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A MORE ACCURATE HALF-DISCRETE HILBERT-TYPE INEQUALITY INVOLVING ONE HIGHER-ORDER DERIVATIVE FUNCTION*

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Abstract By means of the weight functions, Hermite-Hadamards inequality and the techniques of real analysis, a new more accurate half-discrete Hilbert-type inequality involving one higher-order derivative function is given. The equivalent conditions of the best possible constant factor related to a few parameters, the equivalent forms, several particular inequalities and the operator expressions are considered.

Keywords Weight function, Hermite-Hadamards inequality, half-discrete Hilbert-type inequality, higher-order derivative function, parameter, best possible constant factor.

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1. Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$.The following Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$was given by (cf. [4], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1.1}$$

A more accurate form of (1.1) was provided as follows (cf. [4], Theorem 323):.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1.2}$$

By means of Euler-Maclaurin's summation formula, in 2006, Krnic et al. [4]

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provided an extension of (1.1) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{1.3}$$

where, $\lambda_i\in(0,2]$ $(i=1,2), \lambda_1+\lambda_2=\lambda\in(0,4]$, and the constant factor $B(\lambda_1,\lambda_2)$ is the best possible,

$$B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\ (u,v>0)$$

is the beta function. For $\lambda=1,\lambda_1=\frac{1}{q},\lambda_2=\frac{1}{p}$ in (1.3), we have (1.1); for $p=q=2$, $\lambda_1=\lambda_2=\frac{\lambda}{2}$, (1.3) reduces to the published inequality in Yang [25].

In 2019, by applying (1.3) and Abel's summation by parts formula, Adiyasuren et al. [1] gave a Hilbert-type inequality with the kernel as $\frac{1}{(m+n)^{\lambda}}$ involving two partial sums. Inequalities (1.1)-(1.3) play an important role in analysis and its applications (cf. [2, 3, 5, 6, 13, 16, 21–23, 26, 32]).

In 1934, Hardy et al. [4] published a half-discrete Hilbert-type inequality in Theorem 351: If $K(t)$ $(t>0)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^{\infty}K(t)t^{s-1}dt<\infty, a_n\geq 0$, such that $0<\sum_{n=1}^{\infty}a_n^p<\infty$, then

$$\int_0^{\infty}x^{p-2}(\sum_{n=1}^{\infty}K(nx)a_n)^p dx<\phi^p(\frac{1}{p})\sum_{n=1}^{\infty}a_n^p. \tag{1.4}$$

Some new extensions of (1.4) were provided by [17–19, 27, 28].

By the use of the techniques of real analysis, in 2016, Hong et al. [7] gave an equivalent condition of the best possible constant factor related to several parameters in the general form of (1.1). The other similar results were provided by [8–11, 20, 24, 29, 30]. Recently, Yang et al. [31] also gave a new result of the reverse half-discrete Hilbert-type inequality.

In this paper, following the way of [1] and [7], by means of the weight functions, Hermite-Hadamard's inequality and the techniques of real analysis, a new more accurate half-discrete Hilbert-type inequality with the kernel as $\frac{1}{[x+(n-\xi)^{\alpha}]^{\lambda}}$ involving one higher-order derivative function is given. The equivalent conditions of the best possible constant factor related to a few parameters, the equivalent forms, several particular inequalities and the operator expressions are also considered. The lemmas and theorems provided an extensive account of this type of inequalities.

2. Some lemmas

In what follows, we suppose that $p>1, \frac{1}{p}+\frac{1}{q}=1, \mathbf{N}:=\{1,2,\cdots\}, m\in\mathbf{N}\cup\{\mathbf{0}\}, \alpha\in(0,1]$, $\xi\in[0,\frac{1}{2}], \lambda>0, \lambda_1\in(0,\lambda), \lambda_2\in(0,\lambda)\cap(0,\frac{1}{\alpha}], k_{\lambda}(\lambda_i):=B(\lambda_i,\lambda-\lambda_i)$ $(i=1,2)$, $\widehat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \widehat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. We also assume that $f(x):=f^{(0)}(x)$ is a continuous derivative function of m-order unless finite points in $\mathbf{R}_+:=(0,\infty)$, such that $f^{(k)}(x)\geq 0, f^{(k)}(0^+)=0$ $(k=0,1,\cdots,m)$, and $f^{(m)}(x), a_n\geq 0$,

$$0<\int_0^{\infty}x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx<\infty \text{ and } 0<\sum_{n=1}^{\infty}(n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1}a_n^q<\infty.$$

Lemma 2.1. *Define the following weight function:*

$$\varpi_\lambda(\lambda_2, x) := \alpha x^{\lambda-\lambda_2} \sum_{n=1}^{\infty} \frac{(n-\xi)^{\alpha\lambda_2-1}}{[x+(n-\xi)^\alpha]^\lambda} \ (x \in \mathbf{R}_+). \tag{2.1}$$

We have the following inequalities

$$k_\lambda(\lambda_2) - \frac{1}{\lambda_2}\left[\frac{(1-\xi)^\alpha}{x}\right]^{\lambda_2} < \varpi_\lambda(\lambda_2, x) < k_\lambda(\lambda_2) \ (x \in \mathbf{R}_+). \tag{2.2}$$

Proof. For fixed $x \in \mathbf{R}_+$, the function $\frac{(t-\xi)^{\alpha\lambda_2-1}}{[x+(t-\xi)^\alpha]^\lambda}$ is strictly decreasing and strictly convex in $(\frac{1}{2}, \infty)$. In fact, for $\alpha \in (0,1], \xi \in [0, \frac{1}{2}], \lambda_2 \in (0, \lambda) \cap (0, \frac{1}{\alpha}], t \in (\frac{1}{2}, \infty)$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{(t-\xi)^{\alpha\lambda_2-1}}{[x+(t-\xi)^\alpha]^\lambda} = \frac{-(1-\alpha\lambda_2)(t-\xi)^{\alpha\lambda_2-2}}{[x+(t-\xi)^\alpha]^\lambda} - \frac{\lambda\alpha(t-\xi)^{\alpha\lambda_2+\alpha-2}}{[x+(t-\xi)^\alpha]^{\lambda+1}} < 0,\\
&\frac{\partial^2}{\partial t^2}\frac{(t-\xi)^{\alpha\lambda_2-1}}{[x+(t-\xi)^\alpha]^\lambda} = \frac{(1-\alpha\lambda_2)(2-\alpha\lambda_2)(t-\xi)^{\alpha\lambda_2-3}}{[x+(t-\xi)^\alpha]^\lambda}\\
&+\frac{\lambda\alpha(3-2\alpha\lambda_2-\alpha)(t-\xi)^{\alpha\lambda_2+\alpha-3}}{[x+(t-\xi)^\alpha]^{\lambda+1}} + \frac{\lambda\alpha^2(\lambda+1)(t-\xi)^{\alpha\lambda_2+2\alpha-3}}{[x+(t-\xi)^\alpha]^{\lambda+2}} > 0.
\end{aligned}$$

By the decreasingness property of series and Hermite-Hadamard's inequality (cf. [14]), we have

$$\int_1^\infty \frac{(t-\xi)^{\alpha\lambda_2-1}dt}{[x+(t-\xi)^\alpha]^\lambda} < \sum_{n=1}^{\infty}\frac{(n-\xi)^{\alpha\lambda_2-1}}{[x+(n-\xi)^\alpha]^\lambda} < \int_{\frac{3}{2}}^\infty \frac{(t-\xi)^{\alpha\lambda_2-1}dt}{[x+(t-\xi)^\alpha]^\lambda}. \tag{2.3}$$

Setting $v = \frac{(t-\xi)^\alpha}{x}$ $(dt = \frac{1}{\alpha}x^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}-1}dv)$, for $\frac{1}{2} - \xi \geq 0$, we obtain

$$\begin{aligned}
\int_{\frac{3}{2}}^\infty \frac{(t-\xi)^{\alpha\lambda_2-1}dt}{[x+(t-\xi)^\alpha]^\lambda} &= \frac{1}{\alpha x^\lambda}\int_{\frac{(\frac{1}{2}-\xi)^\alpha}{x}}^\infty \frac{(xv)^{\frac{1}{\alpha}(\alpha\lambda_2-1)}}{(1+v)^\lambda}x^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}-1}dv\\
&\leq \frac{1}{\alpha x^{\lambda-\lambda_2}}\int_0^\infty \frac{v^{\lambda_2-1}dv}{(1+v)^\lambda} = \frac{1}{\alpha x^{\lambda-\lambda_2}}k_\lambda(\lambda_2).
\end{aligned}$$

By (2.1) and (2.3), we have

$$\varpi_\lambda(\lambda_2, x) < \alpha x^{\lambda-\lambda_2}\frac{1}{\alpha x^{\lambda-\lambda_2}}k_\lambda(\lambda_2) = k_\lambda(\lambda_2).$$

On the other hand, we find

$$\begin{aligned}
\int_1^\infty \frac{(t-\xi)^{\alpha\lambda_2-1}dt}{[x+(t-\xi)^\alpha]^\lambda} &= \frac{1}{\alpha x^{\lambda-\lambda_2}}\int_{\frac{(1-\xi)^\alpha}{x}}^\infty \frac{v^{\lambda_2-1}}{(1+v)^\lambda}dv\\
&= \frac{1}{\alpha x^{\lambda-\lambda_2}}\left[\int_0^\infty \frac{v^{\lambda_2-1}dv}{(1+v)^\lambda} - \int_0^{\frac{(1-\xi)^\alpha}{x}} \frac{v^{\lambda_2-1}dv}{(1+v)^\lambda}\right]\\
&\geq \frac{1}{\alpha x^{\lambda-\lambda_2}}\left[k_\lambda(\lambda_2) - \int_0^{\frac{(1-\xi)^\alpha}{x}} v^{\lambda_2-1}dv\right]
\end{aligned}$$

$$= \frac{1}{\alpha x^{\lambda-\lambda_2}}\left\{k_\lambda(\lambda_2) - \frac{1}{\lambda_2}[\frac{(1-\xi)^\alpha}{x}]^{\lambda_2}\right\},$$

and then by (2.1) and (2.3), we have

$$\varpi_\lambda(\lambda_2, x) > k_\lambda(\lambda_2) - \frac{1}{\lambda_2}\left[\frac{(1-\xi)^\alpha}{x}\right]^{\lambda_2}.$$

Hence, inequalities (2.2) follow.
The lemma is proved. □

Lemma 2.2. *We have the following Hilbert-type inequality:*

$$I_0 := \int_0^\infty \sum_{n=1}^\infty \frac{a_n f^{(m)}(x)dx}{[x+(n-\xi)^\alpha]^\lambda} < (\frac{1}{\alpha}k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$$
$$\times\left[\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty (n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1}a_n^q\right]^{\frac{1}{q}}. \quad (2.4)$$

Proof. Setting $v = x/(n-\xi)^\alpha$, we can obtain the following another weight function:

$$\omega_\lambda(\lambda_1, n) := (n-\xi)^{\alpha(\lambda-\lambda_1)}\int_0^\infty \frac{x^{\lambda_1-1}dx}{[x+(n-\xi)^\alpha]^\lambda}$$
$$= \int_0^\infty \frac{v^{\lambda_1-1}dv}{(1+v)^\lambda} = k_\lambda(\lambda_1)\ (n\in\mathbf{N}). \quad (2.5)$$

By Hölder's inequality (cf. [14]), we have

$$I_0 := \int_0^\infty \sum_{n=1}^\infty \frac{1}{[x+(n-\xi)^\alpha]^\lambda}\left[\frac{x^{(1-\lambda_1)/q}f^{(m)}(x)}{(n-\xi)^{(1-\alpha\lambda_2)/p}}\right]\left[\frac{(n-\xi)^{(1-\alpha\lambda_2)/p}a_n}{x^{(1-\lambda_1)/q}}\right]dx$$
$$\le \left\{\int_0^\infty \sum_{n=1}^\infty \frac{1}{[x+(n-\xi)^\alpha]^\lambda}\frac{x^{(1-\lambda_1)(p-1)}(f^{(m)}(x))^p}{(n-\xi)^{1-\alpha\lambda_2}}dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=1}^\infty \int_0^\infty \frac{1}{[x+(n-\xi)^\alpha]^\lambda}\frac{(n-\xi)^{(1-\alpha\lambda_2)(q-1)}}{x^{1-\lambda_1}}a_n^q dx\right\}^{\frac{1}{q}}$$
$$= \left\{\frac{1}{\alpha}\int_0^\infty \varpi_\lambda(\lambda_2, x)x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=1}^\infty \omega_\lambda(\lambda_1, n)(n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1}a_n^q\right\}^{\frac{1}{q}}. \quad (2.6)$$

We show that (2.6) does not keep the form of equality. Otherwise (cf. [14]), there exist constants A and B, such that they are not both zero and

$$A\frac{x^{(1-\lambda_1)(p-1)}(f^{(m)}(x))^p}{(n-\xi)^{1-\alpha\lambda_2}} = B\frac{(n-\xi)^{(1-\alpha\lambda_2)(q-1)}}{x^{1-\lambda_1}}a_n^q \text{ a.e. in } \mathbf{R}_+\times\mathbf{N}.$$

Assuming that $A \neq 0$, there exists a $n\in\mathbf{N}$, such that

$$x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p = \frac{B}{A}\frac{(n-\xi)^{q(1-\alpha\lambda_2)}}{x^{1+(\lambda-\lambda_1-\lambda_2)}}a_n^q \text{ a.e. in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx < \infty,$$

based on $\int_0^\infty \cdot \frac{dx}{x^{1+(\lambda-\lambda_1-\lambda_2)}} = \infty$. Then by (2.2) and (2.5), we have (2.4).

The lemma is proved. □

Lemma 2.3. *For $t > 0$, we have the following inequality:*

$$\int_0^\infty e^{-tx} f(x)dx \leq \frac{1}{t^m} \int_0^\infty e^{-tx} f^{(m)}(x)dx. \tag{2.7}$$

Proof. For $f^{(k-1)}(0^+) = 0$ $(k = 1, \cdots, m)$, integration by parts, we find

$$\begin{aligned}\int_0^\infty e^{-tx} f^{(k)}(x)dx &= \int_0^\infty e^{-tx} df^{(k-1)}(x) \\ &= e^{-tx} f^{(k-1)}(x)|_0^\infty - \int_0^\infty f^{(k-1)}(x)de^{-tx} \\ &= \lim_{x\to\infty} e^{-tx} f^{(k-1)}(x) + t\int_0^\infty e^{-tx} f^{(k-1)}(x)dx.\end{aligned}$$

For large enough $x > 0$, we have $e^{-tx} f^{(k-1)}(x) \geq 0$. Then by the increasing property of $f^{(k-1)}(x)$, it follows that $\lim_{x\to\infty} e^{-tx} f^{(k-1)}(x) \geq 0$ and then

$$\int_0^\infty e^{-tx} f^{(k)}(x)dx \leq t\int_0^\infty e^{-tx} f^{(k-1)}(x)dx.$$

Substitution of $i = 1, \cdots, m$ in the above inequality, we have (2.7).

The lemma is proved. □

Note 1. For $m = 0$, in view of $f^{(0)}(x) = f(x)$, (2.7) keeps the form of equality.

3. Main results

Theorem 3.1. *We have the following more accurate half-discrete Hilbert-type inequality involving one higher-order derivative function:*

$$\begin{aligned}I &:= \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)dx}{[x+(n-\xi)^\alpha]^{\lambda+m}} < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}(\frac{1}{\alpha}k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}} \\ &\times \left[\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty (n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1} a_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have

$$0 < \int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx < \infty,\ 0 < \sum_{n=1}^\infty (n-\xi)^{q(1-\alpha\lambda_2)-1} a_n^q < \infty,$$

and the following inequality:

$$I = \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)dx}{[x+(n-\xi)^\alpha]^{\lambda+m}} < \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1, \lambda_2)$$

$$\times \left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty (n-\xi)^{q(1-\alpha\lambda_2)-1} a_n^q\right]^{\frac{1}{q}}. \quad (3.2)$$

Proof. Since we have

$$\frac{1}{[x+(n-\xi)^\alpha]^{\lambda+m}} = \frac{1}{\Gamma(\lambda+m)} \int_0^\infty t^{\lambda+m-1} e^{-[x+(n-\xi)^\alpha]t} dt,$$

by Lebesgue term by term integration theorem (cf. [15]) and (2.7), we obtain

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda+m)} \int_0^\infty \sum_{n=1}^\infty a_n f(x) \int_0^\infty t^{\lambda+m-1} e^{-[x+(n-\xi)^\alpha]t} dt dx \\
&= \frac{1}{\Gamma(\lambda+m)} \int_0^\infty t^{\lambda+m-1} (\int_0^\infty e^{-xt} f(x) dx) \sum_{n=1}^\infty e^{-t(n-\xi)^\alpha} a_n dt \\
&\le \frac{1}{\Gamma(\lambda+m)} \int_0^\infty t^{\lambda+m-1} (t^{-m} \int_0^\infty e^{-xt} f^{(m)}(x) dx) \sum_{n=1}^\infty e^{-t(n-\xi)^\alpha} a_n dt \\
&= \frac{1}{\Gamma(\lambda+m)} \int_0^\infty \sum_{n=1}^\infty a_n f^{(m)}(x) [\int_0^\infty t^{\lambda-1} e^{-[x+(n-\xi)^\alpha]t} dt] dx \\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \int_0^\infty \sum_{n=1}^\infty \frac{a_n f^{(m)}(x) dx}{[x+(n-\xi)^\alpha]^\lambda} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} I_0.
\end{aligned}$$

Then by (2.6), we have (3.1).

The theorem is proved. □

Theorem 3.2. *If $\lambda_1+\lambda_2=\lambda$, then the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} (\frac{1}{\alpha} k_\lambda(\lambda_2))^{\frac{1}{p}} (k_\lambda(\lambda_1))^{\frac{1}{q}}$$

in (3.1) is the best possible. On the other hand, if the same constant factor in (3.1) is the best possible, then for $\lambda-\lambda_1 \le \frac{1}{\alpha}$, we have $\lambda_1+\lambda_2=\lambda$.

Proof. If $\lambda_1+\lambda_2=\lambda$, then (3.1) reduces to (3.2). For any $0<\varepsilon<q\min\{\lambda_1,\lambda_2\}$, we set

$$\widetilde{f}^{(0)}(x) = \widetilde{f}(x) := \begin{cases} 0, & 0<x<1, \\ x^{\lambda_1+m-\frac{\varepsilon}{p}-1}, & x \ge 1, \end{cases}$$

$$\widetilde{a}_n := (n-\xi)^{\alpha(\lambda_2-\frac{\varepsilon}{q})-1} \ (n \in \mathbf{N}),$$

and find

$$\widetilde{f}^{(m)}(x) = \begin{cases} 0, & 0<x<1, \\ \Pi_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p}) x^{\lambda_1-\frac{\varepsilon}{p}-1}, & x>1. \end{cases}$$

For $m=0, \varepsilon \ge 0$, we define $\Pi_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})=1$.

If there exists a constant $M(\le \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} k_\lambda(\lambda_1))$, such that (3.2) is valid when we replace $\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} k_\lambda(\lambda_1)$ by M, then in particular, we have

$$\widetilde{I} := \int_0^\infty \sum_{n=1}^\infty \frac{\widetilde{a}_n \widetilde{f}(x) dx}{[x+(n-\xi)^\alpha]^{\lambda+m}}$$

$$< M\left[\int_0^\infty x^{p(1-\lambda_1)-1}(\widetilde{f}^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\xi)^{q(1-\alpha\lambda_2)-1}\widetilde{a}_n^q\right]^{\frac{1}{q}}. \quad (3.3)$$

By the decreasingness property of series, we find

$$\begin{aligned}
\widetilde{I} &< M\Pi_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})\left[\int_1^\infty x^{p(1-\lambda_1)-1}x^{p(\lambda_1-1)-\varepsilon}dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=1}^{\infty}(n-\xi)^{q(1-\alpha\lambda_2)-1}(n-\xi)^{q\alpha\lambda_2-\alpha\varepsilon-q}\right]^{\frac{1}{q}}\\
&= M\Pi_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})(\int_1^\infty x^{-\varepsilon-1}dx)^{\frac{1}{p}}\\
&\quad\times\left[(1-\xi)^{-\alpha\varepsilon-1}+\sum_{n=2}^{\infty}(n-\xi)^{-\alpha\varepsilon-1}\right]^{\frac{1}{q}}\\
&\le M\Pi_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})(\int_1^\infty x^{-\varepsilon-1}dx)^{\frac{1}{p}}\\
&\quad\times\left[(1-\xi)^{-\alpha\varepsilon-1}+\int_1^\infty(y-\xi)^{-\alpha\varepsilon-1}dy\right]^{\frac{1}{q}}\\
&= \frac{M}{\varepsilon}\Pi_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})\left[\varepsilon(1-\xi)^{-\alpha\varepsilon-1}+\frac{1}{\alpha}(1-\xi)^{-\alpha\varepsilon}\right]^{\frac{1}{q}}.
\end{aligned}$$

Replacing λ by $\lambda+m$, setting $\widetilde{\lambda}_2 := \lambda_2-\frac{\varepsilon}{q}\in(0,\lambda+m)\cap(0,\frac{1}{\alpha}]$, $\widetilde{\lambda}_1 := \lambda_2+m+\frac{\varepsilon}{q}\in(0,\lambda+m)$ in (2.1), by (2.2), we have

$$\begin{aligned}
\widetilde{I} &= \int_1^\infty\left\{x^{\lambda_1+m+\frac{\varepsilon}{q}}\sum_{n=1}^{\infty x}\frac{(n-\xi)^{\alpha(\lambda_2-\frac{\varepsilon}{q})-1}}{[x+(n-\xi)^\alpha]^{\lambda+m}}\right\}x^{-\varepsilon-1}dx\\
&= \int_1^\infty \varpi_{\lambda+m}(\widetilde{\lambda}_2,x)x^{-\varepsilon-1}dx\\
&> \frac{1}{\alpha}\int_1^\infty\{k_{\lambda+m}(\widetilde{\lambda}_2)-\frac{1}{\widetilde{\lambda}_2}[\frac{(1-\xi)^\alpha}{x}]^{\widetilde{\lambda}_2}\}x^{-\varepsilon-1}dx\\
&= \frac{1}{\alpha}[\int_1^\infty k_{\lambda+m}(\widetilde{\lambda}_2)x^{-\varepsilon-1}dx-\int_1^\infty\frac{1}{\widetilde{\lambda}_2}(1-\xi)^{\alpha\widetilde{\lambda}_2}x^{-\widetilde{\lambda}_2-\varepsilon-1}dx]\\
&= \frac{1}{\varepsilon\alpha}(k_{\lambda+m}(\widetilde{\lambda}_2)-\varepsilon O(1)).
\end{aligned}$$

Based on the above results, we find

$$\begin{aligned}
&\frac{1}{\alpha}(k_{\lambda+m}(\widetilde{\lambda}_2)-\varepsilon O(1))\\
&< \varepsilon\widetilde{I} < M\Pi_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})\left[\varepsilon(1-\xi)^{-\alpha\varepsilon-1}+\frac{1}{\alpha}(1-\xi)^{-\alpha\varepsilon}\right]^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon\to 0^+$, in view of the continuity of the beta function, it follows that

$$\frac{\Gamma(\lambda)B(\lambda_1,\lambda_2)}{\alpha^{1/p}\Gamma(\lambda+m)} = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\alpha^{1/p}\Gamma(\lambda+m)} = \frac{B(\lambda_1+m,\lambda_2)}{\alpha^{1/p}\Pi_{i=0}^{m-1}(\lambda_1+i)}\le M.$$

Hence, $M = \frac{\Gamma(\lambda)B(\lambda_1,\lambda_2)}{\alpha^{1/p}\Gamma(\lambda+m)}$is the best possible constant factor in (3.2).

On the other hand, for $\widehat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\widehat{\lambda}_1 + \widehat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda,$$
$$0 < \widehat{\lambda}_1, \widehat{\lambda}_2 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \widehat{\lambda}_2 \leq \frac{1/\alpha}{q} + \frac{1/\alpha}{p} = \frac{1}{\alpha},$$

and $\frac{\Gamma(\lambda)B(\widehat{\lambda}_1,\widehat{\lambda}_2)}{\alpha^{1/p}\Gamma(\lambda+m)} \in \mathbf{R}_+$.Substitution of $\widehat{\lambda}_i = \lambda_i$ $(i = 1, 2)$ in (3.2), we still have

$$I = \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)dx}{[x + (n-\xi)^\alpha]^{\lambda+m}} < \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\widehat{\lambda}_1, \widehat{\lambda}_2)$$
$$\times \left[\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty (n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1} a_n^q\right]^{\frac{1}{q}}. \quad (3.4)$$

By Hölder's inequality (cf. [14]), we have

$$\begin{aligned}
& B(\widehat{\lambda}_1, \widehat{\lambda}_2) = k_\lambda(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}) \\
&= \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}}{(1+u)^\lambda} du = \int_0^\infty \frac{1}{(1+u)^\lambda}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\
&\leq [\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda}du]^{\frac{1}{p}}[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda}du]^{\frac{1}{q}} \\
&= [\int_0^\infty \frac{v^{\lambda_2-1}}{(1+v)^\lambda}dv]^{\frac{1}{p}}[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda}du]^{\frac{1}{q}} \\
&= (k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}. \qquad (3.5)
\end{aligned}$$

In view of

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}(\frac{1}{\alpha}k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$$

is the best possible constant factor in (3.1), by (3.4), we have the following inequality:

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}} \leq \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}B(\widehat{\lambda}_1, \widehat{\lambda}_2) \ (\in \mathbf{R}_+),$$

namely, $B(\widehat{\lambda}_1, \widehat{\lambda}_2) \geq (k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$, and then (3.5) keeps the form of equality.

We observe that (3.5) keeps the form of equality if and only if there exist constants A and B (cf. [14]), such that they are not both zero and $Au^{\lambda-\lambda_2} = Bu^{\lambda_1}$ *a.e.* in $\mathbf{R}_+$. Assuming that $A \neq 0$, we find $u^{\lambda-\lambda_1-\lambda_2} = B/A$ *a.e.* in $\mathbf{R}_+$, namely, $\lambda - \lambda_1 - \lambda_2 = 0$. Hence we have $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. $\square$

4. Equivalent forms and some particular inequalities

Theorem 4.1. *We have the following half-discrete Hilbert-type inequality equivalent to (3.1):*

$$J := \left\{ \sum_{n=1}^{\infty} (n-\xi)^{p\alpha\widehat{\lambda}_2-1} \left[\int_0^{\infty} \frac{f(x)}{[x+(n-\xi)^{\alpha}]^{\lambda+m}} dx \right]^p \right\}^{\frac{1}{p}}$$
$$< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} (\frac{1}{\alpha} k_{\lambda}(\lambda_2))^{\frac{1}{p}} (k_{\lambda}(\lambda_1))^{\frac{1}{q}} \left[\int_0^{\infty} x^{p(1-\widehat{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}}. \quad (4.1)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have the following inequality equivalent to (3.2):

$$\left\{ \sum_{n=1}^{\infty} (n-\xi)^{p\alpha\lambda_2-1} \left[\int_0^{\infty} \frac{f(x)}{[x+(n-\xi)^{\alpha}]^{\lambda+m}} dx \right]^p \right\}^{\frac{1}{p}}$$
$$< \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1, \lambda_2) \left[\int_0^{\infty} x^{p(1-\lambda_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}}. \quad (4.2)$$

Proof. Suppose that (4.1) is valid. By Hölder's inequality (cf. [14]), we have

$$I = \sum_{n=1}^{\infty} \left\{ (n-\xi)^{\alpha\widehat{\lambda}_2-\frac{1}{p}} \int_0^{\infty} \frac{f(x)dx}{[x+(n-\xi)^{\alpha}]^{\lambda+m}} \right\} \left[(n-\xi)^{-\alpha\widehat{\lambda}_2+\frac{1}{p}} a_n \right]$$
$$\leq J \left[\sum_{n=1}^{\infty} (n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1} a_n^q \right]^{\frac{1}{q}}. \quad (4.3)$$

Then by (4.1), we have (3.1).

On the other hand, assuming that (3.1) is valid, we set

$$a_n := (n-\xi)^{p\alpha\widehat{\lambda}_2-1} \left[\int_0^{\infty} \frac{f(x)}{[x+(n-\xi)^{\alpha}]^{\lambda+m}} dx \right]^{p-1}, n \in \mathbf{N}.$$

If $J = 0$, then (4.1) is naturally valid; if $J = \infty$, then it is impossible that makes (4.1) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (3.1), we have

$$0 < \sum_{n=1}^{\infty} (n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1} a_n^q = J^p = I$$
$$< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} (\frac{1}{\alpha} k_{\lambda}(\lambda_2))^{\frac{1}{p}} (k_{\lambda}(\lambda_1))^{\frac{1}{q}} \left[\int_0^{\infty} x^{p(1-\widehat{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} J^{p-1} < \infty.$$
$$J = \left[\sum_{n=1}^{\infty} (n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1} a_n^q \right]^{\frac{1}{p}}$$
$$< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} (\frac{1}{\alpha} k_{\lambda}(\lambda_2))^{\frac{1}{p}} (k_{\lambda}(\lambda_1))^{\frac{1}{q}} \left[\int_0^{\infty} x^{p(1-\widehat{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}},$$

namely, (4.1) follows, which is equivalent to (3.1).

The theorem is proved. □

Theorem 4.2. *If $\lambda_1+\lambda_2=\lambda$, then the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}(\frac{1}{\alpha}k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$$

in (4.1) is the best possible. On the other hand, if the same constant factor in (4.1) is the best possible, then for $\lambda-\lambda_1\le\frac{1}{\alpha}$, we have $\lambda_1+\lambda_2=\lambda$.

Proof. If $\lambda_1+\lambda_2=\lambda$, then by Theorem 3.2, the constant factor

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}(\frac{1}{\alpha}k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$$

in (3.1) is the best possible. By (4.3), the constant factor in (4.1) is still the best possible. Otherwise, we would reach a contradiction that the constant factor in (3.1) is not the best possible.

On the other hand, if the same constant factor in (4.1) is the best possible, then, by the equivalency of (4.1) and (3.1), in view of $J^p=I$ (see the proof of Theorem 4.1), we still can show that the constant factor in (3.1) is the best possible. By the assumption and Theorem 3.2, we have $\lambda_1+\lambda_2=\lambda$.

The theorem is proved. □

Remark 4.1. (i) For $\alpha=1$ in (3.2) and (4.2), we have the following equivalent inequalities:

$$\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)dx}{(x+n-\xi)^{\lambda+m}}$$
$$<\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^pdx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty(n-\xi)^{q(1-\lambda_2)-1}a_n^q\right]^{\frac{1}{q}}, \tag{4.4}$$

$$\left\{\sum_{n=1}^\infty(n-\xi)^{p\lambda_2-1}\left[\int_0^\infty\frac{f(x)}{(x+n-\xi)^{\lambda+m}}dx\right]^p\right\}^{\frac{1}{p}}$$
$$<\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^pdx\right]^{\frac{1}{p}}. \tag{4.5}$$

(ii) For $\xi=0$ in (3.2) and (4.2), we have the following equivalent inequalities:

$$\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)dx}{(x+n^\alpha)^{\lambda+m}}$$
$$<\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^pdx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\alpha\lambda_2)-1}a_n^q\right]^{\frac{1}{q}}, \tag{4.6}$$

$$\left\{\sum_{n=1}^\infty n^{p\alpha\lambda_2-1}\left[\int_0^\infty\frac{f(x)}{(x+n^\alpha)^{\lambda+m}}dx\right]^p\right\}^{\frac{1}{p}}$$
$$<\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^pdx\right]^{\frac{1}{p}}. \tag{4.7}$$

Hence, (3.2) (resp. (4.2)) is a more accurate form of (4.6) (resp. (4.7)).

(ii) For $\xi = \frac{1}{2}$ in (3.2) and (4.2), we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)dx}{[x+(n-\frac{1}{2})^\alpha]^{\lambda+m}}$$
$$< \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1,\lambda_2) \left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty (n-\frac{1}{2})^{q(1-\alpha\lambda_2)-1} a_n^q\right]^{\frac{1}{q}}. \tag{4.8}$$

$$\left\{\sum_{n=1}^\infty (n-\frac{1}{2})^{p\alpha\lambda_2-1}\left[\int_0^\infty \frac{f(x)}{[x+(n-\frac{1}{2})^\alpha]^{\lambda+m}}dx\right]^p\right\}^{\frac{1}{p}}$$
$$< \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}. \tag{4.9}$$

The constant factors in the above inequalities are all the best possible.

5. Operator expressions

Setting $\varphi_i(x) := x^{p(1-i-\widehat{\lambda}_1)-1} (i = 0, \cdots, m), \psi(n) := (n-\xi)^{q(1-\alpha\widehat{\lambda}_2)-1}$, where from

$$\psi^{1-p}(n) := (n-\xi)^{p\alpha\widehat{\lambda}_2-1} \ (x \in \mathbf{R}_+, n \in \mathbf{N}),$$

we define the following normed linear spaces:

$$L_{p,\varphi_i}(\mathbf{R}_+) := \{f = f(x); ||f||_{p,\varphi_i} := (\int_0^\infty \varphi_i(x)|f(x)|^p dx)^{\frac{1}{p}} < \infty\}$$
$$(i = 0, \cdots, m),$$
$$l_{q,\psi} := \{a = \{a_n\}_{n=1}^\infty; ||a||_{q,\psi} := (\sum_{n=1}^\infty \psi(n)|a_n|^q)^{\frac{1}{q}} < \infty\},$$
$$l_{p,\psi^{1-p}} := \{c = \{c_n\}_{n=1}^\infty; ||c||_{p,\psi^{1-p}} := (\sum_{n=1}^\infty \psi^{1-p}(n)|c_n|^p)^{\frac{1}{p}} < \infty\}.$$

For any $f = f(x) \in L_{p,\varphi_i}(\mathbf{R}_+)$, setting

$$c = \{c_n\}_{n=1}^\infty : c_n := \int_0^\infty \frac{f(x)}{[x+(n-\xi)^\alpha]^{\lambda+m}}dx,$$

we can rewrite (4.1) as follows:

$$||c||_{p,\psi^{1-p}} < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}(\frac{1}{\alpha}k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}||f^{(m)}||_{p,\varphi_0},$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 5.1. Define a half-discrete Hilbert-type operator $T : L_{p,\varphi_m}(\mathbf{R}_+) \to l_{p,\psi^{1-p}}$. as follows: For any $f \in L_{p,\varphi_m}(\mathbf{R}_+)$, there exists a unique representation

$c = Tf \in l_{p,\psi^{1-p}}$, such that for any $n \in \mathbf{N}$. $Tf(n) = c_n$ Define the formal inner product of Tf and $a \in l_{q,\psi}$, and the norm of T as follows:

$$(Tf, a) := \sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{[x + (n-\xi)^{\alpha}]^{\lambda+m}} dx = I,$$

$$||T|| := \sup_{f(\neq 0) \in L_{p,\varphi_m}(\mathbf{R}_+)} \frac{||Tf||_{p,\psi^{1-p}}}{||f^{(m)}||_{p,\varphi_0}}.$$

By Theorem 4.1 and Theorem 4.2, we have

Theorem 5.1. *If $f(>0) \in L_{p,\varphi_m}(\mathbf{R}_+), a(>0) \in l_{q,\psi}, ||f||_{p,\varphi_m} > 0, ||a||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Tf, a) < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} (\frac{1}{\alpha} k_{\lambda}(\lambda_2))^{\frac{1}{p}} (k_{\lambda}(\lambda_1))^{\frac{1}{q}} ||f^{(m)}||_{p,\varphi_0} ||a||_{q,\psi}, \tag{5.1}$$

$$||Tf||_{p,\psi^{1-p}} < \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} (\frac{1}{\alpha} k_{\lambda}(\lambda_2))^{\frac{1}{p}} (k_{\lambda}(\lambda_1))^{\frac{1}{q}} ||f^{(m)}||_{p,\varphi_0}. \tag{5.2}$$

Moreover, for $\lambda_1 + \lambda_2 = \lambda$, the constant factorin (5.1) and (5.2) is the best possible, namely,

$$||T|| = \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1, \lambda_2).$$

On the other hand, if the constant factor in (5.1) (or (5.2)) is the best possible, then for $\lambda - \lambda_1 \leq \frac{1}{\alpha}$, we have $\lambda_1 + \lambda_2 = \lambda$.

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