenspace corresponding to λ_N is

$$M_N = \text{span}\{\cos\frac{2N\pi t}{T}, \sin\frac{2N\pi t}{T}\}, \qquad t \in \mathbb{T},\ T \text{ is odd},$$
$$M_N = \text{span}\{\cos\frac{2N\pi t}{T}\} = \text{span}\{\cos \pi t\}, \quad t \in \mathbb{T},\ T \text{ is even}.$$

Semilinear discrete boundary value problems was studied by Atici and Cabada [1], Atici, Cabada and Otero-Espinar [2], Henderson and Thompson [16], Ma and Ma [19,20]. For example, Atici and Cabada [1] concerned with proving the existence of positive solutions of a periodic boundary value problem for a discrete nonlinear equation

$$\begin{cases} -\Delta[p(n-1)\Delta u(n-1)] + q(n)u(n) = \lambda f(n, u(n)), \qquad n \in [1, N]_{\mathbb{Z}}, \\ u(0) = u(N), \quad p(0)\Delta u(0) = p(N)\Delta u(N), \end{cases} \tag{1.4}$$

where f is a continuous function on $\mathbb{R}$. Under certain conditions of f, they obtained that there exists at least one positive solution of (1.4) for λ belonging to a given interval.

Existence and multiplicity of solutions for discrete problems with periodic boundary condition involving mean curvature operator have been studied by some authors, see Bereanu and Thompson [6], Mawhin [22,23] and the references therein. In particular, Bereanu and Thompson [6] considered the forced equation involving the discrete φ-Laplacian

$$D\varphi(Dx_k) = h_k \quad (2 \le k \le n-1), \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}, \tag{1.5}$$

where φ can be the mean curvature operator and $h = (h_2, \cdots, h_{n-1})$ satisfies $\sum\limits_{k=2}^{n-1} h_k = 0$. They obtained the result as following.

Theorem A. *Assume that there exist numbers α, β such that*

$$f_k(\alpha) \ge 0 \ge f_k(\beta) \ (2 \le k \le n-1).$$

Then problem (1.5) has at least one solution with $\alpha \le x_k \le \beta$ $(1 \le k \le n)$.

In fact, most people only obtain the existence of positive solutions, or even the existence of solutions. Very little is known about existence and multiplicity of sign-changing periodic solutions for the problem (1.1). The likely reason is that the multiplicity of higher eigenvalues, $\lambda_k (k \in \{1, 2, \cdots, N-1\})$, is even. However, the key of Rabinowitz global bifurcation theorem is $\dim M_k = 1$, see Rabinowitz [26,27].

In order to apply the bifurcation techniques, we have to work in

$$E_1 := \{u \in H | u(-t) = -u(t), t \in \mathbb{Z}\}$$

and

$$E_2 := \{u \in H | u(-t) = u(t), t \in \mathbb{Z}\},$$

which are two subspaces of

$$H := \{u : \mathbb{Z} \to \mathbb{R} |\ u \text{ is } T\text{-periodic}\}.$$

The above method of constructing invariant subspaces is motivated by Coron [11] and Marlin [21].

Therefore, the purpose of this paper is to use the bifurcation techniques to investigate the existence and multiplicity of sign-changing solutions for (1.1), which is a discrete periodic problem with mean curvature operators in the Minkowski space.

Due to $g_0 = +\infty$, the global bifurcation techniques cannot be used directly in the case. Therefore, we referred to [18] and applied some properties of the superior limit of a certain infinity collection of connected sets. In addition, we also made some estimates on the properties of solutions of (1.1), and proved the existence of a *priori bound* for the solutions of (1.1).

For other nonlinear boundary value problems with odd nonlinearity g, see Naito and Tanaka [24], Garcia-Huidobro and Ubilla [13].

Assume that $g(t, \cdot)$ is odd for all $t \in \hat{\mathbb{T}}$, and satisfies either

(H1) there exists $0 < s_0 < \infty$ such that

$$\begin{aligned} &g(t, s) > 0 \ \text{ for all } \ 0 < s < s_0, \quad t \in \hat{\mathbb{T}}, \\ &g(t, s_0) = 0 \ \text{ for all } \ t \in \hat{\mathbb{T}}, \\ &g(t, s) < 0 \ \text{ for all } \ s > s_0, \quad t \in \hat{\mathbb{T}} \end{aligned}$$

or

(H2) $g(t, s) \geq 0, s \in [0, \infty), t \in \hat{\mathbb{T}}$.

Theorem 1.1. *Assume that*

$$g_0 := \lim_{s \to 0} \frac{g(t, s)}{s} = +\infty \tag{1.6}$$

uniformly for $t \in \hat{\mathbb{T}}$. *Then*

(i) for any $k \in [1, N-1]_{\mathbb{Z}}$, $\nu \in \{+, -\}$, *there exists a connected component* $\mathcal{C}_k^\nu \subset \mathbb{R} \times E_1$ *of sign-changing solutions of (1.1) joins* $(0, 0)$ *with infinity in* λ *direction, and any solution* $(\lambda, u) \in \mathcal{C}_k^\nu$ *satisfies* $\max_{t \in \hat{\mathbb{T}}} |u(t)| < \frac{T}{2}$*;*

(ii) for any $k \in [1, N]_{\mathbb{Z}}$, $\nu \in \{+, -\}$, *if* g *satisfies the condition (H1), then there exists a connected component* $\mathcal{D}_k^\nu \subset \mathbb{R} \times E_2$ *of sign-changing solutions of (1.1) joins* $(0, 0)$ *with infinity in* λ *direction, and any solution* $(\lambda, v) \in \mathcal{D}_k^\nu$ *satisfies* $\max_{t \in \hat{\mathbb{T}}} |v(t)| \leq s_0$*;*

(iii) for any $k \in [1, N]_{\mathbb{Z}}$, $\nu \in \{+, -\}$, *if* g *satisfies the condition (H2), then there exists a connected component* $\mathcal{D}_k^\nu \subset \mathbb{R} \times E_2$ *of sign-changing solutions of (1.1) joins* $(0, 0)$ *with infinity in* λ *direction, and any solution* $(\lambda, v) \in \mathcal{D}_k^\nu$ *satisfies* $\max_{t \in \hat{\mathbb{T}}} |v(t)| < T$.

By a *component* of solution set $\mathcal{S}$ we mean a continuum which is maximal with respect to inclusion ordering.

Corollary 1.1. *Assume that*

$$g_0 := \lim_{s\to 0} \frac{g(t,s)}{s} = +\infty$$

uniformly for $t \in \hat{\mathbb{T}}$. *Then*

(i) for any $\lambda \in (0,+\infty)$, $k \in [1, N-1]_{\mathbb{Z}}$, $\nu \in \{+,-\}$, (1.1) *has odd solutions* u_k^ν *such that* u_k^ν *change their sign* $2k-1$ *times in* $\mathbb{T}$ *and* $u_k^- = -u_k^+$.

(ii) for any $\lambda \in (0,+\infty)$, $k \in [1,N]_{\mathbb{Z}}$, $\nu \in \{+,-\}$, *if* g *satisfies either (H1) or (H2), then* (1.1) *has even solutions* v_k^ν *such that* v_k^ν *change their sign* $2k$ *times in* $\mathbb{T}$ *and* $v_k^- = -v_k^+$.

2. Preliminary results

Definition 2.1 ([17]). Let $y : \hat{\mathbb{T}} \to \mathbb{R}$. If $y(t) = 0$, then t is a zero of $y(t)$. If $y(t) = 0$ and $y(t+1)y(t-1) < 0$, then t is a simple zero of $y(t)$. If $y(t)y(t+1) < 0$, then $s = \frac{ty(t+1)-(t+1)y(t)}{y(t+1)-y(t)}$ is called a nodal point of $y(t)$. The simple zero and the nodal point are called the simple generalized zero of $y(t)$.

The following two lemmas concerning the spectrum properties of the discrete linear problem (1.3). They were established by Ma and Ma [19, 20].

Lemma 2.1 ([19, 20]). *The following are true:*

(i) $$\sum_{t=1}^{T} \cos\frac{2k\pi t}{T} \sin\frac{2k\pi t}{T} = 0, \begin{cases} k = 1,2,\cdots,N, & T \text{ is odd,} \\ k = 1,2,\cdots,N-1, & T \text{ is even;} \end{cases}$$

(ii) $$\sum_{t=1}^{T} \sin\frac{2k\pi t}{T} = 0, \begin{cases} k = 1,2,\cdots,N, & T \text{ is odd,} \\ k = 1,2,\cdots,N-1, & T \text{ is even;} \end{cases}$$

(iii) $$\sum_{t=1}^{T} \sin\frac{2j\pi t}{T} \sin\frac{2k\pi t}{T} = 0, \begin{cases} j\neq k,\ j,k = 1,2,\cdots,N, & T \text{ is odd,} \\ j \neq k,\ j,k = 1,2,\cdots,N-1, & T \text{ is even;} \end{cases}$$

(iv) $$\sum_{t=1}^{T} \cos\frac{2j\pi t}{T} \cos\frac{2k\pi t}{T} = 0, \quad j\neq k, \quad j,k = 1,2,\cdots,N.$$

Set $\varphi_0 = 1$,

$$\varphi_k(t) = \cos\frac{2k\pi t}{T} \Big/ \|\cos\frac{2k\pi t}{T}\|, \ k = 1,2,\cdots,N,$$

$$\psi_k(t) = \sin\frac{2k\pi t}{T} \Big/ \|\sin\frac{2k\pi t}{T}\|, \ k = 1,2,\cdots,N.$$

Then

$$\|\varphi_k\| = 1, \quad \|\psi_k\| = 1, \quad k = 1,2,\cdots,N.$$

Lemma 2.2 ([19, 20]). *The* φ_k *and* ψ_k *defined in Lemma 2.1 satisfy the following*

(i) $\varphi_k (k = 1,2,\cdots,N)$ *changes its sign* $2k$ *times in* $\mathbb{T}$;

(ii) $\psi_k(k = 1, 2, \cdots, N$ *when* T *is odd,* $k = 1, 2, \cdots, N-1$ *when* T *is even) changes its sign* $2k-1$ *times in* $\mathbb{T}$*.*

Definition 2.2 ([29]). Let X be a Banach space and $\{C_n|\ n = 1, 2, \cdots\}$ be a family of subsets of X. Then the the superior limit $\mathcal{D}$ of C_n is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in X | \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x\}.$$

Lemma 2.3 ([18]). *Let X be a Banach space, and let $\{C_n\}$ be a family of connected subsets of X. Assume that*

(a) there exist $z_n \in C_n, n = 1, 2, \cdots,$ and $z^* \in X$, such that $z_n \to z^*$;

(b) $\lim_{n\to\infty} r_n = \infty$, where $r_n = \sup\{\|x\| : x \in C_n\}$;

(c) for every $R > 0$, $\cup_{n=1}^{\infty} C_n \cap B_R$ is a relatively compact set of X, where

$$B_R = \{x \in X|\ \|x\| \le R\}.$$

Then there exists an unbounded component C in $\mathcal{D}$ and $z^* \in C$.

Lemma 2.4. *For any sign-changing solution u of (1.1), if u is odd, then*

$$\|u\| := \max_{t\in\hat{\mathbb{T}}} |u(t)| < \frac{T}{2}.$$

Proof. If $u(0) = u(T) = \|u\|$, then $\|u\| = 0$. The conclusion is clearly correct. If $u(0) = u(T) \neq \|u\|$, we suppose that $|u(t_0)| = \|u\|,\ t_0 \in \hat{\mathbb{T}} \setminus \{0, T, T+1\}$ (Let $t_0 = 1$ if $|u(T+1)| = |u(1)| = \|u\|$).

(i) Assume that $u(t_0) > 0$. Due to u is a sign-changing solution of (1.1), then $|\Delta u(t)| < 1$ for any $t \in \hat{\mathbb{T}}$ and

$$u(t_0) - u(0) = \sum_{i=0}^{t_0-1} \Big(u(i+1) - u(i)\Big) = \sum_{i=0}^{t_0-1} \Delta u(i) \le \sum_{i=0}^{t_0-1} |\Delta u(i)| < \sum_{i=0}^{t_0-1} 1 = t_0,$$

$$u(T) - u(t_0) = \sum_{i=t_0}^{T-1} \Big(u(i+1) - u(i)\Big) = \sum_{i=t_0}^{T-1} \Delta u(i) \ge \sum_{i=t_0}^{T-1} -|\Delta u(i)| > \sum_{i=t_0}^{T-1} -1 = t_0 - T.$$

Therefore,

$$u(t_0) < 0 + \frac{t_0 - (t_0 - T)}{2} < \frac{T}{2}.$$

(ii) Assume that $u(t_0) < 0$. Similarly,

$$|u(t_0)| < \frac{T}{2}.$$

Hence, the proof is completed. □

Lemma 2.5. *Let $u : \hat{\mathbb{T}} \to \mathbb{R}$ such that $\|\Delta u(t)\| < 1$. Then for any $t \in \hat{\mathbb{T}}$,*

$$\nabla\Big(\frac{\Delta u(t)}{\sqrt{1 - (\Delta u(t))^2}}\Big)$$
$$= \nabla(\Delta u(t)) \Big[\frac{\sqrt{1-(\Delta u(t-1))^2}\sqrt{1-(\Delta u(t))^2} + 1 + \Delta u(t-1)\Delta u(t)}{\sqrt{1-(\Delta u(t))^2}\sqrt{1-(\Delta u(t-1))^2}[\sqrt{1-(\Delta u(t))^2} + \sqrt{1-(\Delta u(t-1))^2}]}\Big].$$

Proof. By a simple calculation,

$$\nabla\Big(\frac{u(t)}{v(t)}\Big)=\frac{\nabla u(t)v(t-1)-u(t-1)\nabla v(t)}{v(t)v(t-1)},$$

and

$$\nabla(\sqrt{1-(\Delta u(t))^2})=\frac{-\nabla(\Delta u(t))[\Delta u(t)+\Delta u(t-1)]}{\sqrt{1-(\Delta u(t))^2}+\sqrt{1-(\Delta u(t-1))^2}}.$$

Then

$$\begin{aligned}&\nabla\Big(\frac{\Delta u(t)}{\sqrt{1-(\Delta u(t))^2}}\Big)\\ &=\frac{\nabla(\Delta u(t))\sqrt{1-(\Delta u(t-1))^2}-\Delta u(t-1)\nabla(\sqrt{1-(\Delta u(t))^2})}{\sqrt{1-(\Delta u(t))^2}\sqrt{1-(\Delta u(t-1))^2}}\\ &=\nabla(\Delta u(t))\Big[\frac{\sqrt{1-(\Delta u(t-1))^2}+\Delta u(t-1)\frac{\Delta u(t)+\Delta u(t-1)}{\sqrt{1-(\Delta u(t))^2}+\sqrt{1-(\Delta u(t-1))^2}}}{\sqrt{1-(\Delta u(t))^2}\sqrt{1-(\Delta u(t-1))^2}}\Big]\\ &=\nabla(\Delta u(t))\Big[\frac{\sqrt{1-(\Delta u(t-1))^2}\sqrt{1-(\Delta u(t))^2}+1+\Delta u(t-1)\Delta u(t)}{\sqrt{1-(\Delta u(t))^2}\sqrt{1-(\Delta u(t-1))^2}[\sqrt{1-(\Delta u(t))^2}+\sqrt{1-(\Delta u(t-1))^2}]}\Big].\end{aligned}$$

□

Lemma 2.6. *Assume that g satisfies the condition (H1). Then for any even sign-changing solution u of (1.1),*

$$\max_{t\in\hat{\mathbb{T}}}u(t)\le s_0\quad\text{and}\quad\min_{t\in\hat{\mathbb{T}}}u(t)\ge -s_0.$$

Proof. *Claim.* For all $t\in\hat{\mathbb{T}}$,

$$\nabla(\Delta u(t))<0\quad\text{if}\quad u(t)\in(-\infty,-s_0)\cup(0,s_0);\tag{2.1}$$

$$\nabla(\Delta u(t))>0\quad\text{if}\quad u(t)\in(-s_0,0)\cup(s_0,+\infty).\tag{2.2}$$

Assume that there exists $\bar t\in\mathbb{T}$ such that $u(\bar t)\in(-\infty,-s_0)\cup(0,s_0)$ and $\nabla(\Delta u(\bar t))\ge 0$. Then

$$-\nabla\Big(\frac{\Delta u(\bar t)}{\sqrt{1-(\Delta u(\bar t))^2}}\Big)=\lambda g(\bar t,u(\bar t))>0.$$

From Lemma 2.5 and the fact

$$\Big[\frac{\sqrt{1-(\Delta u(\bar t-1))^2}\sqrt{1-(\Delta u(\bar t))^2}+1+\Delta u(\bar t-1)\Delta u(\bar t)}{\sqrt{1-(\Delta u(\bar t))^2}\sqrt{1-(\Delta u(\bar t-1))^2}[\sqrt{1-(\Delta u(\bar t))^2}+\sqrt{1-(\Delta u(\bar t-1))^2}]}\Big]>0,$$

then

$$\nabla(\Delta u(\bar t))<0.$$

This is a contradiction.

Obviously, (2.2) can be proved by the similar method. Therefore, the claim is proved to be true.

Next, we will show that $\max_{t\in\hat{\mathbb{T}}}u(t)\le s_0$.

Case 1. $\max_{t\in\hat{\mathbb{T}}} u(t) = u(0)$.

Assume that $u(0) > s_0$. Then from (2.2),

$$\nabla(\Delta u(0)) = u(1) - 2u(0) + u(-1) > 0.$$

Due to u is even, then

$$u(1) - 2u(0) + u(-1) = 2(u(1) - u(0)) > 0.$$

This is a contradiction with $\max_{t\in\hat{\mathbb{T}}} u(t) = u(0)$.

Case 2. $\max_{t\in\hat{\mathbb{T}}} u(t) = u(t_0),\ t_0 \in \mathbb{T}$.

Assume that $u(t_0) > s_0$. Due to $u(t_0) = \max_{t\in\hat{\mathbb{T}}} u(t)$, then

$$\Big(u(t_0+1) - u(t_0)\Big) + \Big(u(t_0-1) - u(t_0)\Big) = \nabla(\Delta u(t_0)) \le 0.$$

This is a contradiction with (2.2).

Similarly, $\min_{t\in\hat{\mathbb{T}}} u(t) \ge -s_0$. The proof is completed. □

Lemma 2.7. *Assume that $g(t,\cdot)$ is odd for all $t \in \hat{\mathbb{T}}$, $g(t,s) \ge 0, s \in [0,\infty), t \in \hat{\mathbb{T}}$. Then for any nonconstant solution u of (1.1),*

$$\|u\| < T.$$

Proof. *Claim.* u is a solution with the change of sign in $\hat{\mathbb{T}}$.

Suppose on the contrary that $u(t) > 0, t \in \hat{\mathbb{T}}$. Then $g(t,u) \ge 0$ and $\nabla(\Delta u(t)) \le 0$ for all $t \in \hat{\mathbb{T}}$. This suggests that

$$\Delta u(t) - \Delta u(t-1) \le 0,\ t \in \hat{\mathbb{T}},$$

that is $\Delta u(0) \ge \Delta u(t) \ge \Delta u(T),\ t \in \mathbb{T}$.

By a simple calculation, the boundary conditions in (1.1) are equivalent to the boundary conditions

$$u(0) = u(T), \quad \Delta u(0) = \Delta u(T).$$

Since u is a nonconstant solution that satisfies the boundary condition $\Delta u(0) = \Delta u(T)$, then for any $t \in \mathbb{T}$,

$$\Delta u(0) = \Delta u(t) = \Delta u(T) = c,$$

where $c \in \mathbb{R} \setminus \{0\}$. This contradicts with the boundary conditions $u(0) = u(T)$. Similarly, $u(t) < 0, t \in \hat{\mathbb{T}}$ can derive a contradiction. Therefore, the claim is proved to be true.

Assume that $\max_{t\in\hat{\mathbb{T}}} u(t) = u(t_1),\ \min_{t\in\hat{\mathbb{T}}} u(t) = u(t_2)$. Then $u(t_2) < 0 < u(t_1)$ and

$$u(t_1) - u(t_2) = \sum_{i=t_2}^{t_1-1} \Delta u(i) \le \sum_{i=0}^{T-1} |\Delta u(i)| \le T.$$

Obviously,

$$\|u\| = u(t_1) \le u(t_2) + T < T \quad \text{if} \quad |u(t_1)| \ge |u(t_2)|,$$
$$\|u\| = -u(t_2) \le T - u(t_1) < T \quad \text{if} \quad |u(t_1)| < |u(t_2)|.$$

The proof is completed. □

3. Proof of the main results

Let $\sigma \in (0,1)$ such that $\lambda\sigma \in (0,\infty)$. Let $H := \{u : \mathbb{Z} \to \mathbb{R} |\ u \text{ is } T\text{-periodic}\}$, whose norm is denoted by

$$\|u\| := \max_{t\in\hat{\mathbb{T}}} |u(t)|.$$

Define the operator $L : D(L) \to H$,

$$(Lu)(t) = -\nabla(\Delta u(t)) + \lambda\sigma u(t),$$

where $D(L) = \{u : \mathbb{Z} \to \mathbb{R} \mid u \text{ satisfies } u(0) - u(T) = u(1) - u(T+1) = 0\}$.

We introduce two subspaces of H which will play important roles in the proofs.

(I) Let E_1 be a subspace of H defined by

$$E_1 = \{u \in H |\ u \text{ satisfies } u(-t) = -u(t),\ t \in \mathbb{Z}\}. \tag{3.1}$$

For any $t \in \hat{\mathbb{T}}$, E_1 is invariant by $g(t,\cdot)$: let $u \in E_1$ such that $g(t,u) \in H$, then $g(t,u) \in E_1$.

Let L_1 be the linear operator of E_1 defined by

$$\begin{aligned} &D(L_1) = D(L) \cap E_1, \\ &L_1 u = Lu. \end{aligned}$$

The eigenvalue of L_1 are $\lambda_k + \lambda\sigma,\ k = 1, 2, \cdots, N-1$, and the eigenfunction corresponding to $\lambda_k + \lambda\sigma$ is ψ_k. E_1 is invariant by L_1: let $u \in D(L_1) = D(L) \cap E_1$ such that $L_1 u \in H$, then $L_1 u \in E_1$.

(II) Let E_2 be a subspace of H defined by

$$E_2 = \{u \in H |\ u \text{ satisfies } u(-t) = u(t),\ t \in \mathbb{Z}\}.$$

For any $t \in \hat{\mathbb{T}}$, E_2 is invariant by $g(t,\cdot)$: let $u \in E_1$ such that $g(t,u) \in H$, then $g(t,u) \in E_2$.

Let L_2 be the linear operator of E_2 defined by

$$\begin{aligned} &D(L_2) = D(L) \cap E_2, \\ &L_2 u = Lu. \end{aligned}$$

The eigenvalue of L_2 are $\lambda_k + \lambda\sigma,\ k = 1, 2, \cdots, N$, and the eigenfunction corresponding to $\lambda_k + \lambda\sigma$ is φ_k. E_2 is invariant by L_2: let $u \in D(L_2) = D(L) \cap E_2$ such that $L_2 u \in H$, then $L_2 u \in E_2$.

We define $g^{[n]} : \hat{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ by

$$g^{[n]}(t,s) = \begin{cases} g(t,s), & |s| \in (\frac{1}{n}, \infty),\ t \in \hat{\mathbb{T}}, \\ ng(t, \frac{1}{n})s, & |s| \in [0, \frac{1}{n}],\ t \in \hat{\mathbb{T}}. \end{cases}$$

Then $g^{[n]} \in C(\hat{\mathbb{T}}\times\mathbb{R}, \mathbb{R})$ with $g^{[n]}(t,\cdot)$ is odd, and $(g^{[n]})_0 := \lim_{s\to 0} \frac{g^{[n]}(t,s)}{s} = ng(t,1/n) > 0$. By (1.6), it follows that $\lim_{n\to\infty} (g^{[n]})_0 = +\infty$ uniformly in $t \in \hat{\mathbb{T}}$.

Define $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$q(y,z) = \begin{cases} \frac{\sqrt{1-y^2}\sqrt{1-z^2}[\sqrt{1-y^2}+\sqrt{1-z^2}]}{\sqrt{1-z^2}\sqrt{1-y^2}+1+zy}, & |y|<1, |z|<1, \\ 0, & \max\{|y|,|z|\} \geq 1. \end{cases} \tag{3.2}$$

From Lemma 2.5, it can be verified that (1.1) is equivalent to

$$\begin{cases} -\nabla(\Delta u(t)) = \lambda g(t,u) q(\Delta u(t), \Delta u(t-1)), & t \in \hat{\mathbb{T}}, \\ u(0) = u(T), \quad u(1) = u(T+1). \end{cases} \tag{3.3}$$

Now let us consider the auxiliary family of the equations

$$\begin{cases} -\nabla(\Delta u(t)) = \lambda g^{[n]}(t,u) q(\Delta u(t), \Delta u(t-1)), & t \in \hat{\mathbb{T}}, \\ u(0) = u(T), \quad u(1) = u(T+1). \end{cases} \tag{3.4}$$

Let $\zeta \in C(\hat{\mathbb{T}} \times \mathbb{R})$ be such that

$$g^{[n]}(t,u) = (g^{[n]})_0 u + \zeta^{[n]}(t,u) = ng(t,1/n)u + \zeta^{[n]}(t,u).$$

Note that

$$\lim_{|s|\to 0} \frac{\zeta^{[n]}(t,s)}{s} = 0 \tag{3.5}$$

uniformly for $t \in \hat{\mathbb{T}}$.

Let $\tilde{q}(y,z) = q(y,z) - 1$, $(y,z) \in \mathbb{R}^2$. Because

$$\sqrt{1-x^2} = 1 - x^2 + \circ(x^2), \quad x \to 0.$$

From (3.2), a simple calculation would give the following

$$\tilde{q}(y,z) = \frac{-2y^2 - 2z^2 - zy + \circ(y^2) + \circ(z^2)}{2 - z^2 - y^2 + zy + \circ(y^2) + \circ(z^2)}.$$

Then

$$\lim_{\|\Delta u\|\to 0} \frac{\tilde{q}(\Delta u(t), \Delta u(t-1))}{\|\Delta u\|} = 0. \tag{3.6}$$

3.1. In the subspace E_1 of H

Let us consider

$$\begin{aligned}(L_1 u)(t) =& \lambda\Big((g^{[n]})_0 + \sigma\Big)u(t) + \lambda\Big[\Big(((g^{[n]})_0 + \sigma + \frac{\zeta^{[n]}(t,u)}{u})\tilde{q}(\Delta u(t), \Delta u(t-1)) \\ &+ \frac{\zeta^{[n]}(t,u)}{u}\Big)u(t)\Big]\end{aligned} \tag{3.7}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Eq. (3.7) can be converted to the equivalent equation

$$\begin{aligned} u(t) :=&\lambda \sum_{i=0}^{T} G(t,i)\Big[\Big((g^{[n]})_0+\sigma\Big)u(i) \\ &+\Big(((g^{[n]})_0+\sigma+\frac{\zeta^{[n]}(i,u(i))}{u(i)})\tilde{q}(\Delta u(i),\Delta u(i-1))+\frac{\zeta^{[n]}(i,u(i))}{u(i)}\Big)u(i)\Big] \\ =&\lambda L_1^{-1}\Big[\Big((g^{[n]})_0+\sigma\Big)u(\cdot)\Big](t) \\ &+\lambda L_1^{-1}\Big[\Big(((g^{[n]})_0+\sigma+\frac{\zeta^{[n]}(\cdot,u(\cdot))}{u(\cdot)})\tilde{q}+\frac{\zeta^{[n]}(\cdot,u(\cdot))}{u(\cdot)}\Big)u(\cdot)\Big](t), \end{aligned}$$

where $G(t,i)$ is the Green function of $-\nabla(\Delta u(t))+\lambda\sigma u(t)=0$ with the periodic boundary condition.

Define the operator $\mathcal{H}_1:\mathbb{R}\times\hat{\mathbb{T}}\times E_1\to E_1$ by

$$\mathcal{H}_1(\lambda,t,u)=\lambda L_1^{-1}\Big((((g^{[n]})_0+\sigma+\frac{\zeta^{[n]}(t,u)}{u})\tilde{q}+\frac{\zeta^{[n]}(t,u)}{u})u\Big).$$

Obviously, $\mathcal{H}_1$ is completely continuous. From (3.5) and (3.6), for any $t\in\hat{\mathbb{T}}$,

$$\lim_{\|u\|\to 0}\frac{\|\mathcal{H}_1(\lambda,t,u)\|}{\|u\|}=0$$

uniformly in λ of any bounded set.

In what follows, we use the terminology of Rabinowitz [26,27]. For $k\in\{1,2,\cdots,N-1\}$, let S_k^+ denote the set of functions in E_1 which have exactly $2k-1$ simple generalized zeros in $\mathbb{T}$ and $u(0)=0, u(1)>0$. Set $S_k^-=-S_k^+$, and $S_k=S_k^+\cup S_k^-$. They are disjoint and open in E_1. Finally, let $\Phi_k^{\pm}=\mathbb{R}\times S_k^{\pm}$ and $\Phi_k=\mathbb{R}\times S_k$.

The results of Rabinowitz [26,27] for (3.7) can be stated as follows: For each integer $k\in\{1,2,\cdots,N-1\}$, $\nu\in\{+,-\}$, there exists a continuum $(\mathcal{C}^{[n]})_k^\nu\subseteq\Phi_k^\nu$ of solutions of (3.7) joining $(\frac{\lambda_k+\lambda\sigma}{(g^{[n]})_0+\sigma},0)$ to infinity in Φ_k^ν. Moreover, $(\mathcal{C}^{[n]})_k^\nu/\{(\frac{\lambda_k+\lambda\sigma}{(g^{[n]})_0+\sigma},0)\}\subset\Phi_k^\nu$.

Proof of Theorem 1.1(i). Let us verify that $\{(\mathcal{C}^{[n]})_k^\nu\}$ satisfies all of the conditions of Lemma 2.3. Since

$$\lim_{n\to\infty}\frac{\lambda_k+\lambda\sigma}{(g^{[n]})_0+\sigma}=\lim_{n\to\infty}\frac{\lambda_k+\lambda\sigma}{ng(t,1/n)+\sigma}=0$$

uniformly for $t\in\hat{\mathbb{T}}$. Condition (a) in Lemma 2.3 is satisfied with $z^*=(0,0)$. Obviously

$$r_n=\sup\{|\lambda|+\|y\|\mid(\lambda,y)\in(\mathcal{C}^{[n]})_k^\nu\}=\infty,$$

and accordingly, (b) holds. (c) can be deduced directly from the Arzela-Ascoli Theorem and the definition of $g^{[n]}$. Therefore, the superior limit of $\{(\mathcal{C}^{[n]})_k^\nu\}$, i.e. $\mathcal{D}$, contains an unbounded connected component $\mathcal{C}_k^\nu$ with $(0,0)\in\mathcal{C}_k^\nu$.

Let $(\mu_n,u_n)\in\mathcal{C}_k^\nu$ satisfy

$$\mu_n+\|u_n\|\to\infty.$$

Then $u_n\in E_1$ is odd. From Lemma 2.4, $\|u_n\|<\frac{T}{2}$. Therefore,

$$\sup\{\lambda|(\lambda,u_n)\in\mathcal{C}_k^\nu\}=\infty.$$

The proof is completed. □

Proof of Corollary 1.1(i). From the proof of Theorem 1.1(i), we have

$$(\lambda, u) \in \mathcal{C}_k^\nu \subset \Phi_k^\nu \subset \mathbb{R} \times E_1.$$

Then for any $\lambda \in (0, +\infty), k \in \{1, 2, \cdots, N-1\}, \nu \in \{+, -\}$, (1.1) has odd solutions u_k^ν such that u_k^ν change their sign $2k-1$ times in $\mathbb{T}$ and $u_k^- = -u_k^+$. □

3.2. In the subspace E_2 of H

Let us consider

$$\begin{aligned}(L_2 v)(t) =&\lambda\Big((g^{[n]})_0 + \sigma\Big)v(t) + \lambda\Big((((g^{[n]})_0 + \sigma + \frac{\zeta^{[n]}(x, v)}{v})\tilde{q}(\Delta v(t), \Delta v(t-1)) \\ &+ \frac{\zeta^{[n]}(x, v)}{v})v(t)\Big)\end{aligned} \tag{3.8}$$

as a bifurcation problem from the trivial solution $v \equiv 0$.

For $k \in \{1, 2, \cdots, N\}$, let $\tilde{S}_k^+$ denote the set of functions in E_2 which have exactly $2k$ simple generalized zeros in $\mathbb{T}$ and $v(0) > 0$. Set $\tilde{S}_k^- = -\tilde{S}_k^+$, and $\tilde{S}_k = \tilde{S}_k^+ \cup \tilde{S}_k^-$. They are disjoint and open in E_2. Finally, let $\Psi_k^\pm = \mathbb{R} \times \tilde{S}_k^\pm$ and $\Psi_k = \mathbb{R} \times \tilde{S}_k$.

Similarly, for each $k \in \{1, 2, \cdots, N\}$, $\nu \in \{+, -\}$, there exists a continuum $(\mathcal{D}^{[n]})_k^\nu \subseteq \Psi_k^\nu$ of solutions of (3.8) joining $(\frac{\lambda_k + \lambda\sigma}{(g^{[n]})_0 + \sigma}, 0)$ to infinity in Ψ_k^ν. Moreover, $(\mathcal{D}^{[n]})_k^\nu / \{(\frac{\lambda_k + \lambda\sigma}{(g^{[n]})_0 + \sigma}, 0)\} \subset \Psi_k^\nu$.

Proof of Theorem 1.1(ii). From Lemma 2.3, the superior limit of $\{(\mathcal{D}^{[n]})_k^\nu\}$, i.e. $\mathcal{D}$, contains an unbounded connected component $\mathcal{D}_k^\nu$ with $(0, 0) \in \mathcal{D}_k^\nu$.

Since g satisfies condition (H1) and $v_n \in E_2$ is an even function, then from Lemma 2.6,

$$\max_{t \in \hat{\mathbb{T}}} v_n(t) \le s_0 \quad \text{and} \quad \min_{t \in \hat{\mathbb{T}}} v_n(t) \ge -s_0.$$

That is $\|v_n\| \le s_0$. Therefore,

$$\sup\{\lambda | (\lambda, v_n) \in \mathcal{D}_k^\nu\} = \infty.$$

The proof is completed. □

Proof of Theorem 1.1(iii). From Lemma 2.3, the superior limit of $\{(\mathcal{D}^{[n]})_k^\nu\}$, i.e. $\mathcal{D}$, contains an unbounded connected component $\mathcal{D}_k^\nu$ with $(0, 0) \in \mathcal{D}_k^\nu$.

Since g satisfies condition (H2) and $v_n \in \tilde{S}_k^+$ is a nonconstant solution. Then from Lemma 2.7, $\|v_n\| < T$. Therefore,

$$\sup\{\lambda | (\lambda, v_n) \in \mathcal{D}_k^\nu\} = \infty.$$

The proof is completed. □

Proof of Corollary 1.1(ii). If g satisfies either (H1) or (H2), we can proved $\|v_n\| < \max\{s_0, T\}$. From the proof of Theorem 1.1(ii) and (iii), we have

$$(\lambda, v) \in \mathcal{D}_k^\nu \subset \Psi_k^\nu \subset \mathbb{R} \times E_2.$$

Therefore, for any $\lambda > 0$, $k \in \{1, 2, \cdots, N\}$, $\nu \in \{+, -\}$, (1.1) has even solutions v_k^ν such that v_k^ν change their sign $2k$ times in $\mathbb{T}$ and $v_k^- = -v_k^+$. □

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

References

[1] F. M. Atici and A. Cabada, *Existence and uniqueness results for discrete second-order periodic boundary value problems*, Comput. Math. Appl., 2003, 45(6–9), 1417–1427.

[2] F. M. Atici, A. Cabada and V. Otero-Espinar, *Criteria for existence and nonexistence of positive solutions to a discrete periodic boundary value problem*, J. Difference Equ. Appl., 2003, 9(9), 765–775.

[3] P. Benevieri, J. Marcos do Ó and E. S. de Medeiros, *Periodic solutions for nonlinear equations with mean curvature-like operators*, Appl. Math. Lett., 2007, 20(5), 484–492.

[4] C. Bereanu, P. Jebelean and J. Mawhin, *Periodic solutions of pendulum-Like perturbations of singular and bounded ϕ-Laplacians*, J. Dynam. Differential Equations, 2010, 22(3), 463–471.

[5] C. Bereanu and J. Mawhin, *Multiple periodic solutions of ordinary differential equations with bounded nonlinearities and ϕ-Laplacian*, NoDEA Nonlinear Differential Equations Appl., 2008, 15(1–2), 159–168.

[6] C. Bereanu and H. B. Thompson, *Periodic solutions of second order nonlinear difference equations with discrete ϕ-Laplacian*, J. Math. Anal. Appl., 2007, 330(2), 1002–1015.

[7] C. Bereanu and M. Zamora, *Periodic solutions for indefinite singular perturbations of the relativistic acceleration*, Proc. R. Soc. Edinb. Sect. A., 2018, 148(4), 703–712.

[8] A. Boscaggin and G. Feltrin, *Positive periodic solutions to an indefinite Minkowski-curvature equation*, J. Differential Equations, 2020, 269(7), 5595–5645.

[9] I. Coelho, C. Corsato, F. Obersnel and P. Omari, *Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-curvature equation*, Adv. Nonlinear Stud., 2012, 12(3), 621–638.

[10] I. Coelho, C. Corsato and S. Rivetti, *Positive radial solutions of the Dirichlet problem for the Minkowski-curvature equation in a ball*, Topol. Methods Nonlinear Anal., 2014, 44(1), 23–39.

[11] J. M. Coron, *Periodic solutions of a nonlinear wave equation without monotonicity*, Math. Ann., 1983, 262(2), 273–285.

[12] G. Figueiredo and A. Suarez, *Existence of positive solutions for prescribed mean curvature problems with nonlocal term via sub- and supersolution method*, Math. Methods Appl. Sci., 2020, 43(15), 8496–8505.

[13] M. Garcia-Huidobro and P. Ubilla, *Multiplicity of solutions for a class of nonlinear second-order equations*, Nonlinear Anal., 1997, 28(9), 1509–1520.

[14] C. Gerhardt, *H-surfaces in Lorentzian manifolds*, Comm. Math. Phys., 1983, 89(4), 523–553.

[15] D. Gurban, *Radial non-potential Dirichlet systems with mean curvature operator in Minkowski space*, Positivity, 2021, 25(1), 109–119.

[16] J. Henderson and H. B. Thompson, *Existence of multiple solutions for second-order discrete boundary value problems*, Comput. Math. Appl., 2002, 43(10–11), 1239–1248.

[17] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Second edition, Harcourt/Academic Press, San Diego, CA, 2001.

[18] R. Ma and Y. An, *Global structure of positive solutions for nonlocal boundary value problems involving integral conditions*, Nonlinear Anal., 2009, 71(10), 4364–4376.

[19] R. Ma and H. Ma, *Existence of sign-changing periodic solutions of second order difference equations*, Appl. Math. Comput., 2008, 203(2), 463–470.

[20] R. Ma and H. Ma, *Unbounded perturbations of nonlinear discrete periodic problem at resonance*, Nonlinear Anal., 2009, 70(7), 2602–2613.

[21] J. A. Marlin, *Periodic motions of coupled simple pendulums with periodic disturbances*, Int. J. Nonlinear Mech., 1968, 3, 439–447.

[22] J. Mawhin, *A simple proof of multiplicity for periodic solutions of Lagrangian difference systems with relativistic operator and periodic potential*, J. Difference Equ. Appl., 2016, 22(2), 306–315.

[23] J. Mawhin, *Periodic solutions of second order Lagrangian difference systems with bounded or singular ϕ-Laplacian and periodic potential*, Discrete Contin. Dyn. Syst. Ser. S, 2013, 6(4), 1065–1076.

[24] Y. Naito and S. Tanaka, *On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations*, Nonlinear Anal., 2004, 56(6), 919–935.

[25] P. Omari and E. Sovrano, *Positive solutions of superlinear indefinite prescribed mean curvature problems*, Commun. Contemp. Math., 2021, 23(3), Article ID 2050017, 25 pp.

[26] P. H. Rabinowitz, *Nonlinear Sturm-Liouville problems for second order ordinary differential equations*, Commun. Pure Appl. Math., 1970, 23, 939–961.

[27] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., 1971, 7, 487–513.

[28] S. H. Strogatz, *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering*, Westview Press, New York, 1994.

[29] G. T. Whyburn, *Topological Analysis*, Princeton University Press, Princeton, 1958.