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# BIFURCATIONS OF TRAVELING WAVE SOLUTIONS IN THE HOMOGENEOUS CAMASSA-HOLM TYPE EQUATIONS*

Yan Zhou[1] and Jibin Li[1,†]

**Abstract** This paper studies traveling wave solutions of the homogeneous Camassa-Holm type equations introduced by Hay et al. in 2019. Under given parameter conditions, the corresponding traveling system is a singular system of the first class defined by [16]. The bifurcations of traveling wave solutions in the parameter space are investigated from the perspective of dynamical systems. The existence of solitary wave solution, periodic peakon solution and peakon, pseudo-peakon as well as compacton solution is proved. Possible exact explicit parametric representations of various solutions are given.

**Keywords** Solitary wave, peakon, pseudo-peakon, periodic peakon, compacton, bifurcation, Camassa-Holm type equation.

**MSC(2010)** 34C37, 34C23, 74J30, 58Z05.

## 1. Introduction

Recently, [14] considered the following integrable Camassa-Holm type equations (1.1) with homogeneous nonlinear terms.

$$u_t - u_{xxt} = \alpha u^k u_x + \beta u^k u_{xxx} + \gamma u^{k-1} u_x u_{xx} + \delta u^{k-2} u_x^3. \tag{1.1}$$

This type of nonlinearities have been considered in [1] and [13]. Here $\alpha, \beta, \gamma, \delta, k$ are arbitrary constants. Generally, we assume that $\alpha, k \neq 0$. In [14], Hay, et al. proved that known integrable examples of Camassa-Holm type equations (1.1) correspond to $k = 1$ and $k = 2$, which are the only possible degrees of nonlinearity under the above assumptions.

As a nonlinear generalization of the Camassa-Holm equation with peakon solutions, [2] and [6] discussed the following equation:

$$\begin{aligned} &u_t - u_{xxt} = \tfrac{1}{2}(k+1)(k+2)u^k u_x - \tfrac{1}{2}k(k-1)u^{k-2}u_x^3 - 2ku^{k-1}u_x u_{xx} - u^k u_{xxx}, \\ &k \neq 0, \end{aligned} \tag{1.2}$$

which is the special case of equation (1.1) with $\alpha = \frac{1}{2}(k+1)(k+2), \beta = -1, \gamma = -2k, \delta = -\frac{1}{2}k(k-1)$.

[†]The corresponding author. Email address: lijb@zjnu.cn(J. Li)
[1]School of Mathematical Sciences, Huaqiao University, Quanzhou, Fujian 362021, China
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In [1,13,15], the authors considered the generalized $b$-family equations as follows:

$$u_t - u_{xxt} = -(b+1)u^p u_x + bu^{p-1}u_x u_{xx} + u^p u_{xxx}, \quad b \neq 0, \quad p \neq 0, \tag{1.3}$$

which is the special case of equation (1.1) with $k = p, \alpha = -(b+1), \beta = 1, \gamma = b, \delta = 0$.

For $p = 1$, equation (1.3) includes both the Camassa-Holm(CH) equation ($b = 2$) and the Degasperis-Procesi(DP) equation ($b = 3$). In addition, for $p = 2$, equation (1.3) includes the Novikov equation ($b = 3$).

However, we notice that these authors did not study the bifurcations and possible exact solutions for the corresponding traveling wave systems of equations (1.1), (1.2) and (1.3). In this paper, we consider these problems depending on the parameters of systems for the corresponding traveling wave systems of equations (1.1) and (1.2), respectively.

We have

$$(u^k u_{xx})_x = ku^{k-1}u_x u_{xx} + u^k u_{xxx}, \quad (u^{k-1}u_x^2)_x = (k-1)u^{k-2}u_x^3 + 2u^{k-1}u_x u_{xx}.$$

For equation (1.1), we assume that $\gamma = k\beta + 2\tilde{\delta}, \delta = (k-1)\tilde{\delta}$. Then, (1.1) can be written as

$$u_t - u_{xxt} = \alpha u^k u_x + \beta(u^k u_{xx})_x + \tilde{\delta}(u^{k-1}u_x^2)_x. \tag{1.4}$$

To study the traveling wave solutions of equations (1.1) and (1.2), we set $u(x,t) = u(x+ct) \equiv \phi(\xi)$, where $\xi = x + ct$ and $c$ is the wave speed. We always assume that $c > 0$ in this paper. Substituting $u(x,t) = u(x+ct) \equiv \phi(\xi)$ into (1.4) and (1.2), integrating the obtained equations once, we obtain

$$(\beta\phi^k + c)\phi'' = -\tilde{\delta}\phi^{k-1}(\phi')^2 - \frac{\alpha}{k+1}\phi^{k+1} + c\phi + g, \tag{1.5}$$

and

$$(\phi^k - c)\phi'' = -\frac{1}{2}k\phi^{k-1}(\phi')^2 + \left(\frac{1}{2}k + 1\right)\phi^{k+1} - c\phi + g, \tag{1.6}$$

where $g$ is integral constant, and the prime stands for the derivative with respect to $\xi$. Equations (1.5) and (1.6) are equivalent to the following planar dynamical systems:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\tilde{\delta}\phi^{k-1}y^2 - \frac{\alpha}{k+1}\phi^{k+1} + c\phi + g}{\beta\phi^k + c}, \quad k > 0, \tag{1.7}$$

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\tilde{\delta}y^2 + \phi^2\left(c\phi^p + g\phi^{p-1} + \frac{\alpha}{p-1}\right)}{\phi(c\phi^p + \beta)}, \quad k < 0, p = -k, p \neq 1, \tag{1.8}$$

and

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\frac{1}{2}k\phi^{k-1}y^2 + \left(\frac{1}{2}k + 1\right)\phi^{k+1} - c\phi + g}{\phi^k - c}, \quad k > 0, \tag{1.9}$$

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\frac{1}{2}py^2 + \phi^2\left(c\phi^p - g\phi^{p-1} + \left(\frac{1}{2}p - 1\right)\right)}{\phi(c\phi^p - 1)}, \quad k < 0, p = -k. \tag{1.10}$$

These four systems have the following first integrals, respectively:

$$H_1(\phi,y) = y^2(\beta\phi^k + c)^{\frac{2\tilde{\delta}}{k\beta}} + 2\int(\beta\phi^k + c)^{\frac{2\tilde{\delta}-k\beta}{k\beta}}\left[\frac{\alpha}{k+1}\phi^{k+1} - c\phi - g\right]d\phi = h, \tag{1.11}$$

$$
\begin{aligned}
H_2(\phi,y) =& y^2\phi^{\frac{2\tilde{\delta}}{\beta}}(c\phi^p+\beta)^{-\frac{2\tilde{\delta}}{p\beta}} - 2\int \phi^{1+\frac{2\tilde{\delta}}{\beta}}(c\phi^p+\beta)^{-(1+\frac{2\tilde{\delta}}{p\beta})} \\
&\times\left[c\phi^p+g\phi^{p-1}+\frac{\alpha}{p-1}\right]d\phi = h,
\end{aligned} \tag{1.12}
$$

$$
H_3(\phi,y) = y^2(\phi^k - c) - (\phi^{k+2} - c\phi^2 + 2g\phi) = h, \tag{1.13}
$$

$$
H_4(\phi,y) = \frac{(c\phi^p-1)y^2}{\phi^p} - c\phi^2 + 2g\phi + \phi^{2-p} = h. \tag{1.14}
$$

When $\tilde{\delta} = \frac{1}{2}k\beta$, the first integral $H_1(\phi,y) = h$ becomes the following algebraic integral:

$$
H_{10}(\phi,y) = y^2(\beta\phi^k + c) - 2g\phi - c\phi^2 + \frac{2\alpha}{(k+1)(k+2)}\phi^{k+2} = h. \tag{1.15}
$$

When $\tilde{\delta} = -\frac{1}{2}p\beta, p \neq 1, 2$, the first integral $H_2(\phi,y) = h$ becomes the following algebraic integral:

$$
H_{20}(\phi,y) = \frac{(c\phi^p+\beta)y^2}{\phi^p} + \left[-2g\phi + c\phi^2 + \frac{2\alpha}{(p-1)(p-2)}\phi^{2-p}\right] = h. \tag{1.16}
$$

Because systems (1.9) and (1.10) are the special systems of (1.7) and (1.8) with $\alpha = \frac{1}{2}(k+1)(k+2), \beta = -1, \tilde{\delta} = -\frac{1}{2}k$. The first integrals $H_{10}(\phi,y) = h$ and $H_{20}(\phi,y) = h$ have the similar forms as $H_3(\phi,y) = h$ and $H_4(\phi,y) = h$, respectively. Therefore, we only need to investigate the solutions of systems (1.7) and (1.8) with the parameter group $(k, c, \alpha, \beta, \tilde{\delta})$.

Clearly, for $c\beta < 0$, on the curves $\beta\phi^k + c = 0$ and $c\phi^p + \beta = 0$, respectively, systems (1.7) and (1.8) are discontinuous. Such systems are called the singular traveling wave systems of the first class defined by [16] and [17].

It is interesting to find that the singular traveling systems have peakon, pseudo-peakon, periodic peakon and compacton solution families. Periodic peakon is a classical solution with two time-scales of a singular traveling system. Peakon is a limit solution of a family of periodic peakons or a limit solution of a family of pseudo-peakons under two classes of limit senses (see [20]). Compacton family is a solution family of a singular system, for which all solutions $\phi(\xi)$ have finite sets of support, i.e., the defined region of every $\phi(\xi)$ with respect to $\xi$ is finite and the value region of $\phi$ is bounded. Corresponding to different types of phase orbits, in [16, 17, 19], a classification for different wave profiles of $\phi(\xi)$ was given.

In this paper, the above-mentioned theory of singular traveling wave systems is used to analyze the wave profiles of the wave function $\phi(\xi)$ in the solutions of systems (1.7) and (1.8).

The following relationships of a wave profile of $\phi(\xi)$ with some phase orbits of these planar dynamical systems are known today.

(1) A smooth homoclinic orbit to a saddle point of a traveling wave system gives rise to a smooth solitary wave solution of a PDE.

(2) A smooth heteroclinic loop connecting two saddle points of a traveling wave system gives rise to a kink wave solution and an anti-kink wave solution of a PDE.

(3) For a homoclinic orbit, if there exists a segment which completely lies in a left (or right) small strip neighborhood of a singular straight line, then this homoclinic orbit defines a pseudo-peakon solution of the system.

(4) If there exists a curve triangle connecting saddle points and surrounding a periodic annulus of a center in the corresponding traveling wave system, in the neighborhood of a singular straight line (for which a segment is an edge of the triangle), then as a limit curve of a family of periodic orbits, this curve triangle gives rise to a peakon solution of the system.

(5) For a family of periodic orbits, if there exists a segment of every orbit which completely lies in a left (or right) small strip neighborhood of a singular straight line, then these periodic orbits determine a family of periodic peakon solutions of the system.

(6) For a family of open orbits, if they tend to a singular straight line as $|y| \to \infty$, then this family gives rise to a family of compactons.

By considering the dynamics of the traveling wave solutions determined by the travelling wave system (1.7), all possible exact explicit parametric representations for the traveling wave solutions of equation (1.7) will be given under different parameter conditions. More precisely, more than 8 exact explicit parametric representations are obtained by using the elliptic functions and hyperbolic functions.

This paper is organized as follows. In section 2, we discuss the bifurcations of phase portraits of systems (1.7) and (1.8) depending on the changes of parameter $\alpha$ when $c > 0, \beta, \tilde{\delta}$ are fixed. In sections 3, 4 and 5, we investigate the existence of solitary wave solution, peakon, periodic peakons, pseudo-peakons as well as compacton solutions and give possible exact explicit parametric representations for these solutions.

## 2. Bifurcations of phase portraits of systems (1.7) and (1.8)

Without loss of generality, we assume that the integral constant $g = 0$ and $c > 0$ in this paper.

**2.1** We first consider all possible phase portraits of system (1.7). It is known that system (1.7) has the same invariant curve solutions as the associated regular system:

$$\frac{d\phi}{d\zeta} = y(\beta\phi^k + c), \quad \frac{dy}{d\zeta} = -\tilde{\delta}\phi^{k-1}y^2 - \frac{\alpha}{k+1}\phi^{k+1} + c\phi, \tag{2.1}$$

where $d\xi = (\beta\phi^k + c)d\zeta$, for $\beta\phi^k + c \neq 0$.

Obviously, when $\alpha > 0$ and $k$ is an odd number, system (2.1) has two equilibrium points $O(0,0)$ and $E_1(\phi_1, 0)$ on the $\phi-$axis, where $\phi_1 = \left(\frac{(k+1)c}{\alpha}\right)^{\frac{1}{k}}$; while when $k$ is an even number, system (2.1) has three equilibrium points $O(0,0), E_1(\phi_1, 0)$ and $E_2(-\phi_1, 0)$ on the $\phi$-axis.

When $\beta < 0$ and $k$ is an odd number, on the straight line $\phi = \phi_s = \left(-\frac{c}{\beta}\right)^{\frac{1}{k}}$, system (2.1) has two equilibrium points $S_{\mp}^{+}(\phi_s, \mp y_s)$, where $y_s = \sqrt{Y_s}$, if $Y_s = \left(c - \frac{\alpha}{k+1}\phi_s^k\right)(\tilde{\delta}\phi_s^{k-2})^{-1} \geq 0$. In addition, when $k$ is an even number, on the straight line $\phi = -\phi_s$, system (2.1) has two equilibrium points $S_{\mp}^{-}(-\phi_s, \mp y_s)$. When $\alpha > 0, \beta < 0$, and $\alpha = -\beta(k+1)$, $\phi_1 = \phi_s$.

Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of (2.1) at the equilibrium point $E_j(\phi_j, 0)$. We have

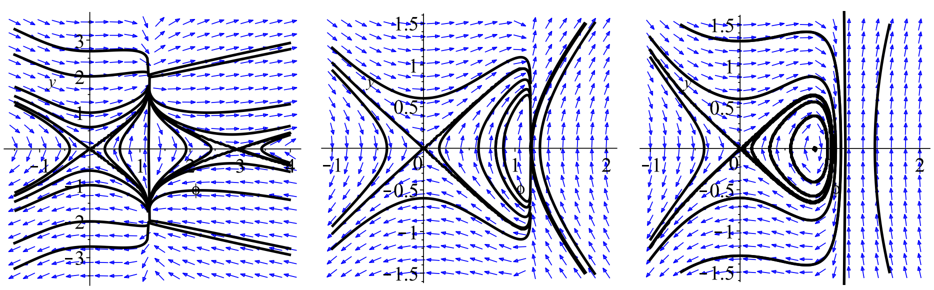
$$J(0,0) = \det M(0,0) = -c^2 < 0, \quad J(\phi_1, 0) = \det M(\phi_1, 0) = kc^2 \left(1 + (k+1)\frac{\beta}{\alpha}\right),$$
$$J(\phi_s, y_s) = \det M(\phi_s, y_s) = -2k\beta\tilde{\delta} y_s^2 \phi_s^{2k-2}. \tag{2.2}$$

By the theory of planar dynamical systems (see [17]), for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; If $J > 0$ and $(\mathrm{Trice}M)^2 - 4J < 0$ $(> 0)$, then it is a center point (a node point); if $J = 0$ and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

We see from the above discussion that the equilibrium point $O(0,0)$ ia a saddle point. When $\alpha > 0, \beta < 0$, if $\alpha < -\beta(k+1)$, then $\phi_s < \phi_1$, the equilibrium point $E_1(\phi_1, 0)$ is also a saddle point. While if $\alpha > -\beta(k+1)$, then $\phi_1 < \phi_s$, the equilibrium point $E_1(\phi_1, 0)$ is a center point. When $\tilde{\delta}\beta > 0$, the equilibrium point $S^{\pm}_{\mp}(\phi_s, \mp y_s)$ are saddle points, otherwise, they are node points.

We write that $h_0 = H_1(0,0), h_1 = H_1(\phi_1, 0), h_s = H_1(\phi_s, y_s)$, where $H_1$ is given by (1.11).

(i) When $\beta\tilde{\delta} < 0$, we have $\tilde{\delta} \neq \frac{1}{2}k\beta$. The equilibrium point $S^{\pm}_{\mp}(\phi_s, \mp y_s)$ are node points of system (2.1). By using the above result, applying the numerical method, and making the parameter $\alpha$ change, for a fixed parameter group $(c, \beta, \tilde{\delta})$, we have the bifurcations of phase portraits of system (1.7) shown in Fig.1.



(a) $0 < \alpha < -\beta(k+1)$ (b) $\alpha = -\beta(k+1)$ (c) $\alpha > -\beta(k+1)$

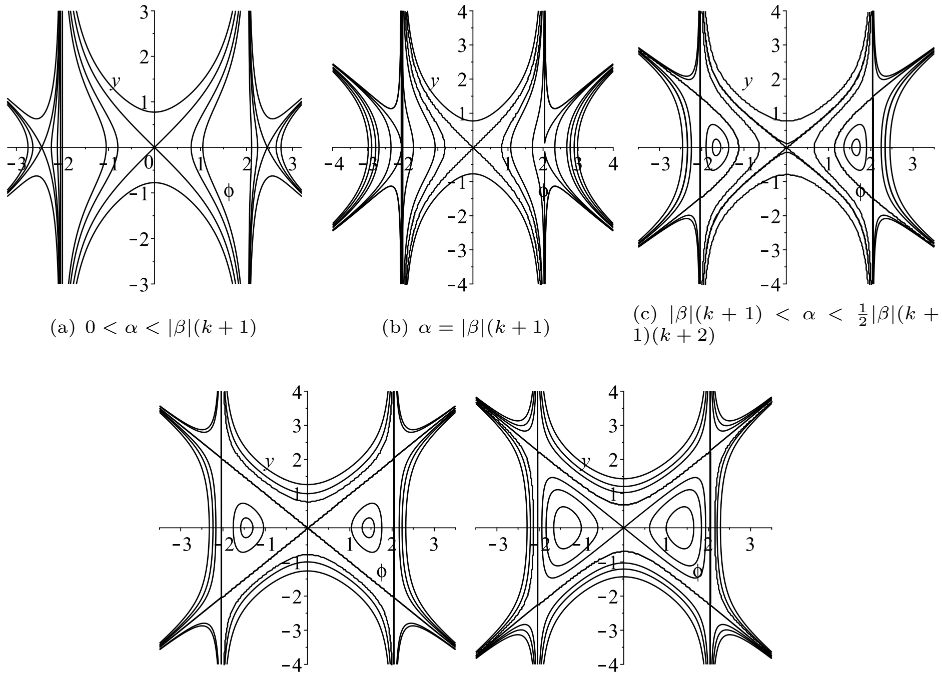
**Figure 1.** The bifurcations of phase portraits of system (1.7) for $\beta < 0, \tilde{\delta} > 0$, $k$ is odd.

(ii) When $\beta\tilde{\delta} > 0$, we consider the case of $\tilde{\delta} = \frac{1}{2}k\beta$. The the equilibrium point $E^{\pm}_{\mp}(\phi_s, \mp y_s)$ are saddle points of system (2.1). In this case, we have the first integral $H_{10}(\phi, y) = h$ of system (1.7) given by (1.15). Hence, we have

$$h_1 = H_{10}(\phi_1, 0) = -c\left(1 - \frac{2}{k+2}\right)\phi_1^2,$$
$$h_s = H_{10}(\phi_s, y_s) = -c\left(1 + \frac{2\alpha}{(k+1)(k+2)\beta}\right)\phi_s^2.$$

Obviously, if and only if $\alpha = -\frac{1}{2}(k+1)(k+2)\beta$, we have $h_s = 0$.

When $k$ is an even number and $\alpha > 0, \beta < 0$, for a fixed parameter group $(c, \beta, \tilde{\delta})$, making the parameter $\alpha$ change, we obtain the bifurcations of phase portraits of system (1.7) shown in Fig.2.




(a) $0 < \alpha < |\beta|(k+1)$ (b) $\alpha = |\beta|(k+1)$ (c) $|\beta|(k+1) < \alpha < \frac{1}{2}|\beta|(k+1)(k+2)$

(d) $\alpha = \frac{1}{2}|\beta|(k+1)(k+2)$ (e) $\alpha > \frac{1}{2}|\beta|(k+1)(k+2)$

**Figure 2.** The bifurcations of phase portraits of system (1.7) for $\tilde{\delta} = \frac{1}{2}k\beta$, and $k$ is even.

(iii) When $k$ is an odd number and $\alpha > 0, \beta < 0$, for a fixed parameter group $(c, \beta, \tilde{\delta})$, making the parameter $\alpha$ change, we obtain the bifurcations of phase portraits of system (1.7) shown in Fig.3.

**2.2** We next consider all possible phase portraits of system (1.8). System (1.8) has the same invariant curve solutions as the associated regular system:

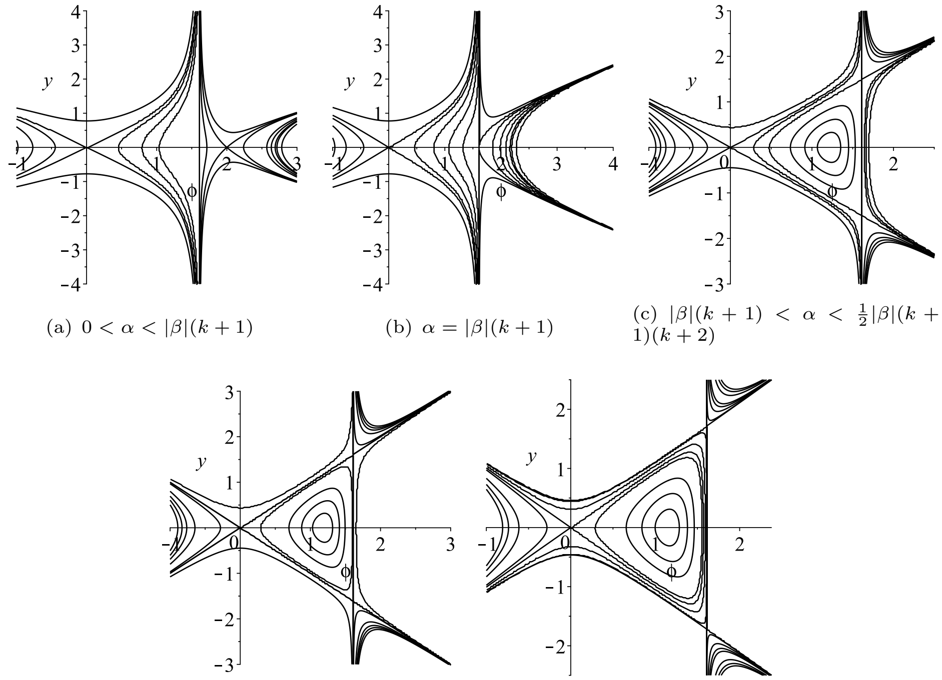
$$\frac{d\phi}{d\zeta} = y\phi(c\phi^p + \beta), \quad \frac{dy}{d\zeta} = -\tilde{\delta}y^2 + \phi^2\left(c\phi^p + \frac{\alpha}{p-1}\right), \tag{2.3}$$

where $p > 2, d\xi = \phi(c\phi^p + \beta)d\zeta$, for $\phi(c\phi^p + \beta) \neq 0$.

Clearly, when $p$ is an odd number and $\alpha < 0$, system (2.3) has the equilibrium points $O(0,0), E_1(\phi_1, 0)$, where $\phi_1 = \left(\frac{-\alpha}{c(p-1)}\right)^{\frac{1}{p}}$. When $\beta < 0$ and $Y_s = \frac{1}{\tilde{\delta}(p-1)}\phi_{s1}^2(-\beta(p-1)+\alpha) > 0$, on the singular straight line $\phi = \phi_{s1} = \left(-\frac{\beta}{c}\right)^{\frac{1}{p}}$, system (2.3) has two equilibrium points $S_{\mp}^{+}(\phi_{s1}, \pm\sqrt{Y_s})$.

When $p$ is an even number and $\alpha < 0$, system (2.3) has the equilibrium points $O(0,0), E_1(\phi_1, 0)$ and $E_2(-\phi_1, 0)$. When $\beta < 0$ and $Y_s > 0$, on the singular straight line $\phi = \pm\phi_{s1}$, respectively, system (2.3) has the equilibrium points $S_{\mp}^{+}(\phi_{s1}, \pm\sqrt{Y_s})$ and $S_{\mp}^{-}(-\phi_{s1}, \pm\sqrt{Y_s})$.

The point $O(0,0)$ is a double equilibrium point of system (2.3). To consider the directions that the orbits of system (2.3) tend to $O(0,0)$ as $\zeta \to \infty$, from $G(\theta) = \left[\frac{\alpha}{p-1}\cos^2\theta - \beta(1 - \frac{1}{2}p)\sin^2\theta\right]\cos\theta = 0$, it follows that $\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{3\pi}{2}$.




(a) $0 < \alpha < |\beta|(k+1)$ (b) $\alpha = |\beta|(k+1)$ (c) $|\beta|(k+1) < \alpha < \frac{1}{2}|\beta|(k+1)(k+2)$

(d) $\alpha = \frac{1}{2}|\beta|(k+1)(k+2)$ (e) $\alpha > \frac{1}{2}|\beta|(k+1)(k+2)$

**Figure 3.** The bifurcations of phase portraits of system (1.7) for $\tilde{\delta} = \frac{1}{2}k\beta$, and $k$ is odd.

Because $\phi = 0$ is a straight line solution of system (2.3), there is no other orbit tending to the origin $O(0,0)$.

We have

$$J(\phi_1, 0) = \det M(\phi_1, 0) = \frac{pc}{p-1}\phi_1^{2+p}(-\beta(p-1)+\alpha), \quad J\left(\phi_{s1}, \sqrt{Y_s}\right) = 2\tilde{\delta}p\beta Y_s. \tag{2.4}$$

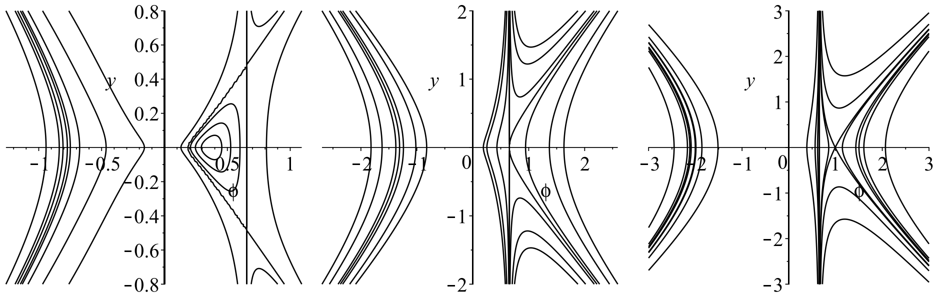
When $\tilde{\delta} = -\frac{1}{2}p\beta$, we write that $h_1 = H_{20}(\phi_1, 0), h_s = H_{20}(\phi_{s1}, y_s)$, where $H_{20}$ is given by (1.16).

When $p$ is an odd number and $\alpha < 0, \beta < 0$, for a fixed parameter group $(c, \beta)$, making the parameter $\alpha$ change, we obtain the bifurcations of phase portraits of system (1.8) shown in Fig.4.

When $p$ is an even number and $\alpha > 0, \beta < 0$, for a fixed parameter group $(c, \beta, \tilde{\delta})$, making the parameter $\alpha$ change, we obtain the bifurcations of phase portraits of system (1.8) shown in Fig.5.

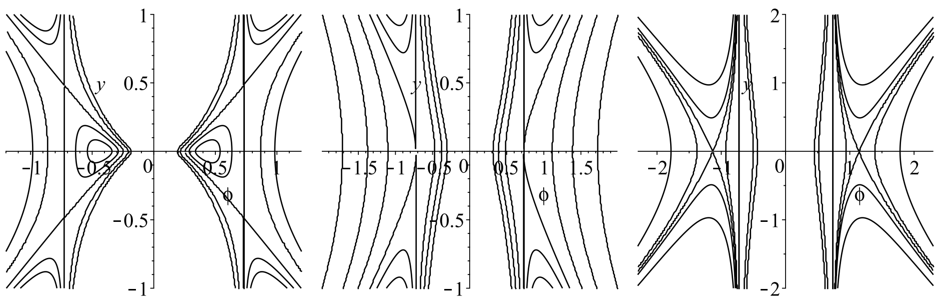
## 3. Existence of solitary wave solution and compacton determined by the orbits of system (1.7) when $\beta\tilde{\delta} < 0$

In this section, we consider the case $\beta\tilde{\delta} < 0$, and $k$ is a odd number in system (1.7).




(a) $\beta(p-1) < \alpha < 0$ (b) $\alpha = \beta(p-1)$ (c) $\alpha < \beta(p-1)$

**Figure 4.** The bifurcations of phase portraits of system (1.8) for $\tilde{\delta} = -\frac{1}{2}p\beta$, and $p$ is odd.




(a) $\beta(p-1) < \alpha < 0$ (b) $\alpha = \beta(p-1)$ (c) $\alpha < \beta(p-1)$

**Figure 5.** The bifurcations of phase portraits of system (1.8) for $\tilde{\delta} = -\frac{1}{2}p\beta$, and $p$ is even.

We know from Fig.1 (a) that for system (1.7) and a fixed paprameter group $(c, \beta, \tilde{\delta})$ with $\beta < 0$, when parameter condition $0 < \alpha < -\beta(k+1)$ holds, then, in the two triangle regions enclosed by the stable and unstable manifolds of the origin $O(0,0)$ and the saddle point $E_1(\phi_1, 0)$ with the segment $S_-^+S_+^+$, there exist two families of bounded orbits of system (1.7),respectively, which give rise to two compaction families of equation (1.4).

It is well known that a smooth homoclinic orbit of a traveling system gives rise to a solitary wave solution of the corresponding nonlinear wave equation. When $0 < -\beta(k+1) < \alpha$, we see from Fig.1 (c) that there exists a homoclinic orbit of system (1.7).

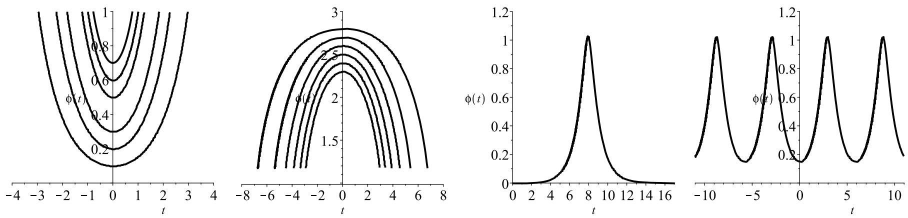
Thus, we immediately obtain the following conclusion.

**Theorem 3.1.** *Assume that the parameters in equation* (1.1) *satisfy the conditions:* $\gamma = k\beta + 2\tilde{\delta}, \delta = (k-1)\tilde{\delta}$. *Then, equation* (1.1) *has the form* (1.4)*, which has the integrable traveling wave system* (1.7) *and* (1.8)*.*

*(i) When* $\beta\tilde{\delta} < 0$ *and* $0 < \alpha < -\beta(k+1)$*, equation* (1.4) *has two families of compacton solutions (see Fig.6 (a),(b)).*

*(ii) When* $\beta\tilde{\delta} < 0$ *and* $0 < -\beta(k+1) < \alpha$*, equation* (1.4) *has a smooth solitary wave solution given by the homoclinic orbit of system* (1.7) *defined by* $H_1(\phi, y) = H(0,0) = h_0$*. In addition, equation* (1.4) *has a family of smooth periodic wave*

*solutions defined by the closed branch of* $H_1(\phi, y) = h,\ h \in (h_0, h_1)$ *(see Fig.6 (c), (d)).*



(a) (b) (c) (d)

**Figure 6.** The profiles of compactons and solitary wave solution of (1.4) for $\beta\tilde{\delta} < 0$.

# 4. Exact Peakon and pseudo-peakon of system (1.7) when $\tilde{\delta} = \frac{1}{2}k\beta$

In order to obtain exact traveling wave solutions, we need to use the following conclusion.

**Proposition 4.1.** *Let* $X(\phi) = A + B\phi + C\phi^2$. *Assume that* $A > 0, \Delta = B^2 - 4AC > 0$. *Considering the integral* $\xi = \int_{\phi_M}^{\phi} \frac{d\phi}{\phi\sqrt{X(\phi)}}$ , *i.e., the solutions of the differential equation* $\frac{d\phi}{d\xi} = \phi\sqrt{X(\phi)}$, *we have*

*(1) When* $X(\phi_M) = 0$,

$$\begin{aligned} \phi(\xi) &= \tfrac{2A}{\sqrt{\Delta}\cosh(\sqrt{A}\xi) - B}, \quad \text{if} \ \ \phi(0) = -\tfrac{B+\sqrt{\Delta}}{2C}, \\ \phi(\xi) &= -\tfrac{2A}{\sqrt{\Delta}\cosh(\sqrt{A}\xi) + B}, \quad \text{if} \ \ \phi(0) = \tfrac{-B+\sqrt{\Delta}}{2C}. \end{aligned} \tag{4.1}$$

*(2) When* $X(\phi_M) \neq 0$,

$$\phi(\xi) = \frac{2A}{P\cosh_q(\sqrt{A}\xi) - B}, \tag{4.2}$$

*where* $P = \frac{1}{\phi_M}\left(2\sqrt{AX(\phi_M)} + B\phi_M + 2A\right), q = \frac{\Delta}{P^2}$, *and* $\cosh_q(\xi)$ *is an Arai q-deformed hyperbolic function (see [7, 8]).*

When $\tilde{\delta} = \frac{1}{2}k\beta$, we see from (1.15) that $y^2 = \frac{h + c\phi^2 - \frac{2\alpha}{(k+1)(k+2)}\phi^{k+2}}{(\beta\phi^k + c)}$. By using the first equation of (1.7), we obtan

$$\begin{aligned} \omega_0\xi &\equiv \sqrt{\frac{2\alpha}{|\beta|(k+1)(k+2)}}\xi \\ &= \int_{\phi_0}^{\phi} \frac{\left(\frac{c}{|\beta|} - \phi^k\right)d\phi}{\sqrt{\left(\frac{c}{|\beta|} - \phi^k\right)\left[\frac{h(k+1)(k+2)}{2\alpha} + \frac{c(k+1)(k+2)}{2\alpha}\phi^2 - \phi^{k+2}\right]}}. \end{aligned} \tag{4.3}$$

**(i) Exact explicit peakon solution**

Suppose that $\alpha = \frac{1}{2}|\beta|(k+1)(k+2)$. We consider the heteroclinic triangle in Fig.3 (d) defined by the level curve of $H_{10}(\phi, y) = 0$ when $k$ is an odd number, and $k > 2$. Now, (4.3) becomes

$$\xi = \int_{\phi}^{\phi_s} \frac{\left(\frac{c}{|\beta|} - \phi^k\right) d\phi}{\phi\sqrt{\left(\frac{c}{|\beta|} - \phi^k\right)^2}} = \int_{\phi}^{\phi_s} \frac{d\phi}{\phi}. \tag{4.4}$$

Thus, it follows from (4.4) that the following peakon solution of Camassa-Holm type equation (1.4) (see [10–12]:

$$\phi(\xi) = \phi_s e^{-|\xi|}. \tag{4.5}$$

When $k$ is an even number, there are two heteroclinic triangles in Fig.2 (d). Besides the peakon solution (4.5), we also have an anti-peakon solution:

$$\phi(\xi) = -\phi_s e^{-|\xi|}. \tag{4.6}$$

**(ii) Exact explicit pseudo-peakon solution and solitary wave solution**

Suppose that $\alpha > \frac{1}{2}|\beta|(k+1)(k+2)$. We next consider the the homoclinic orbit in Fig.3 (e) to the origin $O(0,0)$ defined by $H_{10}(\phi, y) = 0$, when $k$ is an odd number. Now, (4.3) can be written as

$$\omega_0\xi = \int_{\phi}^{\phi_M} \frac{(\phi_s^k - \phi^k)d\phi}{\phi\sqrt{(\phi_s^k - \phi^k)(\phi_M^k - \phi^k)}} = \int_{\psi}^{\psi_M} \frac{(\psi_s - \psi)d\psi}{k\psi\sqrt{(\psi_s - \psi)(\psi_M - \psi)}}, \tag{4.7}$$

where $\psi = \phi^k, \psi_M = \phi_M^k = \frac{c(k+1)(k+2)}{2\alpha}, \psi_s = \frac{c}{|\beta|}$.

By using Proposition 4.1, (4.7) gives rise the following exact solitary wave and pseudo-peakon solution (when $\alpha - \frac{1}{2}|\beta|(k+1)(k+2) \ll 1$) of equation (1.4):

$$\begin{aligned} \phi(\chi) &= (\psi(\chi))^{\frac{1}{k}} = \left(\frac{2\psi_M\psi_s}{(\psi_s - \psi_M)\cosh\left(\sqrt{\psi_M\psi_s}\chi\right) + (\psi_M + \psi_s)}\right)^{\frac{1}{k}}, \quad \chi \in (-\infty, 0), (0, \infty), \\ \xi(\chi) &= \frac{1}{k\omega_0}\left[\psi_s\chi \pm \ln\left(\frac{\sqrt{(\psi_s - \psi(\chi))(\psi_M - \psi(\chi))} + \psi(\chi) - \frac{1}{2}(\psi_s + \psi_M)}{\frac{1}{2}(\psi_s - \psi_M)}\right)\right]. \end{aligned} \tag{4.8}$$

When $k$ is an even number, corresponding to the two homoclinic orbits with "eight figure" (see Fig.2 (e)) defined by $H_{10}(\phi, y) = 0$, there exist a solitary wave solution and an anti-solitary wave solution (or when $\alpha - \frac{1}{2}|\beta|(k+1)(k+2) \ll 1$, a pseudo-peakon solution (4.8) and a pseudo-anti-peakon solution) with the exact form:

$$\phi(\chi) = -(\psi(\chi))^{\frac{1}{k}} = -\left(\frac{2\psi_M\psi_s}{(\psi_s - \psi_M)\cosh\left(\sqrt{\psi_M\psi_s}\chi\right) + (\psi_M + \psi_s)}\right)^{\frac{1}{k}}, \tag{4.9}$$

where the $\xi(\chi)$ is same as (4.8).

By the above discussion, we have the following conclusion.

**Theorem 4.1.** *Assume that the parameters of equation* (1.1) *satisfy the condition* $\gamma = k\beta + 2\tilde{\delta}, \delta = (k-1)\tilde{\delta}$ *and* $\tilde{\delta} = \frac{1}{2}k\beta, \beta < 0$.

*(i) When* $\alpha = \frac{1}{2}|\beta|(k+1)(k+2)$*, and* $k$ *is an odd number, equation* (1.1) *has a peakon solution of Camassa-Holm type given by* (4.5)*. While when* $k$ *is an even number, equation* (1.1) *has a peakon solution and an anti-peakon solutions of Camassa-Holm type given by* (4.5) *and* (4.6)*.*

*(ii) When* $\alpha > \frac{1}{2}|\beta|(k+1)(k+2)$*, and* $k$ *is an odd number, equation* (1.1) *has a solitary solution given by* (4.8)*. While when* $k$ *is an even number, equation* (1.1) *has a solitary wave solution given by* (4.8) *and an anti-solitary wave solution given by* (4.9)*.*

*(iii) When* $\alpha - \frac{1}{2}|\beta|(k+1)(k+2) \ll 1$*, these solitary wave solutions are pseudo-peakon solutions.*

# 5. Exact Peakon and pseudo-peakon of system (1.8) when $\tilde{\delta} = -\frac{1}{2}p\beta$

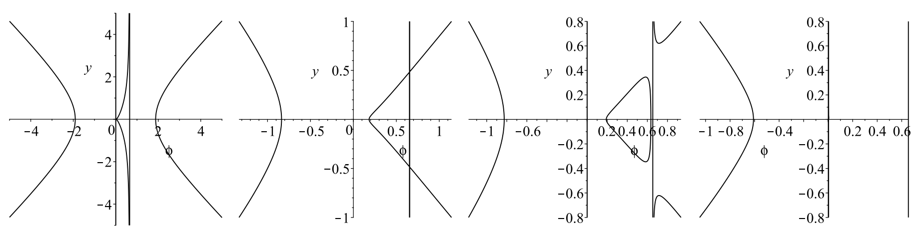
When $\tilde{\delta} = -\frac{1}{2}p\beta$, we see from (1.16) that $y^2 = \frac{\left(h + c\phi^2 + \frac{2\alpha}{(p-1)(2-p)}\phi^{2-p}\right)\phi^p}{c\phi^p + \beta}$. By using the first equation of (1.8), we obtain the integral formula of exact solutions as follows:

$$\xi = \int_{\phi_0}^{\phi} \frac{|\phi^p - \phi_{s1}^p| d\phi}{\phi\sqrt{|\phi^p - \phi_{s1}^p|(\phi^p + \frac{h}{c}\phi^{p-2} + B_0^p)}}, \tag{5.1}$$

where $B_0 = \left(\frac{2\alpha}{c(p-1)(p-2)}\right)^{\frac{1}{p}}$, and $p > 2$.

## 5.1. The case when $p$ is an odd number

When $p$ is an odd number, and $\beta(p-1) < \alpha < 0$, system (1.8) has phase portrait shown in Fig.4 (a). In this case, when $h$ varies, the level curves defined by $H_{20}(\phi, y) = h$ are shown in Fig.7.



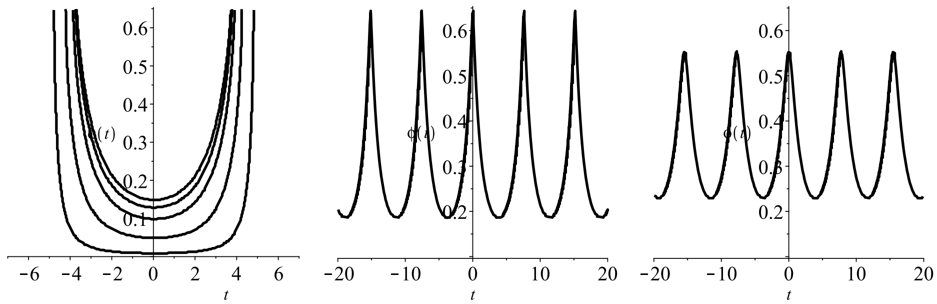
(a) $h < h_s$ (b) $h = h_s$ (c) $h \in (h_s, h_1)$ (d) $h > h_1$

**Figure 7.** The level curves $H_{20}(\phi, y) = h$ of equation (1.8) for $\beta(p-1) < \alpha < 0$.

We see from Fig.7 that the following conclusions hold.

(i) For $h \in (-\infty, h_s)$, there exists a family of open orbit branches defined by $H_{20}(\phi, y) = h$ on the left of the singular straight line $\phi = \phi_s$ (see Fig.7 (a)), which tends to the singular straight line, when $|y| \to \infty$. These open orbits give rise to a family of compacton solutions of equation (1.1) (see Fig.8 (a)).

(ii) For $h = h_s$, the level curve defined by $H_{20}(\phi, y) = h_s$ is an arch which passes through the singular straight line $\phi = \phi_s$ (see Fig.7 (b)). This arch gives rise to a periodic peakon solution (see Fig.8 (b)).




(a) Compactons (b) Periodic peakon (c) Periodic wave

**Figure 8.** Profiles of equation (1.8) for $\beta(p-1) < \alpha < 0$.

(iii) For $h \in (h_s, h_1)$, the level curves defined by $H_{20}(\phi, y) = h$ are a family of closed orbits (see Fig.7 (c)), This family gives rise to periodic wave solutions (see Fig.8 (c)). Specially, when $|h - h_s| \ll 1$, these closed orbits give rise a family of periodic peakons.

Assume that $p = 3$. Then, for the arch orbit in Fig.7 (b), (5.1) can be written as

$$\begin{aligned}\xi &= \int_{\phi_m}^{\phi} \frac{(\phi_{s1}^3 - \phi^3)d\phi}{\phi\sqrt{(\phi_{s1}^3 - \phi^3)(\phi^3 + \frac{h_s}{c}\phi + B_0^3)}} \\ &= \int_{\phi_m}^{\phi} \frac{(\phi_{s1}^2 + \phi_{s1}\phi + \phi^2)d\phi}{\phi\sqrt{(\phi - \phi_m)(\phi - \phi_l)(\phi_{s1}^2 + \phi_{s1}\phi + \phi^2)}} \\ &= \int_{\phi_m}^{\phi} \frac{(\phi_{s1}^2 + \phi_{s1}\phi + \phi^2)d\phi}{\phi\sqrt{(\phi - \phi_m)(\phi - \phi_l)[(\phi - b_1)^2 + a_1^2]}},\end{aligned} \tag{5.2}$$

where $\phi_l < 0 < \phi_m < \phi_1 < \phi_s$.

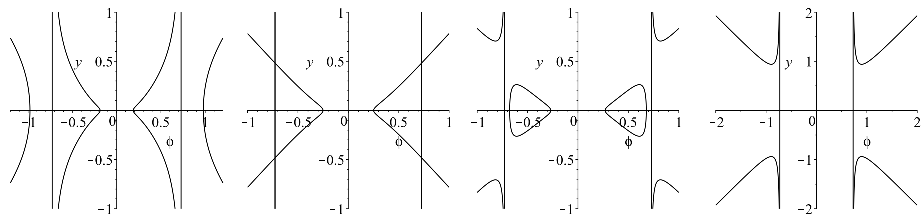
Hence, we obtain the following periodic peakon solution:

$$\begin{aligned}\phi(\chi) &= A_0 + \frac{B_0}{1 - \alpha_0 \mathrm{cn}(\chi, k)}, \quad \chi \in (-\chi_0, \chi_0), \\ \xi(\chi) &= \frac{1}{\sqrt{A_1 B_1}} \left[ \left( \phi_{s1} + \frac{\phi_{s1}^2 (A_1 + B_1)}{\phi_m B_1 + \phi_l A_1} + \frac{\phi_m B_1 \phi_l A_1}{A_1 + B_1} \right) \chi \right. \\ &\quad + \left( \frac{\phi_{s1}^2 (B_1 - A_1)}{\phi_m B_1 + \phi_l A_1} \right) \left( \frac{\tilde{\alpha}_1 - \alpha_0}{1 - \tilde{\alpha}_1^2} \right) \left( \Pi \left( \arccos(\mathrm{cn}(\chi, k)), \frac{\tilde{\alpha}_1^2}{\tilde{\alpha}_1^2 - 1}, k \right) - \tilde{\alpha}_1 f_1 \right) \\ &\quad \left. - \left( \frac{\phi_m B_1 - \phi_l A_1}{A_1 + B_1} \right) \left( \frac{\tilde{\alpha}_1 - \alpha_0}{1 - \alpha_0^2} \right) \left( \Pi \left( \arccos(\mathrm{cn}(\chi, k)), \frac{\alpha_0^2}{\alpha_0^2 - 1}, k \right) - \alpha_0 f_1 \right) \right],\end{aligned} \tag{5.3}$$

where $A_1^2 = (\phi_m - b_1)^2 + a_1^2$, $B_1^2 = (\phi_l - b_1)^2 + a_1^2$, $A_0 = \frac{\phi_m B_1 + \phi_l A_1}{A_1 + B_1}$, $B_0 = \frac{-2A_1 B_1 (\phi_m - \phi_l)}{A_1^2 - B_1^2}$, $k^2 = \frac{(A_1 + B_1)^2 - (\phi_m - \phi_l)^2}{4A_1 B_1}$, $\alpha_0 = \frac{A_1 + B_1}{B_1 - A_1}$, $\tilde{\alpha}_1 = \frac{\phi_m B_1 + \phi_l A_1}{\phi_m B_1 - \phi_l A_1}$, $\chi_0 = \mathrm{cn}^{-1}\left( \frac{1}{\alpha_0} \left( 1 - \frac{B_0}{\phi_{sl} - A_0} \right) \right)$ and $f_1$ is a special function (see [9], page 215).

## 5.2. The case when $p$ is an even number

When $p$ is an even number, and $\alpha > |\beta|(p-1)$, system (1.8) has phase portrait shown in Fig.5 (c). In this case, when $h$ varies, the level curves defined by $H_{20}(\phi, y) = h$ are shown in Fig.9.



(a) $h < h_s$ (b) $h = h_s$ (c) $h \in (h_s, h_1)$ (d) $h > h_1$

**Figure 9.** The level curves $H_{20}(\phi, y) = h$ of equation (1.8) for $\beta(p-1) < \alpha < 0$

(i) For $h \in (-\infty, h_s)$, there exist two families of open orbit branches defined by $H_{20}(\phi, y) = h$ near the two singular straight lines $\phi = \pm\phi_{s1}$ (see Fig.9 (a)), which tend to two singular straight linse, when $|y| \to \infty$, respectively. These open orbits give rise to two families of compacton solutions of equation (1.1).

(ii) For $h = h_s$, the level curves defined by $H_{20}(\phi, y) = h_s$ are two arches (see Fig.9 (b)), which passing through the singular straight line $\phi = \pm\phi_{s1}$, respectively. Two arches give rise to two periodic peakon solutions.

(iii) For $h \in (h_s, h_1)$, the level curves defined by $H_{20}(\phi, y) = h$ are two families of closed orbits (see Fig.9 (c)), These families give rise to two families of periodic wave solutions. Specially, when $|h - h_s| \ll 1$, these closed orbits give rise two families of periodic peakon solutions.

Assume that $p = 4$. For the right arch orbit, (5.1) can be written as

$$\begin{aligned}\xi &= \int_{\phi_m}^{\phi} \frac{(\phi_{s1}^4 - \phi^4)d\phi}{\phi\sqrt{(\phi_{s1}^4 - \phi^4)(\phi^4 + \frac{h_s}{c}\phi^2 + B_0^4)}} = \int_{\phi_m}^{\phi} \frac{(\phi_{s1}^2 + \phi^2)d\phi}{\phi\sqrt{(\phi_{s1}^2 + \phi^2)(\phi^2 - \phi_m^2)}} \\ &= \int_{\psi_m}^{\psi} \frac{(\psi_{s1} + \psi)d\psi}{2\psi\sqrt{(\psi - \psi_m)(\psi_{s1} + \psi)}}, \end{aligned} \tag{5.4}$$

where $0 < \phi_m < \phi_1 < \phi_{s1}, \psi = \phi^2, \psi_m = \phi_m^2, \psi_{s1} = \phi_{s1}^2$.

(5.4) implies that the following two periodic peakon solutions:

$$\begin{aligned}\phi(\chi) &= \pm(\psi(\chi))^{\frac{1}{2}} = \pm\left(\frac{2\psi_m\psi_{s1}}{|(\psi_{s1}-\psi_m)+(\psi_{s1}+\psi_m)\cos\left(\sqrt{\psi_m\psi_{s1}}\chi\right)|}\right)^{\frac{1}{2}}, \quad \chi \in (-\pi, \pi), \\ \xi(\chi) &= \frac{1}{2}\left[\psi_{s1}\chi \pm \ln\left(\frac{\sqrt{(\psi_{s1}-\psi(\chi))(\psi(\chi)-\psi_m)}+\psi(\chi)+\frac{1}{2}(\psi_{s1}-\psi_m)}{\frac{1}{2}(\psi_{s1}+\psi_m)}\right)\right]. \end{aligned} \tag{5.5}$$

**Theorem 5.1.** *Assume that the parameters of equation* (1.1) *satisfy the condition* $\gamma = k\beta + 2\tilde{\delta}, \delta = (k-1)\tilde{\delta}$ *and* $\tilde{\delta} = \frac{1}{2}k\beta = -\frac{1}{2}p\beta, \beta < 0$.

*(i) When* $\beta(p-1) < \alpha < 0$*, and* $p$ *is an odd number, corresponding to the arch branch of the level curves* $H_{20}(\phi, y) = h_s$*, equation* (1.1) *has a periodic peakon*

*solution. Specially, for $p = 3$, this periodic peakon has the exact parametric representation given by* (5.3)*.*

*(ii) When $\beta(p-1) < \alpha < 0$, and $p$ is an even number, corresponding to two arch branches of the level curves $H_{20}(\phi, y) = h_s$, equation* (1.1) *has two periodic peakon solutions. Specially, for $p = 4$, this periodic peakon has the exact parametric representation given by* (5.5)*.*

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