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EXISTENCE AND UNIQUENESS OF DISCONTINUOUS PERIODIC ORBITS IN SECOND ORDER DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT IMPULSES*

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Abstract In this paper, we are concerned with the existence and uniqueness of discontinuous periodic orbits for a class of second order impulsive differential equations with state-dependent jumps. we apply geometric method to estimate the time mapping of the equation, and then by using Poincaré-Bohl fixed point theorem to obtain some existence criteria under assumptions that the nonlinear term satisfies linear growth conditions. And, the uniqueness of the discontinuous periodic orbit is further proved. Finally, several specific impulsive functions are presented in examples to illustrate the obtained results.

Keywords State-dependent impulse, Poincaré-Bohl fixed point theorem, discontinuous periodic orbit, existence, uniqueness.

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1. Introduction

In this paper, we study the existence and uniqueness of discontinuous periodic orbits for the following second order impulsive differential equation with state-dependent jumps

$$\begin{cases} x'' + g(x) = p(t, x, x'), & x \neq 0, \\ \Delta x = 0, & \\ \Delta x' = J(x, x'), & x = 0, \end{cases} \tag{1.1}$$

where $g \in C(\mathbb{R}, \mathbb{R})$, $p : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, bounded, τ-periodic in the first variable and $\tau > 0$ is the minimum period, $\Delta x, \Delta x'$ are called as impulsive jumps. Note that (1.1) is a model of dynamical systems with impulsive effects, the continuous case is derived from a family of impact oscillators. See [21] for example, it is clear that the model is a particular case of system (1.1).

System (1.1) is included in a family of second order differential equations with

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state-dependent impulses given by

$$\begin{cases} x'' + g(x) = p(t, x, x'), & (x, x') \notin \Lambda, \\ \Delta x = I(x, x'), \\ \Delta x' = J(x, x'), & (x, x') \in \Lambda, \end{cases}$$

where $I, J \in C(\Lambda, \mathbb{R})$ and Λ is a set in $\mathbb{R}^2$. This type of differential equations depending state jumps is a good mathematical model. It is often applied to optimal management and accurate control in biological models, see [27–29] for example and the references therein. Moreover, it is used to define a class of semicontinuous dynamical systems/impulsive semidynamical systems, for more information see [2, 4, 6, 14] and the references therein.

When $p = p(t)$, system (1.1) without impulses is the Duffing differential equation

$$x'' + g(x) = p(t). \tag{1.2}$$

It is a simple mathematical-physical equation model, which possesses many important applications in mechanical and electronic engineering. For example, it can be used to describe resonance phenomena, (sub)harmonic oscillation and almost/quasi-periodic vibration etc. There is a wide literatures concerning the existence of periodic solutions of second order differential equations, see for example [7–10, 20–22, 24–26] and the references therein.

Impulsive phenomena exist widely in many dynamical systems due to abrupt changes at certain instants during evolution processes. The mathematical description leads to impulsive differential equations. It has attracted the attention of many researchers because it is used to describe population dynamics, biological phenomena and some physical behaviors. And, it can generate some new and particular dynamics or control evolution of processes. There have been a lot of achievements on the theory and applications, see for example [1, 3, 5, 11–13, 15, 17–19, 23] and the references therein. The pervasiveness of impulsive effects, a natural question rises: when (1.2) possesses some type of impulses, how to ensure the existence and the number of periodic solutions for the differential equations with impulsive effects. It is a challenging and interesting topic. Recently, there are a few results on the existence of periodic solutions of the Duffing differential equations with impulses. In [19], Qian et al. considered the following second order impulsive differential equation

$$\begin{cases} x'' + g(x) = p(t, x, x'), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j-), x'(t_j-)), \\ \Delta x'(t_j) = J_j(x(t_j-), x'(t_j-)), & j = \pm 1, \pm 2, \cdots, \end{cases} \tag{1.3}$$

where g satisfies the superlinear growth condition $\lim_{|x|\to+\infty} \frac{g(x)}{x} = +\infty$, and $0 \leq t_1 < \cdots < t_k < 2\pi$, $\Delta x(t_j) = x(t_j+) - x(t_j-), \Delta x'(t_j) = x'(t_j+) - x'(t_j-)$, $I_j, J_j : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous mappings and $t_{j+k} = t_j + 2\pi, I_{j+k} = I_j, J_{j+k} = J_j$ for $j = \pm 1, \pm 2, \cdots$. By a geometric approach and Poincaré-Birkhoff twist theorem, the authors proved the existence of infinitely many periodic solutions of (1.3) when $p = p(t)$, and the existence of periodic solutions for the non-conservative case by developing a twist fixed point theorem. In [17], by applying the same way as [10] Niu and Li also studied the existence of infinitely many periodic solutions for a class of semilinear impulsive Duffing equations

$$\begin{cases} x'' + g(x) = p(t), & t \neq t_j, \\ x(t_j+) = I(x(t_j-), x'(t_j-)), \\ x'(t_j+) = J(x(t_j-), x'(t_j-)), & j \in \mathbb{Z}, \end{cases}$$

where $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfying $M \leq g'(x) \leq K$ with M, K being two positive constants, $p \in C(\mathbb{R}, \mathbb{R})$ with $p(t) = p(t+2\pi)$, $I, J : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous mappings, and $0 \leq t_1 < 2\pi$, $t_{j+1} = t_j + 2\pi$ for $j \in \mathbb{Z}$. As everyone knows, semilinear Duffing equations can be resonant and then may have no periodic solution. Hence the existence problem of periodic solutions challenges more attention for the resonance phenomena. In addition, we observe that the impulses involved in the above papers are impulses at fixed time. Compared with the differential equations with impulses of fixed time, the study of state-dependent impulses is more difficult due to the uncertainty of collision time between solution mappings and collision surfaces. And, the number of works devoted to the state-dependent impulses is comparatively low. In this paper, we are concerned with the problem of discontinuous periodic orbits of (1.1). It can be written as the following planar impulsive system

$$\begin{cases} x' = y, \\ y' = -g(x) + p(t, x, y), & x \neq 0, \\ \Delta x = 0, \quad \Delta y = J(x, y), & x = 0. \end{cases} \tag{1.4}$$

In (1.4), we assume that $q_1 \leq \frac{g(x)}{x} \leq p_1$ for $x \leq -A_1$, $q_2 \leq \frac{g(x)}{x} \leq p_2$ for $x \geq A_1$, $A_1 > 0$ is a constant, and $J : \{0\} \times \mathbb{R} \to \mathbb{R}$ is a continuous and bounded impulsive function. In what follows, we denote by $J(y) = J(x, y)$ for $x = 0$. From the theoretical point of view, we strive to estimate the time mapping of (1.4). Then, under assumptions that the abstract function g satisfies various linear growth conditions, by applying Poincaré-Bohl fixed point theorem to obtain the existence criteria of periodic orbits with impulsive jumps (*i.e.* discontinuous periodic orbits defined in Definition 2.1). And, the uniqueness of the discontinuous periodic orbit is further presented.

This paper is organized as follows. In section 2, we present some preliminaries on planar impulsive differential systems with state-dependent jumps. In section 3, we state and prove the main results. First, by geometric analysis we estimate the time mapping of the impulsive differential system. Then, by Poincaré-Bohl fixed point theorem (see Lemma 3.1), we obtain some existence criteria of discontinuous periodic orbits. And the uniqueness is further proved. In section 4, several specific impulsive functions are given in examples to illustrate the obtained results. Concluding remarks are outlined in section 5.

2. Preliminaries

Consider the following planar differential system with state-dependent impulses

$$\begin{cases} x' = y, \\ y' = -g(x) + p(t, x, y), & x \neq 0, \\ \Delta x = 0, \quad \Delta y = J(y), & x = 0, \end{cases} \tag{2.1}$$

and the initial condition

$$x(0^+) = x_0, \qquad y(0^+) = y_0. \tag{2.2}$$

In this paper, we assume that there exists a sequence of impact times $\{t_k\}, k \in \mathbb{Z}_+$ with collision surface, satisfying $0 < t_k < t_{k+1} \uparrow +\infty$ such that a solution of the initial value problem (2.1)-(2.2) intersects with $\{x = 0\}$ at $t = t_k, k \in \mathbb{Z}_+$, and there is not accumulation point. Then $x(t_k) = 0$, $\Delta x = x(t_k^+) - x(t_k) = 0$ and $\Delta y = y(t_k^+) - y(t_k), k \in \mathbb{Z}_+$. Let $\Phi : (0, y(t_k)) \to (0, y(t_k^+))$ denote an impulsive mapping. Moreover, we further assume the following hypotheses for (2.1).

$(H1)$ $g(x)$ is Lipschitz continuous satisfying

$$(g_0): \lim_{|x|\to+\infty} g(x)\text{sign}(x) = +\infty.$$

$(H2)$ $J(y)$ is continuous, bounded and it satisfies $yJ(y) > 0$ for $y \neq 0$.

$(H3)$ There exist a constant $A_1 > 0$ and an integer $m > 0$ such that

$$q_1 \leq \frac{g(x)}{x} \leq p_1, \quad x \leq -A_1, \qquad q_2 \leq \frac{g(x)}{x} \leq p_2, \quad x \geq A_1, \tag{2.3}$$

with

$$\frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} = \frac{\tau}{m\pi}, \qquad \frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} = \frac{\tau}{(m+1)\pi}, \tag{2.4}$$

where p_1, q_1, p_2, q_2 are positive constants.

$(H4)$ There exist constants $A_2 > 0$ and $B > \frac{\max(1,p_1^2,p_2^2)}{\min(1,q_1^2,q_2^2)}(1 + \frac{J}{R_0\gamma})\gamma^2 E\pi$ such that

(i) $g(x) - p_1 x \geq B, \quad x \leq -A_2;$ or $g(x) - p_2 x \leq -B, \quad x \geq A_2,$

(ii) $g(x) - q_1 x \leq -B, \quad x \leq -A_2;$ or $g(x) - q_2 x \geq B, \quad x \geq A_2,$

where $E = \max_{[0,\tau]\times\mathbb{R}\times\mathbb{R}} |p(t, x, y)|, J = \max_{y\in\mathbb{R}} |J(y)|$, and R_0, γ are from Lemmas 3.2 and 3.3 in the section 3 respectively.

By $(H1)$, it is easy to show the existence and uniqueness of solutions of the initial value problem (2.1)-(2.2). And it is defined on the whole t-axis. Let $\varphi(P_0, t)$ denote a solution mapping of (2.1)-(2.2), which starts from the initial point $P_0(x_0, y_0)$ at $t = 0$. By the similar way to [11], we next describe how to decide $\varphi(P_0, t)$ for $t > 0$.

When $t \in (0, t_1]$, (2.1)-(2.2) are identical to the following initial value problem of an ordinary differential equation

$$\begin{cases} x' = y, \\ y' = -g(x) + p(t, x, y), \\ \varphi(P_0, 0^+) = P_0. \end{cases}$$

Hence there exists a unique solution denoted by $\varphi_1(P_0, t), 0 < t \leq t_1$. Further, $\varphi(P_0, t) = \varphi_1(P_0, t)$ for $t \in [0, t_1]$ and

$$\varphi(P_0, t_1^+) = \Phi(\varphi_1(P_0, t_1)) = P_1^+.$$

Similarly, for $t \in (t_1, t_2]$ then (2.1)-(2.2) are identical to the initial value problem

$$\begin{cases} x' = y, \\ y' = -g(x) + p(t, x, y), \\ \varphi(P_0, t_1^+) = P_1^+. \end{cases}$$

Hence there exists a unique solution denoted by $\varphi_2(P_1^+, t), t_1 < t \le t_2$ such that $\varphi(P_0, t) = \varphi_2(P_1^+, t)$ for $t \in (t_1, t_2]$ and

$$\varphi(P_0, t_2^+) = \Phi(\varphi_2(P_1^+, t_2)) = P_2^+.$$

By induction, we obtain the unique solution $\varphi(P_0, t)$ of (2.1)-(2.2), and it is defined for all $t > 0$ due to (g_0). The solution mapping $\varphi(P_0, t)$ for $t \ge 0$ can be written as

$$\varphi(P_0, t) = \begin{cases} \varphi_1(P_0, t), & 0 \le t \le t_1, \\ \varphi_2(P_1^+, t), & t_1 < t \le t_2, \\ \quad \vdots & \\ \varphi_n(P_{n-1}^+, t), & t_{n-1} < t \le t_n, \\ \quad \vdots & \end{cases}$$

where $P_n^+ = \Phi(\varphi_n(P_{n-1}^+, t_n))$. The case of $t < 0$ is similar and so omitted.

Now, let $\varphi(P, t)$ be the solution mapping of (2.1)-(2.2) with the initial point P, L_P^+ is the positive orbit starting from P at $t = 0$, and denote by

$$\Lambda^+ = \{x = 0,\ y > 0\}, \qquad \Lambda^- = \{x = 0,\ y < 0\};$$
$$\Sigma^+ = \{(x, y) : x > 0\}, \qquad \Sigma^- = \{(x, y) : x < 0\}.$$

In what follows, we consider a positive orbit L_P^+ of (2.1) starting from $P(0, y) \in \Lambda^+$ at $t = 0$. By the direction of the vector field, L_P^+ enters into Σ^+ for $t > 0$ small. Next, from $(H1), (H3)$ and $(H4)$ we assume that L_P^+ moves in a clockwise fashion in Σ^+, and there exists a moment $T_1 > 0$ such that L_P^+ then intersects with Λ^- for the first time at $t = T_1$. Note that $(0, T_1)$ corresponds the time of $\varphi(P, t)$ running in Σ^+, and denote by $\varphi(P, T_1) = P_1(0, y_0)$ with $y_0 < 0$. Later, when Φ is applied, by $(H2)$ and the direction of the vector field, L_P^+ starting from $\Phi(P_1) \in \Lambda^-$ continuous to enter into Σ^-. Similarly, we assume that L_P^+ moves forward in a clockwise fashion in Σ^-, and there exists another moment $T_2 (> T_1)$ such that L_P^+ then comes back to Λ^+ for the first time at $t = T_2$. Further, $\varphi(\Phi(P_1), T_2) \in \Lambda^+$ and $\Phi(\varphi(\Phi(P_1), T_2)) \in \Lambda^+$.

Based on the above assumptions, and inspired by relevant definitions in [4,12,16], we make the following definitions.

Definition 2.1. For a given $P \in \Lambda$, if there exist T_1 and T_2 with $T_2 > T_1 > 0$ such that $\Phi(\varphi(\Phi(\varphi(P, T_1)), T_2)) = P$, we call the solution mapping $\varphi(P, [0, T_2])$ a discontinuous periodic solution with period T_2. And the corresponding orbit is called as a discontinuous periodic orbit.

Definition 2.2. For a given $(0, y) \in \Lambda^+$, let $(0, -p_R(y))$ be the first intersection point of the orbit starting from $(0, y)$ with Λ^-. Then we call $P_R : (0, y) \to (0, -p_R(y))$ a *right Poincaré mapping*. Similarly, for a given $(0, -y) \in \Lambda^-$, let $(0, p_L(y))$ be the first intersection point of the orbit starting from $(0, -y)$ with Λ^+. Then $P_L : (0, -y) \to (0, p_L(y))$ is called as a *left Poincaré mapping*.

By the continuous dependence of solutions on initial values, P_L and P_R are continuous. Definition 2.2 shows that an orbit of (2.1) which starts from $(0, y) \in \Lambda^+$/or $(0, -y) \in \Lambda^-$, then P_R/or P_L maps it to $(0, -p_R(y)) \in \Lambda^-$/or $(0, p_L(y)) \in \Lambda^+$. Later, Φ is applied such that the orbit occurs a jump on Λ satisfying $\Phi((0, -p_R(y))) =$

$(0, -p_R(y) + J(-p_R(y)))$/or $\Phi((0, p_L(y))) = (0, p_L(y) + J(p_L(y)))$. Further, the solution mapping rotates a circle on the phase plane be of the form

$$\Phi \circ P_L \circ \Phi \circ P_R \quad \text{or} \quad \Phi \circ P_R \circ \Phi \circ P_L. \tag{2.5}$$

See Figure 1 for example: a discontinuous periodic orbit of rotating a circle on the (x, y)-plane. Note that Φ is continuous, so by the continuity of function of functions, the solution mapping $\varphi(P, t)$ is continuous for $P \in \mathbb{R}^2$ and $t \in [0, \tau]$.

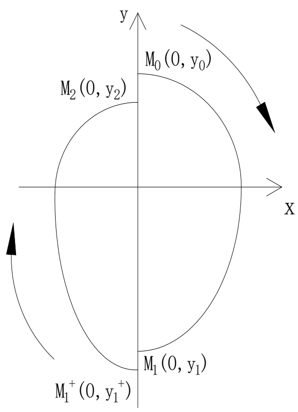

Figure 1. A discontinuous periodic orbit which starts from the point M_0 and rotates clockwise a circle on the (x, y)-plane: $M_0(0, y_0) \xrightarrow{P_R} M_1(0, y_1) \xrightarrow{\Phi} M_1^+(0, y_1^+) \xrightarrow{P_L} M_2(0, y_2) \xrightarrow{\Phi} M_0(0, y_0)$, where $y_1 = -p_R(y_0)$, $\Phi(M_1) = M_1^+$, $y_1^+ = -p_R(y_0) + J(-p_R(y_0))$, $y_2 = p_L(p_R(y_0) - J(-p_R(y_0)))$, $\Phi(M_2) = M_0$.

3. Main results

In this section, by estimating the time of a solution mapping of (2.1)-(2.2) rotating a circle on the phase plane and applying the Poincaré-Bohl fixed point theorem, we obtain the existence of discontinuous periodic orbits. And the uniqueness is followed proved. Moreover, we present several existence criteria under the assumptions of g satisfying linear growth conditions.

First, we recall an existence result of periodic solutions from the Poincaré-Bohl fixed point theorem.

Lemma 3.1 ([7]). *Suppose that $\mathcal{F} : \mathbb{D} \to \mathbb{R}^2$ is a continuous mapping, where $\mathbb{D} \subset \mathbb{R}^2$ is a bounded closed region including the origin O as an interior point, and the boundary $\partial\mathbb{D}$ is star-shaped about the origin. If for every $p \in \partial\mathbb{D}$, the image $q = \mathcal{F}(p)$ satisfies $\overrightarrow{Oq} \neq \lambda\overrightarrow{Op}$, where $\lambda > 0$ is a constant. Then $\mathcal{F}$ has at least one fixed point in $\mathbb{D}$.*

It is sufficient to prove that for any solution mapping of (2.1) which starts from the point $p = (x(0), y(0)) \in \partial\mathbb{D}$, when it moves to $q = (x(\tau), y(\tau)) \in \mathbb{R}^2$, then the points p and q are not on the same ray starting from the origin.

Making the polar coordinates transformation

$$x(t) = r(t)\cos\theta(t), \qquad y(t) = r(t)\sin\theta(t), \tag{3.1}$$

then the resulting equation of (2.1) for $r(t) > 0$ is of the form

$$\begin{cases} \dfrac{dr(t)}{dt} = r\cos\theta\sin\theta + [p(t,x,y) - g(r\cos\theta)]\sin\theta, \\ \dfrac{d\theta(t)}{dt} = -\sin^2\theta + \dfrac{1}{r}[p(t,x,y) - g(r\cos\theta)]\cos\theta, \qquad t \neq t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{3.2}$$

and $r(t_k^+) = |y(t_k^+)|,\ r(t_k) = |y(t_k)|$ and $\theta(t_k^+) = \theta(t_k) = \frac{\pi}{2} + n\pi$ for $k \in \mathbb{Z}_+,\ n \in \mathbb{Z}$.

Let $r(t) = r(t; r_0, \theta_0), \theta(t) = \theta(t; r_0, \theta_0)$ be the solution of (3.2) with $(r(0^+), \theta(0^+)) = (r_0, \theta_0)$. Note that (r_0, θ_0) can be chosen such that $r(t; r_0, \theta_0) > 0$ for $t \in [0, \tau]$, and then $\theta(t; r_0, \theta_0)$ is well defined.

For a given $R_0 > 0$, denote by $\mathbb{S}_{R_0} = \{(r, \theta) : r = R_0\}$ and $\mathbb{B}_{R_0} = \{(r, \theta) : r > R_0\}$. Assume that $(r(t), \theta(t))$ is a solution of (3.2) satisfying $r(0^+) = R_0, \theta(0^+) = \theta_0$ (where θ_0 is arbitrary), and let $L(R_0)$ denote the corresponding orbit starting from (R_0, θ_0). For convenience, we further assume that there exist $0 < t_1 < t_2 < \cdots < t_q \leq \tau$ satisfying $x(t_k) = 0, k = 1, 2, \cdots, q$. It is easy to prove the following results.

Lemma 3.2. *Assume that $(H1)$-$(H3)$ hold, then there exists $R_0(> 0)$ sufficiently large such that*

$$\frac{d\theta}{dt}(t; R_0, \theta_0) < 0, \qquad t \in [0, \tau]\backslash\{t_k\}, k = 1, 2, \cdots, q.$$

Lemma 3.3. *Assume that $(H1)$-$(H3)$ hold, then there exists $R_1(> 0)$ such that for $r_0 = R_0 \geq R_1$*

$$R_0/\gamma < r(t) < R_0\gamma + J, \qquad t \in [0, \tau], \tag{3.3}$$

where R_0 is given in Lemma 3.2, J comes from $(H4)$ and $\gamma > 0$ is a constant.

Proof. By $(H3)$, $\overline{\lim}_{|x|\to+\infty} |\frac{g(x)}{x}| = b < +\infty$. Further, there exists $A > 0$ such that $|\frac{g(x)}{x}| < b + 1$ for $|x| \geq A$. Let $r(t) > 2\delta$, where $\delta = E + \max_{|x|\leq A} |g(x)|$ and $E = \max_{[0,\tau]\times\mathbb{R}\times\mathbb{R}} |p(t, x, y)|$. When $t \in [0, \tau]\backslash\{t_k\},\ k = 1, 2, \cdots, q$, by the first equality of (3.2) one has that

$$\Big|\frac{dr(t)}{dt}\Big| < \frac{1}{2}(b + 3)r(t).$$

Choosing $R_1 = 2\delta\gamma$ with $\gamma = e^{(b+3)\pi}$, when $R_0 \geq R_1$ one has that

$$R_0/\gamma < r(t) < R_0\gamma, \qquad t \in [0, \tau]\backslash\{t_k\}. \tag{3.4}$$

While for $t = t_k, k = 1, 2, \cdots, q$, by $yJ(y) > 0$ for $y \neq 0$ it follows that

$$R_0/\gamma < r(t_k) < r(t_k^+) < r(t_k) + J < R_0\gamma + J.$$

Hence (3.3) holds. The proof is complete. □

Now we are ready to state and prove one of the main results.

Theorem 3.1. *Assume that $(H1)$-$(H4)$ hold, then (2.1) has at least one discontinuous periodic orbit.*

Proof. By $(H3)$, there exists $R_1' > 0$ such that $\theta'(t) < 0$ for $t \in [0, \tau]\backslash\{t_k\}, k = 1, 2, \cdots, q$. By Lemma 3.3, there exists $R_1 > 0$ such that for any $R_0 \geq R_1$ the orbit $L(R_0)$ is located in $\mathcal{D} = \{R_0/\gamma \leq r(t) \leq R_0\gamma + J\}$. Let $R_0 > \max(\gamma R_1', R_1, \gamma A)$, where $\gamma = e^{(b+3)\pi}$ and $A \geq \max(A_1, A_2)$. In the following, we estimate the time of $L(R_0)$ rotating a circle on the (x, y)-plane.

Choosing $M_0(R_0, \theta_0) \in \mathcal{D}$ with $M_0 \in \{(x, y) : x > A\}$, and considering $L(R_0)$ which starts from M_0 at $t = 0$. Assume that $L(R_0)$ intersects with $\{x = A\}, \Lambda^-$ successively at M_1, M_2, Φ is applied such that $\Phi(M_2) = M_2^+ \in \Lambda^-$. Later, it starting from M_2^+ enters into Σ^-, intersects with $\{x = -A\}, \{x = -A\}, \Lambda^+$ successively at M_3, M_4, M_5, and Φ maps M_5 to $M_5^+ \in \Lambda^+$. Next, $L(R_0)$ continues to enter into Σ^+ and intersect with $\{x = A\}, \{\theta = \theta_0\}$ at M_6, M_7 successively (see a schematic diagram in Figure 2). Denote the corresponding moments and arguments by $\theta_1 = \theta(\widetilde{t}_1), \theta_2 = \theta(\widetilde{t}_2), \theta_3 = \theta(\widetilde{t}_3), \theta_4 = \theta(\widetilde{t}_4), \theta_5 = \theta(\widetilde{t}_5), \theta_6 = \theta(\widetilde{t}_6), \theta_7 = \theta(\widetilde{t}_7)$. Moreover, we let

$$g(x) = \begin{cases} p_1 x + f_1(x), & x < 0, \\ p_2 x + f_2(x), & x \geq 0; \end{cases} \qquad g(x) = \begin{cases} q_1 x + h_1(x), & x < 0, \\ q_2 x + h_2(x), & x \geq 0. \end{cases} \tag{3.5}$$

Then it follows from $(H4)$ that

(i) $f_1(x) \geq B, \quad x \leq -A;$ or $f_2(x) \leq -B, \quad x \geq A.$

(ii) $h_1(x) \leq -B \quad x \leq -A;$ or $h_2(x) \geq B, \quad x \geq A.$

Assume that

$$f_2(x) \leq -B, \qquad h_2(x) \geq B, \qquad x \geq A. \tag{3.6}$$

By (3.2), (3.5)-(3.6) and Lemma 3.3, one has that

$$\begin{aligned} -\theta' &= \sin^2\theta + \frac{1}{r}[g(r\cos\theta) - p(t, r\cos\theta, r\sin\theta)]\cos\theta \\ &= \sin^2\theta + p_2\cos^2\theta + \frac{1}{r}[f_2(r\cos\theta) - p(t, r\cos\theta, r\sin\theta)]\cos\theta \\ &\leq \sin^2\theta + p_2\cos^2\theta - \frac{B}{R_0\gamma + J}\cos\theta + \frac{\gamma E}{R_0}. \end{aligned}$$

Further, it follows that

$$\begin{aligned} \widetilde{t}_1 = \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} &\geq \int_{\theta_1}^{\theta_0} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{R_0\gamma+J}\cos\theta + \frac{\gamma E}{R_0}} \\ &\geq \int_{\theta_1}^{\theta_0} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{R_0\gamma+J}\cos\theta + \frac{\gamma E}{R_0}}. \end{aligned}$$

Similarly,

$$\widetilde{t}_7 - \widetilde{t}_6 \geq \int_{\theta_7}^{\theta_6} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{R_0\gamma+J}\cos\theta + \frac{\gamma E}{R_0}}.$$

Note that $\theta_7 + 2\pi = \theta_0$, so

$$
\begin{aligned}
\widetilde{t}_1 + \widetilde{t}_7 - \widetilde{t}_6 &\geq \int_{\theta_1}^{\theta_6+2\pi} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{R_0\gamma+J}\cos\theta + \frac{\gamma E}{R_0}} \\
&= \int_{\theta_1}^{\theta_6+2\pi} \Big[\frac{1}{\sin^2\theta + p_2\cos^2\theta} + \frac{\frac{B\cos\theta}{R_0\gamma+J} - \frac{\gamma E}{R_0}}{(\sin^2\theta + p_2\cos^2\theta)^2}\Big] d\theta + \Delta(R_0) \\
&\geq \int_{\theta_1}^{\theta_6+2\pi} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta} + \frac{B}{\max(1,p_2^2)(R_0\gamma+J)} \int_{-\frac{\pi}{2}+\psi}^{\frac{\pi}{2}-\psi} \cos\theta d\theta \\
&\quad - \frac{\gamma E\pi}{\min(1,p_2^2)R_0} + \Delta(R_0) \\
&= \int_{\theta_1}^{\theta_6+2\pi} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta} + \frac{2B}{\max(1,p_2^2)(R_0\gamma+J)} \\
&\quad - \frac{\gamma E\pi}{\min(1,p_2^2)R_0} + \Delta(R_0) + o\Big(\frac{1}{R_0}\Big),
\end{aligned}
$$

where

$$
\Delta(R_0) = \int_{\theta_1}^{\theta_6+2\pi} \frac{\Big(\frac{B\cos\theta}{R_0\gamma+J} - \frac{\gamma E}{R_0}\Big)^2 d\theta}{(\sin^2\theta + p_2\cos^2\theta)^2(\sin^2\theta + p_2\cos^2\theta - \frac{B\cos\theta}{R_0\gamma+J} + \frac{\gamma E}{R_0})},
$$

ψ is an argument (see Figure 2) satisfying

$$
\psi = \alpha(R_0) = \arcsin\frac{\gamma A}{R_0} = \frac{\gamma A}{R_0} + o(\frac{1}{R_0^2}),
$$

and $o(\cdot)$ is infinitesimal of higher order. Since

$$
|\Delta(R_0)| \leq \frac{\pi}{R_0^2} \times \frac{\Big(\frac{B}{M\gamma} + \frac{\gamma E}{M}\Big)^2}{\min(1,p_2^2)(\min(1,p_2) - \frac{B}{R_0\gamma+J})},
$$

it follows that $\Delta(R_0) = o\Big(\frac{1}{R_0}\Big)$.

Similarly, by $f_1(x) = (\frac{g(x)}{x} - p_1)x \geq 0$ for $x \leq -A$, it is easy to obtain that

$$
-\theta' \leq \sin^2\theta + p_1\cos^2\theta + \frac{\gamma E}{R_0}.
$$

Further, one has that

$$
\widetilde{t}_4 - \widetilde{t}_3 \geq \int_{\theta_4}^{\theta_3} \frac{d\theta}{\sin^2\theta + p_1\cos^2\theta} - \frac{\gamma E\pi}{\min(1,p_1^2)R_0} + o\Big(\frac{1}{R_0}\Big).
$$

On the other hand, let $f_1 = \max_{-A\leq x\leq 0}|f_1(x)|$, $f_2 = \max_{0\leq x\leq A}|f_2(x)|$. Then

$$
\begin{aligned}
-\theta' &= \sin^2\theta + p_2\cos^2\theta + \frac{1}{r}[f_2(r\cos\theta) - p(t, r\cos\theta, r\sin\theta)]\cos\theta \\
&\leq \sin^2\theta + p_2\cos^2\theta + \frac{\gamma(f_2+E)}{R_0}, \qquad 0 < x < A,
\end{aligned}
$$

and $-\theta' \le \sin^2\theta + p_1\cos^2\theta + \frac{\gamma(f_1+E)}{R_0}$ for $-A < x < 0$. Hence one has that

$$\begin{aligned}
\widetilde{t}_3 - \widetilde{t}_1 + \widetilde{t}_6 - \widetilde{t}_4 =& \Big(\int_{\theta_3}^{-\frac{\pi}{2}} + \int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{-\frac{\pi}{2}}^{\theta_1} + \int_{\theta_6}^{-\frac{3\pi}{2}}\Big)\frac{d\theta}{-\theta'} \\
\ge& \Big(\int_{-\frac{\pi}{2}}^{\theta_1} + \int_{\theta_6}^{-\frac{3\pi}{2}}\Big)\frac{d\theta}{\sin^2\theta + p_2\cos^2\theta + \frac{\gamma(f_2+E)}{R_0}} \\
&+ \Big(\int_{\theta_3}^{-\frac{\pi}{2}} + \int_{-\frac{3\pi}{2}}^{\theta_4}\Big)\frac{d\theta}{\sin^2\theta + p_1\cos^2\theta + \frac{\gamma(f_1+E)}{R_0}} \\
=& \Big(\int_{-\frac{\pi}{2}}^{\theta_1} + \int_{\theta_6}^{-\frac{3\pi}{2}}\Big)\frac{d\theta}{\sin^2\theta + p_2\cos^2\theta} \\
&+ \Big(\int_{\theta_3}^{-\frac{\pi}{2}} + \int_{-\frac{3\pi}{2}}^{\theta_4}\Big)\frac{d\theta}{\sin^2\theta + p_1\cos^2\theta} + o\Big(\frac{1}{R_0}\Big).
\end{aligned}$$

Combing with the above several inequalities and (2.4), then the time T of $L(R_0)$ rotating a circle on the (x, y)-plane satisfies

$$\begin{aligned}
T \ge& \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_1\cos^2\theta} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta} \\
&+ \frac{2B}{\max(1, p_2^2)(R_0\gamma + J)} - \frac{\gamma E\pi}{\min(1, p_1^2)R_0} - \frac{\gamma E\pi}{\min(1, p_2^2)R_0} + o\Big(\frac{1}{R_0}\Big) \\
=& \Big(\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}}\Big)\pi + \frac{2B}{\max(1, p_2^2)(R_0\gamma + J)} - \frac{\gamma E\pi}{\min(1, p_1^2)R_0} \\
&- \frac{\gamma E\pi}{\min(1, p_2^2)R_0} + o\Big(\frac{1}{R_0}\Big) \\
=& \frac{\tau}{m+1} + \frac{2B}{\max(1, p_2^2)(R_0\gamma + J)} - \frac{\gamma E\pi}{\min(1, p_1^2)R_0} - \frac{\gamma E\pi}{\min(1, p_2^2)R_0} + o\Big(\frac{1}{R_0}\Big).
\end{aligned} \tag{3.7}$$

By the similar arguments, it follows that

$$T \le \frac{\tau}{m} - \Big[\frac{2B}{\max(1, q_2^2)(R_0\gamma + J)} - \frac{\gamma E\pi}{\min(1, q_1^2)R_0} - \frac{\gamma E\pi}{\min(1, q_2^2)R_0}\Big] + o\Big(\frac{1}{R_0}\Big). \tag{3.8}$$

Due to $p_1 \ge q_1, p_2 \ge q_2$ and $B > \frac{\max(1, p_1^2, p_2^2)(1+\frac{J}{R_0\gamma})}{\min(1, q_1^2, q_2^2)}\gamma^2 E\pi$, it follows from $(H4)$ that

$$\begin{aligned}
&\frac{2B}{\max(1, p_2^2)(R_0\gamma + J)} - \frac{\gamma E\pi}{\min(1, p_1^2)R_0} - \frac{\gamma E\pi}{\min(1, p_2^2)R_0} > 0, \\
&\frac{2B}{\max(1, q_2^2)(R_0\gamma + J)} - \frac{\gamma E\pi}{\min(1, q_1^2)R_0} - \frac{\gamma E\pi}{\min(1, q_2^2)R_0} > 0.
\end{aligned} \tag{3.9}$$

Further, by (3.7)-(3.9) when $R_0 > 0$ is sufficiently large we have that

$$\frac{\tau}{m+1} < T < \frac{\tau}{m}. \tag{3.10}$$

This implies that the number of rotation of $L(R_0)$ in $[0, \tau]$ is greater than m but less than $m + 1$. Moreover, it follows from $(H1)$ and (2.5) that the Poincaré mapping $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$

$$(R_0, \theta_0) \to (r(\tau; R_0, \theta_0), \theta(\tau; R_0, \theta_0))$$

is well defined, and it is continuous. Since the solution $(r(t; R_0, \theta_0), \theta(t; R_0, \theta_0))$ of (3.2) is τ-periodic, which is equivalent to the initial point (R_0, θ_0) being the fixed point of $\mathcal{P}$. Therefore, by Lemma 3.1, (2.1) has at least one discontinuous periodic orbit with period τ. The proof is complete. □

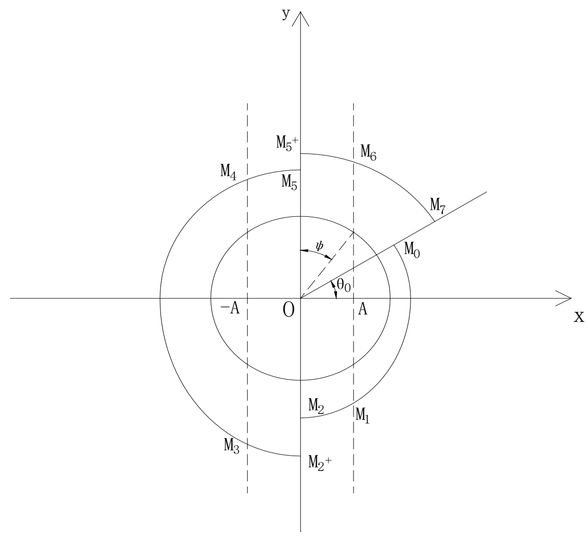

Figure 2. Schematic diagram

Next, we further show the uniqueness of the discontinuous periodic orbit.

Theorem 3.2. *Suppose that* $l \le \frac{g(x)-g(y)}{x-y} \le L$ *for* $x, y \in \mathbb{R}, x \ne y$. *Then system* (2.1) *with* $p(t, x, x') = p(t)$ *has at most one discontinuous periodic orbit, where* $\frac{2n\pi}{\tau} \le \min\{l, 1\} < \max\{L, 1\} \le \frac{2(n+1)\pi}{\tau}$ *with* $2n\pi < \tau \le 2(n+1)\pi$ *for* $n \in \mathbb{N}$.

Proof. Assume that (2.1) has two discontinuous periodic solutions with period τ, *i.e.* $x = x_1(t)$ and $x = x_2(t)$. And, denote by $\bar{t}_1, \bar{t}_2, \cdots, \bar{t}_{2k} \in (0, \tau]$ such that $x_i(\bar{t}_j) = 0$ for $i = 1, 2$ and $j = 1, 2, \cdots, 2k$. Let

$$u(t) = x_1(t) - x_2(t), \qquad v(t) = x_1'(t) - x_2'(t).$$

When $t \ne \bar{t}_j, j = 1, 2, \cdots, 2k$, one has that

$$\begin{cases} \dfrac{du(t)}{dt} = v(t), \\ \dfrac{dv(t)}{dt} = -g(x_1(t)) + g(x_2(t)). \end{cases} \tag{3.11}$$

Taking $u(t) = \rho(t)\cos\varphi(t), v(t) = \rho(t)\sin\varphi(t)$, then for $\rho(t) > 0$

$$\frac{d\varphi(t)}{dt} = -\sin^2\varphi(t) + \frac{1}{\rho(t)}[g(x_2(t)) - g(x_1(t))]\cos\varphi(t).$$

Further, it follows from the condition $l \le \frac{g(x)-g(y)}{x-y} \le L$ that

$$-\max\{L, 1\} < \varphi'(t) = -\sin^2\varphi - \frac{g(x_1) - g(x_2)}{x_1 - x_2}\cos^2\varphi < -\min\{l, 1\}.$$

By integrating the above inequality from 0 to τ, we have that

$$-\max\{L,1\}\tau < \varphi(\tau)-\varphi(0) = \int_0^\tau \varphi'(t)dt < -\min\{l,1\}\tau$$

due to $\varphi(\bar{t}_j^+) = \varphi(\bar{t}_j), j = 1,2,\cdots,2k$. Further, $-2(n+1)\pi < \varphi(\tau)-\varphi(0) < -2n\pi$ for every $n \in \mathbb{N}$. Therefore, (3.11) has no nontrivial discontinuous periodic solution, and then $u(t) \equiv 0$. The proof is complete. □

Corollary 3.1. *In Theorem 3.1, $(H4)$ is replaced by the following condition*

$$\begin{aligned}&\lim_{x\to-\infty}(g(x)-p_1x) = +\infty \quad \text{or} \quad \lim_{x\to+\infty}(g(x)-p_2x) = -\infty,\\ &\lim_{x\to-\infty}(g(x)-q_1x) = -\infty \quad \text{or} \quad \lim_{x\to+\infty}(g(x)-q_2x) = +\infty.\end{aligned} \tag{3.12}$$

Then (2.1) *has at least one discontinuous periodic orbit.*

$(H3)'$ There exists $A_1 > 0$ such that

$$\frac{g(x)}{x} \le p_1, \quad x \le -A_1, \qquad \frac{g(x)}{x} \le p_2, \quad x \ge A_1,$$

where $p_1 > 0, p_2 > 0$ are constants satisfying $\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} = \frac{\tau}{\pi}$, and

$$\overline{\lim}_{x\to-\infty} g(x) < \underline{E} \le \overline{E} < \underline{\lim}_{x\to+\infty} g(x)$$

with $\overline{E} = \max_{[0,\tau]\times\mathcal{D}} p(t,x,y)$, $\underline{E} = \min_{[0,\tau]\times\mathcal{D}} p(t,x,y)$.

$(H4)'$ There exist $A_2 > 0$ and $B > \frac{\max(1,p_1^2,p_2^2)}{\min(1,q_1^2,q_2^2)}(1+\frac{J}{R_0\gamma})\gamma^2 E\pi$ such that

$$g(x) - p_1x \ge B, \quad x \le -A_2 \quad \text{or} \quad g(x)-p_2x \le -B, \quad x \ge A_2.$$

Theorem 3.3. *Assume that $(H1)$-$(H2)$ and $(H3)'$-$(H4)'$ hold, then* (2.1) *has at least one discontinuous periodic orbit.*

Proof. By the similar analysis to Theorem 3.1, it follows from $(H3)'$-$(H4)'$ that

$$\begin{aligned}T \ge &\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_1\cos^2\theta} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta}\\ &+ \frac{2B}{\max(1,p_2^2)(R_0\gamma+J)} - \frac{\gamma E\pi}{\min(1,p_1^2)R_0} - \frac{\gamma E\pi}{\min(1,p_2^2)R_0} + o\Big(\frac{1}{R_0}\Big) > \tau.\end{aligned}$$

Further, the number of rotation of any solution trajectory which moves in $\mathcal{D}$ during $[0,\tau]$ is less than one. Therefore, by Lemma 3.1 the conclusion holds. □

Corollary 3.2. *In Theorem 3.3, $(H4)'$ is replaced by the following condition*

$$\lim_{x\to-\infty}(g(x)-p_1x) = +\infty \qquad \text{or} \qquad \lim_{x\to+\infty}(g(x)-p_2x) = -\infty.$$

Then (2.1) *has at least one discontinuous periodic orbit.*

Theorem 3.4. *Assume that $(H1)$-$(H2)$ and the following condition hold*

$(H5)$ *There exist* $A_1 > 0$ *and an integer* $m > 0$ *such that*

$$q_1 \le \frac{g(x)}{x} \le p_1, \quad x \le -A_1, \qquad q_2 \le \frac{g(x)}{x} \le p_2, \quad x \ge A_1,$$

with

$$\frac{\tau}{(m+1)\pi} < \frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} < \frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} < \frac{\tau}{m\pi}, \tag{3.13}$$

where p_1, q_1, p_2, q_2 *are positive constants. Then* (2.1) *has at least one discontinuous periodic orbit.*

Proof. Note that there exist positive constants $p_1^*(> p_1), p_2^*(> p_2), q_1^*(< q_1), q_2^*(< q_2)$ such that $\frac{1}{\sqrt{q_1^*}} + \frac{1}{\sqrt{q_2^*}} = \frac{\tau}{m\pi}$ and $\frac{1}{\sqrt{p_1^*}} + \frac{1}{\sqrt{p_2^*}} = \frac{\tau}{(m+1)\pi}$. So $(H3)$ holds for such constants $p_1^*, p_2^*, q_1^*, q_2^*$. Moreover,

$$g(x) - p_2^* x \le (p_2 - p_2^*)x, \quad g(x) - q_2^* x \ge (q_2 - q_2^*)x, \quad x \ge A_1.$$

Further, (3.12) is satisfied. By Corollary 3.1, the conclusion holds. □

Corollary 3.3. *In Theorem 3.4,* $(H5)$ *is replaced by the following condition*

$(H5)'$ *There exists* $A_1 > 0$ *such that*

$$\frac{g(x)}{x} \le p_1, \quad x \le -A_1, \qquad \frac{g(x)}{x} \le p_2, \quad x \ge A_1,$$

where $p_1 > 0$ *and* $p_2 > 0$ *are constants satisfying* $\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} > \frac{\tau}{m}$. *And*

$$\overline{\lim}_{x\to-\infty} g(x) < \underline{E} \le \overline{E} < \underline{\lim}_{x\to+\infty} g(x).$$

Then (2.1) *has at least one discontinuous periodic orbit.*

Proof. Similarly, there exist $p_1^*(> p_1)$ and $p_2^*(> p_2)$ such that $\frac{1}{\sqrt{p_1^*}} + \frac{1}{\sqrt{p_2^*}} = \frac{\tau}{m}$. Moreover, $g(x) - p_2^* x \le (p_2 - p_2^*)x$ for $x \ge A_1$. So $(H3)'$-$(H4)'$ of Theorem 3.3 hold. □

Remark 3.1. When $yJ(y) < 0$ but $y\Phi(y) > 0$ for $y \ne 0$. Then $0 < |y(t_k)| - |J(y(t_k))| < |y(t_k^+)| < |y(t_k)| + J, k \in \mathbb{Z}_+$. Further, by (3.1) and (3.4) one has that

$$R_0/\gamma - J < r(t) < R_0\gamma + J, \qquad t \in [0, \tau].$$

Since Φ is applied such that the solution trajectory occurs jumps in Λ, as long as $R_0/\gamma - J > \varepsilon$ for some $\varepsilon > 0$ (where ε is a constant), then solutions of (2.1) satisfy the elastic property (*i.e.* for any $b > 0$, there exists $r_b > 0$ such that $|P_0| \ge r_b$ implies that $|\varphi(P_0, t)| \ge b$ for $t \in (0, s], s > 0$, where $|P_0|$ denotes the distance of P_0 in $\mathbb{R}^2$). Further, the follow-up work is similar to the case $yJ(y) > 0$ for $y \ne 0$ (see Example 4.1 for example).

Remark 3.2. When $y\Phi(y) < 0$ for $y \ne 0$. Then solutions of (2.1) move only in Σ_+/Σ_-. Further, if there exists $(0, y_0)/(0, -y_0) \in \Lambda$ for $y_0 > 0$ such that

$$(0, y_0) \xrightarrow{P_R} (0, -p_R(y_0)) \xrightarrow{\Phi} (0, y_0)/(0, -y_0) \xrightarrow{P_L} (0, p_L(y_0)) \xrightarrow{\Phi} (0, -y_0),$$

then the orbit starting from $(0, y_0)/(0, -y_0)$ is a discontinuous periodic orbit of (2.1) (see Figure 3 and Example 4.3 for example).

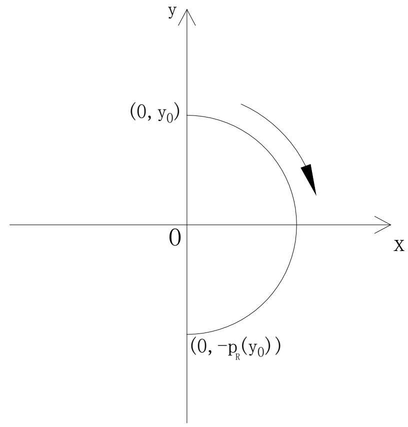

Figure 3. Schematic diagram: $(0, y_0) \xrightarrow{P_R} (0, -p_R(y_0)) \xrightarrow{\Phi} (0, y_0)$.

Remark 3.3. In [17], Niu and Li studied the multiplicity of 2π-periodic solutions of a semilinear impulsive Duffing equation via Poincaré-Birkhoff twist theorem. But, we observe that there is only one impulsive jump in $[0, 2\pi]$, the impulsive effects are impulses at fixed time, and it is required that $g \in C^1$ satisfying $M \le g'(x) \le K$. In this paper, we study a second order semilinear differential equation with state-dependent impulses. There are a finite number of collision times in $[0, \tau]$. And, we require that g is Lipschitz continuous satisfying $q_i \le \frac{g(x)}{x} \le p_i$ for $|x| \ge A_1, i = 1, 2$. By applying the Poincaré-Bohl fixed point theorem, we obtain some existence criteria of discontinuous periodic orbits under assumptions that g satisfies various linear growth conditions. Further, the uniqueness is also proved under the condition $l \le \frac{g(x)-g(y)}{x-y} \le L$ for $x, y \in \mathbb{R}$ and $x \ne y$. Hence in some sense, our paper generalizes and improves some existing results.

4. Examples

In this section, we give several specific impulsive functions to illustrate the obtained results.

Example 4.1. Consider the following impulsive system

$$\begin{cases} x' = y, & \\ y' = -g(x) + p(t, x, y), & x \ne 0, \\ \Delta x = 0, \quad \Delta y = cy, & x = 0, \end{cases} \tag{4.1}$$

where c is an arbitrary constant, and $g(x)$ is given by

$$g(x) = \begin{cases} \mu_1 x + \nu, & x < 0, \\ \mu_2 x + \nu, & x \ge 0, \end{cases} \tag{4.2}$$

with $\mu_1 > 0, \mu_2 > 0$ and ν being constants.

Theorem 4.1. *Assume that* $\frac{1}{\sqrt{\mu_1}} + \frac{1}{\sqrt{\mu_2}} \ne \frac{\tau}{m\pi}$ *for an integer* $m > 0$, *then* (4.1) *with* (4.2) *has at least one discontinuous periodic orbit with period* τ.

Proof. Choosing $p_1 = q_1 = \mu_1$ and $p_2 = q_2 = \mu_2$, it follows that there must be an integer $m > 0$ such that $\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} > \frac{\tau}{m}$ or

$$\frac{\tau}{(m+1)\pi} < \frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} = \frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} < \frac{\tau}{m\pi}.$$

Let $g^*(x) = g(x) - \nu$ and $p^*(t,x,y) = p(t,x,y) - \nu$, when $A_1 > 0$ is large sufficiently, all conditions of Theorem 3.4 and Corollary 3.3 hold.

Moreover, for $J(y) = cy$ we consider the following three possible cases.

(i) When $c > 0$, *i.e.* $yJ(y) > 0$ for $y \neq 0$.

Since $|y^+| > |y|$, it follows from (3.4) that $R_0/\gamma < r(t_k) < r(t_k^+) < (1+c)R_0\gamma$ for $k = 1, 2, \cdots, q$. Further, one has that

$$R_0/\gamma < r(t) < (1+c)R_0\gamma, \qquad t \in [0, \tau].$$

(ii) When $-1 < c < 0$, *i.e.* $yJ(y) < 0$ but $y\Phi(y) > 0$ for $y \neq 0$.

Since $|y^+| < |y|$, from (3.4) then $0 < (1+c)R_0/\gamma < r(t_k^+) < r(t_k) < R_0\gamma,\ k = 1, 2, \cdots, q$. Further

$$(1+c)R_0/\gamma < r(t) < R_0\gamma, \qquad t \in [0, \tau].$$

(iii) When $c < -1$, *i.e.* $yJ(y) < 0$ and $y\Phi(y) < 0$ for $y \neq 0$.

It follows from (3.4) that $|1+c|R_0/\gamma < r(t_k^+) < (1+|c|)r(t_k) < (1+|c|)R_0\gamma,\ k = 1, 2, \cdots, q$. So

$$\min\{1, |1+c|\}R_0/\gamma < r(t) < (1+|c|)R_0\gamma, \qquad t \in [0, \tau].$$

In conclusion, let $d_1(R_0) < r(t) < d_2(R_0)$ with $\lim_{R_0 \to +\infty} d_1(R_0) = \lim_{R_0 \to +\infty} d_2(R_0) = +\infty$ for simplicity, then by the similar analysis to Theorem 3.1 one has that

$$\begin{aligned}&\frac{\tau}{m+1} + \frac{2B}{\max(1, p_2^2)d_2(R_0)} - \frac{E\pi}{\min(1, p_1^2)d_1(R_0)} - \frac{E\pi}{\min(1, p_2^2)d_1(R_0)} + o\Big(\frac{1}{R_0}\Big)\\ \le\, &T \le \frac{\tau}{m} - \Big[\frac{2B}{\max(1, q_2^2)d_2(R_0)} - \frac{E\pi}{\min(1, q_1^2)d_1(R_0)} - \frac{E\pi}{\min(1, q_2^2)d_1(R_0)}\Big] + o\Big(\frac{1}{R_0}\Big).\end{aligned}$$

Further, $\frac{\tau}{m+1} < T < \frac{\tau}{m}$ when $R_0 > 0$ is sufficiently large. The proof is complete. □

Example 4.2. In (2.1), assume that $g(x)$ is as in (4.2) and $J(y) = d$ with $0 < |d| < R_0/\gamma$. When $\frac{1}{\sqrt{\mu_1}} + \frac{1}{\sqrt{\mu_2}} \neq \frac{\tau}{m\pi}$ for an integer $m > 0$, the corresponding impulsive system has at least one discontinuous periodic orbit with period τ.

In fact, when $d > 0$, it follows that $|y^+| < |y| + d$. Further,

$$R_0/\gamma < r(t) < R_0\gamma + d, \qquad t \in [0, \tau].$$

While for $-R_0/\gamma < d < 0$, it follows that $||y| + d| < |y^+| < |y|$. Then

$$R_0/\gamma + d < r(t) < R_0\gamma, \qquad t \in [0, \tau].$$

Example 4.3. Consider the planar center

$$x' = y, \qquad y' = -x. \tag{4.3}$$

Obviously, solution trajectories are a family of closed curves surrounding the origin. Now consider a class of impulses as follows

$$\Delta y = 2, \qquad \text{for} \quad x = 0,\ y \leq 0. \tag{4.4}$$

Let $(x(t; x_0, y_0), y(t; x_0, y_0))$ be a solution of (4.3)-(4.4) satisfying $x_0 = 0,\ y_0 = 1$. It is easy to verify that the orbit is a discontinuous periodic orbit be of the form (see the left diagram in Figure 4)

$$(0, 1) \xrightarrow{P_R} (0, -1) \xrightarrow{\Phi} (0, 1).$$

And, from (4.4) then the orbit starting from $(0, 2)$ is also a discontinuous periodic orbit as follows

$$(0, 2) \xrightarrow{P_R} (0, -2) \xrightarrow{\Phi} (0, 0) \xrightarrow{\Phi} (0, 2). \tag{4.5}$$

Note that in (4.5), the impulsive mapping Φ is applied twice, but it is still defined as a discontinuous periodic orbit of (4.3)-(4.4) (see the right diagram in Figure 4).

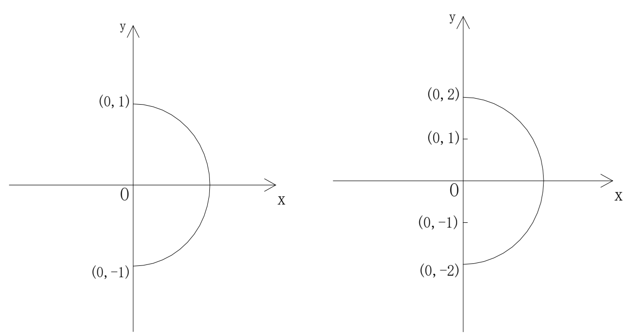

Figure 4. Left: $\Phi(0, -1) = (0, 1)$; Right: $\Phi(\Phi(0, -2)) = (0, 2)$.

Remark 4.1. If (4.4) is replaced by $\Delta y = -2$ for $x = 0, y \geq 0$, we have the similar results as follows

$$(0, -1) \xrightarrow{P_L} (0, 1) \xrightarrow{\Phi} (0, -1),$$
$$(0, -2) \xrightarrow{P_L} (0, 2) \xrightarrow{\Phi} (0, 0) \xrightarrow{\Phi} (0, -2).$$

5. Concluding remarks

In this paper, we have investigated the existence and uniqueness of discontinuous periodic orbits for a semilinear second order impulsive differential equation with state-dependent impulses. By geometric analysis, we estimated the time mapping of a solution trajectory rotating a circle on the phase plane. Then, under the growth conditions of g, some existence criteria were obtained via the Poincaré-Bohl fixed point theorem. Moreover, the uniqueness of the discontinuous periodic orbit is further proved. Finally, we presented several specific impulsive functions in examples to illustrate the effectiveness of the obtained results.

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