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STOCHASTICALLY PERMANENT ANALYSIS OF A NON-AUTONOMOUS HOLLING II PREDATOR-PREY MODEL WITH A COMPLEX TYPE OF NOISES

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Abstract This paper is considered a non-autonomous stochastic Holling II predator-prey model with a complex type of noises. By constructing a Lyapunov function and applying the dominated convergence theorem, stochastically permanent is proved. More importantly, two values λ_1, λ_2 are expressed by using the density function of the Falk Planck equation and some parameters in the system. Among them, $\lambda_1 > 0$ is proved to be the sufficient condition for the persistence in mean. Then, applying the strong law of large number and exponential martingale inequality, two necessary lemmas are introduced. Furthermore, utilizing the lemmas and $\lambda_2 < 0$, the sufficient conditions for extinction of the system are obtained. Actually, the two sufficiency conditions obtained are approached to the necessary conditions. Finally, some numerical simulations are carried out to verify the influence of the complex type of noises on the system.

Keywords Stochastic Holling II predator-prey model, complex noises, time-related parameters, persistence in mean, extinction.

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1. Introduction

Predator-prey is one of the the basic relationships among species in ecosystem. It refers to the interspecies relationship in which one organism feeds on another organism, such as cats and mice, eagles and rabbits, wolves and sheep, etc. In the study of the predator-prey model, the model with functional response is one of the popular subjects in the field of biomathematics [3]. Many scholars have done a lot of researches on various functional responses, such as the classical Lotka Volterra model [19,34], Beddington-DeAngelsis model [5,35] and Leslie-Gower model [1,36]. It is worth to mention that Holling [12] discussed three different functional responses to simulate predation in 1959. Based on this, Liu etc [20] introduced a predator-prey

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model with Holling II functional response:

$$\begin{cases} dx(t) = \left(r_1 x(t) - b_1 x^2(t) - \dfrac{c_1 x(t)}{1+x(t)} y(t) \right) dt, \\ dy(t) = \left(-r_2 y(t) - b_2 y^2(t) + \dfrac{c_2 x(t)}{1+x(t)} y(t) \right) dt, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ respectively represent the population density of the prey and predator at time t, $r_i, b_i, c_i (i = 1, 2)$ are positive constants. r_1 stands for the intrinsic growth rate of $x(t)$, r_2 denotes the death rate of $y(t)$, $b_i (i = 1, 2)$ are density dependent coefficients of $x(t)$, $y(t)$ respectively, $\frac{c_i x(t)}{1+x(t)} y(t) (i = 1, 2)$ represent Holling II functional responses, c_1 is the capturing rate of the predator and c_2 is the rate at which nutrients are converted into the predator's reproduction.

For the past few years, there have been many excellent results on Holling II predator-prey model [10, 31]. However, in the actual ecosystem, the population system will be disturbed by various random factors, such as natural disasters and human factors. They will cause more or less changes in population numbers. In order to simulate the actual situation, many scholars have introduced stochastic perturbations into the deterministic models to show the influence of environmental fluctuations on population dynamics [2, 13, 15, 16, 21–23, 28]. For example, Mao and Marion et al. [28] discussed a classic Lotka-Volterra stochastic model. They concluded that even a small intensity of environmental noise will inhibit the potential population explosion. Liu and Mandal [22] proposed a stochastic population model with one prey and two predators. They analysed the stability of the system and gave the sufficient conditions of extinction. Li etc [23] explored some asymptotic behaviors of a mutualism model with stochastic perturbations. Lv and Wang [24] considered the effect of environmental fluctuation on growth rate by replacing r_i to $r_i + \sigma_i dB_i(t) (i = 1, 2)$. And they proposed the following predator-prey model:

$$\begin{cases} dx(t) = \left(r_1 x(t) - b_1 x^2(t) - \dfrac{c_1 x(t)}{1+x(t)} y(t) \right) dt + \sigma_1 x(t) dB_1(t), \\ dy(t) = \left(r_2 y(t) - b_2 y^2(t) + \dfrac{c_2 x(t)}{1+x(t)} y(t) \right) dt + \sigma_2 y(t) dB_2(t), \end{cases} \tag{1.2}$$

where r_2 represents the birth rate of the predator. They showed that the system (1.2) is stochastically ultimately bounded. Moreover, they concluded that system (1.2) will be stochastically permanent, persistent in mean and extinct under some conditions.

In addition, due to seasonal variations, the temperature and humidity in the ecosystem will be fluctuant. These factors will lead many parameters to change over time, such as the natural growth rate, death rate and capturing rate of the population. In this regard, many scholars introduced time-related parameters to describe this phenomenon. For example, they substituted parameter r in the system to $r(t)$ and discussed some dynamical properties of the system with stochastic disturbance [7, 32].

In recent years, some authors researched the stochastic models with two random perturbations. Most of them introduced the impact of linear environmental noises on population dynamics [25, 33]. They mainly explored the effect of stochastic perturbations on the part of non-functional responses. Authors rarely considered the influence of environmental noises on the parameters of the functional response [8, 11].

For instance, Du and Nhu [8] focused on a stochastic model with Beddington–DeAngelis functional response and a complex type of noises. They researched permanence and extinction of the system. By superseding μ_i to $\mu_i+\sigma_i dB_i(t)(i=1,2)$ and changing β to $\beta+\sigma_3 dB_3(t)$, they set up the following stochastic model with a complex type of noises:

$$\begin{cases} dS(t)=\left(\alpha-\mu_1 S(t)-\dfrac{\beta S(t)I(t)}{1+m_1S(t)+m_2I(t)}\right)dt+\sigma_1 S(t)dB_1(t) \\ \qquad\qquad -\dfrac{\sigma_3 S(t)I(t)}{1+m_1S(t)+m_2I(t)}dB_3(t), \\ dI(t)=\left(-\mu_2 I(t)+\dfrac{\beta S(t)I(t)}{1+m_1S(t)+m_2I(t)}\right)dt+\sigma_2 I(t)dB_2(t) \\ \qquad\qquad +\dfrac{\sigma_3 S(t)I(t)}{1+m_1S(t)+m_2I(t)}dB_3(t), \end{cases}$$

where $S(t), I(t)$ are the numbers of the susceptible, infective individuals, respectively; α is the recruitment rate of the population, μ_1,μ_2 are the natural death rates. $\beta>0$ denote the infection coefficient. $m_1,m_2>0$ represent the parameters of the inhibitory effect.

Moreover, Guo etc [11] made a first attempt to discuss the predator-prey model with Crowly-Martin type functional response and perturbed by a complex type of noises. They similarly discussed the effect of stochastic perturbations on parameters which is in the response function. In the current, almost no one has discussed the non-autonomous Holling II predator-prey model with a complex type of noises.

Therefore, inspired by the above literatures. In one hand, We replace r_i, b_i, c_i of the system (1.1) to $r_i(t), b_i(t), c_i(t)(i=1,2)$. They are used to express the effects of seasonal variation on birth rate, death rate, density dependent coefficients, the capturing rate and the rate of nutrient conversion. In the other hand, we consider the influences of natural disasters and human factors on populations. Changing $c_i(t)$ to $c_i(t)+\sigma_j(t)dB_j(t)(j=3,4)$. Then we establish the following stochastic non-autonomous model with a complex type of noises:

$$\begin{cases} dx(t)=\left(r_1(t)x(t)-b_1(t)x^2(t)-\dfrac{c_1(t)x(t)}{1+x(t)}y(t)\right)dt \\ \qquad\qquad +\sigma_1(t)x(t)dB_1(t)-\dfrac{\sigma_3(t)x(t)}{1+x(t)}y(t)dB_3(t), \\ dy(t)=\left(-r_2(t)y(t)-b_2(t)y^2(t)+\dfrac{c_2(t)x(t)}{1+x(t)}y(t)\right)dt \\ \qquad\qquad +\sigma_2(t)y(t)dB_2(t)+\dfrac{\sigma_4(t)x(t)}{1+x(t)}y(t)dB_4(t), \end{cases} \tag{1.3}$$

where $r_i(t), b_i(t), c_i(t)(i=1,2)$ and $\sigma_j(t)(j=1,2,3,4)$ are non-negative continuous and bounded functions on $\mathbb{R}_+$. $B_i(t)(i=1,2,3,4)$ are mutually independent Brownian motions defined on the complete probability space $\left(\Omega, F, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P}\right)$. Filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions $\mathrm{Z}(t)=(x(t),y(t))$ and $|\mathrm{Z}(t)|=\sqrt{x^2(t)+y^2(t)}$. $\sigma_j(t)$ indicate the intensity of white noises.

The main purpose of this paper is to discuss the sufficiency and almost necessary conditions for persistence in mean and extinction of system (1.3). And this paper mainly contains the following contents, in Section 2, we introduce some preliminary results and two values λ_1, λ_2. Section 3 discusses the stochastically permanent and persistence in mean of the system (1.3). We get that if $\lambda_1 > 0$, the system (1.3) is persistence in mean. In Section 4, $\lambda_2 < 0$ are proved to be the sufficiency condition for the extinction. Finally, we verify the theoretical results by numerical simulation, and comparing with the deterministic system. The results of numerical simulation show that small random disturbances will lead to volatility changes in population numbers. When the intensity of perturbation satisfies $\lambda_2 < 0$, the population will be extinct.

2. Preliminaries

For the convenience of the rest of the paper, simplifying the notation firstly and some lemmas are provided which will be used in this paper. We indicate $z = (u, v) \in \mathbb{R}_+^2$ and $a \wedge b = \min\{a, b\}$. According to Theorem 2.1 in [38], we can obtained that $Z(t) = (x_z(t), y_z(t)) = (x(t), y(t))$ is the unique global positive solution of system (1.3) with initial value z. And $Z(t)$ is a strong homogeneous Markov process. Moreover, we define

$$r_i^u = \sup_{t\in\mathbb{R}_+} r(t),\ r_i^l = \inf_{t\in\mathbb{R}_+} r(t),\ (i = 1, 2).$$

Through out this paper, the same symbols are also applied to $b_i, c_i (i = 1, 2)$ and $\sigma_j(t)(j = 1, 2, 3, 4)$.

Firstly, we give the sufficient condition for the extinction of $x(t)$. Using Itô formula, we discuss

$$\begin{aligned} d\ln x(t) &= \left(r_1(t) - b_1(t)x(t) - \frac{c_1(t)y(t)}{1+x(t)} - \frac{1}{2}\sigma_1^2(t) - \frac{\sigma_3^2(t)y(t)^2}{2(1+x(t))^2}\right) dt \\ &\quad + \sigma_1(t)dB_1(t) - \frac{\sigma_3(t)y(t)}{1+x(t)} dB_3(t) \\ &\le \left(r_1^u - \frac{1}{2}(\sigma_1^l)^2\right) dt + \sigma_1^u dB_1(t) - \frac{\sigma_3^l y(t)}{1+x(t)} dB_3(t). \end{aligned}$$

Integrating the above equation from 0 to t yields:

$$\frac{\ln x(t)}{t} \le \frac{\ln x(0)}{t} + r_1^u - \frac{1}{2}(\sigma_1^l)^2 + \frac{\sigma_1^u B_1(t)}{t} - \frac{1}{t}\int_0^t \frac{\sigma_3^l y(s)}{1+x(s)} dB_3(s).$$

According to the strong law of large numbers [29], we get

$$\lim_{t\to\infty} \frac{\sigma_i^u B_i(t)}{t} = 0, \quad i = 1, 2 \tag{2.1}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\sigma_3^l y(s)}{1+x(s)} dB_3(s) = 0. \tag{2.2}$$

From above, we can obtain that if $r_1^u < \frac{1}{2}(\sigma_1^l)^2$, then $\lim\limits_{t\to\infty} x(t) = 0$ *a.s.*. It means that $x(t)$ is extinction. When the prey dies out, so does the predator $y(t)$, vice versa. Thus, to consider the dynamic behaviors of $y(t)$ when $x(t)$ will not die out, we assume $r_1^u > \frac{1}{2}(\sigma_1^l)^2$.

In addition, we provid the following equation on the boundary:

$$\mathrm{d}\widetilde{x}(t) = \widetilde{x}(t)\left(r_1^u - b_1^l \widetilde{x}(t)\right)\mathrm{d}t + \sigma_1^u \widetilde{x}(t)\mathrm{d}B_1(t), \tag{2.3}$$

with initial $\widetilde{x}(0) = x(0) > 0$, $\forall t \geq 0$ a.s. Further, we get the expression of a positive solution $\widetilde{x}(t)$ of equation (2.3) from Theorem 2.1 in [26].

By solving the Fokker-Planck equation [11], we receive density $\varphi(x)$ of stationary distribution π of the process $\widetilde{x}(t)$.

$$\varphi(x) = \frac{l^q}{\Gamma(q)} x^{q-1} e^{-lx}, \quad x > 0, \tag{2.4}$$

where $l = \frac{2b_1^l}{(\sigma_1^u)^2}$, $q = \frac{2r_1^u}{(\sigma_1^u)^2} - 1 > 0$ and $\Gamma(\cdot)$ is the Gamma function [17]. From strong ergodicity theorem [37], for any measurable function $\Psi(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\int_0^\infty |\Psi(x)| f^*(x)dx < \infty$. Applying to $\widetilde{x}_u(t)$ which is the solution of (2.3) with initial value u, we obtain

$$P\left\{\lim_{t\to\infty} \frac{1}{t}\int_0^t \Psi\left(\widetilde{x}_u(t)\right)dt = \int_0^\infty \Psi(x)\varphi(x)dx\right\} = 1, \quad \forall u > 0. \tag{2.5}$$

Further, using $\Psi\left(\widetilde{x}_u(t)\right) = \widetilde{x}_u^p(t)$ to equation (2.4), when $p = 1$, we have,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{x}_u(t)dt = \frac{r_1^u - (\sigma_1^u)^2/2}{b_1^l} > 0. \tag{2.6}$$

3. Stochastic Permanence

Before studying the stochastic permanence of system (1.3), the definition of stochastic permanence is provided.

Definition 3.1 ([29]). If for arbitrary $\varepsilon \in (0,1)$, there are two positive constants β_1 and β_2 such that for positive initial data $z_0 = (x_0, y_0)$, the solution $x(t)$ of problem (1.3) has the property that

$$\liminf_{t\to\infty} P\left\{|Z(t)| \geq \beta_1\right\} \geq 1-\varepsilon, \quad \liminf_{t\to\infty} P\left\{|Z(t)| \leq \beta_2\right\} \geq 1-\varepsilon.$$

Then, system (1.3) is said to be stochastically permanent.

Lemma 3.1. *If* $(x(t), y(t))$ *be a positive solution of* (1.3) *with any initial value* $z \in \mathbb{R}_+^2$*, then there exist* $L_1(p)$ *and* $L_2(p)$ *such that*

$$E\left[x^p(t)\right] \leq L_1(p), \quad E\left[y^p(t)\right] \leq L_2(p), \quad p > 0.$$

Proof. The method we are going to use is learned from lemma 2.1 in [4]. And we directly work out

$$\begin{aligned} dx^p(t) = {} & px^p(t)\left[r_1(t) - b_1(t)x(t) - \frac{c_1(t)y(t)}{1+x(t)} + (p-1)\left(\frac{1}{2}\sigma_1^2(t) + \frac{\sigma_3^2(t)y^2(t)}{2(1+x(t))^2}\right)\right]dt \\ & + p\sigma_1(t)x^p(t)dB_1(t) - p\sigma_3(t)\frac{x^p(t)y(t)}{1+x(t)}dB_3(t) \end{aligned}$$

$$\leq px^p(t)\left[r_1^u - b_1^l x(t) + \frac{p}{2}\left((\sigma_1^u)^2 + (\sigma_3^u)^2 M_1^2\right)\right]dt$$
$$+ p\sigma_1^u x^p(t)dB_1(t) - p\sigma_3^l \frac{x^p(t)y(t)}{1+x(t)}dB_3(t), \tag{3.1}$$

where $M_1 := \max_{x,y\in R_+} \frac{y(t)}{1+x(t)} < \infty$ is a positive constant, because $y(t)$ is is bounded. And

$$dy^p(t) = py^p(t)\left[-r_2(t) - b_2(t)y(t) + \frac{c_2(t)x(t)}{1+x(t)} + (p-1)\left(\frac{1}{2}\sigma_2^2(t) + \frac{\sigma_4^2(t)x^2(t)}{2(1+x(t))^2}\right)\right]dt$$
$$+ p\sigma_2(t)y^p(t)dB_2(t) + p\sigma_4(t)\frac{x(t)y^p(t)}{1+x(t)}dB_4(t)$$
$$\leq py^p(t)\left[-b_2^l y(t) + c_2^u + \frac{p}{2}\left((\sigma_2^u)^2 + (\sigma_4^u)^2\right)\right]dt$$
$$+ p\sigma_2^u y^p(t)dB_2(t) + p\sigma_4^u \frac{x(t)y^p(t)}{1+x(t)}dB_4(t). \tag{3.2}$$

Based on the expectation of equation (3.1) and the Hölder's inequality [29], we acquire

$$\frac{dE\left[x^p(t)\right]}{dt} \leq p\left(r_1^u + \frac{(\sigma_1^u)^2}{2}p + \frac{(\sigma_3^u)^2 M_1^2}{2}p\right)E\left[x^p(t)\right] - pb_1^l E\left[x^{p+1}(t)\right]$$
$$\leq p\left(r_1^u + \frac{(\sigma_1^u)^2}{2}p + \frac{(\sigma_3^u)^2 M_1^2}{2}p\right)E\left[x^p(t)\right] - pb_1^l\left(E\left[x^p(t)\right]\right)^{1+1/p}.$$

In the same way,

$$\frac{dE\left[y^p(t)\right]}{dt} \leq p\left(c_2^u + \frac{(\sigma_2^u)^2}{2}p + \frac{(\sigma_4^u)^2}{2}p\right)E\left[y^p(t)\right] - pb_2^l\left(E\left[y^p(t)\right]\right)^{1+1/p}.$$

From Lemma 2.1 in [2], we have

$$\limsup_{t\to\infty} E\left[x^p(t)\right] \leq \left(\frac{r_1^u + \frac{(\sigma_1^u)^2}{2}p + \frac{(\sigma_3^u)^2 M_1^2}{2}p}{b_1^l}\right)^p,$$

and

$$\limsup_{t\to\infty} E\left[y^p(t)\right] \leq \left(\frac{c_2^u + \frac{(\sigma_2^u)^2}{2}p + \frac{(\sigma_4^u)^2}{2}p}{b_2^l}\right)^p.$$

Thus, there exists a positive constant $L(p)$, such that

$$E\left[x^p(t)\right] \leq L(p), \quad E\left[y^p(t)\right] \leq L(p), \quad p > 0, \quad t \in [0,\infty).$$

At this point, Lemma 3.1 has been proved. □

Theorem 3.1. *System* (3.1) *is stochastically permanent.*

Proof. Considering Lemma 3.1 and the dominated convergence theorem, we have

$$E\left[\lim_{t\to\infty}\frac{1}{t}\int_0^t x^p(s)ds\right] = \lim_{t\to\infty}\frac{1}{t}\int_0^t E\left[x^p(s)\right]ds \leq L(p),$$

$$E\left[\lim_{t\to\infty}\frac{1}{t}\int_0^t y^p(s)ds\right]=\lim_{t\to\infty}\frac{1}{t}\int_0^t E\left[y^p(s)\right]ds\le L(p).$$

Using Chebyshev inequality [29]. Then we get that for arbitrarily small $\epsilon\in(0,1)$, there exist constants $\overline{\beta}=\overline{\beta}(\epsilon,p)>1$ and $\widetilde{\beta}_p=\widetilde{\beta}(\epsilon,p)>1$, such that

$$P\{|Z|\ge\overline{\beta}\}\le\overline{\beta}^{-p}E|Z|^p,$$

and

$$P\left\{x(t)<\widetilde{\beta}_p\right\}\le\epsilon,\quad P\left\{y(t)<\widetilde{\beta}_p\right\}\le\epsilon.$$

Therefore, we get

$$\limsup_{t\to\infty}P\{|Z|\ge\overline{\beta}\}\le\overline{\beta}^{-p}E|Z|^p=\epsilon,$$

and

$$\limsup_{t\to\infty}P\{|Z|\le\widetilde{\beta}_p\}\le\epsilon.$$

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$$\liminf_{t\to\infty}P\{|Z|\le\overline{\beta}\}\ge1-\epsilon,\quad\liminf_{t\to\infty}P\{|Z|\ge\widetilde{\beta}_p\}\ge1-\epsilon.$$

According to Definition 3.1, the proof is completed. □

From the proof of Theorem 3.1, we have

$$\limsup_{t\to\infty}P\{0<x\le\overline{\beta},\quad 0<y\le\overline{\beta}\}\ge1-\epsilon,\tag{3.3}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x^p(s)ds\le\widetilde{\beta}_p<\infty\quad\text{a.s.},\qquad\lim_{t\to\infty}\frac{1}{t}\int_0^t y^p(s)ds\le\widetilde{\beta}_p<\infty\quad\text{a.s.}\tag{3.4}$$

Definition 3.2 ([6]). If $x(t),y(t)$ satisfy the following condition

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds>0,\quad\lim_{t\to\infty}\frac{1}{t}\int_0^t y(s)ds>0.$$

Then the system (1.3) is said to be persistence in mean.

On the basis of equation (2.4), we set

$$\lambda_1=-r_2^u-\frac{1}{2}(\sigma_2^u)^2-\frac{1}{2}\left(\sigma_4^u\right)^2+\int_0^\infty\left(\frac{c_2^l x}{1+x}-\frac{(\sigma_4^u)^2x^2}{2(1+x)^2}\right)\varphi(x)dx.\tag{3.5}$$

Theorem 3.2. *If $\lambda_1>0$, for any initial value $z\in\mathbb{R}_+^2$, the system* (1.3) *is persistence in mean.*

Proof. The method is similar to Theorem 2.2 in [9]. Applying Itô formula, we get

$$\frac{1}{t}\ln x(t)=\frac{1}{t}\ln x(0)+\frac{1}{t}\int_0^t\left(r_1(s)-\frac{1}{2}\sigma_1^2(s)-b_1(s)x(s)-\frac{c_1(s)y(s)}{1+x(s)}\right)ds$$

$$+\frac{\sigma_1(t)B_1(t)}{t}-\frac{1}{t}\int_0^t\frac{\sigma_3^2(s)y^2(s)}{2(1+x(s))^2}ds-\frac{1}{t}\int_0^t\frac{\sigma_3(s)y(s)}{1+x(s)}dB_3(s).$$

Introducing $\widetilde{x}(s)$, adding and subtracting the same terms, the above formula becomes

$$\begin{aligned}\frac{1}{t}\ln x(t)\geq{}&\frac{1}{t}\ln x(0)+\frac{1}{t}\int_0^t(r_1^l-\frac{1}{2}(\sigma_1^u)^2-b_1^u\widetilde{x}(s))ds\\&+\frac{1}{t}\int_0^t b_1^u(\widetilde{x}(s)-x(s))ds-\frac{1}{t}\int_0^t\frac{c_1^u y(s)}{1+x(s)}ds-\frac{1}{t}\int_0^t\frac{(\sigma_3^u)^2y^2(s)}{2(1+x(s))^2}ds\\&+\frac{\sigma_1^l B_1(t)}{t}-\frac{1}{t}\int_0^t\frac{\sigma_3^u y(s)}{1+x(s)}dB_3(s).\end{aligned}\tag{3.6}$$

Likewise,

$$\begin{aligned}\frac{1}{t}\ln y(t)\geq{}&\frac{1}{t}\ln y(0)-r_2^u-\frac{1}{2}(\sigma_2^u)^2-\frac{1}{t}\int_0^t b_2^u y(s)ds-\frac{1}{t}\int_0^t\frac{(\sigma_4^u)^2x^2(s)}{2(1+x(s))^2}ds\\&+\frac{1}{t}\int_0^t\left(\frac{c_2^l\widetilde{x}(s)}{1+\widetilde{x}(s)}-\frac{(\sigma_4^u)^2\widetilde{x}^2(s)}{2(1+\widetilde{x}(s))^2}\right)ds-\frac{1}{t}\int_0^t c_2^u(\widetilde{x}(s)-x(s))ds\\&+\frac{\sigma_2^l B_2(t)}{t}+\frac{1}{t}\int_0^t\frac{\sigma_4^l x(s)}{1+x(s)}dB_4(s).\end{aligned}\tag{3.7}$$

By using the comparison theorem, equation (2.1) and (2.6) and the footnotes in [9], we get

$$\lim_{t\to\infty}\frac{\ln x(t)}{t}\leq 0,\quad \lim_{t\to\infty}\frac{\ln y(t)}{t}\leq 0.\tag{3.8}$$

Then, taking (2.1), (2.2), and using (2.5), (3.5) and (3.8), we conclude

$$\liminf_{t\to\infty}-\frac{1}{t}\int_0^t b_1^u(\widetilde{x}(s)-x(s))ds+\frac{1}{t}\int_0^t c_1^u y(s)ds\geq 0\quad\text{a.s.}\tag{3.9}$$

From (2.5), (3.7) and (3.8), we get

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t c_2^u(\widetilde{x}(s)-x(s))ds+\frac{1}{t}\int_0^t b_2^u y(s)ds\\&\geq -r_2^u-\frac{1}{2}(\sigma_2^u)^2-\frac{1}{2}\left(\sigma_4^u\right)^2+\frac{1}{t}\int_0^t\left(\frac{c_2^l\widetilde{x}(s)}{1+\widetilde{x}(s)}-\frac{(\sigma_4^u)^2\widetilde{x}^2(s)}{2(1+\widetilde{x}(s))^2}\right)ds\\&=\lambda_1\quad a.s.\end{aligned}\tag{3.10}$$

By dividing (3.9) by b_1^u, (3.10) by c_2^u. And adding the each side of them respectively, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t y(s)ds\geq\frac{\lambda_1 b_1^u c_2^u}{b_1^u b_2^u+c_1^u c_2^u}>0\quad a.s.\tag{3.11}$$

It proves that the system (1.3) is persistence in mean. □

4. Extinction

In ecosystems, researching the extinction of species is significant for maintaining ecological balance and species diversity. So, in this section, we discuss the stability of the predator $y(t)$ when $x(t)$ will not be extinct.

Lemma 4.1. *If p is a positive constant, and satisfies that*

$$c_1^u M_1 + \frac{(p+1)}{2}\left((\sigma_1^u)^2 + (\sigma_3^u)^2 M_1^2\right) < r_1^l, \tag{4.1}$$

where $M_1 := \max\limits_{x,y\in R_+} \frac{y(t)}{1+x(t)} < \infty$, $\forall t > 0$. Then, for any initial value $z \in \mathbb{R}_+^2$, the solution $(x(t), y(t))$ of (1.3) satisfies that

$$\limsup_{t\to\infty} E\left(\frac{1}{Z^p(t)}\right) \le K.$$

Proof. Referring to the method of Lemma 3.6 in [27]. We denote $F(t) = \frac{1}{x(t)}$, then

$$dF(t) = \mathcal{L}F(t)dt - F(t)\left(\sigma_1(t)dB_1(t) - \frac{\sigma_3(t)y(t)}{1+x(t)}dB_3(t)\right).$$

where

$$\mathcal{L}F(t) = F(t)\left(-r_1(t) + \sigma_1^2(t) + b_1(t)x(t) + \frac{c_1(t)y(t)}{1+x(t)} + \frac{\sigma_3^2(t)y^2(t)}{(1+x(t))^2}\right).$$

$\mathcal{L}$ is the differential operator. Applying the generalized Itô formula. And introducing $p > 0$ which satisfies equation (4.1), we have

$$\mathcal{L}(1+F(t))^p = p(1+F(t))^{p-1}\mathcal{L}F(t) + \frac{p(p-1)}{2}F^2(t)(1+F(t))^{p-2}\left(\sigma_1^2(t) + \frac{\sigma_3^2(t)y^2(t)}{(1+x(t))^2}\right).$$

Then, we arbitrarily choose a constant $h > 0$ to satisfies

$$h + pc_1^u M_1 + \frac{p(p+1)}{2}\left((\sigma_1^u)^2 + (\sigma_3^u)^2 M_1^2\right) < pr_1^l. \tag{4.2}$$

Furthermore, we get

$$\begin{aligned}\mathcal{L}e^{ht}(1+F(t))^p &= he^{ht}(1+F(t))^p + e^{ht}\mathcal{L}(1+F(t))^p \\ &= e^{ht}(1+F(t))^{p-2}\left[h(1+F(t))^2 + V\right],\end{aligned}$$

where

$$V = p(1+F(t))\mathcal{L}F(t) + \frac{p(p-1)}{2}F^2(t)\left(\sigma_1^2(t) + \frac{\sigma_3^2(t)y^2(t)}{(1+x(t))^2}\right).$$

As a result,

$$\begin{aligned}&\mathcal{L}e^{ht}(1+F(t))^p \\ \le & e^{ht}(1+F(t))^{p-2}\Bigg\{h + b_1^u p + pF(t)\left(-r_1^l + \frac{c_1^u y(t)}{1+x(t)} + (\sigma_1^u)^2 + \frac{(\sigma_3^u)^2 y^2(t)}{(1+x(t))^2} + \frac{2h}{p} + b_1^u\right)\end{aligned}$$

$$+ F^2(t)\left[h - pr_1^l + p\frac{c_1^u y(t)}{1+x(t)} + \frac{p(p+1)}{2}\left((\sigma_1^u)^2 + \frac{(\sigma_3^u)^2 y^2(t)}{(1+x(t))^2}\right)\right]\Bigg\}.$$

According to Lemma 3.1, we have $\forall t > 0$, $M_1 := \max\limits_{x,y\in R_+} \frac{y(t)}{1+x(t)} < \infty$. Then,

$$\begin{aligned}
&\mathcal{L}e^{ht}(1+F(t))^p \\
\leq& e^{ht}(1+F(t))^{p-2}\Bigg\{h + b_1^u p + pF(t)\left(-r_1^l + c_1^u M_1 + (\sigma_1^u)^2 + (\sigma_3^u)^2 M_1^2 + \frac{2h}{p} + b_1^u\right) \\
&+ F^2(t)\left[h - pr_1^l + pc_1^u M_1 + \frac{1}{2}p(p+1)((\sigma_1^u)^2 + (\sigma_3^u)^2 M_1^2)\right]\Bigg\}.
\end{aligned}$$

By equation (4.2), we can find a positive constant H so that $\mathcal{L}e^{ht}(1+F(t))^p \leq He^{ht}$. Hence, integrating and taking expectations yield,

$$E\left[e^{ht}(1+F(t))^p\right] \leq (1+F(0))^p + \frac{H}{h}E(e^{ht}).$$

Therefore, we get

$$\limsup_{t\to\infty} E\left[F(t)^p\right] \leq \limsup_{t\to\infty} E\left[(1+F(t))^p\right] \leq \frac{H}{h} := K.$$

To sum up, this Lemma holds. □

Considering equation (3.3) and Chebyshev inequality. For any $\epsilon \in (0,1)$, there exists a constant $\beta = \beta(\epsilon) > 1$ such that

$$\limsup_{t\to\infty} P\left\{\beta^{-1} \leq x \leq \overline{\beta}, 0 < y \leq \overline{\beta}\right\} \geq 1-\epsilon. \tag{4.3}$$

In order to proof Lemma 4.2, we firstly define

$$\lambda_2 = -r_2^l - \frac{1}{2}(\sigma_2^l)^2 + \int_0^\infty \left(\frac{c_2^u x}{1+x} - \frac{(\sigma_4^l)^2 x^2}{2(1+x)^2}\right)\varphi(x)dx, \tag{4.4}$$

where $\varphi(x)$ is given in equation (2.4).

Lemma 4.2. *If $\lambda_2 < 0$, for any $\epsilon > 0$, $\overline{\beta}, \beta > 1$ and $\rho > 0$, there exists $\hat{\delta} = \hat{\delta}(\epsilon, \overline{\beta}) > 0$ such that*

$$P\left\{\lim_{t\to\infty} y_z(t) = 0,\ |\ln x_z(s) - \ln \widetilde{x}_u(s)| < \theta,\ \forall t \geq 0\right\} \geq 1-\epsilon,\ \forall z \in [\beta^{-1}, \overline{\beta}] \times (0, \hat{\delta}].$$

Proof. Inspired by [11], define the function $V : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$V(x,y) = -r_2(t) - \frac{1}{2}\sigma_2^2(t) - b_2(t)y + \frac{c_2(t)e^x}{1+e^x} - \frac{\sigma_4^2(t)e^{2x}}{2\left(1+e^x\right)^2}.$$

For $\forall (x,y) \in \mathbb{R}\times\mathbb{R}_+$, we compute

$$\begin{aligned}
&\left|\frac{\partial V(x,y)}{\partial x}\right| + \left|\frac{\partial V(x,y)}{\partial y}\right| \\
=&\left|\frac{c_2(t)e^x}{1+e^x} - \frac{c_2(t)e^x}{\left(1+e^x\right)^2} - \frac{\sigma_4^2(t)e^{2x}}{\left(1+e^x\right)^2} + \frac{\sigma_4^2(t)e^{2x}}{\left(1+e^x\right)^3}\right| + |-b_2(t)| \\
\leq& c_2^u + \frac{1}{3}(\sigma_4^u)^2 + b_2^u \\
:=& M_2,
\end{aligned} \tag{4.5}$$

and

$$V\left(\ln \widetilde{x}_u, 0\right) = -r_2(t) - \frac{1}{2}\sigma_2^2(t) + \frac{c_2(t)\widetilde{x}_u}{1+\widetilde{x}_u} - \frac{\sigma_4^2(t)\widetilde{x}_u^2}{2\left(1+\widetilde{x}_u\right)^2}.$$

According to the expression of the solution $\widetilde{x}(t)$, we obtain that for any $u > 0$, if $\beta^{-1} \leq u \leq \overline{\beta}$, then $\widetilde{x}_{\beta^{-1}}(t) \leq \widetilde{x}_u(t) \leq \widetilde{x}_{\overline{\beta}}(t)$. Therefore, applying equation (2.5), we have that for any $x > 0$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{c_2^u \widetilde{x}_{\overline{\beta}}}{1+\widetilde{x}_{\overline{\beta}}} ds = \lim_{t\to\infty}\int_0^\infty \frac{c_2^u x}{1+x}\varphi(x)dx \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{(\sigma_4^l)^2 \widetilde{x}_{\beta^{-1}}^2}{2(1+\widetilde{x}_{\beta^{-1}})^2} ds = \lim_{t\to\infty}\int_0^\infty \frac{(\sigma_4^l)^2 x}{2(1+x)^2}\varphi(x)dx \quad a.s.$$

Thus, using equation (4.4), for any $u \in [\beta^{-1}, \overline{\beta}]$, we get

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\int_0^t V\left(\ln \widetilde{x}_u, 0\right) ds \\
&= -r_2(t) - \frac{1}{2}\sigma_2^2(t) + \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{c_2(t)\widetilde{x}_u}{1+\widetilde{x}_u}ds - \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\sigma_4^2(t)\widetilde{x}_u^2}{2\left(1+\widetilde{x}_u\right)^2}ds \\
&\leq -r_2^l - \frac{1}{2}(\sigma_2^l)^2 + \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{c_2^u\widetilde{x}_{\overline{\beta}}}{1+\widetilde{x}_{\overline{\beta}}}ds - \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{(\sigma_4^l)^2\widetilde{x}_{\beta^{-1}}^2}{2(1+\widetilde{x}_{\beta^{-1}})^2}ds \\
&= \lambda_2 \quad a.s.
\end{aligned} \tag{4.6}$$

Furthermore, combining equation (4.6) and the strong law of large numbers [29], for any $\epsilon > 0$ and $u \in [\beta^{-1}, \overline{\beta}]$, there exists $T = \max\{T_1, T_2\}$ and Ω_z^1 such that $P(\Omega_z^1) \geq 1 - \frac{\varepsilon}{3}$. For all $\omega \in \Omega_z^1$, we have

$$\int_0^t V\left(\ln \widetilde{x}_u, 0\right) ds \leq \frac{\lambda_2 t}{2}, \qquad \forall t \geq T_1, \tag{4.7}$$

$$\sigma_2(t)B_2(t) + \int_0^t \frac{\sigma_4(t)x(s)}{1+x(s)}dB_4(s) \leq \frac{|\lambda_2|t}{6}, \qquad \forall t \geq T_2. \tag{4.8}$$

By Itô's formula, and considering the supremum and infimum of the parameters, we get

$$\begin{aligned}
&d\left(\ln x_z(t) - \ln \widetilde{x}_u(t)\right)^2 \\
&\leq \Big[2b_1^l\left(\ln x_z(t) - \ln \widetilde{x}_u(t)\right)\left(\widetilde{x}_u(t) - x_z(t)\right) \\
&\quad - \frac{2c_1^l y_z(t)\left(\ln x_z(t) - \ln \widetilde{x}_u(t)\right)}{1+x_z(t)} - \frac{(\sigma_3^l)^2 y_z^2(t)\left(\ln x_z(t) - \ln \widetilde{x}_u(t)\right)}{\left(1+x_z(t)\right)^2} \\
&\quad + \frac{(\sigma_3^u)^2 y_z^2(t)}{\left(1+x_z(t)\right)^2}\Big] dt - \frac{2\sigma_3^l y_z(t)\left(\ln x_z(t) - \ln \widetilde{x}_u(t)\right)}{1+x_z(t)}dB_3(t).
\end{aligned}$$

Moreover, using exponential martingale inequality [30], we have $P(\Omega_z^2) \geq 1 - \frac{\varepsilon}{3}$, where

$$\begin{aligned}\Omega_z^2 = \Bigg\{ \int_0^t -\frac{2\sigma_3^l y_z(s)\left(\ln x_z(s) - \ln \widetilde{x}_u(s)\right)}{1 + x_z(s)} dB_3(s) \leq \frac{\theta^2}{2} \\ + g_1 \int_0^t \frac{4(\sigma_3^l)^2 y_z^2(s)\left(\ln x_z(s) - \ln \widetilde{x}_u(s)\right)^2}{\left(1 + x_z(s)\right)^2} ds \Bigg\},\end{aligned} \tag{4.9}$$

$g_1 = \frac{1}{\theta^2} \ln \frac{3}{\varepsilon}$ and $\theta = \frac{1}{2} M_2 \left(\varepsilon \wedge \frac{|\lambda_2|}{6}\right)$. By virtue of the stochastic boundedness of $x_z(t)$ and $y_z(t)$, we define

$$g_2 = 2c_1^l \theta + (\sigma_3^l)^2 M_1 \theta + (\sigma_3^u)^2 M_1 + 4g_1 (\sigma_3^l)^2 M_1, \quad \alpha = \frac{\theta^2}{6g_2 T} \wedge \theta. \tag{4.10}$$

Then, according to Lemma 2.2 in [8], there exists $\delta > 0$ such that $\forall z \in [H^{-1}, \overline{H}] \times (0, \delta]$ and $P(\Omega_z^3) \geq 1 - \frac{\varepsilon}{3}$ where

$$\Omega_z^3 = \left\{ |\ln x_z(t) - \ln \widetilde{x}_u(t)| < \theta, y_z(t) < \alpha, \forall t \in [0, T] \right\}. \tag{4.11}$$

Set $\xi_z = \inf\{t \geq 0 : |\ln x_z(t) - \ln \widetilde{x}_u(t)| \geq \theta\}$ and $\tau_z = \inf\{t \geq 0 : y_z(t) \geq \alpha\}$. From (4.10), we have $\xi_z \wedge \tau_z > T$. We define $\omega \in \Omega_z := \Omega_z^1 \cap \Omega_z^2 \cap \Omega_z^3$. Combining (4.5), (4.7), (4.8) and (4.9), applying Itô's formula and the mean-value theorem, we get that if $z \in [\beta^{-1}, \overline{\beta}] \times (0, \delta]$, $\omega \in \Omega_z$ and $t \in [T, \xi_z \wedge \tau_z]$,

$$\begin{aligned}\ln y_z(t) &= \ln v + \int_0^t V\left(\ln x_z(s), y_z(s)\right) ds + \sigma_2(t) B_2(t) + \int_0^t \frac{\sigma_4(t) x_z(s)}{1 + x_z(s)} dB_4(s) \\ &\leq \ln v + \int_0^t V\left(\ln \widetilde{x}_u(s), 0\right) ds + M_2 \int_0^t \left[|\ln x_z(s) - \ln \widetilde{x}_u(s)| + y_z(s)\right] ds + \frac{|\lambda_2| t}{6} \\ &\leq \ln v - \frac{|\lambda_2| t}{6}.\end{aligned} \tag{4.12}$$

Furthermore, from (4.9) to (4.12), we obtain that

$$\begin{aligned}&\left(\ln x_z(t) - \ln \widetilde{x}_u(t)\right)^2 \\ \leq &\int_0^t \Bigg(\frac{2c_1^l y_z(s)\left|\ln x_z(s) - \ln \widetilde{x}_u(s)\right|}{1 + x_z(s)} + \frac{(\sigma_3^l)^2 y_z^2(s)\left|\ln x_z(s) - \ln \widetilde{x}_u(s)\right|}{\left(1 + x_z(s)\right)^2} \\ &+ \frac{(\sigma_3^u)^2 y_z^2(s)}{\left(1 + x_z(s)\right)^2} + g_1 \frac{4(\sigma_3^l)^2 y_z^2(s)\left(\ln x_z(s) - \ln \widetilde{x}_u(s)\right)^2}{\left(1 + x_z(s)\right)^2} \Bigg) ds + \frac{\theta^2}{2} \\ \leq &\, g_2 \int_0^T y_z(s) ds + g_2 \int_T^t y_z(s) ds + \frac{\theta^2}{2} \\ < &\, g_2 \alpha T + g_2 \frac{6v}{|\lambda_2|} + \frac{\theta^2}{2},\end{aligned}$$

and $g_2 \alpha T \leq \frac{\theta^2}{6}$. If $z \in [\beta^{-1}, \overline{\beta}] \times (0, \hat{\delta}]$, where $\hat{\delta} := \delta \wedge \left(\frac{\theta^2 |\lambda_2|}{30 g_2}\right) \wedge \left(\alpha e^{\frac{|\lambda_2| T}{6}}\right)$, $\omega \in \Omega_z$, we get

$$\left(\ln x_z(t) - \ln \widetilde{x}_u(t)\right)^2 < \frac{5}{6}\theta^2, \quad y_z(t) \leq v e^{\frac{|\lambda_2| T}{6}} < \alpha, \qquad t \in [T, \xi_z \wedge \tau_z].$$

Hence, according to equation (3.10) and the definition of $\xi_z \wedge \tau_z$, we get $\xi_z \wedge \tau_z = \infty$. For $z \in [\beta^{-1}, \overline{\beta}] \times (0, \hat{\delta}]$, if $\omega \in \Omega_z$, then we have

$$\Omega_z \subset \left\{ |\ln x_z(t) - \ln \widetilde{x}_u(t)| < \theta, \ \lim_{t\to\infty} y_z(t) < \alpha, \forall t > 0 \right\}.$$

As a result, for $z \in [\beta^{-1}, \overline{\beta}] \times (0, \hat{\delta}]$,

$$P \left\{ \lim_{t\to\infty} y_z(t) = 0, \ |\ln x_z(s) - \ln \widetilde{x}_u(s)| < \theta, \ \forall t \geq 0 \right\} \geq P(\Omega_z) \geq 1 - \epsilon.$$

The proof of this Lemma is completed. □

From Lemma 4.1 and Lemma 4.2, we can get the following theorem.

Theorem 4.1. *If $\lambda_2 < 0$, for any initial value $z_0 \in \mathbb{R}^2_+$, we have*

$$\mathbb{P} \left\{ \lim_{t\to\infty} \frac{\ln y_{z_0}(t)}{t} = \lambda_2 \right\} = 1.$$

Proof. The idea of this proof comes from Theorem 2.2 in [8]. For any $z_0 \in \mathbb{R}^2_+$, from equation (4.3), we get

$$\limsup_{t\to\infty} P \left\{ \beta^{-1} \leq x_{z_0} \leq \overline{\beta}, 0 < y_{z_0} \leq \overline{\beta} \right\} \geq 1 - \epsilon. \tag{4.13}$$

It can be derived from Lemma 4.2 that the process $(x(t), y(t))$ is not recurrent in $\mathbb{R}^2_+$. And we can obtain that the process must be transient from [18]. Then, we define a compact set

$$\Phi = \left\{ (u_1, v_1) : \beta^{-1} \leq u_1 \leq \overline{\beta}, \hat{\delta} \leq v_1 \leq \overline{\beta} \right\} \in \mathbb{R}^2_+,$$

where $\overline{\beta}$, $\hat{\delta}$ and ϵ are given in Lemma 4.2. From the definition of transience,

$$\lim_{t\to\infty} P \left\{ x_{z_0}(t) \in \Phi \right\} = 0. \tag{4.14}$$

Using (4.13) and (4.14), we get that there exists $T_3 > 0$ such that

$$P \left\{ Z_{z_0}(T_3) \in [\beta^{-1}, \overline{\beta}] \times (0, \hat{\delta}] \right\} \geq 1 - 2\varepsilon. \tag{4.15}$$

From Lemma 4.2 and (4.12), applying the strong law of large numbers, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \left| \frac{\ln y_z(t)}{t} - \lambda_2 \right| \\
\leq &\limsup_{t\to\infty} \frac{1}{t} \int_0^t |V(\ln x_z(s), y_z(s)) - V(\ln \widetilde{x}_u(s), 0)| \, ds \\
&+ \limsup_{t\to\infty} \frac{\ln v}{t} + \limsup_{t\to\infty} \left(\frac{1}{t} \int_0^t \frac{\sigma_4^u x_z(s)}{1 + x_z(s)} dB_4(s) + \frac{\sigma_2^u B_2(t)}{t} \right) \\
\leq &\varepsilon \quad \text{a.s.} \quad \text{in} \quad \Omega_z.
\end{aligned}$$

Utilizing Lemma 4.2, equation (4.15) and Markov property, we get

$$P \left\{ \limsup_{t\to\infty} \left| \frac{\ln y_{z_0}(t)}{t} - \lambda_2 \right| \leq \varepsilon, \quad \forall t > 0 \right\} \geq 1 - 3\varepsilon.$$

Because ϵ is arbitrarily,

$$P \left\{ \limsup_{t\to\infty} \frac{\ln y_{z_0}(t)}{t} = \lambda_2 \right\} = 1.$$

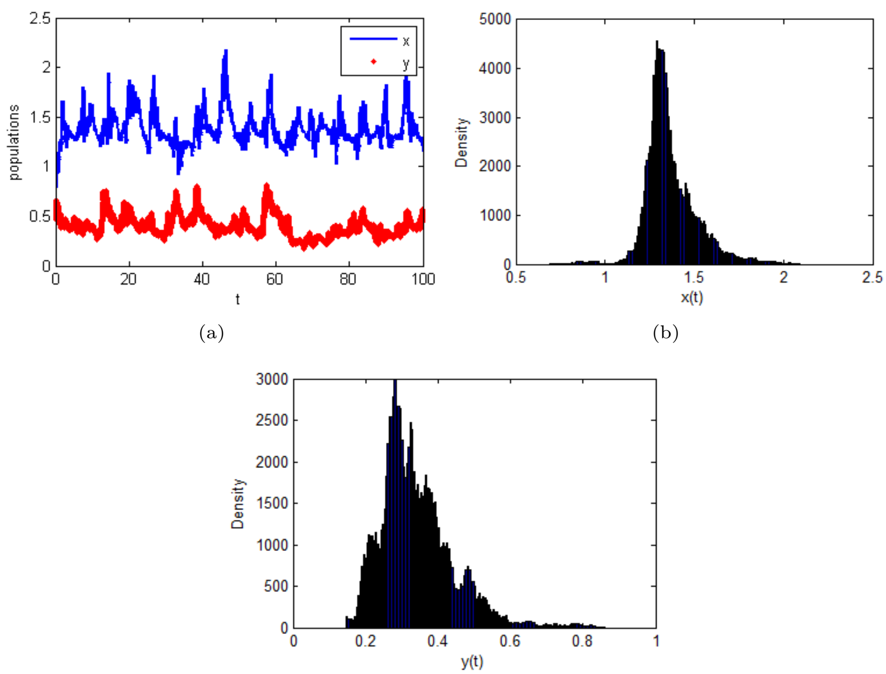
It means that the predator $y_{z_0}(t)$ will be extinct. □

5. Numerical Simulations

In this section, Milstein's method in literature [14] was used for numerical simulation to verify the theoretical results of this paper. And comparing them with the deterministic system (1.1). To illustrate the main result, we set the initial value $(x(0), y(0)) = (0.7, 0.5)$, and parameters $r_1 = 1.2 + 0.1\sin(t)$, $r_2 = 0.05 + 0.1\cos(t)$, $b_1 = 0.8 + 0.1\sin(t)$, $b_2 = 0.7 + 0.1\cos(t)$, $c_1 = 0.7 + 0.1\sin(t)$, $c_2 = 0.6 + 0.1\cos(t)$. By choosing the different values of $\sigma_i(t)(i = 1, 2, 3, 4)$, we can observe the changes in the number of populations.

First, we select appropriate disturbance values to verify the stochastically permanence of the system (1.3). Then by changing values of random perturbation, we substantiate the sufficient conditions for extinction and compare the the population of system (1.3) with the deterministic system (1.1). From the results, we found that random disturbances will generate fluctuant changes in population numbers, and even lead to extinction. In the end, we get the result shown in the following figures.

In Figure 1, we choose $\sigma_1(t) = 0.1 + 0.1\sin(t)$, $\sigma_2(t) = 0.2 + 0.1\cos(t)$, $\sigma_3(t) = 0.3 + 0.1\sin(t)$, $\sigma_4(t) = 0.1 + 0.1\cos(t)$. Then we get $r_1^u > \frac{1}{2}(\sigma_1^l)^2$ and $\lambda_1 > 0$, which satisfies the condition of Theorem 3.2. Moreover, it can be seen that the number of the two populations gradually tends to be stable. It shows that the system (1.3) is stochastically permanent, which is the same as Theorem 3.2.



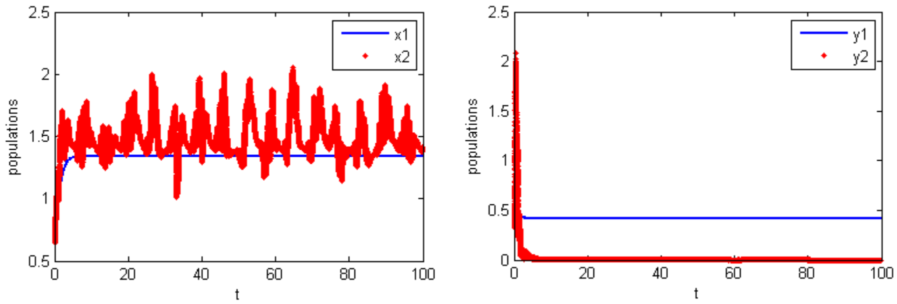
(a) (b)

(c)

Figure 1. (a) reflects the change in the population of $x(t)$ and $y(t)$ of system (1.3) under a small stochastic disturbance. (b), (c) is the frequency histograms of $x(t)$ and $y(t)$.

In Figure 2, by alterring the intensity of noises, we consider the impact of environmental noises on system (1.3). Choosing $\sigma_2(t) = 1 + 0.1\sin(t)$, $\sigma_4(t) = 1 + 0.1\cos(t)$. $\sigma_1(t)$ and $\sigma_3(t)$ are the same in Figure 1. In this situation, we get

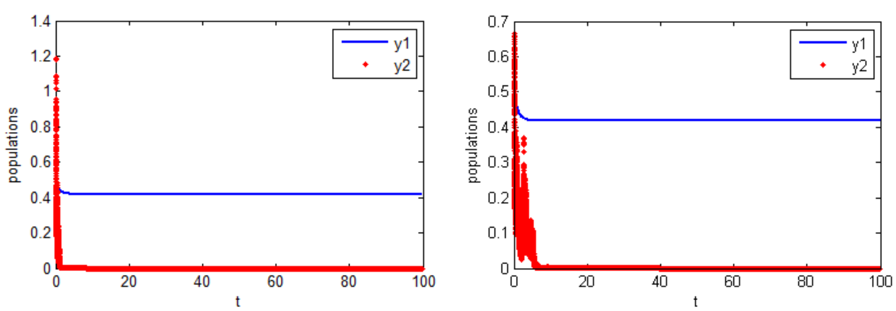
$r_1^u > \frac{1}{2}(\sigma_1^l)^2$ and $\lambda_2 < 0$. From (a), we notice that the prey of system (1.1) and system (1.3) are still stable and stochastically permanent. But due to the influence of complex type of noise, the number of prey in system (1.3) changes more drastically. In Figure 2(b), $y_1(t)$ is persistent. But the predator $y_2(t)$ tends to be extinct, which is consistent with Theorem 4.1. It shows that environmental disturbance has a noticeable impact on the population.



(a) (b)

Figure 2. (a) $x_1(t)$ represent the prey of the system (1.1) without stochastic disturbance, and $x_2(t)$ is the prey of system (1.3). (b) The representation of $y_1(t)$ and $y_2(t)$ are corresponded to $x_1(t)$ and $x_2(t)$.

In Figure 3, we change the values of $\sigma_2(t)$ and $\sigma_4(t)$ respectively to observe the number of predators. And we leave other values unchanged from Figure 1. In Figure 3(a), we increase the value of $\sigma_2(t)$. Choosing $\sigma_2(t) = 3 + 0.1\cos(t)$, $\sigma_4(t) = 0.1 + 0.1\cos(t)$. But in Figure 3(b), we individually change the value of $\sigma_4(t)$. Setting $\sigma_2(t) = 0.2 + 0.1\cos(t)$, $\sigma_4(t) = 3 + 0.1\cos(t)$. In the two cases, $\lambda_2 < 0$ which satisfies the condition of Theorem 4.1. Then we obtain that $y_1(t)$ is still persistent. The predator of the system (1.3) are going to be extinct in both situations. It is correspond with Theorem 4.1.



(a) (b)

Figure 3. Changes in the number of populations in two cases. $y_1(t)$ and $y_2(t)$ have the same representation as Figure 2. (a) $\sigma_2(t) = 3 + 0.1\cos(t)$, $\sigma_4(t) = 0.1 + 0.1\cos(t)$. (b) $\sigma_2(t) = 0.2 + 0.1\cos(t)$, $\sigma_4(t) = 3 + 0.1\cos(t)$.

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