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NEW BLOW-UP CRITERIA FOR 3D CHEMOTAXIS-NAVIER-STOKES EQUATIONS*

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Abstract In this paper, we consider new blow-up criteria for the chemotaxis-Navier-Stokes equations in three dimensions. Specifically, by combining the Prodi-Serrin condition for oxygen concentration ∇c with some condition on the velocity or vorticity of fluid in Besov space, we establish new blow-up criteria for local existence of classical solutions for chemotaxis-Navier-Stokes equations. The scaling invariant blow-up criterion involving cell density n and gradient of velocity is also investigated.

Keywords Chemotaxis-Navier-Stokes equations, blow-up criteria, Besov spaces.

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1. Introduction

In this paper, we consider the following 3D chemotaxis-Navier-Stokes equations:

$$\begin{cases} \partial_t n + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), \\ \partial_t c + u\cdot\nabla c = \Delta c - k(c)n, \\ \partial_t u + (u\cdot\nabla)u + \nabla P = \Delta u - n\nabla\phi, \\ \nabla\cdot u = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \end{cases} \tag{1.1}$$

where the unknowns $n(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^+, c(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^+, u(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^3$ and $P(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$ denote the cell density, the oxygen concentration, the fluid velocity, and the corresponding scalar pressure, respectively. The nonnegative function $k(c)$ denotes the oxygen consumption rate, and the nonnegative function $\chi(c)$ denotes chemotactic sensitivity. The time-independent function $\phi=\phi(x)$ is the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force.

The system (1.1) was firstly proposed by Tuval [19] to describe a biological process, in which bacteria move by swimming towards higher concentration of oxygen according to mechanism of chemotaxis while the movement of fluid is under the influence of gravitational force generated by bacteria themselves. Many researchers made great efforts to establish the existence theory for system (1.1). Duan etc [7, p11] were the first to prove the global existence of weak solutions to the Cauchy problems of (1.1) under some smallness assumptions on potential function and initial data,

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and later Liu and Lorz [14] removed the smallness assumption and obtained global existence of weak solutions with large data. Winkler [21] established the existence of a unique global classical solution with arbitrary large initial data in a convex domain Ω of $\mathbb{R}^2$. Subsequently, Winkler proved that the global classical solution converges to a constant state $(n_\infty, 0, 0)$ as time goes to infinity in [22]. Based on Winkler's work, Jiang etc [12, p11] generalized the result by removing the assumption of convex domain. What is more relevant to this paper is that Chae, Kang and Lee [4] established local existence of classical solutions (u, n, c) for system (1.1) in $\mathbb{R}^d, d = 2, 3$, which satisfies

$$
\begin{aligned}
&(u, n, c) \in L^\infty(0, T; H^m(\mathbb{R}^d) \times H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d)), \\
&(\nabla u, \nabla n, \nabla c) \in L^2(0, T; H^m(\mathbb{R}^d) \times H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d)), \quad \text{for some} \quad T > 0
\end{aligned}
\tag{1.2}
$$

with the assumptions

$$
\begin{cases}
\chi, k \in C^m(\mathbb{R}^+),\ k(0) = 0,\ \|\nabla^l \phi\|_{L^\infty} < \infty, \quad \text{for} \quad 1 \le |l| \le m, \\
(u_0, n_0, c_0) \in H^m(\mathbb{R}^d) \times H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d), \quad \text{with} \quad m \ge 3.
\end{cases}
\tag{1.3}
$$

Under the same assumptions as in (1.3), the authors in [5] considered the hyperbolic oxygen concentration equation, and obtained the same results mentioned before. For other interesting results on system (1.1), we refer to [2, 3, 10, 11, 15, 16, 23] and the reference therein.

The blow-up criteria on the global regularity of solutions for system (1.1) got much attention (see [2,4,5,8,24,25]). We collect these blow-up criteria and into two parts. The first one is based on the concentration term ∇c. For example, Chae, Kang and Lee [4] proposed a blow-up criterion

$$
\|\nabla u\|_{L^1(0,T;L^\infty)} + \|\nabla c\|_{L^2(0,T;L^\infty)} < \infty. \tag{1.4}
$$

Moreover, they proposed a Prodi-Serrin type regularity criterion

$$
\|u\|_{L^q(0,T;L^p)} + \|\nabla c\|_{L^2(0,T;L^\infty)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, 3 < p \le \infty. \tag{1.5}
$$

Recently, Dai and Liu [8] proposed a low modes blow-up criterion

$$
\int_0^T \|\nabla c_{\le Q_c(t)}(t)\|_{L^\infty}^2 + \|u_{\le Q_u(t)}(t)\|_{B^1_{\infty,\infty}} \mathrm{d}t < \infty, \tag{1.6}
$$

where $u_{\le Q_u}$ and $c_{\le Q_c}$ low frequency part of u and c below the wavenumber 2^{Q_u} and 2^{Q_c} respectively, both of which can be clearly defined within the framework of Littlewood-Paley theory in the upcoming sections. The other one is based on the density term n. For example, Chae, Kang and Lee [5] obtained a regularity criterion

$$
\|u\|_{L^\gamma(0,T^*;L^\beta)} + \|n\|_{L^q(0,T^*;L^p)} < \infty, \tag{1.7}
$$

where

$$
\frac{3}{\beta} + \frac{2}{\gamma} \le 1,\ 3 < \beta \le \infty,\ \frac{3}{p} + \frac{2}{q} = 2,\ \frac{3}{2} < p \le \infty.
$$

As for blow-up criteria in Besov space, Choe and Lkhagvasuren [2] gave the extension for the local in time solution

$$\|u\|_{L^\infty(0,T;\dot{B}^{\rho}_{\beta,\infty})} + \|n\|_{L^\infty(0,T;\dot{B}^{\sigma}_{\lambda,\infty})} < C, \tag{1.8}$$

with

$$\frac{3}{p} - \frac{3}{\beta} + \rho > 0, \quad \frac{3}{p} - \frac{3}{\lambda} + \sigma - 1 > 0,$$

where $1 \le p < \frac{6}{5}$ and $(n_0, c_0, u_0) \in \dot{B}^{2+\frac{3}{p}}_{p,1}(\mathbb{R}^3) \times \dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3) \times \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$. Later, Zhai and Yin [25] extended the local solutions to global solutions under the smallness assumptions on $\|n_0\|_{\dot{B}^{-2+\frac{d}{p}}_{q,1}}$, $\|c_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}}$ and $\|u_0^h\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}}$ with $u_0^h = (u_0^1, u_0^2)$ proposed a blow-up criterion:

$$\int_0^{T^*} (\|\nabla \times u\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|n\|_{\dot{B}^{\frac{d}{q}}_{q,1}})\mathrm{d}t = \infty. \tag{1.9}$$

Note that, if the Laplacian term Δc is removed, then $(1.1)_2$ becomes a hyperbolic equation. Xie and Ma [24] proposed several blow-up criteria for this parabolic-hyperbolic type of system (1.1). Their results are as follows: If u satisfies any one of the following conditions:

$$u \in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.10}$$

$$u \in L^{\frac{2}{1-\theta}}(0,T;\dot{B}^{-\theta}_{\infty,\infty}(\mathbb{R}^3)) \quad \text{with} \quad 0 < \theta < 1, \tag{1.11}$$

$$\omega := \operatorname{curl} u \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.12}$$

$$P \in L^{\frac{2}{2+r}}(0,T;\dot{B}^r_{\infty,\infty}(\mathbb{R}^3)) \quad \text{with} \quad -1 \le r \le 1 (r = -1) \tag{1.13}$$

and n satisfies

$$n \in L^2(0,T;L^\infty(\mathbb{R}^3)), \tag{1.14}$$

then the solution (n, c, u) cannot blow up. In addition, other forms of chemotaxis-Navier-Stokes equations with the linear diffusion term Δn is replaced by Δn^m have also been extensively studied, and we refer to [9, 14, 17]. For the chemotaxis-Navier-Stokes equations with logistic source, rotating flux term and slow p-Laplacian diffusion which are shown in [6, 20] and [18], respectively.

The main objectives of this paper are to obtain new blow-up criteria for the chemotaxis-Navier-Stokes equations (1.1) and to improve or generalize some of the previous results of [2, 4, 5]. Now we are in position to give our main conclusions.

Theorem 1.1. *Let $n_0 \ge 0, c_0 \ge 0$ and (1.3) hold, and T^* be the maximal time for local existence of classical solutions (u, n, c) of (1.1) as given by (1.2). For any $0 < T < T^*$, Assume there holds for ∇c that*

$$\int_0^T \|\nabla c\|^\beta_{L^\alpha}\mathrm{d}t < \infty \quad \textit{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} = 1, \quad 3 < \alpha \le \infty \tag{1.15}$$

and one of the following conditions is true:

(1) The velocity u satisfies

$$\int_0^T \|u(t)\|^\beta_{\dot{B}^0_{\alpha,\infty}}\mathrm{d}t < \infty \quad \textit{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} = 1, \quad 3 < \alpha \le \infty. \tag{1.16}$$

(2) The gradient of velocity satisfies

$$\int_0^T \|\nabla u(t)\|_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t < \infty. \tag{1.17}$$

(3) The vorticity of velocity $\omega = \nabla \times u$ satisfies

$$\int_0^T \|\omega\|^{\beta}_{\dot{B}^0_{\alpha,\infty}} \mathrm{d}t < \infty, \quad \frac{3}{\alpha} + \frac{2}{\beta} = 2, \quad \frac{3}{2} < \alpha < 3. \tag{1.18}$$

Then the solution (u, n, c) can be survived exceed T^*.

Remark 1.1. The main contribution of this theorem is to improve the condition $\nabla c \in L^2(0, T; L^\infty)$ in [4] (see (1.4) and (1.5)) to the general form $\nabla c \in L^\beta(0, T; L^\alpha)$ with $\frac{3}{\alpha} + \frac{2}{\beta} = 1$. Besides, the Prodi-Serrin condition for velocity u and vorticity ω is also generalized in some suitable Besov space.

Theorem 1.2. *Let $n_0 \geq 0, c_0 \geq 0$ and $k'(c)$ be non-negative satisfying* (1.3)*, and T^* be the maximal time for local existence of classical solutions (u, n, c) of* (1.1) *as given by* (1.2)*. Suppose further that n_0 satisfies*

$$\int_{\mathbb{R}^3} n_0 |\ln n_0| \mathrm{d}x < \infty$$

and for any $0 < T < T*$, there holds

$$\sup_{0<t<T} \int_{\mathbb{R}^3} n \ln n dx > 0$$

$$\int_0^T (\|n\|^q_{L^p} + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}) \mathrm{d}t < \infty \quad \textit{with} \quad \frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{2} < p \leq \infty. \tag{1.19}$$

Then the solution (n, c, u) can be survived exceed T^*

2. Preliminary and Notations

We begin this section with some notations and lemmas, which are useful for us to prove Theorem 1.1 and Theorem 1.2. In order to define Besov spaces, we first introduce the Littlewood-Paley decomposition theory. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function, given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transformation $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) \mathrm{d}x,$$

and its inverse Fourier transform $\mathcal{F}^{-1}f = \check{f}$ is defined by

$$\check{f}(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} f(\xi) \mathrm{d}\xi.$$

More generally, the Fourier transform of any $f \in \mathcal{S}'(\mathbb{R}^3)$, the space of tempered distributions, is given by

$$\langle \hat{f}, g \rangle = \langle f, \hat{g} \rangle,$$

for any $g \in \mathcal{S}(\mathbb{R}^3)$. The Fourier transform is a bounded linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also bounded. We fix the notation

$$\mathcal{S}_h = \{\phi \in \mathcal{S}, \int_{\mathbb{R}^3} \phi(x) x^\gamma \mathrm{d}x = 0, |\gamma| = 0, 1, 2, \cdots\}.$$

Its dual is given by

$$\mathcal{S}_h' = \mathcal{S}'/\mathcal{S}_h^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ is the space of polynomial. In other words, two distributions in $\mathcal{S}_h'$ are identified as the same if their difference is a polynomial. Let us choose two non-negative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported in $\mathfrak{B} = \{\xi \in \mathbb{R}^3 : |\xi| \le 4/3\}$ and $\mathfrak{C} = \{\xi \in \mathbb{R}^3 : 3/4 \le |\xi| \le 8/3\}$, respectively, such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^3 \backslash \{0\},$$

and

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^3.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, and then we define the homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operator $\dot{S}_j$ as follows:

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) u(x-y) \mathrm{d}y,$$

and

$$\dot{S}_j u = \chi(2^{-j}D) = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) u(x-y) \mathrm{d}y.$$

Informally, $\dot{\Delta}_j$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, while $\dot{S}_j$ is a frequency projection to the ball $\{|\xi| \sim 2^j\}$. It is straightforward to verify that $\dot{\Delta}_j \dot{\Delta}_k f = 0$ if $|j-k| \ge 2$. Especially for any $f \in L^2(\mathbb{R}^3)$, we have the Littlewood-Paley decomposition:

$$f = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j f. \tag{2.1}$$

We now give the definitions of Besov spaces. Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ defined via the full-dyadic decomposition. We say that $f \in \dot{B}^s_{p,q}(\mathbb{R}^3)$, if $f \in S_h'$ and

$$\sum_{j=-\infty}^{+\infty} (2^{js} \|\dot{\Delta}_j f\|_{L^p})^q < \infty,$$

with the norm

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j=-\infty}^{+\infty} 2^{qjs} \|\dot{\Delta}_j f\|_{L^p}^q)^{\frac{1}{q}}, \ 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}, \ q = \infty. \end{cases}$$

It is of interest to note that the homogeneous Besov space $\dot{B}^s_{2,2}(\mathbb{R}^3)$ is equivalent to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$. The following Bernstein inequalities will be used in the next section.

Lemma 2.1 (Lemma 2.1, [1]). *Let $\mathfrak{B}$ be a ball and $\mathfrak{C}$ an annulus. A constant C exists such that for any nonnegative integer k, and couple (p,q) in $[1,\infty]^2$ with $1 \le p \le q$, and any function u of $L^p(\mathbb{R}^d)$, we have*

$$\mathtt{Supp}\ \hat{u} \subset \lambda\mathfrak{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \le C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}, \tag{2.2}$$

$$\mathtt{Supp}\ \hat{u} \subset \lambda\mathfrak{C} \Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \le \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \le C^{k+1}\lambda^k\|u\|_{L^p}. \tag{2.3}$$

Lemma 2.2 (Theorem 2.42, [1]). *Let $1 \le q < p < \infty$ and α be a positive real number. A constant C exists such that*

$$\|f\|_{L^p} \le C\|f\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}}\|f\|^{\theta}_{\dot{B}^{\beta}_{q,q}}, \ \textit{with } \beta = \alpha\left(\frac{p}{q}-1\right) \textit{ and } \theta = \frac{q}{p}. \tag{2.4}$$

3. Proof of Main results

In this section, we prove our main results. In view of the conditions of 1.1, 1.2 and (1.3), we see the unique classical solution (n,c,u) satisfying (1.2) for time interval $(0,T^*)$ with $(n_0,c_0,u_0) \in H^m(\mathbb{R}^3)\times H^{m-1}(\mathbb{R}^3)\times H^m(\mathbb{R}^3)$. Our strategy the local classical solution can be survived the maximal interval $(0,T^*)$ if the conditions (1.15) and (1.16), (1.15) and (1.17), (1.15) and (1.18) in Theorem 1.1 or (1.19) in Theorem 1.2 is satisfied. We prove Theorem 1.1 and Theorem 1.2 in turn.

3.1. Proof of Theorem 1.1

First of all, we give the proof of Theorem 1.1 when the conditions (1.15) and (1.16) simultaneously hold. Multiplying n to both sides of the equation $(1.1)_1$ and using integration by parts, we apply the Hölder, Young's and Gagliardo-Nirenberg inequalities to get the L^2 estimate of n that

$$\begin{aligned}
\frac{d}{dt}\|n\|^2_{L^2} + 2\|\nabla n\|^2_{L^2} &= -\int_{\mathbb{R}^3} \nabla\cdot(n\chi(c)\nabla c)\, n\mathrm{d}x \\
&\le \|\nabla c\|_{L^\alpha}\|n\|_{L^{\frac{2\alpha}{\alpha-2}}}\|\nabla n\|_{L^2} \\
&\le \|\nabla c\|_{L^\alpha}\|n\|^{\frac{\alpha-3}{\alpha}}_{L^2}\|\nabla n\|^{\frac{\alpha+3}{\alpha}}_{L^2} \\
&\le C\|\nabla c\|^{\frac{2\alpha}{\alpha-3}}_{L^\alpha}\|n\|^2_{L^2} + \frac{1}{2}\|\nabla n\|^2_{L^2}.
\end{aligned} \tag{3.1}$$

Similar to above, by testing $-\Delta c$ to the equation $(1.1)_2$, we get the estimates of $\|\nabla c\|_{L^2}$ as follows:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c\|^2_{L^2} + \|\nabla^2 c\|^2_{L^2} &\le \int_{\mathbb{R}^3}|\nabla u\nabla c\nabla c|\mathrm{d}x + \int_{\mathbb{R}^3}|k(c)n\Delta c|\mathrm{d}x \\
&\le \|\nabla c\|_{L^\alpha}\|\nabla c\|_{L^{\frac{2\alpha}{\alpha-2}}}\|\nabla u\|_{L^2} + C\|n\|_{L^2}\|\nabla^2 c\|_{L^2} \\
&\le C\|\nabla c\|^{\frac{2\alpha}{\alpha-3}}_{L^\alpha}\|\nabla c\|^2_{L^2} + C\left(\|\nabla u\|^2_{L^2} + \|n\|^2_{L^2}\right) + \frac{1}{2}\|\nabla^2 c\|^2_{L^2}.
\end{aligned} \tag{3.2}$$

As to the equation $(1.1)_3$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2 &\le \left|\int_{\mathbb{R}^3} u\cdot\nabla u\Delta u\mathrm{d}x\right| + \int_{\mathbb{R}^3}|\nabla\phi n\Delta u|\mathrm{d}x\\ &\le I_1 + C\|n\|_{L^2}\|\nabla^2 u\|_{L^2},\end{aligned}\tag{3.3}$$

where I_1 is bounded

$$\begin{aligned}I_1 &\le \|u\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}}\|\nabla^2 u\|_{L^2}\\ &\le C\|u\|_{\dot{B}^{-\frac{3}{\alpha}}_{\infty,\infty}}^{1-\frac{2}{p}}\|u\|_{\dot{B}^{\beta_1}_{2,2}}^{\frac{2}{p}}\|\nabla u\|_{\dot{B}^{-1-\frac{3}{\alpha}}_{\infty,\infty}}^{\frac{2}{p}}\|\nabla u\|_{\dot{B}^{\beta_2}_{2,2}}^{1-\frac{2}{p}}\|\nabla^2 u\|_{L^2}\\ &\le C\|u\|_{\dot{B}^0_{\alpha,\infty}}\|u\|_{\dot{B}^{\beta_1}_{2,2}}^{\frac{2}{p}}\|\nabla u\|_{\dot{B}^{\beta_2}_{2,2}}^{1-\frac{2}{p}}\|\nabla^2 u\|_{L^2},\end{aligned}\tag{3.4}$$

where β_1 and β_2 is given by

$$\beta_1=\frac{3}{\alpha}\left(\frac{p}{2}-1\right)\quad\text{and}\quad\beta_2=\left(1+\frac{3}{\alpha}\right)\left(\frac{p}{p-2}-1\right).\tag{3.5}$$

Now, by the fact $3<\alpha<\infty$, we choose suitable p such that

$$\max\left\{\frac{6}{\alpha}+4,\frac{2\alpha}{3}+2\right\}<p<\frac{4\alpha}{3}+2.\tag{3.6}$$

It is easy to see that $1<\beta_1<2$ and $0<\beta_2<1$. Thus by interpolation inequality, one has

$$\|u\|_{\dot{B}^{\beta_1}_{2,2}}\le C\|\nabla u\|_{L^2}^{2-\beta_1}\|\nabla^2 u\|_{L^2}^{\beta_1-1},\tag{3.7}$$

$$\|\nabla u\|_{\dot{B}^{\beta_2}_{2,2}}\le C\|\nabla u\|_{L^2}^{1-\beta_2}\|\nabla^2 u\|_{L^2}^{\beta_2}.\tag{3.8}$$

Inserting (3.7) and (3.8) into (3.4), we obtain

$$\begin{aligned}I_1 &\le C\|u\|_{\dot{B}^0_{\alpha,\infty}}\|\nabla u\|_{L^2}^{\frac{2}{p}(2-\beta_1)+(1-\frac{2}{p})(1-\beta_2)}\|\nabla^2 u\|_{L^2}^{\frac{2}{p}(\beta_1-1)+(1-\frac{2}{p})\beta_2}\|\nabla^2 u\|_{L^2}\\ &= C\|u\|_{\dot{B}^0_{\alpha,\infty}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}}\|\nabla^2 u\|_{L^2}^{1+\frac{3}{\alpha}}\\ &\le C\|u\|_{\dot{B}^0_{\alpha,\infty}}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\nabla^2 u\|_{L^2}^2.\end{aligned}$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\le C\|u\|_{\dot{B}^0_{\alpha,\infty}}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^2+C\|n\|_{L^2}^2+\frac{1}{2}\|\nabla^2 u\|_{L^2}^2.\tag{3.9}$$

Combining (3.1), (3.2) with (3.9), absorbing the small terms on the right hand of the inequality, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|n\|_{L^2}^2+\|\nabla c\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+\left(\|\nabla n\|_{L^2}^2+\|\nabla^2 c\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\right)\\ \le& C\left(1+\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}+\|u\|_{\dot{B}^0_{\alpha,\infty}}^{\frac{2\alpha}{\alpha-3}}\right)\left(\|n\|_{L^2}^2+\|\nabla c\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right).\end{aligned}\tag{3.10}$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|n\|_{L^2}^2+\|\nabla c\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+\int_0^T\left(\|\nabla n\|_{L^2}^2+\|\nabla^2 c\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\right)\mathrm{d}t\\ \le& C\left(\|n_0\|_{L^2}^2+\|\nabla c_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2\right)\exp\left(\int_0^T\left(1+\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}+\|u\|_{\dot{B}^0_{\alpha,\infty}}^{\frac{2\alpha}{\alpha-3}}\right)\mathrm{d}t\right).\end{aligned}\tag{3.11}$$

Thus, we get

$$(n,\nabla u,\nabla c)\in L^\infty(0,T;L^2(\mathbb{R}^3))\quad\text{and}\quad(\nabla n,\nabla^2 u,\nabla^2 c)\in L^2(0,T;L^2(\mathbb{R}^3)).\tag{3.12}$$

Next, we multiply both sides of the equation $(1.1)_2$ by $\Delta^2 c$, thus yielding

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 c\|_{L^2}^2+\|\nabla^3 c\|_{L^2}^2\le&\left|\int_{\mathbb{R}^3}\Delta(u\cdot\nabla c)\Delta c\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}\Delta(k(c)n)\Delta c\mathrm{d}x\right|\\ \le& C\int_{\mathbb{R}^3}\left|\nabla u\,\nabla c\,\nabla^3 c\right|\mathrm{d}x+C\int_{\mathbb{R}^3}|n\,\nabla c\,\nabla\Delta c|\,\mathrm{d}x\\ &+C\int_{\mathbb{R}^3}|\nabla n\,\nabla\Delta c|\,\mathrm{d}x\\ =&J_1+J_2+J_3.\end{aligned}$$

We deal with $J_i, i=1,2,3$ as follows:

$$\begin{aligned}J_1\le&\|\nabla c\|_{L^\alpha}\|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2}}}\|\nabla^3 c\|_{L^2}\\ \le& C\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^2+C\|\nabla^2 u\|_{L^2}^2+\frac{1}{6}\|\nabla^3 c\|_{L^2}^2,\\ J_2\le& C\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\|n\|_{L^2}^2+C\|\nabla n\|_{L^2}^2+\frac{1}{6}\|\nabla^3 c\|_{L^2}^2,\\ J_3\le& C\|\nabla n\|_{L^2}^2+\frac{1}{6}\|\nabla^3 c\|_{L^2}^2.\end{aligned}$$

Therefore, by (3.12) one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 c\|_{L^2}^2+\|\nabla^3 c\|_{L^2}^2\le& C\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\left(\|\nabla u\|_{L^2}^2+\|n\|_{L^2}^2\right)+C\left(\|\nabla n\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\right)\\ \le& C\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}+C\left(\|\nabla n\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\right),\end{aligned}\tag{3.13}$$

and then applying (1.15) and (3.10) yields

$$\sup_{0\le t\le T}\|\Delta c\|_{L^2}^2+\int_0^T\|\nabla^3 c(t)\|_{L^2}^2\mathrm{d}t\le C<\infty.$$

Finally, on one hand, by the embedding inequality

$$\|u\|_{L^6}\le C\|\nabla u(t)\|_{L^2},$$

it is easy to see that

$$u\in L^4(0,T;L^6(\mathbb{R}^3))\ \text{ with }\ \frac{3}{6}+\frac{2}{4}=1.\tag{3.14}$$

On the other hand, by (3.10) and (3.13), we have

$$\begin{aligned}\int_0^T \|\nabla c(t)\|_{L^\infty}^2 \mathrm{d}t &\le \int_0^T \|\nabla c(t)\|_{L^2}^{\frac{1}{2}} \|\nabla \Delta c(t)\|_{L^2}^{\frac{3}{2}} \mathrm{d}t \\ &\le C(T)\left(\int_0^T \|\nabla \Delta c(t)\|_{L^2}^2 \mathrm{d}t\right)^{\frac{3}{4}} \\ &\le C(T) < \infty. \end{aligned} \tag{3.15}$$

Thus, we apply the blow-up criterion (1.5) to finish the proof when (1.15) and (1.16) hold.

Secondly, if the assumptions of (1.15) and(1.17) hold, then it is enough for us to give a new estimate for I_1 in (3.3). By using the Littlewood-Paley decomposition (2.1), we decompose u as follows

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u = \sum_{j<-N} \dot{\Delta}_j u + \sum_{j=-N}^{N} \dot{\Delta}_j u + \sum_{j>N} \dot{\Delta}_j u,$$

where N is a positive integer to be chosen later. Substituting this into I_1, we have

$$\begin{aligned} I_1 \le& \sum_{j<-N} \left|\int_{\mathbb{R}^3} \dot{\Delta}_j u \cdot \nabla u \Delta u \mathrm{d}x\right| + \sum_{j=-N}^{N} \left|\int_{\mathbb{R}^3} \dot{\Delta}_j u \cdot \nabla u \Delta u \mathrm{d}x\right| \\ &+ \sum_{j>N} \left|\int_{\mathbb{R}^3} \dot{\Delta}_j u \cdot \nabla u \Delta u \mathrm{d}x\right| \\ =& I_{11} + I_{12} + I_{13}. \end{aligned} \tag{3.16}$$

For I_{11}, by Hölder's and Bernstein inequalities (2.3) and (2.2), one has

$$\begin{aligned} I_{11} &\le C \sum_{j<-N} \int_{\mathbb{R}^3} |\dot{\Delta}_j u| |\nabla u| |\nabla^2 u| \mathrm{d}x \\ &\le \left(\sum_{j<-N} \|\dot{\Delta}_j u\|_{L^\infty}\right) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \\ &\le C \left(\sum_{j<-N} 2^{\frac{1}{2}j} \|\dot{\Delta}_j \nabla u\|_{L^2}\right) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \\ &\le C 2^{-\frac{1}{2}N} \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \\ &\le C_0 2^{-N} \|\nabla u\|_{L^2}^4 + \frac{1}{16} \|\nabla^2 u\|_{L^2}^2. \end{aligned} \tag{3.17}$$

Similar to I_{11}, we apply Hölder's and Bernstein inequalities (2.3) again to get

$$\begin{aligned} I_{13} &\le \left(\sum_{j>N} \|\dot{\Delta}_j u\|_{L^3}\right) \|\nabla u\|_{L^6} \|\nabla^2 u\|_{L^2} \\ &\le C \left(\sum_{j>N} 2^{\frac{1}{2}j} \|\dot{\Delta}_j u\|_{L^2}\right) \|\nabla^2 u\|_{L^2}^2 \end{aligned}$$

$$\leq C_1 2^{-\frac{1}{2}N} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}^2. \tag{3.18}$$

As to I_{12}, by using integration by parts, the property of divergence free for u, it is easy to see that

$$I_{12} \leq \left(\sum_{j=-N}^{N} \|\dot{\Delta}_j \nabla u\|_{L^\infty} \right) \|\nabla u\|_{L^2}^2 \leq CN \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}. \tag{3.19}$$

Next, we choose a suitable N such that

$$\max\{C_0, C_1\} 2^{-\frac{1}{2}N} \|\nabla u\|_{L^2} \leq \frac{1}{8}, \tag{3.20}$$

i.e.

$$N \geq \frac{\log^+(\max\{C_0, C_1\}\|\nabla u\|_{L^2})}{\log 2} + 2, \tag{3.21}$$

where $\log^+ t = \log t$ for $1 \leq t$ and $\log^+ t = 0$ for $0 < t < 1$. Thus, by combining (3.16)-(3.21), we have

$$I_1 \leq C \log(\|\nabla u\|_{L^2} + e) \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla^2 u\|_{L^2}^2.$$

Then (3.9) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \leq C \log(\|\nabla u\|_{L^2} + e) \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \|\nabla u\|_{L^2}^2 + C\|n\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 u\|_{L^2}^2.$$

Consequently, it yields that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + \left(\|\nabla n\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) \\
\leq& C \log(\|\nabla u\|_{L^2} + e) \left(\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}} + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \right) \left(\|n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right).
\end{aligned}$$

Applying Gronwall's inequality, we obtain for any $t < T$

$$\begin{aligned}
&\|n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \int_0^T \left(\|\nabla n\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) \mathrm{d}t \\
\leq& C\left(\|n_0\|_{L^2}^2 + \|\nabla c_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2\right) \\
&\times \exp\left(\int_0^t \log(\|\nabla u\|_{L^2} + e) \left(\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}} + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \right) \mathrm{d}\tau \right).
\end{aligned}$$

Denote $W(t) := \log(\|n(t)\|_{L^2}^2 + \|\nabla c(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + e)$, then we have

$$W(t) \leq C_1 + C_2 \int_0^t \left(\|\nabla c\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}} + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \right) W(\tau)\mathrm{d}\tau.$$

Applying Gronwall's inequality again, we get the same results of (3.12). We omit the remainder of the proof as it can be obtained with obvious modifications from the previous analysis, and then we finish the proof when (1.15) and (1.17) hold.

Thirdly, in view of (3.1), we notice that (1.15) implies $n \in L^\infty(0,T;L^2(\mathbb{R}^3))$ and $\nabla n \in L^2(0,T;L^2(\mathbb{R}^3))$, and then by the equation $(1.1)_3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le C\|n\|_{L^2}\|u\|_{L^2},$$

then we obtain $u \in L^\infty(0,T;L^2(\mathbb{R}^3))$, $\nabla u \in L^2(0,T;L^2(\mathbb{R}^3))$. From (3.2), we see that $\nabla c \in L^\infty(0,T;L^2(\mathbb{R}^3))$, $\nabla^2 c \in L^2(0,T;L^2(\mathbb{R}^3))$.
Next, we recall the vorticity equation

$$\omega_t - \Delta\omega + u\nabla\omega = \omega\nabla u - \nabla\times(n\nabla\phi). \tag{3.22}$$

By testing the equations (3.22) with ω and applying Lemma 2.2, Hölder's inequality and integration by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 &\le \|u\|_{L^p}\|\omega\|_{L^{\frac{2p}{p-2}}}\|\nabla\omega\|_{L^2} + C\|n\|_{L^2}\|\nabla\omega\|_{L^2}\\ &\le \|u\|_{\dot B^{1-\frac{3}{\alpha}}_{\infty,\infty}}^{1-\frac{2}{p}}\|u\|_{\dot B^{\beta_1}_{2,2}}^{\frac{2}{p}}\|\omega\|_{\dot B^{-\frac{3}{\alpha}}_{\infty,\infty}}^{\frac{2}{p}}\|\omega\|_{\dot B^{\beta_2}_{2,2}}^{1-\frac{2}{p}}\|\nabla\omega\|_{L^2} + C\|n\|_{L^2}\|\nabla\omega\|_{L^2}\\ &\le C\|\omega\|_{\dot B^0_{\alpha,\infty}}\|u\|_{\dot B^{\beta_1}_{2,2}}^{\frac{2}{p}}\|\omega\|_{\dot B^{\beta_2}_{2,2}}^{1-\frac{2}{p}}\|\nabla\omega\|_{L^2} + C\|n\|_{L^2}\|\nabla\omega\|_{L^2},\end{aligned} \tag{3.23}$$

where α is a real number which belongs to $(\frac{3}{2},3)$, β_1 and β_2 are given, respectively, by $\beta_1 = (\frac{3}{\alpha}-1)(\frac{p}{2}-1)$ and $\beta_2 = \frac{3}{\alpha}(\frac{p}{p-2}-1)$. We can choose a suitable p such that

$$\max\{\frac{6}{\alpha}+2, \frac{6}{3-\alpha}\} < p < \frac{2\alpha+6}{3-\alpha}.$$

It's easy to see that $1 < \beta_1 < 2$ and $0 < \beta_2 < 1$. Thus by interpolation inequality,

$$\|u\|_{B^{\beta_1}_{2,2}} \le C\|\omega\|_{L^2}^{2-\beta_1}\|\nabla\omega\|_{L^2}^{\beta_1-1}, \tag{3.24}$$

$$\|\omega\|_{B^{\beta_2}_{2,2}} \le C\|\omega\|_{L^2}^{1-\beta_2}\|\nabla\omega\|_{L^2}^{\beta_2}. \tag{3.25}$$

Then inserting (3.24) and (3.25) into (3.23), we use the Gargliardo-Nirenberg's inequality and Young's inequality to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 &\le C\|\omega\|_{B^0_{\alpha,\infty}}\|\omega\|_{L^2}^{\frac{2}{p}(2-\beta_1)+(1-\frac{2}{p})(1-\beta_2)}\|\nabla\omega\|_{L^2}^{\frac{2}{p}(\beta_1-1)+(1-\frac{2}{p})\beta_2+1}\\ &\quad + C\|n\|_{L^2}\|\nabla\omega\|_{L^2}\\ &\le C\|\omega\|_{B^0_{\alpha,\infty}}\|\omega\|_{L^2}^{2-\frac{3}{\alpha}}\|\nabla\omega\|_{L^2}^{\frac{3}{\alpha}} + C\|n\|_{L^2}\|\nabla\omega\|_{L^2}\\ &\le \|\omega\|_{B^0_{\alpha,\infty}}^{\frac{2\alpha}{2\alpha-3}}\|\omega\|_{L^2}^2 + \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|n\|_{L^2}^2 + \frac{1}{4}\|\nabla\omega\|_{L^2}^2.\end{aligned}$$

Thanks to the boundedness of n, we obtain

$$\omega \in L^\infty(0,T;L^2(\mathbb{R}^3)) \quad \text{and} \quad \nabla\omega \in L^2(0,T;L^2(\mathbb{R}^3)).$$

Therefore, we obtain the same results as (3.14) and (3.15). We omit the rest of the proof, which is analogous to the previous one. Finally, we use a known blow-up criterion (1.5) to finish the proof when (1.15) and (1.18) hold.

3.2. Proof of Theorem 1.2

Now, we prove the result of the theorem when assumption (1.19) hold. We denote vorticity as $\omega := \nabla \times u$, and recall the vorticity equation (3.22). Then we multiply both sides of (3.22) by ω and integrate over $\mathbb{R}^3$ yielding

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 = \int_{\mathbb{R}^3} \omega \cdot \nabla u \omega \mathrm{d}x - \int_{\mathbb{R}^3} \nabla \times (n\nabla\phi)\omega \mathrm{d}x := K_1 + K_2. \quad (3.26)$$

For K_1, we use the Littlewood-Paley decomposition (2.1), we decompose u as follows:

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u = \sum_{j<-L} \dot{\Delta}_j u + \sum_{j=-L}^{L} \dot{\Delta}_j u + \sum_{j>L} \dot{\Delta}_j u, \quad (3.27)$$

where N is a positive integer to be chosen later. Substituting this into K_1, we have

$$\begin{aligned} K_1 &\le \sum_{j<-L} \left| \int_{\mathbb{R}^3} \omega \cdot \nabla \dot{\Delta}_j u \, \omega \mathrm{d}x \right| + \sum_{j=-L}^{L} \left| \int_{\mathbb{R}^3} \omega \cdot \nabla \dot{\Delta}_j u \, \omega \mathrm{d}x \right| \\ &\quad + \sum_{j>L} \left| \int_{\mathbb{R}^3} \omega \cdot \nabla \dot{\Delta}_j u \, \omega \mathrm{d}x \right| \\ &= K_{11} + K_{12} + K_{13}. \end{aligned} \quad (3.28)$$

For K_{11}, integration by parts, the property of divergence free, Hölder's and Bernstein inequalities (2.3), one has

$$\begin{aligned} K_{11} &\le C \sum_{j<-L} \int_{\mathbb{R}^3} |\omega||\dot{\Delta}_j u||\nabla\omega| \mathrm{d}x \\ &\le \left(\sum_{j<-L} \|\dot{\Delta}_j u\|_{L^\infty} \right) \|\omega\|_{L^2} \|\nabla\omega\|_{L^2} \\ &\le C \left(\sum_{j<-L} 2^{\frac{1}{2}j} \|\dot{\Delta}_j \nabla u\|_{L^2} \right) \|\omega\|_{L^2} \|\nabla\omega\|_{L^2} \\ &\le C 2^{-\frac{1}{2}L} \|\nabla u\|_{L^2} \|\omega\|_{L^2} \|\nabla\omega\|_{L^2} \\ &\le C_2 2^{-L} \|\nabla u\|_{L^2}^2 \|\omega\|_{L^2}^2 + \frac{1}{16}\|\nabla\omega\|_{L^2}^2. \end{aligned} \quad (3.29)$$

Similar to K_{11}, we apply Hölder's and Bernstein inequalities (2.3) again to get

$$\begin{aligned} K_{13} &\le \left(\sum_{j>L} \|\dot{\Delta}_j u\|_{L^3} \right) \|\omega\|_{L^6} \|\nabla\omega\|_{L^2} \\ &\le C \left(\sum_{j>L} 2^{\frac{1}{2}j} \|\dot{\Delta}_j u\|_{L^2} \right) \|\nabla\omega\|_{L^2}^2 \\ &\le C_3 2^{-\frac{1}{2}L} \|\nabla u\|_{L^2} \|\nabla\omega\|_{L^2}^2. \end{aligned} \quad (3.30)$$

As to K_{12}, it is easy to see that

$$K_{12} \leq \left(\sum_{j=-L}^{L} \|\dot{\Delta}_j \nabla u\|_{L^\infty} \right) \|\omega\|_{L^2}^2 \leq CL \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \|\omega\|_{L^2}^2. \tag{3.31}$$

Next, we choose a suitable N such that

$$\max\{C_2, C_3\} 2^{-\frac{1}{2}L} \|\nabla u\|_{L^2} \leq \frac{1}{8}, \tag{3.32}$$

i.e.

$$N \geq \frac{\log^+(\max\{C_0, C_1\}\|\nabla u\|_{L^2})}{\log 2} + 2, \tag{3.33}$$

where $\log^+ t = \log t$ for $1 \leq t$ and $\log^+ t = 0$ for $0 < t < 1$. Thus, by combining (3.26)-(3.33), we have

$$K_1 \leq C \log(\|\nabla u\|_{L^2} + e) \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \|\omega\|_{L^2}^2 + \frac{1}{4}\|\omega\|_{L^2}^2 + \frac{1}{4}\|\nabla \omega\|_{L^2}^2.$$

For K_2, we have

$$\begin{aligned} K_2 &\leq C \int_{\mathbb{R}^3} |\nabla n||\omega| \mathrm{d}x \\ &\leq C \|\nabla n^{\frac{1}{2}}\|_{L^2} \|n^{\frac{1}{2}}\|_{L^{2p}} \|\omega\|_{L^{\frac{2p}{p-1}}} \\ &\leq C \|\nabla n^{\frac{1}{2}}\|_{L^2} \|n\|_{L^p}^{\frac{1}{2}} \|\omega\|_{L^2}^{\frac{2p-3}{2p}} \|\nabla \omega\|_{L^2}^{\frac{3}{2p}} \\ &\leq C \|n\|_{L^p}^{\frac{2p}{2p-3}} \|\omega\|_{L^2}^2 + \frac{1}{16}\|\nabla n^{\frac{1}{2}}\|_{L^2}^2 + \frac{1}{4}\|\nabla \omega\|_{L^2}^2. \end{aligned}$$

Combining above two inequalities, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 \leq \log(\|\nabla u\|_{L^2} + e) \left(\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|n\|_{L^p}^{\frac{2p}{2p-3}} + 1 \right) \|\omega\|_{L^2}^2 \\ + \frac{1}{16}\|\nabla n^{\frac{1}{2}}\|_{L^2}^2. \end{aligned} \tag{3.34}$$

Next, we deal with the equation of n. We begin with the equation of $n \ln n$

$$\partial_t(n \ln n) = -(u \cdot \nabla n)(\ln n + 1) + \Delta(n \ln n) - \frac{|\nabla n|^2}{n} - \nabla \cdot (\chi(c) n \nabla c)(\ln n + 1).$$

It implies that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} n \ln n \mathrm{d}x + 4 \int_{\mathbb{R}^3} |\nabla n^{\frac{1}{2}}|^2 \mathrm{d}x \\ = &\int_{\mathbb{R}^3} \chi(c) \nabla n \nabla c \mathrm{d}x \leq C \int_{\mathbb{R}^3} n^{\frac{1}{2}} \nabla n^{\frac{1}{2}} \nabla c \mathrm{d}x \\ \leq &C \|n^{\frac{1}{2}}\|_{L^{2p}} \|\nabla n^{\frac{1}{2}}\|_{L^2} \|\nabla c\|_{L^{\frac{2p}{p-1}}} \\ \leq &C \|n\|_{L^p}^{\frac{1}{2}} \|\nabla n^{\frac{1}{2}}\|_{L^2} \|\nabla c\|_{L^{\frac{2p}{p-1}}} \end{aligned}$$

$$
\begin{aligned}
&\le C\|n\|_{L^p}\|\nabla c\|^2_{L^{\frac{2p}{p-1}}} + \frac{1}{4}\|\nabla n^{\frac{1}{2}}\|^2_{L^2} \\
&\le C\|n\|_{L^p}^{\frac{2p}{2p-3}}\|\nabla c\|^2_{L^2} + \frac{1}{4}\|\nabla^2 c\|^2_{L^2} + \frac{1}{4}\|\nabla n^{\frac{1}{2}}\|^2_{L^2}. \qquad (3.35)
\end{aligned}
$$

By testing equation $(1.1)_2$ against Δc, we get another estimates of $\|\nabla c\|_{L^2}$ that

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c\|^2_{L^2} + \|\nabla^2 c\|^2_{L^2} &= \int_{\mathbb{R}^3} u\cdot\nabla c\Delta c\mathrm{d}x + \int_{\mathbb{R}^3} k(c)n\Delta c\mathrm{d}x \\
&= H_1 + H_2.
\end{aligned}
$$

For H_2, by using the same computations as in (3.35), we have

$$
\begin{aligned}
H_2 &= -\int_{\mathbb{R}^3} k'(c)n|\nabla c|^2\mathrm{d}x - \int_{\mathbb{R}^3} k(c)\nabla n\nabla c\mathrm{d}x \\
&\le -\int_{\mathbb{R}^3} k(c)\nabla n\nabla c\mathrm{d}x \\
&\le C\|n\|_{L^p}^{\frac{2p}{2p-3}}\|\nabla c\|^2_{L^2} + \frac{1}{4}\|\nabla^2 c\|^2_{L^2} + \frac{1}{4}\|\nabla n^{\frac{1}{2}}\|^2_{L^2}, \qquad (3.36)
\end{aligned}
$$

where we use the fact that $k' \ge 0$. For H_1, we use the Littlewood-Paley decomposition for u as follows in (3.27). Then, we have

$$
\begin{aligned}
H_1 \le &\left|\sum_{j<-M}\int_{\mathbb{R}^3}\dot{\Delta}_j u\cdot\nabla c\Delta c\mathrm{d}x\right| + \left|\sum_{j=-M}^{M}\int_{\mathbb{R}^3}\dot{\Delta}_j u\cdot\nabla c\Delta c\mathrm{d}x\right| \\
&+ \left|\sum_{j>M}\int_{\mathbb{R}^3}\dot{\Delta}_j u\cdot\nabla c\Delta c\mathrm{d}x\right| \\
= &H_{11} + H_{12} + H_{13}. \qquad (3.37)
\end{aligned}
$$

By using (3.2), Hölder's and Bernstein inequalities (2.3), we estimate $H_{1i}, i = 1,2,3$ in turn

$$
\begin{aligned}
H_{11} &\le \left(\sum_{j<-M}\|\dot{\Delta}_j u\|_{L^\infty}\right)\|\nabla c\|_{L^2}\|\nabla^2 c\|_{L^2} \\
&\le C\left(\sum_{j<-M}2^{\frac{1}{2}j}\|\dot{\Delta}_j\nabla u\|_{L^2}\right)\|\nabla c\|_{L^2}\|\nabla^2 c\|_{L^2} \\
&\le C2^{-\frac{1}{2}M}\|\nabla u\|_{L^2}\|\nabla c\|_{L^2}\|\nabla^2 c\|_{L^2} \\
&\le C_4 2^{-M}\|\nabla u\|^2_{L^2}\|\nabla c\|^2_{L^2} + \frac{1}{16}\|\nabla^2 c\|^2_{L^2}, \qquad (3.38) \\
H_{12} &\le \sum_{j=-M}^{M}\int_{\mathbb{R}^3}|\dot{\Delta}_j\nabla u||\nabla c|^2\mathrm{d}x \\
&\le \left(\sum_{j=-M}^{M}\|\dot{\Delta}_j\nabla u\|_{L^\infty}\right)\|\nabla c\|^2_{L^2}
\end{aligned}
$$

$$\leq CM\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}\|\nabla c\|^2_{L^2}, \tag{3.39}$$

$$\begin{aligned} H_{13} &\leq \left(\sum_{j>M}\|\dot{\Delta}_j u\|_{L^3}\right)\|\nabla c\|_{L^6}\|\nabla^2 c\|_{L^2} \\ &\leq C\left(\sum_{j>N}2^{\frac{1}{2}j}\|\dot{\Delta}_j u\|_{L^2}\right)\|\nabla^2 c\|^2_{L^2} \\ &\leq C_5 2^{-\frac{1}{2}M}\|\nabla u\|_{L^2}\|\nabla^2 c\|^2_{L^2}. \end{aligned} \tag{3.40}$$

Next, we choose a suitable N such that

$$\max\{C_4, C_5\}2^{-\frac{1}{2}M}\|\nabla u\|_{L^2} \leq \frac{1}{8}. \tag{3.41}$$

Thus, by a similar argument as before, we deduce from (3.37)-(3.41),

$$H_1 \leq C\log(\|\nabla u\|_{L^2}+e)\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}\|\nabla c\|^2_{L^2} + \frac{1}{4}\|\nabla c\|^2_{L^2} + \frac{1}{4}\|\nabla^2 c\|^2_{L^2}.$$

Therefore, in view of (3.36), we can deduce that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c\|^2_{L^2} + \|\nabla^2 c\|^2_{L^2} \leq C\log(\|\omega\|_{L^2}+e)\left(\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|n\|^{\frac{2p}{2p-3}}_{L^p} + 1\right)\|\nabla c\|^2_{L^2} \\ + \frac{1}{4}\|\nabla n^{\frac{1}{2}}\|^2_{L^2}. \end{aligned} \tag{3.42}$$

Combining (3.34), (3.35) with (3.42), and then absorbing the small term on the right hand side, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla c\|^2_{L^2} + \|\omega\|^2_{L^2} + \int_{\mathbb{R}^3} n\ln n\mathrm{d}x\right) + \|\nabla^2 c\|^2_{L^2} + \|\nabla\omega\|^2_{L^2} + 2\int_{\mathbb{R}^3}|\nabla n^{\frac{1}{2}}|^2\mathrm{d}x \\ \leq& C\log(\|\omega\|_{L^2}+\|\nabla c\|_{L^2}+e)\left(\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|n\|^{\frac{2p}{2p-3}}_{L^p} + 1\right)\left(\|\omega\|^2_{L^2} + \|\nabla c\|^2_{L^2}\right), \end{aligned}$$

which implies that

$$\begin{aligned} &\|\nabla c(t)\|^2_{L^\infty_t(L^2)} + \|\omega(t)\|^2_{L^\infty_t(L^2)} + \sup_{0<\tau<t}\int_{\mathbb{R}^3} n\ln n\mathrm{d}x \\ \leq& C\int_0^t \left(\|\nabla c(t)\|^2_{L^\infty_t(L^2)} + \|\omega(t)\|^2_{L^\infty_t(L^2)}\right)\log(\|\omega(\tau)\|_{L^2} + \|\nabla c(\tau)\|_{L^2} + e) \\ &\times\left(\|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|n(\tau)\|^{\frac{2p}{2p-3}}_{L^p} + 1\right)\mathrm{d}\tau + C(c_0, u_0, n_0), \end{aligned}$$

then by (1.19) and Gronwall's inequality, we have

$$\begin{aligned} &\|\nabla c(t)\|^2_{L^2} + \|\omega(t)\|^2_{L^2} \\ \leq& C\exp\left[\int_0^t \log(\|\omega(\tau)\|_{L^2}+\|\nabla c(\tau)\|_{L^2}+e)\left(\|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\infty}}+\|n(\tau)\|^{\frac{2p}{2p-3}}_{L^p}+1\right)\mathrm{d}\tau\right], \end{aligned}$$

where $C = C(n_0, \omega_0, c_0)$. Defined $Z(t) := \log(\|\omega(t)\|_{L^2} + \|\nabla c(t)\|_{L^2} + e)$, above inequality implies that

$$Z(t) \leq C_1 + C_2 \int_0^t \left(\|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|n(\tau)\|_{L^p}^{\frac{2p}{2p-3}} + 1 \right) Z(\tau) \mathrm{d}\tau.$$

Applying the Gronwall's inequality to $Z(t)$ again, we have

$$(\omega, \nabla c) \in L^\infty(0, T^*; L^2(\mathbb{R}^3)) \quad \text{and} \quad (\nabla\omega, \nabla^2 c) \in L^2(0, T^*; L^2(\mathbb{R}^3)). \tag{3.43}$$

Following the above argument by applying Gronwall's inequality twice, it yields (3.43). Next, we can show that $\|\nabla c\|_{L^2(0,T;L^\infty)} < \infty$ by using the same argument as (3.14) and (3.15), and then finish the proof when assumption (1.19) holds.

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