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TWO NEW INERTIAL RELAXED GRADIENT CQ ALGORITHMS ON THE SPLIT EQUALITY PROBLEM*

Yuanheng Wang[1,†], Xueting Li[1] and Bingnan Jiang[1]

Abstract In order to better solve the split equality problems in Hilbert spaces, we propose two new algorithms. Combining inertial iteration methods, we construct an inertial simultaneous relaxed gradient CQ algorithm with adaptive step size, and prove its weak convergence under simpler and more straightforward conditions. Combining viscous iteration in the above algorithm, we construct an inertial viscosity simultaneous relaxed gradient CQ algorithm, and prove the strong convergence of the algorithm under simpler and more straightforward conditions. We also give some numerical experiments to compare with some known algorithms, which demonstrate the rationality and superiority of our algorithms in several rates of convergence.

Keywords Split equality problem, inertial relaxed gradient CQ algorithm, viscosity algorithm, weak and strong convergence, split equality problem.

MSC(2010) 47H09, 47H10, 47H04.

1. Introduction

The split equality problem is one of the hot study topics in nonlinear analysis, which was first proposed in 2013 by Moudafi [11]. It is widely used in practical problems such as analog intensity modulated radiotherapy, game theory, medical image reconstruction and partial differential equation decomposition method, etc [7, 12, 14, 15, 17]. So, the split equality problem has very important research value.

Let C and Q be two nonempty closed and convex subsets of real Hilbert spaces E_1 and E_2, respectively. Let $A_1 : E_1 \to E_3$, $A_2 : E_2 \to E_3$ be two bounded linear operators, where E_3 is also a real Hilbert space. Then the split equality problem (SEP) is expressed as follows:

$$\text{Find } x \in C, y \in Q, \text{ such that } A_1x = A_2y. \tag{1.1}$$

All the solutions of SEP (1.1) can denote as Ω, i.e.,

$$\Omega := \{(x, y) \in C \times Q \mid A_1x = A_2y\}. \tag{1.2}$$

†The corresponding author. Email address: yhwang@zjnu.cn(Y. Wang)
1Department of Mathematics, Zhejiang Normal University, Jinhua 321000, China
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It can be observed that when $E_2 = E_3$ and $A_2 = I$ is identity mapping, the SEP (1.1) degenerates into the famous split feasibility problem (SFP) which was first proposed by Censor et al. [3] in Euclidean spaces in 1994.

Later, Byrne [1] proposed the CQ algorithm to solve the SFP (1.1) as follows: For any $x_1 \in E_1$, let

$$x_{n+1} = P_C(x_n - \gamma A_1^*(I - P_Q A_1 x_n), \quad \forall n \geq 1,$$

where the stepsize γ is a constant, A_1^* is the adjoint operator of A_1, $P_C : E_1 \to C$ and $P_Q : E_2 \to Q$ are the metric projections onto C and Q, respectively.

But the CQ algorithm can be used to solve the SFP only when the P_C, P_Q and $\|A_1\|$ are calculated. and these calculations are also very difficult to carry out. For overcoming these difficulties and solving the the SFP , many scholars have proposed some new algorithms. For example, Kesornprom et al. [8] suggested two gradient-CQ algorithms in 2019 and they proved the weak and strong convergence of their algorithms under certain conditions.

While about SEP (1.1), Moudafi [11] gave the following alternating CQ algorithm to solve it, and he proved the weak convergence of this algorithm.

Algorithm 1.1. The algorithm of Moudafi

Initialization: Choose $x_0 \in E_1$, $y_0 \in E_2$ arbitrarily.
Iterative step: Compute x_{n+1}, y_{n+1} via

$$\begin{cases} x_{n+1} = P_{C_n}[x_n - \tau A_1^*(A_1 x_n - A_2 y_n)], \\ y_{n+1} = P_{Q_n}[y_n + \tau A_2^*(A_1 x_{n+1} - A_2 y_n)], \end{cases}$$

where

$$C_n = \{x \in E_1 \mid \lambda(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \quad \xi_n \in \partial\lambda(x_n),$$
$$Q_n = \{y \in E_2 \mid \delta(y_n) \leq \langle \zeta_n, y_n - y \rangle\}, \quad \zeta_n \in \partial\delta(y_n),$$

and $\tau \in \left(0, \min\left\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right\}\right)$.

Similar to the CQ algorithm in the SFP, the step size in the Algorithm 1.1 depends on $\|A_1\|$ and $\|A_2\|$, which is not easy to calculate in practice. In order to better solve the SEP (1.1), Shi et al. [14] improved Algorithm 1.1 and provided Algorithm 1.2 as follows; and they proved that the sequence generated by the Algorithm 1.2 is strongly convergent.

Algorithm 1.2. The algorithm of Shi et al.

Initialization: Choose $x_0 \in E_1$, $y_0 \in E_2$ arbitrarily.
Iterative step: Compute x_{n+1}, y_{n+1} via

$$\begin{cases} x_{n+1} = P_C\left\{(1-\beta_n)[x_n - \gamma A_1^*(A_1 x_n - A_2 y_n)]\right\}, \\ y_{n+1} = P_Q\left\{(1-\beta_n)[y_n + \gamma A_2^*(A_1 x_n - A_2 y_n)]\right\}, \end{cases}$$

where $\beta_n \in (0,1)$ and satisfied $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$, and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ or $\lim_{n\to\infty} \frac{|\beta_{n+1}-\beta_n|}{\beta_n} = 0$, $\gamma \in \left(0, \min\left\{\frac{1}{\|A_1\|^2}, \frac{1}{\|A_2\|^2}\right\}\right)$.

Recently, a new Algorithm 1.3 was proposed by Tian et al. [17] to solve the SEP (1.1) as follows:

Algorithm 1.3. The algorithm of Tian et al.

Initialization: Choose $x_0 \in E_1$, $y_0 \in E_2$ arbitrarily.

Iterative step: Compute x_{n+1}, y_{n+1} via

$$\begin{cases} u_n = P_C[x_n - \gamma_n A_1^*(A_1x_n - A_2y_n)], \\ v_n = P_Q[y_n + \gamma_n A_2^*(A_1x_n - A_2y_n)], \\ x_{n+1} = P_C[x_n - \gamma_n A_1^*(A_1u_n - A_2v_n)], \\ y_{n+1} = P_Q[y_n + \gamma_n A_2^*(A_1u_n - A_2v_n)], \end{cases}$$

where $\gamma_n = \sigma\rho^{m_n}$, $\sigma > 0$, $\rho \in (0,1)$, m_n is the smallest nonnegative integer and $\mu \in (0,1)$ satisfying the following

$$\begin{aligned} &\gamma_n[\|A_1^*(A_1x_n - A_2y_n) - A_1^*(A_1u_n - A_2v_n)\|^2 \\ &+\|A_2^*(A_2y_n - A_1x_n) - A_2^*(A_2v_n - A_1u_n)\|^2]^{\frac{1}{2}} \\ \leq\ &\mu(\|x_n - u_n\|^2 + \|y_n - v_n\|^2)^{\frac{1}{2}}. \end{aligned}$$

However, these algorithms described above are not good enough and their conditions are too complicated. So, inspired by the above algorithms, we study the split equality problem (SEP (1.1)) further and construct more effective algorithms in this paper. We propose two new improved inertial relaxed iterative CQ algorithms for solving SEP under simpler and more straightforward conditions. We give the proofs of the weak convergence or strong convergence of the two algorithms. And We use numerical experiments to reflect the rationality and superiority of the convergence rates of our algorithms.

2. Preliminaries

Throughout the article, we all suppose that E is a real Hilbert space with its inner product $\langle\cdot,\cdot\rangle$ and its norm $\|\cdot\|$ and $C \neq \emptyset, C \subset E$ is closed convex. Some mathematical symbols, definitions and lemma are given as follows.

Definition 2.1 ([19]). Let $S : E \to E$ be an operator. Then S is called a $\rho-$contractive mapping ($\rho \in [0,1)$), if

$$\|Sx - Sy\| \leq \rho\|x - y\|,\ \forall x, y \in E.$$

Definition 2.2 ([20]). For any $u \in E$, if there is a unique point $z \in C$ satisfying

$$\|u - z\| = \inf\{\|u - y\| : y \in C\},$$

then $z \in C$ is called the metric projection of u on C, denoted as $z = P_Cu$.

Definition 2.3 ([8]). Let $g : E \to \mathbb{R}$ be a convex function. $\xi \in E$ is said to be a subgradient of g at $z \in E$ if

$$g(u) \geq g(z) + \langle\xi, u - z\rangle,\ \forall u \in E.$$

The subdifferential of g at point z is the set of all subgradients of g at point z, which can be denoted by $\partial g(z)$, i.e.,

$$\partial g(z) = \{\xi \in E \mid g(u) \geq g(z) + \langle\xi, u - z\rangle,\ \forall u \in E\}.$$

Definition 2.4 ([8]). A function $G : E \to \mathbb{R}$ is called weakly lower semi-continuous (w-lsc) at x_0, if for any $\{x_n\} \subset E$ and $x_n \rightharpoonup x_0$,

$$G(x_0) \leq \liminf_{n\to\infty} G(x_n).$$

Lemma 2.1 ([2,4,6]). *For all $u \in E$, the following assertions hold:*

(i) $\langle u - P_C u, z - P_C u\rangle \leq 0$, $\forall z \in C$,

(ii) $\|z - P_C u\|^2 \leq \|u - z\|^2 - \|u - P_C u\|^2$, $\forall z \in C$,

(iii) $\langle u - z, P_C u - P_C z\rangle \geq \|P_C u - P_C z\|^2$, $\forall z \in E$.

Lemma 2.2 ([9,10,16,18]). *The following equalities or inequalities hold:*

(i) $|\langle u, z\rangle| \leq \|u\|\|z\|$, $\forall u, z \in E$.

(ii) $\sqrt{(a+b)^2 + (h+k)^2} \leq \sqrt{a^2 + h^2} + \sqrt{b^2 + k^2}$, $\forall a, b, h, k \in \mathbb{R}$.

(iii) $2\langle u - b, u - v\rangle = \|u - b\|^2 + \|u - v\|^2 - \|b - v\|^2$, $\forall u, b, v \in E$.

(iv) $\|ru + (1-r)z\|^2 = r\|u\|^2 + (1-r)\|z\|^2 - r(1-r)\|u - z\|^2$, $\forall r \in \mathbb{R}$, $\forall u, z \in E$.

(v) $\|u + z\|^2 \leq \|u\|^2 + 2\langle u + z, z\rangle$, $\forall u, z \in E$.

Lemma 2.3 ([5]). *Assume that $\{\zeta_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ are three non-negative real number sequences, and satisfy*

$$\zeta_{n+1} \leq \zeta_n + \alpha_n(\zeta_n - \zeta_{n-1}) + \delta_n,\ n \geq 1,$$

$\sum_{n=0}^{\infty} \delta_n < \infty$. *If there is a constant $\alpha \in (0, 1)$ such that $0 \leq \alpha_n \leq \alpha < 1$, then the following assertions hold:*

(i) $\sum_{n=1}^{\infty} [\zeta_n - \zeta_{n-1}]_+ < \infty$, *where* $r \in \mathbb{R}, [r]_+ = \max\{r, 0\}$;

(ii) *There is* $\zeta^* \in [0, +\infty)$, *such that* $\lim_{n\to\infty} \zeta_n = \zeta^*$.

Lemma 2.4 ([20]). *Suppose that a sequence $\{x_n\} \subset E$ satisfies the following conditions:*

(i) $\lim_{n\to\infty} \|x_n - z\|$ *exists*, $\forall z \in C$;

(ii) $W_w(x_n) = \{x \mid x_{n_k} \rightharpoonup x,\ \{x_{n_k}\} \subset \{x_n\}\} \subset C$.

Then $x_n \rightharpoonup p \in C$.

Lemma 2.5 ([13]). *Let $\{\Lambda_n\} \subseteq \mathbb{R} : \Lambda_n \geq 0$, satisfy*

$$\Lambda_{n+1} \leq (1 - \varsigma_n)\Lambda_n + \varsigma_n\delta_n, n \geq 0,$$
$$\Lambda_{n+1} \leq \Lambda_n - \sigma_n + \gamma_n, n \geq 0,$$

where $\{\sigma_n\} \subset \mathbb{R}$, $\{\delta_n\} \subset \mathbb{R}$, $\{\gamma_n\} \subset \mathbb{R}$ and $\{\varsigma_n\} \subset (0, 1)$, such that

(i) $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=0}^{\infty} \varsigma_n = \infty$,

(ii) *For all* $\{n_k\} \subset \{n\}$, $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *if* $\lim_{k\to\infty} \sigma_{n_k} = 0$.

Then $\lim_{n\to\infty} \Lambda_n = 0$.

3. Weak convergence theorem

Assume that E_1, E_2 and E_3 are three real Hilbert spaces. Let $C \subset E_1$ and $Q \subset E_2$ be nonempty closed and convex subsets, respectively. Suppose that $A : E_1 \to E_3$, $B : E_2 \to E_3$ are two bounded linear operators. Let

$$f(x, y) := \frac{1}{2}\|Ax - By\|^2.$$

Then

$$\nabla f(x, y) = (A^*(Ax - By), -B^*(Ax - By)),$$

and

$$\|\nabla f(x, y)\|^2 = \|A^*(Ax - By)\|^2 + \|B^*(Ax - By)\|^2.$$

We next propose our weak convergence algorithm and theorem.

Algorithm 3.1. Inertial simultaneous relaxed gradient-CQ algorithm

Initialization: Choose $x_0, x_1 \in E_1$, $y_0, y_1 \in E_2$ arbitrarily.

Iterative step: Compute x_{n+1}, y_{n+1} via

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ w_n = y_n + \alpha_n(y_n - y_{n-1}), \\ v_n = z_n - \tau_n A^*(Az_n - Bw_n), \\ u_n = w_n + \tau_n B^*(Az_n - Bw_n), \\ x_{n+1} = P_{C_n}[v_n - \varphi_n A^*(Av_n - Bu_n)], \\ y_{n+1} = P_{Q_n}[u_n + \varphi_n B^*(Av_n - Bu_n)], \end{cases}$$

where

$$C_n = \{x \in E_1 \mid \lambda(x_n) \le \langle \xi_n, x_n - x\rangle\}, \quad \xi_n \in \partial\lambda(x_n),$$
$$Q_n = \{y \in E_2 \mid \delta(y_n) \le \langle \zeta_n, y_n - y\rangle\}, \quad \zeta_n \in \partial\delta(y_n),$$

$\alpha_n \subset [0, \alpha]$ for some $\alpha \in (0, 1)$ and

$$\tau_n = \frac{\rho_n f(z_n, w_n)}{\|\nabla f(z_n, w_n)\|^2 + \theta_n},$$
$$\varphi_n = \frac{\rho_n f(v_n, u_n)}{\|\nabla f(v_n, u_n)\|^2 + \theta_n},$$

$0 < \rho_n < 4$,$0 < \theta_n < 1$.

Theorem 3.1. *Assume that* $\inf_n \rho_n(4 - \rho_n) > 0$, $\lim_{n\to\infty} \theta_n = 0$ *and* $0 \le \alpha_n \le \min\{\bar{\alpha}_n, \hat{\alpha}_n\}$ *with*

$$\bar{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|^2}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \ne x_{n-1}, \\ \alpha, & \text{if } x_n = x_{n-1}, \end{cases}$$

$$\hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|y_n - y_{n-1}\|^2}, \frac{\epsilon_n}{\|y_n - y_{n-1}\|}\right\}, & \text{if } y_n \ne y_{n-1}, \\ \alpha, & \text{if } y_n = y_{n-1}, \end{cases}$$

where $\{\epsilon_n\} \subset [0,\infty)$, $\sum\limits_{n=1}^{\infty} \epsilon_n < \infty$.

Then the sequence pairs $\{(x_n, y_n)\}$ *generated by Algorithm 3.1 weakly converge to a solution* $(q,p) \in \Omega$ *of the SEP (1.1).*

Proof. Let's take $(q,p) \in \Omega$. By Lemma 2.1 (ii) and Lemma 2.2 (iii), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
= &\ \|P_{C_n}[v_n - \varphi_n A^*(Av_n - Bu_n)] - q\|^2 \\
\le &\ \|v_n - \varphi_n A^*(Av_n - Bu_n) - q\|^2 - \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
= &\ \|v_n - q\|^2 + \varphi_n^2 \|A^*(Av_n - Bu_n)\|^2 - 2\varphi_n \langle v_n - q, A^*(Av_n - Bu_n)\rangle \\
&- \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
= &\ \|v_n - q\|^2 + \varphi_n^2 \|A^*(Av_n - Bu_n)\|^2 - 2\varphi_n \langle Av_n - Aq, Av_n - Bu_n\rangle \\
&- \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
= &\ \|v_n - q\|^2 + \varphi_n^2 \|A^*(Av_n - Bu_n)\|^2 - \varphi_n(\|Av_n - Aq\|^2 + \|Av_n - Bu_n\|^2 \\
&- \|Bu_n - Aq\|^2) - \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2. \qquad (3.1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|y_{n+1} - p\|^2 \\
\le &\ \|u_n - p\|^2 + \varphi_n^2 \|B^*(Av_n - Bu_n)\|^2 - \varphi_n(\|Bu_n - Bp\|^2 + \|Av_n - Bu_n\|^2 \\
&- \|Av_n - Bp\|^2) - \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2. \qquad (3.2)
\end{aligned}$$

Adding (3.1) and (3.2), since $Aq = Bp$, we get

$$\begin{aligned}
&\|x_{n+1} - q\|^2 + \|y_{n+1} - p\|^2 \\
\le &\ \|v_n - q\|^2 + \|u_n - p\|^2 - 2\varphi_n \|Av_n - Bu_n\|^2 + \varphi_n^2 [\|A^*(Av_n - Bu_n)\|^2 \\
&+ \|B^*(Av_n - Bu_n)\|^2] - \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
&- \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2 \\
= &\ \|v_n - q\|^2 + \|u_n - p\|^2 - 4\varphi_n f(v_n, u_n) + \varphi_n^2 \|\nabla f(v_n, u_n)\|^2 \\
&- \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 - \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2 \\
= &\ \|v_n - q\|^2 + \|u_n - p\|^2 - \frac{4\rho_n (f(v_n,u_n))^2}{\|\nabla f(v_n,u_n)\|^2 + \theta_n} \\
&+ \frac{\rho_n^2 (f(v_n,u_n))^2}{(\|\nabla f(v_n,u_n)\|^2 + \theta_n)^2} \|\nabla f(v_n,u_n)\|^2 \\
&- \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 - \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2 \\
\le &\ \|v_n - q\|^2 + \|u_n - p\|^2 - \frac{4\rho_n (f(v_n,u_n))^2}{\|\nabla f(v_n,u_n)\|^2 + \theta_n} + \frac{\rho_n^2 (f(v_n,u_n))^2}{\|\nabla f(v_n,u_n)\|^2 + \theta_n} \\
&- \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 - \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2 \\
= &\ \|v_n - q\|^2 + \|u_n - p\|^2 - \rho_n(4 - \rho_n) \frac{(f(v_n,u_n))^2}{\|\nabla f(v_n,u_n)\|^2 + \theta_n} \\
&- \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
&- \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2. \qquad (3.3)
\end{aligned}$$

Moreover, we see that

$$\|v_n - q\|^2$$

$$
\begin{aligned}
&= \|z_n - \tau_n A^*(Az_n - Bw_n) - q\|^2 \\
&= \|z_n - q\|^2 + \tau_n^2\|A^*(Az_n - Bw_n)\|^2 - 2\tau_n\langle z_n - q, A^*(Az_n - Bw_n)\rangle \\
&= \|z_n - q\|^2 + \tau_n^2\|A^*(Az_n - Bw_n)\|^2 - 2\tau_n\langle Az_n - Aq, Az_n - Bw_n\rangle \\
&= \|z_n - q\|^2 + \tau_n^2\|A^*(Az_n - Bw_n)\|^2 - \tau_n(\|Az_n - Aq\|^2 + \|Az_n - Bw_n\|^2 \\
&\quad -\|Bw_n - Aq\|^2).
\end{aligned} \tag{3.4}
$$

Similarly,

$$
\begin{aligned}
&\|u_n - p\|^2 \\
= &\|w_n - p\|^2 + \tau_n^2\|B^*(Az_n - Bw_n)\|^2 - \tau_n(\|Bw_n - Bp\|^2 + \|Az_n - Bw_n\|^2 \\
&-\|Az_n - Bp\|^2).
\end{aligned} \tag{3.5}
$$

Adding (3.4) and (3.5), since $Aq = Bp$, we obtain

$$
\begin{aligned}
&\|v_n - q\|^2 + \|u_n - p\|^2 \\
= &\|z_n - q\|^2 + \|w_n - p\|^2 - 2\tau_n\|Az_n - Bw_n\|^2 + \tau_n^2[\|A^*(Az_n - Bw_n)\|^2 \\
&+\|B^*(Az_n - Bw_n)\|^2] \\
= &\|z_n - q\|^2 + \|w_n - p\|^2 - 4\tau_n f(z_n, w_n) + \tau_n^2\|\nabla f(z_n, w_n)\|^2 \\
\leq &\|z_n - q\|^2 + \|w_n - p\|^2 - \frac{4\rho_n(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} + \frac{\rho_n^2(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} \\
= &\|z_n - q\|^2 + \|w_n - p\|^2 - \rho_n(4 - \rho_n)\frac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n}.
\end{aligned} \tag{3.6}
$$

Combining (3.3) and (3.6), we deduce

$$
\begin{aligned}
&\|x_{n+1} - q\|^2 + \|y_{n+1} - p\|^2 \\
\leq &\|z_n - q\|^2 + \|w_n - p\|^2 - \rho_n(4 - \rho_n)\frac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} \\
&-\rho_n(4 - \rho_n)\frac{(f(v_n, u_n))^2}{\|\nabla f(v_n, u_n)\|^2 + \theta_n} - \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
&-\|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2.
\end{aligned} \tag{3.7}
$$

From Lemma 2.2 (iv), we see that

$$
\begin{aligned}
&\|z_n - q\|^2 \\
&= \|x_n + \alpha_n(x_n - x_{n-1}) - q\|^2 \\
&= \|(1 + \alpha_n)(x_n - q) - \alpha_n(x_{n-1} - q)\|^2 \\
&= (1 + \alpha_n)\|x_n - q\|^2 - \alpha_n\|x_{n-1} - q\|^2 + \alpha_n(1 + \alpha_n)\|x_n - x_{n-1}\|^2 \\
&\leq (1 + \alpha_n)\|x_n - q\|^2 - \alpha_n\|x_{n-1} - q\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - q\|^2 + \alpha_n(\|x_n - q\|^2 - \|x_{n-1} - q\|^2) + 2\epsilon_n.
\end{aligned} \tag{3.8}
$$

Similarly,

$$
\|w_n - p\|^2 \leq \|y_n - p\|^2 + \alpha_n(\|y_n - p\|^2 - \|y_{n-1} - p\|^2) + 2\epsilon_n. \tag{3.9}
$$

Adding (3.8) and (3.9), we get

$$
\|z_n - q\|^2 + \|w_n - p\|^2
$$

$$\leq \|x_n - q\|^2 + \|y_n - p\|^2 + \alpha_n(\|x_n - q\|^2 + \|y_n - p\|^2 - \|x_{n-1} - q\|^2 - \|y_{n-1} - p\|^2) + 4\epsilon_n. \tag{3.10}$$

Using (3.10) and (3.7), we can obtain

$$\begin{aligned} &\|x_{n+1} - q\|^2 + \|y_{n+1} - p\|^2 \\ \leq\ &\|z_n - q\|^2 + \|w_n - p\|^2 \\ \leq\ &\|x_n - q\|^2 + \|y_n - p\|^2 + \alpha_n(\|x_n - q\|^2 + \|y_n - p\|^2 - \|x_{n-1} - q\|^2 \\ &-\|y_{n-1} - p\|^2) + 4\epsilon_n. \end{aligned} \tag{3.11}$$

By using Lemma 2.3, we obtain that there exists $\lim_{n\to\infty}(\|x_n - q\|^2 + \|y_n - p\|^2)$. So $\{x_n\}$ and $\{y_n\}$ are all bounded.

Again from (3.7), it follows that

$$\begin{aligned} &\rho_n(4-\rho_n)\frac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} + \rho_n(4-\rho_n)\frac{(f(v_n, u_n))^2}{\|\nabla f(v_n, u_n)\|^2 + \theta_n} \\ &+\|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 + \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2 \\ \leq\ &\|z_n - q\|^2 + \|w_n - p\|^2 - \|x_{n+1} - q\|^2 - \|y_{n+1} - p\|^2 \\ \leq\ &\|x_n - q\|^2 + \|y_n - p\|^2 - \|x_{n+1} - q\|^2 - \|y_{n+1} - p\|^2 + \alpha_n(\|x_n - q\|^2 \\ &+\|y_n - p\|^2 - \|x_{n-1} - q\|^2 - \|y_{n-1} - p\|^2) + 4\epsilon_n. \end{aligned} \tag{3.12}$$

So we obtain

$$\lim_{n\to\infty} \rho_n(4-\rho_n)\frac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} = 0, \tag{3.13}$$

$$\lim_{n\to\infty} \rho_n(4-\rho_n)\frac{(f(v_n, u_n))^2}{\|\nabla f(v_n, u_n)\|^2 + \theta_n} = 0, \tag{3.14}$$

$$\lim_{n\to\infty} \|x_{n+1} - v_n + \varphi_n A^*(Av_n - Bu_n)\| = 0, \tag{3.15}$$

and

$$\lim_{n\to\infty} \|y_{n+1} - u_n - \varphi_n B^*(Av_n - Bu_n)\| = 0. \tag{3.16}$$

From our assumptions, we conclude that

$$\lim_{n\to\infty} \frac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2} = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty} \frac{(f(v_n, u_n))^2}{\|\nabla f(v_n, u_n)\|^2} = 0. \tag{3.18}$$

Obviously $\{\|\nabla f(z_n, w_n)\|\}$ is bounded. So $\lim_{n\to\infty} f(z_n, w_n) = 0$. This means $\lim_{n\to\infty}\|Az_n - Bw_n\| = 0$. Similarly, we also get $\lim_{n\to\infty} f(v_n, u_n) = 0$. So

$$\varphi_n\|\nabla f(v_n, u_n)\| = \frac{\rho_n f(v_n, u_n)}{\|\nabla f(v_n, u_n)\|^2 + \theta_n}\|\nabla f(v_n, u_n)\| \to 0, \text{ as } n \to \infty,$$

which means

$$\lim_{n\to\infty} \varphi_n\|A^*(Av_n - Bu_n)\| = 0, \tag{3.19}$$

and

$$\lim_{n\to\infty} \varphi_n \|B^*(Av_n - Bu_n)\| = 0. \tag{3.20}$$

Hence, by (3.15), (3.16), (3.19) and (3.20), we obtain

$$x_{n+1} - v_n \to 0, \text{ as } n \to \infty, \tag{3.21}$$

and

$$y_{n+1} - u_n \to 0, \text{ as } n \to \infty, \tag{3.22}$$

Similarly, we have $\tau_n\|A^*(Az_n - Bw_n)\| \to 0$ and $\tau_n\|B^*(Az_n - Bw_n)\| \to 0$ as $n \to \infty$. Hence, by the expression of v_n and u_n, we have

$$v_n - z_n \to 0, \text{ as } n \to \infty, \tag{3.23}$$

and

$$u_n - w_n \to 0, \text{ when } n \to \infty. \tag{3.24}$$

On the other side,

$$\|z_n - x_n\| = \alpha_n\|x_n - x_{n-1}\| \le \epsilon_n, \tag{3.25}$$

and

$$\|w_n - y_n\| = \alpha_n\|y_n - y_{n-1}\| \le \epsilon_n. \tag{3.26}$$

Hence

$$z_n - x_n \to 0, \text{ as } n \to \infty, \tag{3.27}$$

and

$$w_n - y_n \to 0, \text{ as } n \to \infty. \tag{3.28}$$

Combining (3.21), (3.22), (3.23), (3.24), (3.27) and (3.28), we know

$$x_{n+1} - x_n \to 0, \text{ as } n \to \infty, \tag{3.29}$$

and

$$y_{n+1} - y_n \to 0, \text{ when } n \to \infty. \tag{3.30}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, we conclude that $W_w(x_n, y_n) \neq \emptyset$. Taking $(x^0, y^0) \in W_w(x_n, y_n)$, there exist $\{x_{n_k}\} \subseteq \{x_n\}$ and $\{y_{n_k}\} \subseteq \{y_n\}$ such that $x_{n_k} \rightharpoonup x^0$ and $y_{n_k} \rightharpoonup y^0$. Next, we show that $(x^0, y^0) \in \Omega$. Since $x_{n_k+1} \in C_{n_k}$, by the definition of C_{n_k}, we get

$$\lambda(x_{n_k}) \le \langle \xi_{n_k}, x_{n_k} - x_{n_k+1} \rangle \tag{3.31}$$

where $\xi_{n_k} \in \partial\lambda(x_{n_k})$. It follows that, by the boundedness of $\partial\lambda$,

$$\lambda(x_{n_k}) \le \|\xi_{n_k}\|\|x_{n_k} - x_{n_k+1}\| \to 0, \text{ as } k \to \infty. \tag{3.32}$$

By the w-lsc of λ, $x_{n_k} \rightharpoonup x^0$ and (3.32), we conclude that

$$\lambda(x^0) \le \liminf_{k\to\infty} \lambda(x_{n_k}) \le 0. \tag{3.33}$$

Thus $x^0 \in C$. Similarly, we have $y^0 \in Q$. By (3.25), (3.26) and $\lim\limits_{n\to\infty} \|Az_n - Bw_n\| = 0$, we obtain $\lim\limits_{n\to\infty} \|Ax_n - By_n\| = 0$. We can easily see that $Ax_{n_k} - By_{n_k} \rightharpoonup Ax^0 - By^0$. It follows from the w-lsc of the norm that

$$\|Ax^0 - By^0\| \le \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0, \tag{3.34}$$

hence $(x^0, y^0) \in \Omega$. By using Lemma 2.4, the sequence $\{(x_n, y_n)\}$ weakly converges to a solution in Ω. □

4. Strong convergence theorem

We next give our strong convergence theorem about our second new Algorithm 4.1 as following.

Algorithm 4.1 Inertial viscosity simultaneous relaxed gradient-CQ algorithm

Initialization: Choose $x_0, x_1 \in E_1$, $y_0, y_1 \in E_2$ arbitrarily.

Iterative step: Compute x_{n+1}, y_{n+1} via

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ w_n = y_n + \alpha_n(y_n - y_{n-1}), \\ v_n = z_n - \tau_n A^*(Az_n - Bw_n), \\ u_n = w_n + \tau_n B^*(Az_n - Bw_n), \\ x_{n+1} = \beta_n h(x_n) + (1-\beta_n)P_{C_n}[v_n - \varphi_n A^*(Av_n - Bu_n)], \\ y_{n+1} = \beta_n h(y_n) + (1-\beta_n)P_{Q_n}[u_n + \varphi_n B^*(Av_n - Bu_n)], \end{cases}$$

where

$$C_n = \{x \in E_1 \mid \lambda(x_n) \le \langle \xi_n, x_n - x\rangle\}, \quad \xi_n \in \partial\lambda(x_n),$$
$$Q_n = \{y \in E_2 \mid \delta(y_n) \le \langle \zeta_n, y_n - y\rangle\}, \quad \zeta_n \in \partial\delta(y_n),$$

h is ρ-contractive mapping, $\{\alpha_n\} \subset [0, \alpha]$ for some $\alpha > 0$, $\{\beta_n\} \subset (0,1)$ and

$$\tau_n = \frac{\rho_n f(z_n, w_n)}{\|\nabla f(z_n, w_n)\|^2 + \theta_n},$$

and

$$\varphi_n = \frac{\rho_n f(v_n, u_n)}{\|\nabla f(v_n, u_n)\|^2 + \theta_n},$$

$0 < \rho_n < 4$, $0 < \theta_n < 1$.

Theorem 4.1. *Assume that* $\lim_{n\to\infty} \theta_n = 0$, $\inf_n \rho_n(4-\rho_n) > 0$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $0 \le \alpha_n \le \tilde{\alpha}_n$ *with*

$$\tilde{\alpha}_n = \begin{cases} \alpha, & \text{if } x_n = x_{n-1} \text{ and } y_n = y_{n-1}, \\ \min\left\{\alpha, \epsilon_n / \sqrt{\|x_n - x_{n-1}\|^2 + \|y_n - y_{n-1}\|^2}\right\}, & \text{otherwise}, \end{cases}$$

where $\epsilon_n = o(\beta_n)$*:* $\lim_{n\to\infty} \epsilon_n/\beta_n = 0$.

Then the sequence pairs $\{(x_n, y_n)\}$ *generated by Algorithm 4.1 strongly converge to a solution* $(x^0, y^0) \in \Omega$*, and the solution* (x^0, y^0) *is also the unique solution of the following variational inequality problem:*

$$\begin{cases} \langle x^0 - h(x^0), x - x^0\rangle \ge 0, \\ \langle y^0 - h(y^0), y - y^0\rangle \ge 0, \end{cases} \quad \forall (x, y) \in \Omega.$$

Proof. Since $\lim_{n\to\infty} \epsilon_n/\beta_n = 0$, so there exists $M_1 > 0$ such that $\epsilon_n \le \beta_n M_1$. Let $(q, p) \in \Omega$, from Algorithm 4.1 and Minkowski Inequality (Lemma 2.2 (ii)), we have

$$\sqrt{\|z_n - q\|^2 + \|w_n - p\|^2}$$

$$\begin{aligned}
&\leq \sqrt{(\|x_n - q\| + \alpha_n\|x_n - x_{n-1}\|)^2 + (\|y_n - p\| + \alpha_n\|y_n - y_{n-1}\|)^2} \\
&\leq \sqrt{\|x_n - q\|^2 + \|y_n - p\|^2} + \epsilon_n \\
&\leq \sqrt{\|x_n - q\|^2 + \|y_n - p\|^2} + \beta_n M_1. \qquad (4.1)
\end{aligned}$$

Set $s_n = P_{C_n}[v_n - \varphi_n A^*(Av_n - Bu_n)]$ and $t_n = P_{Q_n}[u_n + \varphi_n B^*(Av_n - Bu_n)]$. It can be known from the proof of Theorem 3.1

$$\begin{aligned}
&\|s_n - q\|^2 + \|t_n - p\|^2 \\
\leq\ & \|z_n - q\|^2 + \|w_n - p\|^2 - \rho_n(4-\rho_n)\frac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} \\
&-\rho_n(4-\rho_n)\frac{(f_n(v_n, u_n))^2}{\|\nabla f(v_n, u_n)\|^2 + \theta_n} - \|s_n - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
&-\|t_n - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2, \qquad (4.2)
\end{aligned}$$

hence

$$\sqrt{\|s_n - q\|^2 + \|t_n - p\|^2} \leq \sqrt{\|z_n - q\|^2 + \|w_n - p\|^2}. \qquad (4.3)$$

Combining (4.1), (4.3) and using Algorithm 4.1, Minkowski Inequality (Lemma 2.2 (ii)), we get

$$\begin{aligned}
&\sqrt{\|x_{n+1} - q\|^2 + \|y_{n+1} - p\|^2} \\
\leq\ & [(\beta_n\|h(x_n) - q\| + (1-\beta_n)\|s_n - q\|)^2 \\
&+(\beta_n\|h(y_n) - p\| + (1-\beta_n)\|t_n - p\|)^2]^{\frac{1}{2}} \\
\leq\ & \beta_n\sqrt{\|h(x_n) - q\|^2 + \|h(y_n) - p\|^2} \\
&+(1-\beta_n)\sqrt{\|s_n - q\|^2 + \|t_n - p\|^2} \\
\leq\ & [1-(1-\rho)\beta_n]\sqrt{\|x_n - q\|^2 + \|y_n - p\|^2} \qquad (4.4) \\
&+\beta_n(\sqrt{\|h(q) - q\|^2 + \|h(p) - p\|^2} + M_1) \\
\leq\ & \max\left\{\sqrt{\|x_n - q\|^2 + \|y_n - p\|^2}, M_2\right\}, \qquad (4.5)
\end{aligned}$$

where $M_2 = (\sqrt{\|h(q) - q\|^2 + \|h(p) - p\|^2} + M_1)/(1-\rho)$. We can use induction to obtain

$$\sqrt{\|x_n - q\|^2 + \|y_n - p\|^2} \leq \max\left\{\sqrt{\|x_1 - q\|^2 + \|y_1 - p\|^2}, M_2\right\}. \qquad (4.6)$$

So, $\{x_n\}$ and $\{y_n\}$ are bounded.

From Algorithm 4.1 and Lemma 2.2 (v), we have

$$\begin{aligned}
&\|z_n - q\|^2 \\
\leq\ & \|x_n - q\|^2 + 2\alpha_n\langle x_n - x_{n-1}, z_n - q\rangle \\
\leq\ & \|x_n - q\|^2 + 2\epsilon_n\|z_n - q\|. \qquad (4.7)
\end{aligned}$$

Similarly,

$$\|w_n - p\|^2 \leq \|y_n - p\|^2 + 2\epsilon_n\|w_n - p\|. \qquad (4.8)$$

From the expression of x_{n+1}, we have

$$\|x_{n+1} - x^0\|^2$$

$$
\begin{aligned}
&= \|\beta_n(h(x_n) - x^0) + (1-\beta_n)(s_n - x^0)\|^2 \\
&= (1-\beta_n)^2\|s_n - x^0\|^2 + \beta_n^2\|h(x_n) - x^0\|^2 + 2\beta_n(1-\beta_n)\langle h(x_n) - x^0, s_n - x^0\rangle \\
&= (1-\beta_n)^2\|s_n - x^0\|^2 + \beta_n^2\|h(x_n) - x^0\|^2 + 2\beta_n\langle h(x_n) - x^0, s_n - x^0\rangle \\
&\quad -2\beta_n^2\langle h(x_n) - x^0, s_n - x^0\rangle \\
&\le (1-\beta_n)^2\|s_n - x^0\|^2 + \beta_n^2\|h(x_n) - x^0\|^2 + 2\beta_n\langle h(x_n) - h(x^0), s_n - x^0\rangle \\
&\quad +2\beta_n\langle h(x^0) - x^0, s_n - x^0\rangle + 2\beta_n^2\|h(x_n) - x^0\|\|s_n - x^0\| \\
&\le (1-\beta_n)^2\|s_n - x^0\|^2 + \beta_n^2\|h(x_n) - x^0\|^2 + \beta_n\rho\|x_n - x^0\|^2 + \beta_n\rho\|s_n - x^0\|^2 \\
&\quad +2\beta_n\langle h(x^0) - x^0, s_n - x^0\rangle + 2\beta_n^2\|h(x_n) - x^0\|\|s_n - x^0\|. \qquad (4.9)
\end{aligned}
$$

Similarly, from the expression of y_{n+1}, we get

$$
\begin{aligned}
&\|y_{n+1} - y^0\|^2 \\
\le\ & (1-\beta_n)^2\|t_n - y^0\|^2 + \beta_n^2\|h(y_n) - y^0\|^2 + \beta_n\rho\|y_n - y^0\|^2 + \beta_n\rho\|t_n - y^0\|^2 \\
& +2\beta_n\langle h(y^0) - y^0, t_n - y^0\rangle + 2\beta_n^2\|h(y_n) - y^0\|\|t_n - y^0\|. \qquad (4.10)
\end{aligned}
$$

Adding (4.9) and (4.10), we can obtain

$$
\begin{aligned}
&\|x_{n+1} - x^0\|^2 + \|y_{n+1} - y^0\|^2 \\
\le\ & (1-\beta_n)^2(\|s_n - x^0\|^2 + \|t_n - y^0\|^2) + \beta_n^2(\|h(x_n) - x^0\|^2 + \|h(y_n) - y^0\|^2) \\
& +\beta_n\rho(\|x_n - x^0\|^2 + \|y_n - y^0\|^2) + \beta_n\rho(\|s_n - x^0\|^2 + \|t_n - y^0\|^2) \\
& +2\beta_n(\langle h(x^0) - x^0, s_n - x^0\rangle + \langle h(y^0) - y^0, t_n - y^0\rangle) \\
& +2\beta_n^2(\|h(x_n) - x^0\|\|s_n - x^0\| + \|h(y_n) - y^0\|\|t_n - y^0\|). \qquad (4.11)
\end{aligned}
$$

Substituting (4.3), (4.7) and (4.8) into (4.11), we deduce

$$
\begin{aligned}
&\|x_{n+1} - x^0\|^2 + \|y_{n+1} - y^0\|^2 \\
\le\ & [1 - 2(1-\rho)\beta_n](\|x_n - x^0\|^2 + \|y_n - y^0\|^2) \\
& +2\beta_n(\langle h(x^0) - x^0, s_n - x^0\rangle + \langle h(y^0) - y^0, t_n - y^0\rangle) \\
& +\beta_n^2 M_3 + 4\epsilon_n(\|z_n - x^0\| + \|w_n - y^0\|) \\
=\ & [1 - 2(1-\rho)\beta_n](\|x_n - x^0\|^2 + \|y_n - y^0\|^2) \\
& +\beta_n\Big[2\langle h(x^0) - x^0, s_n - x^0\rangle + 2\langle h(y^0) - y^0, t_n - y^0\rangle + \beta_n M_3 \\
& +\frac{4\epsilon_n}{\beta_n}(\|z_n - x^0\| + \|w_n - y^0\|)\Big] \\
=\ & (1-\varsigma_n)(\|x_n - x^0\|^2 + \|y_n - y^0\|^2) \\
& +\varsigma_n\frac{1}{2(1-\rho)}\Big[2\langle h(x^0) - x^0, s_n - x^0\rangle + 2\langle h(y^0) - y^0, t_n - y^0\rangle \\
& +\beta_n M_3 + \frac{4\epsilon_n}{\beta_n}(\|z_n - x^0\| + \|w_n - y^0\|)\Big] \qquad (4.12)
\end{aligned}
$$

where $M_3 \ge 0$ is constant, $\varsigma_n = 2(1-\rho)\beta_n$.

On the other side,

$$
\begin{aligned}
&\|x_{n+1} - x^0\|^2 \\
=\ & \|\beta_n h(x_n) + (1-\beta_n)s_n - x^0\|^2 \\
=\ & \|(s_n - x^0) + \beta_n(h(x_n) - s_n)\|^2
\end{aligned}
$$

$$\leq \|s_n - x^0\|^2 + 2\beta_n \|h(x_n) - s_n\| \|x_{n+1} - x^0\|. \tag{4.13}$$

Similarly, we have

$$\|y_{n+1} - y^0\|^2 \leq \|t_n - y^0\|^2 + 2\beta_n \|h(y_n) - t_n\| \|y_{n+1} - y^0\|. \tag{4.14}$$

Combining (4.2), (4.7), (4.8), (4.13) and (4.14), we know

$$\begin{aligned}
& \|x_{n+1} - x^0\|^2 + \|y_{n+1} - y^0\|^2 \\
\leq\ & \|s_n - x^0\|^2 + \|t_n - y^0\|^2 + 2\beta_n (\|h(x_n) - s_n\| \|x_{n+1} - x^0\| \\
& + \|h(y_n) - t_n\| \|y_{n+1} - y^0\|) \\
\leq\ & \|x_n - x^0\|^2 + \|y_n - y^0\|^2 - \rho_n (4 - \rho_n) \frac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} \\
& - \rho_n (4 - \rho_n) \frac{(f(v_n, u_n))^2}{\|\nabla f(v_n, u_n)\|^2 + \theta_n} - \|s_n - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 \\
& - \|t_n - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2 + 2\epsilon_n (\|z_n - x^0\| + \|w_n - y^0\|) \\
& + 2\beta_n (\|h(x_n) - s_n\| \|x_{n+1} - x^0\| + \|h(y_n) - t_n\| \|y_{n+1} - y^0\|).
\end{aligned} \tag{4.15}$$

Set

$$\begin{aligned}
\Lambda_n &= \|x_n - x^0\|^2 + \|y_n - y^0\|^2; \\
\delta_n &= \tfrac{1}{2(1-\rho)} \Big[2\langle h(x^0) - x^0, s_n - x^0 \rangle + 2\langle h(y^0) - y^0, t_n - y^0 \rangle + \beta_n M_3 \\
& \quad + \tfrac{4\epsilon_n}{\beta_n} (\|z_n - x^0\| + \|w_n - y^0\|) \Big]; \\
\sigma_n &= \rho_n (4 - \rho_n) \tfrac{(f(z_n, w_n))^2}{\|\nabla f(z_n, w_n)\|^2 + \theta_n} + \rho_n (4 - \rho_n) \tfrac{(f(v_n, u_n))^2}{\|\nabla f(v_n, u_n)\|^2 + \theta_n} \\
& \quad + \|s_n - v_n + \varphi_n A^*(Av_n - Bu_n)\|^2 + \|t_n - u_n - \varphi_n B^*(Av_n - Bu_n)\|^2; \\
\gamma_n &= 2\epsilon_n (\|z_n - x^0\| + \|w_n - y^0\|) + 2\beta_n (\|h(x_n) - s_n\| \|x_{n+1} - x^0\| \\
& \quad + \|h(y_n) - t_n\| \|y_{n+1} - y^0\|).
\end{aligned}$$

Then (4.12) and (4.15) can be rewritten as

$$\Lambda_{n+1} \leq (1 - \varsigma_n) \Lambda_n + \varsigma_n \delta_n, \tag{4.16}$$

and

$$\Lambda_{n+1} \leq \Lambda_n - \sigma_n + \gamma_n. \tag{4.17}$$

It is easy to know that $\lim\limits_{n\to\infty} \varsigma_n = 0$, $\sum\limits_{n=1}^{\infty} \varsigma_n = \infty$ and $\lim\limits_{n\to\infty} \gamma_n = 0$. So, we can seen from Lemma 2.5 that we prove that $\lim\limits_{n\to\infty} \Lambda_n = 0$ if we show that $\limsup\limits_{k\to\infty} \delta_{n_k} \leq 0$ whenever $\lim\limits_{k\to\infty} \sigma_{n_k} = 0$ for any $\{n_k\} \subset \{n\}$.

Let's take $\{n_k\} \subseteq \{n\}$ with $\lim\limits_{k\to\infty} \sigma_{n_k} = 0$. From assumptions, we can get

$$\lim_{k\to\infty} \frac{(f(z_{n_k}, w_{n_k}))^2}{\|\nabla f(z_{n_k}, w_{n_k})\|^2} = 0, \tag{4.18}$$

$$\lim_{k\to\infty} \frac{(f(v_{n_k}, u_{n_k}))^2}{\|\nabla f(v_{n_k}, u_{n_k})\|^2} = 0, \tag{4.19}$$

$$\lim_{k\to\infty} \|s_{n_k} - v_{n_k} + \varphi_{n_k} A^*(Av_{n_k} - Bu_{n_k})\| = 0, \tag{4.20}$$

and

$$\lim_{k\to\infty} \|t_{n_k} - u_{n_k} - \varphi_{n_k} B^*(Av_{n_k} - Bu_{n_k})\| = 0. \tag{4.21}$$

Obviously $\{\|\nabla f(z_n, w_n)\|\}$ is bounded, so $\lim_{k\to\infty} f(z_{n_k}, w_{n_k}) = 0$, i.e. $\lim_{k\to\infty} \|Az_{n_k} - Bw_{n_k}\| = 0$. Similarly, we also get $\lim_{k\to\infty} f(v_{n_k}, u_{n_k}) = 0$. Note that

$$\varphi_{n_k} \|\nabla f(v_{n_k}, u_{n_k})\| = \frac{\rho_{n_k} f(v_{n_k}, u_{n_k})}{\|\nabla f(v_{n_k}, u_{n_k})\|^2 + \theta_{n_k}} \|\nabla f(v_{n_k}, u_{n_k})\| \to 0, \text{ as } k \to \infty,$$

which means

$$\lim_{k\to\infty} \varphi_{n_k} \|A^*(Av_{n_k} - Bu_{n_k})\| = 0, \tag{4.22}$$

and

$$\lim_{k\to\infty} \varphi_{n_k} \|B^*(Av_{n_k} - Bu_{n_k})\| = 0. \tag{4.23}$$

Hence, by (4.20), (4.21), (4.22) and (4.23), we obtain

$$s_{n_k} - v_{n_k} \to 0, \text{ as } k \to \infty, \tag{4.24}$$

and

$$t_{n_k} - u_{n_k} \to 0, \text{ as } k \to \infty. \tag{4.25}$$

Similarly, we have $\tau_{n_k} \|A^*(Az_{n_k} - Bw_{n_k})\| \to 0$ and $\tau_{n_k} \|B^*(Az_{n_k} - Bw_{n_k})\| \to 0$ as $n \to \infty$. Hence

$$v_{n_k} - z_{n_k} \to 0, \text{ as } k \to \infty, \tag{4.26}$$

and

$$u_{n_k} - w_{n_k} \to 0, \text{ as } k \to \infty. \tag{4.27}$$

In addition,

$$\|z_n - x_n\| = \alpha_n \|x_n - x_{n-1}\| \le \epsilon_n, \tag{4.28}$$

and

$$\|w_n - y_n\| = \alpha_n \|y_n - y_{n-1}\| \le \epsilon_n. \tag{4.29}$$

Hence

$$z_{n_k} - x_{n_k} \to 0, \text{ as } k \to \infty, \tag{4.30}$$

and

$$w_{n_k} - y_{n_k} \to 0, \text{ as } k \to \infty. \tag{4.31}$$

Combining (4.24), (4.25), (4.26), (4.27), (4.30) and (4.31), we obtain

$$s_{n_k} - x_{n_k} \to 0, \text{ as } k \to \infty, \tag{4.32}$$

and

$$t_{n_k} - y_{n_k} \to 0, \text{ as } k \to \infty. \tag{4.33}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, we conclude that $W_w(x_{n_k}, y_{n_k}) \neq \emptyset$. Taking $(\bar{x}, \bar{y}) \in W_w(x_{n_k}, y_{n_k})$, there exist subsequences $\{x_{n_{k_j}}\} \subseteq \{x_{n_k}\}$ and $\{y_{n_{k_j}}\} \subseteq \{y_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \bar{x}$, $y_{n_{k_j}} \rightharpoonup \bar{y}$, $\lim_{j\to\infty} \langle h(x^0) - x^0, x_{n_{k_j}} \rangle = \limsup_{k\to\infty} \langle h(x^0) - x^0, x_{n_k} \rangle$ and

$\lim_{j\to\infty}\langle h(y^0)-y^0, y_{n_{k_j}}\rangle = \limsup_{k\to\infty}\langle h(y^0)-y^0, y_{n_k}\rangle$. As the same proof in Theorem 3.1, $W_w(x_{n_k}, y_{n_k}) \subset \Omega$, i.e., $(\bar{x}, \bar{y}) \in \Omega$. From assumption, we have

$$\begin{aligned}
&\limsup_{k\to\infty}\langle h(x^0)-x^0, s_{n_k}-x^0\rangle \\
&= \limsup_{k\to\infty}\langle h(x^0)-x^0, x_{n_k}-x^0\rangle \\
&= \lim_{j\to\infty}\langle h(x^0)-x^0, x_{n_{k_j}}-x^0\rangle \\
&= \langle h(x^0)-x^0, \bar{x}-x^0\rangle \\
&\leq 0.
\end{aligned} \tag{4.34}$$

Similarly, we have

$$\limsup_{k\to\infty}\langle h(y^0)-y^0, t_{n_k}-y^0\rangle \leq 0. \tag{4.35}$$

Hence $\limsup_{k\to\infty}\delta_{n_k} \leq 0$. So we obtain $\lim_{n\to\infty}\Lambda_n = 0$. It means $(x_n, y_n) \to (x^0, y^0)$ as $n \to \infty$. □

5. Numerical results

Now, by using an example, we compare our weak convergence Algorithm 3.1 with Moudafi' Algorithm 1.1 and Tian's Algorithm 1.3, and our strong convergence Algorithm 4.1 with Shi's Algorithm 1.2 in this section. All the programmes are written in Matlab 9.0 and performed on PC Desktop Intel(R) Core(TM) i5-1035G1 CPU @ 1.00GHz 1.19GHz, RAM 16.0GB.

Example 5.1. Let $E_1 = E_2 = \mathbb{R}^2$, $E_3 = \mathbb{R}^4$. Define two linear operators $A : E_1 \to E_3$ and $B : E_2 \to E_3$ by the following 4×2 matrixes

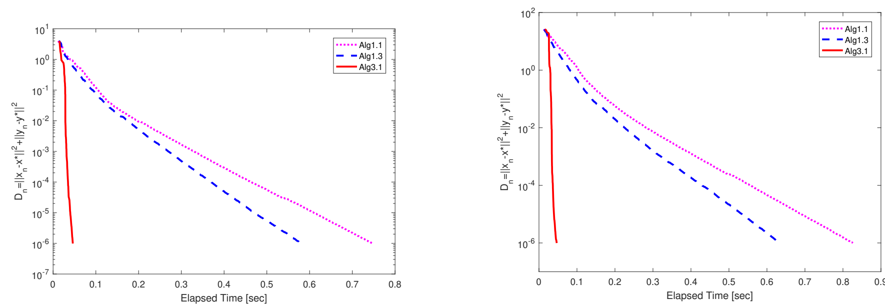
$$A = \begin{pmatrix} 2 & 1 \\ 1 & -3 \\ 0 & 2 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & -1 \\ 0 & 6 \\ 1 & -2 \\ 7 & -6 \end{pmatrix}.$$

Where, the operator A (similar for B) is defined by, for any $x = (x_1, x_2) \in E_1$,

$$Ax = \left(\begin{pmatrix} 2 & 1 \\ 1 & -3 \\ 0 & 2 \\ 1 & 4 \end{pmatrix} (x_1, x_2)^{\mathrm{T}} \right)^{\mathrm{T}} \in E_3 = \mathbb{R}^4.$$

$c((x_{(1)}, x_{(2)})^{\mathrm{T}}) = x_{(1)}^2 + x_{(2)}^2 - 25$, and $q((y_{(1)}, y_{(2)})^{\mathrm{T}}) = y_{(1)}^2 + y_{(2)}^2 - 100$. Then $C = \{x \in \mathbb{R}^2 \mid \|x\| \leq 5\}$ and $Q = \{y \in \mathbb{R}^2 \mid \|y\| \leq 10\}$. It is obvious that $\Omega = \{((0,0)^{\mathrm{T}}, (0,0)^{\mathrm{T}})\}$. Denote $x^* = (0,0)^{\mathrm{T}}$ and $y^* = (0,0)^{\mathrm{T}}$. We use $\|x_n - x^*\|^2 + \|y_n - y^*\|^2 \leq \varepsilon$ for stopping criterion.

Firstly, we compare Algorithm 3.1 with Algorithm 1.1 and Algorithm 1.3. Let $x_0 = (2,2)^{\mathrm{T}}$, $y_0 = (1,1)^{\mathrm{T}}$, $\alpha = 0.5$, $\epsilon_n = \frac{1}{n^2}$, $\rho_n = \frac{n}{n+1}$, $\theta_n = \frac{1}{n}$ in Algorithm 3.1, $\tau = \frac{1}{4}\min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\}$ in Algorithm 1.1, and $\sigma = 1$, $\rho = \mu = 0.3$ in Algorithm 3. The numerical results are shown in Figure 1 and Table 1.



(a) $x_1 = (1,1)^{\mathrm{T}}, y_1 = (-1,-1)^{\mathrm{T}}$ (b) $x_1 = (2,3)^{\mathrm{T}}, y_1 = (-2,-3)^{\mathrm{T}}$

Figure 1. The error versus the elapsed time of Algorithm 1.1, Algorithm 1.3 and Algorithm 3.1 for Example 5.1

Table 1. Numerical results of Algorithm 3.1, Algorithm 1.1 and Algorithm 1.3 as regards Example 5.1

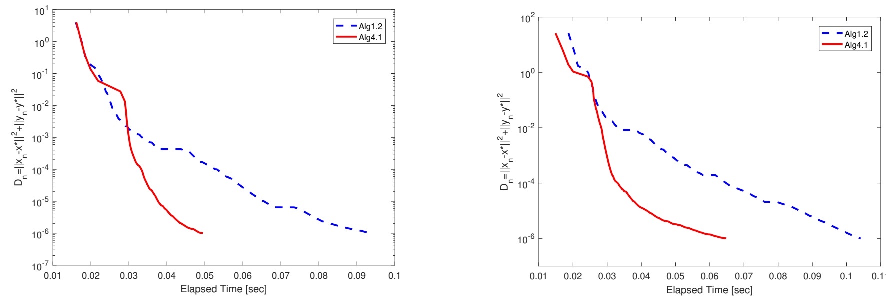
(x_1, y_1)	ε	Alg. 3.1		Alg. 1.1		Alg. 1.3	
		Iter.	Time [s]	Iter.	Time [s]	Iter.	Time [s]
$x_1 = (1,1)^{\mathrm{T}}$ $y_1 = (-1,-1)^{\mathrm{T}}$	10^{-4}	115	0.0352	7459	0.4457	2930	0.3545
	10^{-5}	184	0.0396	9675	0.5831	3785	0.4463
	10^{-6}	270	0.0472	11892	0.7096	4640	0.5523
$x_1 = (2,3)^{\mathrm{T}}$ $y_1 = (-2,-3)^{\mathrm{T}}$	10^{-4}	94	0.0349	8858	0.5829	3384	0.4264
	10^{-5}	143	0.0389	10805	0.6863	4239	0.5241
	10^{-6}	239	0.0450	13022	0.8358	5094	0.6312

Secondly, we compare Algorithm 4.1 with Algorithm 1.2. Let $x_0 = (2,2)^{\mathrm{T}}$, $y_0 = (1,1)^{\mathrm{T}}$, $\alpha = 0.5$, $\epsilon_n = \frac{1}{n^2}$, $\rho_n = \frac{n}{n+1}$, $\theta_n = \frac{1}{n}$ and $h \equiv 0$ in Algorithm 3.1, $\gamma = \frac{1}{2}\min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\}$ in Algorithm 1.2, and $\beta_n = \frac{1}{2n}$ in these two algorithms. The numerical results are shown in Figure 2 and Table 2.

It can be clearly observed from Figure 1 and Table 1 that the weak convergence algorithm we proposed has shorter elapsed time and less iterative number than Algorithm 1.1 and Algorithm 1.3 under the same conditions. Similarly, it can be seen that our strong convergence algorithm is more superior than Algorithm 1.2 in terms of elapsed time and iterative number from Figure 2 and Table 2.

6. Conclusions

In this paper, we propose two new inertial relaxed iterative CQ algorithms for solving the split equality problem (SEP) under simpler and more straightforward conditions. Our main content is divided into three modules:



(a) $x_1 = (1,1)^{\mathrm{T}}, y_1 = (-1,-1)^{\mathrm{T}}$ (b) $x_1 = (2,3)^{\mathrm{T}}, y_1 = (-2,-3)^{\mathrm{T}}$

Figure 2. The error versus the elapsed time of Algorithm 1.2 and Algorithm 4.1 for Example 5.1

Table 2. Numerical results of Algorithm 4.1 and Algorithm 1.2 as regards Example 5.1

(x_1, y_1)	ε	Alg. 4.1		Alg. 1.2	
		Iter.	Time [s]	Iter.	Time [s]
$x_1 = (1,1)^{\mathrm{T}}$ $y_1 = (-1,-1)^{\mathrm{T}}$	10^{-4}	28	0.0319	425	0.0487
	10^{-5}	84	0.0376	779	0.0641
	10^{-6}	256	0.0493	1260	0.0926
$x_1 = (2,3)^{\mathrm{T}}$ $y_1 = (-2,-3)^{\mathrm{T}}$	10^{-4}	110	0.0325	670	0.0637
	10^{-5}	235	0.0422	1054	0.0851
	10^{-6}	581	0.0647	1562	0.1043

1) By introducing inertia, we propose the inertial simultaneous relaxed gradient-CQ algorithm with adaptive step size (Algorithm 3.1). Compare with the algorithm of Tian et al. (Algorithm 1.3), we extend the algorithm from two-step iteration to three-step iteration, and at the same time, greatly simplify the conditions that the step-size needs to meet, and still get the weak convergence of the Algorithm 3.1.

2) On the basis of Algorithm 3.1, combining with viscous iteration, we propose the inertial viscosity simultaneous relaxed gradient-CQ algorithm (Algorithm 4.1), and prove the strong convergence of the Algorithm 4.1.Compare with the algorithm of Shi et al. (Algorithm 1.2), we extend the algorithm from two-step iteration to many-step iteration with a contractive mapping h, and at the same time, greatly simplify the conditions that the step-size needs to meet, and still get the strong convergence of the Algorithm 4.1.

3) We use an example to compare Algorithm 3.1 with the algorithm of Moudafi (Algorithm 1.1) and the algorithm of Tian et al. (Algorithm 1.3), and compare Algorithm 4.1 with the algorithm of Shi et al. (Algorithm 1.2). The numerical results in the figures and tables show that our algorithms have faster convergence speed. Therefor our results here generalize and improve on the corresponding results of many authors (such as [8, 11, 14, 15, 17, 19]) in recent times.

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