Journal of Applied Analysis and Computation
Volume 12, Number 2, April 2022, 497–516
Website:http://www.jaac-online.com
DOI:10.11948/20210101

ψ-SHIFTED OPERATIONAL MATRIX SCHEME FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract In this paper, we present a numerical method to solve space-time fractional partial differential equations. We introduce ψ-shifted Chebyshev polynomials to construct operational matrices of fractional differentiation in the Caputo sense. These operational matrices are then used to find the solution of fractional partial differential equations. The efficiency and applicability of introduced numerical scheme is tested by comparing the proposed numerical approximations with the results obtained from existing numerical methods.

Keywords Fractional partial differential equations, ψ-shifted Chebyshev polynomials, operational matrices, fractional derivatives.

1. Introduction

In recent years, the investigation and application of fractional partial differential equations for solving several real world problems in various areas is becoming popular. These areas include fluid and continuum mechanics, mathematical physics, viscoplastic and viscoelastic flows, acoustics, biology, psychology, chemistry and engineering. Many problems can be modelled with the aid of fractional derivatives and fractional differential equations more easily and elegantly [29]. It is well known fact that most of fractional differential equations do not have exact analytical solution, so we have to adopt numerical and approximation techniques to solve these equations. These methods include Homotopy analysis method [14], Generalized differential transform method [20], Chebyshev spectral approximation [16], Adomian decomposition method [10], Green-Haar wavelets method [23] and many more.

The trend of formulating operational matrices by using various types of orthogonal polynomials is emerging rapidly. These polynomials are chosen according to their spectral properties making them more suitable for the solution of the problems under consideration. In [19], authors formulated operational matrices of fractional integration using shifted Chebyshev polynomials and then used these matrices in the approximation of the solution of fractional differential equations. Fukang Yin et al. [31] formulated operational matrices of fractional integration and differentiation with the help of two-dimensional Legendre functions of fractional order to approximate the solution of partial differential equations. A discrete method using Genocchi polynomials and Laguerre functions of fractional order in the large interval to solve multi-term fractional differential equation of variable order is presented

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in [6].

In the literature, derivatives and integrals of fractional order are defined in numerous ways, so the idea of presenting a general definition emerged. Ricardo Almeida [1] introduced the idea of fractional derivatives and integrals of a function with respect to the function ψ, where some particular choices of ψ recovers some of derivatives and integrals of fractional type like the Hadamard, Riemann-Liouville, and the Erdelyi-kober. M. Awadalla and Y. Y. Yameni in [4] presented the solution of exponential growth and decay models adopting the concept of ψ-Caputo fractional derivatives by using different kernel functions for performance evaluation. Ziadane Baitiche et al. in [5] used monotone iterative technique to formulate the extremal solution of ψ-Caputo fractional differential equation with non-linear boundary conditions. In [18] M. M. Matar et al. investigated ρ-Laplacian non-periodic non-linear boundary value problem in the form of generalized Caputo fractional derivative. The authors in [26], presented the solution of fractional boundary value problem of thermostat control model with ψ-Hilfer fractional operator.

The main objective of this paper is to develop operational matrices of fractional order partial derivatives with respect to a function to find numerical solution of a certain class of partial differential equations of fractional order. Consider

$$\eta(x,t)\frac{\partial^{\alpha,\psi}u(x,t)}{\partial t^{\alpha}}+\mu(x,t)\frac{\partial^{\beta,\psi}u(x,t)}{\partial x^{\beta}}=\mathfrak{g}(x,t), \tag{1.1}$$

on the domain $x\in[0,L]$, $t\in[0,T]$ where the parameters α and β refers to fractional or integer order partial derivatives with $0<\alpha,\beta\leq 2$, the coefficients $\eta(x,t)$ and $\mu(x,t)$ may be constants or variables and $\mathfrak{g}(x,t)$ is known function. We take the initial conditions as

$$u(x,0)=u_o(x), \qquad\qquad 0\leq x\leq L,$$

and the boundary conditions as

$$u(0,t)=q_o(t) \text{ and } u(L,t)=q_1(t), \qquad 0\leq t\leq T,$$

where $u_o(x)$, $q_o(t)$ and $q_1(t)$ are known functions on the given domain.

The ψ-Caputo derivative is modified concept of fractional derivatives which is more suitable to real-world phenomenons in many situations. Generalized ψ-Caputo operators of fractional differentiation and integration unifies some other definitions for different values of ψ [1]. In our proposed technique, we will formulate operational matrices of fractional differentiation by using introduced ψ-shifted Chebyshev orthogonal polynomials taking the derivatives in the Caputo sense and we obtain an algebraic system of equations. The solution of this algebraic system provides the solution of the problem. The paper has been organized as follows.

Section 2 presents some basic definitions from fractional calculus. Section 3 deals with the introduction of ψ-shifted Chebyshev polynomials. In Section 4, the approximation for function of one and two variables is presented. Section 5 is devoted for numerical formulation. Section 6 deals with the theoretical analysis of purposed technique. Some illustrative numerical examples are discussed in Section 7.

2. Preliminaries

This section deals with some basic notations and preliminaries that will be used further in our work.

Definition 2.1 ([1]). Let u be the integrable function defined on $K = [a, b]$ where K may be finite or infinite. Suppose $\psi \in C^n(K; \mathbb{R})$ is an increasing function and $\psi'(x) \neq 0$ for $x \in K$ with $n \in \mathbb{N}$. Let $\delta \in \mathbb{R}^+$ then ψ-fractional integral of function u is defined as

$$I_a^{\delta,\psi} u(x) := \frac{1}{\Gamma(\delta)} \int_a^x (\psi(x) - \psi(\rho))^{\delta-1} u(\rho)\psi'(\rho)d\rho,$$

where Γ is the famous Gamma function. Further, ψ-fractional integral operator satisfies semi group property.

$$I_a^{\delta,\psi} I_a^{\lambda,\psi} u(x) = I_a^{\delta+\lambda,\psi} u(x), \qquad \delta, \lambda > 0.$$

Definition 2.2 ([2]). Let $\delta > 0$, $n \in \mathbb{N}$, $K = [a, b]$, $\psi \in C^n(K; \mathbb{R})$ and ψ is increasing with $\psi'(x) \neq 0$ for $x \in K$, then

For $u \in C^n(K; \mathbb{R})$, ψ-Caputo fractional derivative is defined as

$$D_a^{\delta,\psi} u(x) = I_a^{n-\delta,\psi} u_\psi^{[n]}(x),$$

where

$$u_\psi^{[n]}(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n u(x), \tag{2.1}$$

so that, we have

$$D_a^{\delta,\psi} u(x) = \begin{cases} u_\psi^{[m]}(x), & \text{if } \delta = m \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\delta)} \int_a^x (\psi(x) - \psi(\rho))^{n-\delta-1} \psi'(\rho) u_\psi^{[n]}(\rho)d\rho, & \text{if } \delta \notin \mathbb{N}. \end{cases}$$

For $u \in C^{n-1}(K; \mathbb{R})$, ψ-Caputo fractional derivative is defined as

$$D_a^{\delta,\psi} u(x) = D_a^{\delta,\psi}\Big[u(x) - \sum_{s=0}^{n-1} \frac{u_\psi^{[s]}(a)}{s!}(\psi(x) - \psi(a))^s\Big],$$

where

$$n = \begin{cases} \delta, & \text{if } \delta \in \mathbb{N}, \\ [\delta] + 1, & \text{if } \delta \notin \mathbb{N}, \end{cases}$$

and $[\delta]$ represents integer part of δ.

Lemma 2.1 ([1]). *Let $\nu \in \mathbb{R}$, $\nu > n$, $\delta > 0$, then fractional derivative of $u(x) = (\psi(x) - \psi(a))^{(\nu-1)}$ is*

$$D_a^{\delta,\psi} u(x) = \frac{\Gamma(\nu)}{\Gamma(\nu - \delta)} (\psi(x) - \psi(a))^{(\nu-\delta-1)}.$$

Theorem 2.1 ([3]). *Let $\delta \in \mathbb{R}^+$, $\nu > -1$, then fractional integral of $u(x) = (\psi(x) - \psi(a))^{(\nu-1)}$ is*

$$I_a^{\delta,\psi} u(x) = \frac{\Gamma((\nu)}{\Gamma(\nu+\delta)} (\psi(x) - \psi(a))^{\delta+\nu-1}.$$

Theorem 2.2 ([2]). *Let $u : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$ and $n - 1 < \delta < n$ then following hold.*
(i) If $u \in C^1[a, b]$, then ${}^cD_a^{\delta,\psi} I_a^{\delta,\psi} u(x) = u(x)$,
(ii) If $u \in C^{n-1}[a, b]$, then $I_a^{\delta,\psi}\ {}^cD_a^{\delta,\psi} u(x) = u(x) - \sum_{s=0}^{n-1} \Xi_s (\psi(x) - \psi(a))^s$, where $\Xi_s = \frac{u_\psi^{[s]}(a)}{s!}$ and $u_\psi^{[s]}(a)$ is the same function as defined in (2.1).

Definition 2.3 ([3]). Suppose ψ is an increasing function with $\psi' \neq 0$ on the interval $K = [a, b]$ and ϱ is the weight function then the space defined as

$$H_\psi^2(K; \mathbb{R}) = \left\{ h : K \to \mathbb{R}\ : h \text{ is measurable and } \int_K |h(x)|^2 \varrho(x) \psi^{'}(x) dx < \infty \right\},$$

endowed with the inner product

$$(g, h)_{H_\psi^2(K;\mathbb{R})} = \int_K g(x) h(x) \varrho(x) \psi'(x) dx, \qquad g, h \in H_\psi^2(K; \mathbb{R}),$$

and the induced norm

$$\|h\|_{H_\psi^2(K;\mathbb{R})} = \left(\int_K |h(x)|^2 \varrho(x) \psi^{'}(x) dx \right)^{\frac{1}{2}}, \quad h \in H_\psi^2(K; \mathbb{R}),$$

is a Hilbert space. Also, for a given function $h : K \to \mathbb{R}$, we define $\mathfrak{S} : [0, 1] \to \mathbb{R}$ by

$$\mathfrak{S}(t) = \mathfrak{S}(\psi^{-1}(t)), \qquad 0 \leq t \leq 1.$$

3. ψ-shifted Chebyshev polynomials

The orthogonal polynomials play a vital role in the approximation of functions to solve fractional differential and integral equations. The technique of approximation by orthogonal polynomials appear in many areas of Mathematics which is the subject of interest for researchers. Different types of orthogonal polynomials can be used in their original form as well as some modified forms to solve variety of mathematical problems. Due to the orthogonal properties of polynomials, the solution of fractional differential equation often reduces to algebraic equation system which are then easy to solve.

Russian Mathematician Pafnuty Chebyshev was the man who introduced nth degree orthogonal polynomials named Chebyshev polynomials in the interval $[-1, 1]$ with leading coefficient unity especially suitable for the approximation of other functions. These polynomials are widely used in numerical analysis, uniform approximation, least square approximation, solving ordinary and partial differential equations, spectral and pseudo-spectral methods and so on [22, 27]. Most recently, for numerical purpose shifted Chebyshev polynomials are used in which the range of independent variable is $[0, 1]$ instead of $[-1, 1]$. Several researchers are using these polynomials in the approximation of the solution for different types of differential and integral equations [12, 32]. First kind Chebyshev polynomials on the interval $[-1,1]$ are defined as

$$T_n(x) = \cos(n \arccos(x)), \qquad \text{when} \qquad x = \cos\theta.$$

Chebyshev polynomials are orthogonal upon weight function $\varrho(x) = \frac{1}{\sqrt{1-x^2}}$ and

$$\int_{-1}^{1} T_l(x)\ T_k(x)\ \varrho(x)dx = \kappa_l \delta_{lk},$$
$$\kappa_l = \begin{cases} \frac{\vartheta_l}{2}\pi, & l = k, \\ 0, & l \neq k, \end{cases}$$

where $\vartheta_0 = 2$, $\vartheta_l = 1$ for $l \geq 1$ and δ_{lk} is Kronecker delta. In numerical computation, the range $[0,1]$ is beneficial than the range $[-1,1]$, so the independent variable t in $[-1,1]$ is mapped to the variable x in $[0,1]$ by using transformation $t = 2x-1$. The first kind shifted Chebyshev polynomials $T_n^*(x)$ of degree n in x on the interval $[0,1]$ are given as

$$T_n^*(x) = T_n(t) = T_n(2x-1),$$

and the shifted Chebyshev extreme points for $T_n^*(x)$ are defined as [9]

$$x_m = \frac{1}{2}\big(\cos(\frac{m\pi}{N}) + 1\big), \qquad 0 \leq m \leq N,\ \ N = 1, 2, 3, \cdots .$$

The orthogonality relation with weight function $\varrho^*(x) = \frac{1}{\sqrt{x-x^2}}$ takes the form

$$\int_{0}^{1} T_l^*(x)\ T_k^*(x)\ \varrho^*(x)dx = \kappa_l \delta_{lk}.$$

For numerical purpose, we use power series representation of shifted Chebyshev polynomials [12]

$$T_n^*(x) = \sum_{l=0}^{n} \frac{n(-1)^{(n-l)}(2)^{2l}(n+l-1)!}{(2l)!(n-l)!} x^l, \qquad n > 0. \tag{3.1}$$

For the presented numerical scheme, we introduce ψ-shifted Chebyshev polynomials as

$$\chi_n^{*\psi}(x) = \sum_{l=0}^{n} \frac{n(-1)^{(n-l)}(2)^{2l}(n+l-1)!}{(2l)!(n-l)!} (\psi(x))^l, \quad n > 0,$$

where

$$\chi_n^{*\psi}(x) = T_n^*(\psi(x)), \qquad \text{for all } x. \tag{3.2}$$

ψ-shifted Chebyshev polynomials may be written as

$$\chi_n^{*\psi}(x) = \sum_{l=0}^{n} \lambda_{nl} (\psi(x))^l, \qquad n > 0,$$

where

$$\lambda_{nl} = n\frac{(-1)^{(n-l)}(2)^{2l}(n+l-1)!}{(2l)!(n-l)!}.$$

The orthogonality of ψ-shifted Chebyshev polynomials with weight function $\varrho^{*\psi}(x)$ is

$$\int_0^1 \chi_l^{*\psi}(x)\chi_k^{*\psi}(x)\varrho^{*\psi}(x)\psi'(x)dx = \kappa_l\delta_{lk}, \tag{3.3}$$

where $\varrho^{*\psi}(x) = \varrho^*(\psi(x))$.

Fractional derivative of shifted Chebyshev polynomials is given as [17]

$$^cD_a^\delta\ T_n^* = \begin{cases} 0, & n < \lceil\delta\rceil, \\ \sum\limits_{l=\lceil\delta\rceil}^{n} \frac{\lambda_{nl}l!}{(l-\delta)!}(x)^{l-\delta}, & n \geq \lceil\delta\rceil. \end{cases}$$

Using Lemma 2.3, we get a similar formula for fractional derivative of ψ-shifted Chebyshev polynomials as

$$^cD_a^{\delta,\psi}\ \chi_n^{*\psi} = \begin{cases} 0, & n < \lceil\delta\rceil, \\ \sum\limits_{l=\lceil\delta\rceil}^{n} \frac{\lambda_{nl}l!}{(l-\delta)!}(\psi(x))^{l-\delta}, & n \geq \lceil\delta\rceil, \end{cases} \tag{3.4}$$

where $\lceil\delta\rceil$ represents the smallest integer greater than or equal to δ and is called ceiling function.

4. Approximation of functions

Chebyshev and shifted Chebyshev polynomials have a wide range of application in numerical computation. The series expansion representation of a function using orthogonal polynomials is significant in approximation theory which forms basis of the solution for different types of differential equations. The solution of fractional partial differential equations by developing the operational matrices with the aid of these orthogonal polynomials produce reliable results.

A function $u(x)$ for $x \in [a,b]$ can be represented as

$$u(x) = \sum_{n=0}^{\infty} b_n\ \chi_n^{*\psi}(x), \tag{4.1}$$

where b_n are expansion coefficients which can be calculated as

$$b_n = \frac{(u, \chi_n^{*\psi})}{\|\chi_n^{*\psi}\|}.$$

For practical purposes, we use first m-terms of ψ-shifted Chebyshev polynomials and approximate $u(x)$ as

$$u(x) \simeq u_m(x) = \sum_{n=0}^{m-1} b_n\ \chi_n^{*\psi}(x) = \mathfrak{B}^T\Omega_m^{*\psi}(x),$$

where

$$\mathfrak{B} = [b_0, b_1, ..., b_{m-1}]^T,$$

and

$$\Omega_m^{*\psi}(x) = \left[\chi_0^{*\psi}(x), \chi_1^{*\psi}(x), ..., \chi_{m-1}^{*\psi}(x)\right]^T, \qquad x \in K. \tag{4.2}$$

While dealing with two dimensional problems, it is crucial to introduce functions of two variables. Suppose, we use ψ-shifted Chebyshev polynomials for both variables x and t, then

$$\mathfrak{G}_{mn}^{*\psi}(x,t) = \chi_m^{*\psi}(x)\chi_n^{*\psi}(t), \qquad (x,t) \in \nabla = [0,1] \times [0,1]. \tag{4.3}$$

Next, we will show that two variable functions $\mathfrak{G}_{mn}^{*\psi}(x,t)$ are orthogonal.

Theorem 4.1. *The orthogonality of $\mathfrak{G}_{mn}^{*\psi}(x,t)$ is*

$$\int_0^1 \int_0^1 \mathfrak{G}_{mn}^{*\psi}(x,t)\mathfrak{G}_{ij}^{*\psi}(x,t)\varrho^{*\psi}(x,t)\psi'(x)\psi'(t)dxdt = \kappa\delta_{mi}\delta_{nj},$$

where $\varrho^{*\psi}(x,t) = \varrho^{*\psi}(x)\varrho^{*\psi}(t)$ is the weight function and $\kappa = \kappa_m\kappa_n$.

Proof. For the proof of the theorem, we use orthogonality of ψ-shifted Chebyshev polynomials.

$$\begin{aligned}
&\int_0^1 \int_0^1 \mathfrak{G}_{mn}^{*\psi}(x,t)\mathfrak{G}_{ij}^{*\psi}(x,t)\varrho^{*\psi}(x,t)\psi'(x)\psi'(t)dxdt \\
=&\int_0^1 \int_0^1 \chi_m^{*\psi}(x)\chi_n^{*\psi}(t)\chi_i^{*\psi}(x)\chi_j^{*\psi}(t)\varrho^{*\psi}(x)\varrho^{*\psi}(t)\psi'(x)\psi'(t)dxdt \\
=&\int_0^1 \chi_m^{*\psi}(x)\chi_i^{*\psi}(x)\varrho^{*\psi}(x)\psi'(x)dx \int_0^1 \chi_n^{*\psi}(t)\chi_j^{*\psi}(t)\varrho^{*\psi}(t)\psi'(t)dt \\
=&\kappa_m\delta_{mi}\kappa_n\delta_{nj} \\
=&\kappa\delta_{mi}\delta_{nj}.
\end{aligned}$$

□

The function $u(x,t)$ with independent variables x and t on the interval ∇ can be expanded in term of double ψ-shifted Chebyshev polynomials as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{kl}\mathfrak{G}_{kl}^{*\psi}(x,t),$$

where the coefficients c_{kl} are the expansion coefficients and can be obtained by solving

$$c_{kl} = \frac{(u, \mathfrak{G}_{kl}^{*\psi})}{\|\mathfrak{G}_{kl}^{*\psi}\|}.$$

So that, the truncated series for $u(x,t)$ is

$$u(x,t) \simeq u_{mn}(x,t) = \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} c_{kl}\mathfrak{G}_{kl}^{*\psi}(x,t) = (\Omega_m^{*\psi}(x))^T M_{kl}\Omega_n^{*\psi}(t),$$

where the ψ-shifted Chebyshev vectors $\Omega_m^{*\psi}(x)$ and $\Omega_n^{*\psi}(t)$ are the same as defined in (4.2) and the matrix M_{kl} is the matrix of expansion coefficients c_{kl}.

5. Formulation of numerical method

Approximation of continuous functions in different numerical techniques is vital in scientific computing for finding the solution of mathematical problems involving fractional type operators. Interpolation is the estimation or determination of a function from certain known values of the function. Polynomial interpolation is used to approximate the function u with some polynomial function provided that its values at distinct points are known. These polynomials can be represented using different basis such as monomial, Newton and Lagrange etc. Relationship between different forms of interpolating polynomial can be made by suitable coordinate transformations. Irrespective of the fact that function is a polynomial or not, polynomial interpolation is useful [7,11,28].

The interpolating polynomial of monomial-type basis is

$$\begin{aligned} P_{m,n}(x,t) =& \sum_{r=0}^{m} \sum_{s=0}^{n} \varsigma_{r,s}(\psi(x))^r(\psi(t))^s, \\ =& \mathbb{X}^T \mathcal{C}\ \mathbb{T}, \end{aligned} \tag{5.1}$$

where $\mathbb{X} = [1, \psi(x), (\psi(x))^2, ...(\psi(x))^m]^T$, $\mathbb{T} = [1, \psi(t), (\psi(t))^2, ...(\psi(t))^n]^T$ and $\mathcal{C}$ is a square matrix defined as

$$\mathcal{C} = \begin{bmatrix} \varsigma_{0,0} & \varsigma_{0,1} & \cdots & \varsigma_{0,n} \\ \varsigma_{1,0} & \varsigma_{1,1} & \cdots & \varsigma_{1,n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \varsigma_{m,0} & \varsigma_{m,1} & \cdots & \varsigma_{m,n} \end{bmatrix}.$$

Consider the interpolating points

$$\mathfrak{P}_{m,n} = \{(x_r, t_s) | r = 0, 1, 2, ..., m, s = 0, 1, 2, ..., n\},$$

with $x_r \neq x_s$ and $t_r \neq t_s$ and function values are defined as

$$u_{r,s} := u(x_r, t_s),$$

where u is a square matrix. A unique two-variable polynomial for some specific selection of interpolating points $\mathfrak{P}_{m,n}$ is

$$P_{m,n}(x_r, t_s) \equiv u(x_r, t_s) =: u_{r,s}.$$

At interpolating points, another form is defined as [28]

$$P_{m,n}(x,t) = \text{vec}(\mathcal{C})^T\ \mathfrak{m}(x,t),$$

where $\mathfrak{m}(x,t)$ is two-variable monomial basis and is defined as

$$\begin{aligned} \mathfrak{m}(x,t)^T =& [1\ \ \psi(x)\ \ (\psi(x))^2...(\psi(x))^m\ \ \psi(t)\ \ \psi(x)\psi(t)\ \ (\psi(x))^2\psi(t)...(\psi(x))^m\psi(t)... \\ & (\psi(t))^n\ \ \psi(x)(\psi(t))^n...(\psi(x))^m(\psi(t))^n], \end{aligned}$$

and $\text{vec}(\mathcal{C})^T = [\varsigma_{0,0}\ \ \varsigma_{1,0} \cdots \varsigma_{m,0}\ \ \varsigma_{0,1}\ \ \varsigma_{1,1} \cdots \varsigma_{m,1} \cdots \varsigma_{0,n}\ \ \varsigma_{1,n} \cdots \varsigma_{m,n}]$.

In order to calculate the coefficients $\varsigma_{r,s}$, we solve $\mathfrak{N}\mathcal{A}=\mathfrak{U}$, where $\mathfrak{N}=[\mathfrak{m}(x_r,t_s)]$ is a square matrix, $\mathcal{A}$=vec($\mathcal{C}$) and $\mathfrak{U}$=$[u_{0,0}\ \ u_{1,0}...u_{m,0}\ \ u_{0,1}\ \ u_{1,1}...u_{m,1}...u_{0,n}\ \ u_{1,n}...u_{m,n}]^T$. The interpolating polynomial in terms of double ψ-shifted Chebyshev polynomial basis is

$$P_{m,n}(x,t)=\sum_{r=0}^{m}\ \sum_{s=0}^{n}\ \xi_{r,s}\chi_r^{*\psi}(x)\chi_s^{*\psi}(t), \tag{5.2}$$

where the coefficients $\xi_{r,s}$ can be calculated by solving $\mathfrak{N}\mathcal{A}=\mathfrak{U}$.

In the next steps, we will relate these polynomial basis with the Lagrangian basis. The interpolating polynomials in term of Lagrangian basis at specified interpolating points is

$$\begin{aligned} P_{m,n}(x,t)=&\sum_{r=0}^{m}\ \sum_{s=0}^{n}u_{r,s}L_{r,m}(x)L_{n,s}(t), \\ =&\mathbb{L}^T\ u\ \mathbb{L}^*, \end{aligned} \tag{5.3}$$

so that $\mathbb{L}=[L_{0,m}(x)\ \ L_{1,m}(x)...L_{m,m}(x)]^T$ and $\mathbb{L}^*=[L_{n,0}(t)\ \ L_{n,1}(t)...L_{n,n}(t)]^T$.

Also, we define $L_{r,m}(x)=\frac{\mathbb{Q}(x)}{x_r-x_k}$ for $r=0,1,2,...,m$ and $L_{n,s}(t)=\frac{\mathbb{Q}^*(t)}{t_s-t_k}$ for $s=0,1,2,...,n$, where

$$\mathbb{Q}(x)=\prod_{k=0,k\neq r}^{m}(x-x_k)\quad\text{and}\quad\mathbb{Q}^*(t)=\prod_{k=0,k\neq s}^{n}(t-t_k).$$

In another form [28]

$$P_{m,n}(x,t)=vec(\mathfrak{U})^T\ \mathfrak{L}(x,t),$$

where

$$\mathfrak{L}(x,t)=\begin{bmatrix} L_{0,m}(x).L_{n,0}(t)\\ L_{1,m}(x).L_{n,0}(t)\\ .\\ .\\ L_{m,m}(x).L_{n,0}(t)\\ L_{0,m}(x).L_{n,1}(t)\\ .\\ .\\ L_{m,m}(x).L_{n,n}(t)\end{bmatrix}.$$

Combining Equation (5.2) and (5.3)

$$\sum_{r=0}^{m}\ \sum_{s=0}^{n}\ \xi_{r,s}\chi_r^{*\psi}(x)\chi_s^{*\psi}(t)=\sum_{r=0}^{m}\ \sum_{s=0}^{n}u_{r,s}L_{r,m}(x)L_{n,s}(t). \tag{5.4}$$

In matrix form, we have $(\mathfrak{G}^{*\psi})^T\ \mathcal{A}=\mathfrak{L}^T\ \mathfrak{U}$ where $\mathfrak{G}^{*\psi}$ represents two-variable ψ-shifted Chebyshev polynomial basis and $\mathfrak{L}=\mathfrak{W}\ \mathfrak{G}^{*\psi}$ with $\mathfrak{W}=(\mathfrak{N}^T)^{-1}$. Thus, for

$m = n$, we have

$$u(x,t) = \sum_{r=0}^{n} \sum_{s=0}^{n} u(x_r, t_s)\mathfrak{W}_{rs}\mathfrak{G}_{rs}^{*\psi}(x,t). \tag{5.5}$$

To solve fractional partial differential equation, we have to take partial derivatives with respect to both independent variables. Taking partial derivative of Equation (5.5) with respect to x

$$\begin{aligned}\frac{\partial^{\alpha,\psi} u(x,t)}{\partial x^{\alpha}} &= \sum_{r=0}^{n} \sum_{s=0}^{n} u(x_r, t_s)\mathfrak{W}_{rs}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{rs}^{*\psi}(x,t)}{\partial x^{\alpha}}, \\ &= \sum_{r=0}^{n} \sum_{s=0}^{n} u(x_r, t_s)\mathbb{G}^{*\psi,\alpha},\end{aligned}$$

where

$$\mathbb{G}^{*\psi,\alpha}(x,t) = \begin{bmatrix} \sum_{s=0}^{n}\mathfrak{W}_{os}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{so}^{*\psi}(x,t)}{\partial x^{\alpha}} & \sum_{s=0}^{n}\mathfrak{W}_{1s}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{so}^{*\psi}(x,t)}{\partial x^{\alpha}} & \dots & \sum_{s=0}^{n}\mathfrak{W}_{ns}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{s0}^{*\psi}(x,t)}{\partial x^{\alpha}} \\ \sum_{s=0}^{n}\mathfrak{W}_{os}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{s1}^{*\psi}(x,t)}{\partial x^{\alpha}} & \sum_{s=0}^{n}\mathfrak{W}_{1s}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{s1}^{*\psi}(x,t)}{\partial x^{\alpha}} & \dots & \sum_{s=0}^{n}\mathfrak{W}_{ns}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{s1}^{*\psi}(x,t)}{\partial x^{\alpha}} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \sum_{s=0}^{n}\mathfrak{W}_{os}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{sn}^{*\psi}(x,t)}{\partial x^{\alpha}} & \sum_{s=0}^{n}\mathfrak{W}_{1s}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{sn}^{*\psi}(x,t)}{\partial x^{\alpha}} & \dots & \sum_{s=0}^{n}\mathfrak{W}_{ns}\frac{\partial^{\alpha,\psi}\mathfrak{G}_{sn}^{*\psi}(x,t)}{\partial x^{\alpha}} \end{bmatrix}.$$

Now, taking partial derivative of Equation (5.5) with respect to t

$$\begin{aligned}\frac{\partial^{\beta,\psi} u(x,t)}{\partial t^{\beta}} &= \sum_{r=0}^{n} \sum_{s=0}^{n} u(x_r, t_s)\mathfrak{W}_{rs}\frac{\partial^{\beta,\psi}\mathfrak{G}_{rs}^{*\psi}(x,t)}{\partial t^{\beta}}, \\ &= \sum_{r=0}^{n} \sum_{s=0}^{n} u(x_r, t_s)\mathbb{G}^{*\psi,\beta},\end{aligned}$$

where

$$\mathbb{G}^{*\psi,\beta}(x,t) = \begin{bmatrix} \sum_{s=0}^{n}\mathfrak{W}_{os}\frac{\partial^{\beta,\psi}\mathfrak{G}_{so}^{*\psi}(x,t)}{\partial t^{\beta}} & \sum_{s=0}^{n}\mathfrak{W}_{1s}\frac{\partial^{\beta,\psi}\mathfrak{G}_{so}^{*\psi}(x,t)}{\partial t^{\beta}} & \dots & \sum_{s=0}^{n}\mathfrak{W}_{ns}\frac{\partial^{\beta,\psi}\mathfrak{G}_{so}^{*\psi}(x,t)}{\partial t^{\beta}} \\ \sum_{s=0}^{n}\mathfrak{W}_{os}\frac{\partial^{\beta,\psi}\mathfrak{G}_{s1}^{*\psi}(x,t)}{\partial t^{\beta}} & \sum_{s=0}^{n}\mathfrak{W}_{1s}\frac{\partial^{\beta,\psi}\mathfrak{G}_{s1}^{*\psi}(x,t)}{\partial t^{\beta}} & \dots & \sum_{s=0}^{n}\mathfrak{W}_{ns}\frac{\partial^{\beta,\psi}\mathfrak{G}_{s1}^{*\psi}(x,t)}{\partial t^{\beta}} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \sum_{s=0}^{n}\mathfrak{W}_{os}\frac{\partial^{\beta,\psi}\mathfrak{G}_{sn}^{*\psi}(x,t)}{\partial t^{\beta}} & \sum_{s=0}^{n}\mathfrak{W}_{1s}\frac{\partial^{\beta,\psi}\mathfrak{G}_{sn}^{*\psi}(x,t)}{\partial t^{\beta}} & \dots & \sum_{s=0}^{n}\mathfrak{W}_{ns}\frac{\partial^{\beta,\psi}\mathfrak{G}_{sn}^{*\psi}(x,t)}{\partial t^{\beta}} \end{bmatrix}.$$

So, we get the required differentiation matrices. For numerical computations, all these calculations are executed in Matlab. To find out the approximate solution, we further proceed as

$$[\mathrm{diag}(\tilde{\eta})\mathbb{G}^{*\psi,\alpha} + \mathrm{diag}(\tilde{\mu})\mathbb{G}^{*\psi,\beta}]\ U = G, \tag{5.6}$$

where $\mathrm{diag}(\tilde{\eta}) = [\eta(x_o, t_o), \eta(x_1, t_1), ..., \eta(x_n, t_n)]$, $\mathrm{diag}(\tilde{\mu}) = [\mu(x_o, t_o), \mu(x_1, t_1), ..., \mu(x_n, t_n)]$, $U = [U(x_r, t_s)]_{(n+1)\times(n+1)}$ and $G = [G(x_r, t_s)]_{(n+1)\times(n+1)}$.

The system in Equation (5.6) is an algebraic system.

6. Theoretical analysis

In this section first we find the error for neglected terms and then we deal with the estimation of the best approximation error norm for two-variable function, whose expansion is carried out in terms of ψ-shifted Chebyshev polynomials. For the purpose, we find upper bound of the absolute error with the aid of Lagrange interpolation polynomials.

Let $E_m^{*\psi}(x)$ be the error in the approximation of $u(x)$ in terms of ψ-shifted Chebyshev polynomials then we have following theorem.

Theorem 6.1 ([29]). *Let $u(x)$ be approximated by ψ-shifted Chebyshev polynomials as defined in (4.1), then error in the approximation of $u(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all neglected coefficients.*

$$E_m^{*\psi}(x) = |u(x) - u_m(x)| \leq \sum_{k=m}^{\infty} |b_k|, \quad \textit{where} \quad |\chi_k^{*\psi}(x)| \leq 1.$$

To proceed further, we consider $u(x,t)$ a smooth function on $\nabla = [0,1]\times[0,1]$. Let us define

$$¥_{m,n}^{*\psi} = Span\{\chi_m^{*\psi}(x)\chi_n^{*\psi}(t), \quad m = 0,1,...,M-1,\ n = 0,1,...,N-1\},$$

where $\chi_m^{*\psi}(x)$ and $\chi_n^{*\psi}(t)$ are ψ-shifted Chebyshev polynomials of M and N degree respectively.

Theorem 6.2. *Let us consider a smooth function $u(x,t)$ on ∇ and $\breve{u}(x,t)$ its best approximation out of $¥_{m,n}^{*\psi}$, then there exists positive constants $\hbar_1, \hbar_2$ and $\hbar_3$ such that*

$$\max_{(x,t)\in\nabla}\left|\frac{\partial^M u(x,t)}{\partial x^M}\right| \leq \hbar_1,\ \max_{(x,t)\in\nabla}\left|\frac{\partial^N u(x,t)}{\partial t^N}\right| \leq \hbar_2,\ \max_{(x,t)\in\nabla}\left|\frac{\partial^{M+N} u(x,t)}{\partial x^M \partial t^N}\right| \leq \hbar_3,$$

then

$$\|u(x,t) - \breve{u}(x,t)\|_\infty \leq \aleph_1\aleph_2,$$

where $\aleph_1 = \left(\frac{1}{M!}\cdot\frac{\hbar_1}{2^{2M-1}} + \frac{1}{N!}\cdot\frac{\hbar_2}{2^{2N-1}} + \frac{1}{M!N!}\cdot\frac{\hbar_3}{2^{2M+2N-2}}\right)$ *and* $\aleph_2 = 2\Big(\sqrt{\psi(1)-\psi(a)} - \sqrt{\psi(0)-\psi(a)}\Big)$.

Proof. Let us define the interpolating polynomial for $u(x,t)$ at the points (x_m, t_n) as $P_{m,n}^{*\psi}(x,t)$ where x_m and t_n are the roots of ψ-shifted Chebyshev polynomials $\chi_m^{*\psi}(x)$ and $\chi_n^{*\psi}(t)$. Then, by the definition of best approximation

$$\|u(x,t) - \breve{u}(x,t)\|_\infty \leq \|u(x,t) - P_{m,n}^{*\psi}(x,t)\|_\infty \quad \forall\ P_{m,n}^{*\psi}(x,t) \in ¥_{m,n}^{*\psi}. \tag{6.1}$$

Similar to the procedure in [8,13,21], we have

$$u(x,t) - P^{*\psi}_{m,n}(x,t) = \frac{\partial^M u(\theta,t)}{\partial x^M} . \frac{\prod_{m=0}^{M-1}(x-x_m)}{M!} + \frac{\partial^N u(x,\zeta)}{\partial t^N} . \frac{\prod_{n=0}^{N-1}(t-t_n)}{N!} - \frac{\partial^{M+N} u(\tilde{\theta},\tilde{\zeta})}{\partial x^M \partial t^N} . \frac{\prod_{m=0}^{M-1}(x-x_m) \prod_{n=0}^{N-1}(t-t_n)}{M!N!},$$

where (θ,ζ) and $(\tilde{\theta},\tilde{\zeta}) \in \nabla$.

$$\begin{aligned} \left|u(x,t) - P^{*\psi}_{m,n}(x,t)\right| \leq & \frac{1}{M!} \max_{(x,t)\in\nabla}\left|\frac{\partial^M u(x,t)}{\partial x^M}\right| . \left|\prod_{m=0}^{M-1}(x-x_m)\right| \\ & + \frac{1}{N!} \max_{(x,t)\in\nabla}\left|\frac{\partial^N u(x,t)}{\partial t^N}\right| . \left|\prod_{n=0}^{N-1}(t-t_n)\right| \\ & + \frac{1}{M!N!} \max_{(x,t)\in\nabla}\left|\frac{\partial^{M+N} u(x,t)}{\partial x^M \partial t^N}\right| . \left|\prod_{m=0}^{M-1}(x-x_m) \prod_{n=0}^{N-1}(t-t_n)\right|. \end{aligned} \tag{6.2}$$

By hypothesis on u, there exists constant $\hbar_1$ such that $\max_{(x,t)\in\nabla}\left|\frac{\partial^M u(x,t)}{\partial x^M}\right| \leq \hbar_1$. Next, we derive the bound of the factor $\left|\prod_{m=0}^{M-1}(x-x_m)\right|$. The transformation $x = \frac{1}{2}(\wp+1)$ maps the interval $[-1,1]$ to $[0,1]$. Therefore

$$\min_{x_m\in[0,1]} \max_{x\in[0,1]}\left|\prod_{m=0}^{M-1}(x-x_m)\right| = \min_{\wp_m\in[-1,1]} \max_{\wp\in[-1,1]}\left|\prod_{m=0}^{M-1}\frac{1}{2}(\wp-\wp_m)\right| = \frac{1}{2^{2M-1}},$$

where $\wp_m{}'s$ are the roots of Chebyshev polynomials. Similarly, we have

$$\max_{(x,t)\in\nabla}\left|\frac{\partial^N u(x,t)}{\partial t^N}\right| \leq \hbar_2, \quad \max_{(x,t)\in\nabla}\left|\frac{\partial^{M+N} u(x,t)}{\partial x^M \partial t^N}\right| \leq \hbar_3$$

and

$$\min_{t_n\in[0,1]} \max_{t\in[0,1]}\left|\prod_{n=0}^{N-1}(t-t_n)\right| = \frac{1}{2^{2N-1}}.$$

Thus

$$\left|u(x,t) - P^{*\psi}_{m,n}(x,t)\right| \leq \frac{1}{M!}.\frac{\hbar_1}{2^{2M-1}} + \frac{1}{N!}.\frac{\hbar_2}{2^{2N-1}} + \frac{1}{M!N!}.\frac{\hbar_3}{2^{2M+2N-2}}.$$

By Equation (6.1)

$$\begin{aligned} \|u(x,t) - \breve{\mathfrak{u}}(x,t)\|_\infty \leq & \left(\int_0^1\int_0^1 |u(x,t) - P^{*\psi}_{m,n}(x,t)|^2 \varrho^{*\psi}(x,t)\psi'(x)\psi'(t)dxdt\right)^{\frac{1}{2}} \\ \leq & \aleph_1\left[\left(\int_0^1 \varrho^{*\psi}(x)\psi'(x)dx\right)\left(\int_0^1 \varrho^{*\psi}(t)\psi'(t)dt\right)\right]^{\frac{1}{2}} \\ = & \aleph_1\left[\left(\int_0^1 \frac{1}{(\psi(x)-\psi(a))^{\frac{1}{2}}\sqrt{(1-(\psi(x)-\psi(a)))}}\psi'(x)dx\right)\right. \end{aligned}$$

$$\times\Big(\int_0^1 \frac{1}{(\psi(t)-\psi(a))^{\frac{1}{2}}\sqrt{(1-(\psi(t)-\psi(a)))}}\psi'(t)dt\Big)\Big]^{\frac{1}{2}}$$
$$\leq \aleph_1\aleph_2,$$

where $\aleph_1 = \big(\frac{1}{M!}\cdot\frac{\hbar_1}{2^{2M-1}} + \frac{1}{N!}\cdot\frac{\hbar_2}{2^{2N-1}} + \frac{1}{M!N!}\cdot\frac{\hbar_3}{2^{2M+2N-2}}\big)$ and $\aleph_2 = 2\big(\sqrt{\psi(1)-\psi(a)} - \sqrt{\psi(0)-\psi(a)}\big)$.

So, by increasing the terms of ψ-shifted Chebyshev functions, the approximate solution converges to exact solution. □

Remark. Precisely, for ψ-shifted Chebyshev polynomials $\chi_m^{*\psi}(x)$ and $\chi_n^{*\psi}(t)$ of M and N degree if $M = N$ and we increase the terms then we have the convergence of approximate solution to exact solution. So, whether the degree of ψ-shifted Chebyshev polynomials $\chi_m^{*\psi}(x)$ and $\chi_n^{*\psi}(t)$ is same or different, convergence is obtained.

Thus, for $M = N$, we have

$$\|u(x,t) - \breve{u}(x,t)\|_\infty \leq 2\Big(\sqrt{\psi(1)-\psi(a)} - \sqrt{\psi(0)-\psi(a)}\Big) \times\Big(\frac{1}{M!}\cdot\frac{(\hbar_1+\hbar_2)}{2^{2M-1}} + \frac{1}{(M!)^2}\cdot\frac{\hbar_3}{2^{4M-2}}\Big).$$

Corollary 6.1. *Suppose that $u(x,t)$ is smooth function defined on ∇ with bounded derivatives as defined in Theorem (6.2) then if $(\Omega_m^{*\psi}(x))^T M_{kl}\Omega_n^{*\psi}(t)$ is expansion of $u(x,t)$ in terms of ψ-shifted Chebyshev polynomials then*

$$\|u(x,t) - (\Omega_m^{*\psi}(x))^T M_{kl}\Omega_m^{*\psi}(t)\| \to 0, \qquad as \quad m\to\infty.$$

7. Numerical results

In this section, we present some fractional partial differential equations whose solution will be carried out by the proposed numerical scheme. The purpose of presenting these numerical examples is to illustrate the accuracy and applicability of the proposed numerical method. We will discuss the applicability of proposed technique numerically and graphically by comparing the proposed numerical results with exact solutions reported in the literature. Matlab is used for the execution of all the computations.

Example 7.1. In this example, we will apply proposed technique on two different cases taking different boundary conditions. In each case, we will consider three different chosen functions ψ and different fractional and integral values of α and β. Let $\psi, f \in C^n(K)$, $\psi : K \to K$, $K = [a,b]$ where $0 \leq a, b \leq 1$, ψ is increasing, $\psi' \neq 0$ for all $x, t \in K$ and $n \in \mathbb{N}$. Here, we test the method for following choices of ψ.

- $\psi_1(x) = x,\ \psi_1(t) = t,$
- $\psi_2(x) = \frac{1}{2}x(x+1),\ \psi_2(t) = \frac{1}{2}t(t+1),$
- $\psi_3(x) = \frac{x(e^x+2)}{e+2},\ \psi_3(t) = \frac{t(e^t+2)}{e+2}.$

Case 1: Consider partial differential equation (1.1) with zero initial and boundary conditions, $\eta(x,t) = \mu(x,t) = 1$ and $0 < \alpha, \beta \leq 2$. The exact solution is

$$u_{ex} = (\psi(t))^{(5-\alpha)}\ (\psi(x))^{(5-\beta)}\ (1-(\psi(x)))(1-(\psi(t))).$$

The function $\mathfrak{g}(x,t)$ is as under

$$\begin{aligned}\mathfrak{g}(x,t) =& \left[\frac{\Gamma(6-\alpha)}{\Gamma(6-2\alpha)}(\psi(t))^{(5-2\alpha)} - \frac{\Gamma(7-\alpha)}{\Gamma(7-2\alpha)}(\psi(t))^{(6-2\alpha)}\right] \times \left[(\psi(x))^{(5-\beta)}\right.\\ &\left. - (\psi(x))^{(6-\beta)}\right] + \left[\frac{\Gamma(6-\beta)}{\Gamma(6-2\beta)}(\psi(x))^{(5-2\beta)} - \frac{\Gamma(7-\beta)}{\Gamma(7-2\beta)}(\psi(x))^{(6-2\beta)}\right]\\ &\times \left[(\psi(t))^{(5-\alpha)} - (\psi(t))^{(6-\alpha)}\right].\end{aligned}$$

Table 1 contains the absolute errors of exact and proposed numerical solution for fractional order derivatives of α and β taking three different choices of function ψ when N=10. From the data we obtained, it is observed that error is almost negligible at many points for integer order derivatives, so, we present the error analysis for fractional derivatives only. From the Table 1, it can be seen that when both the fractional derivatives are same the error is less as compared to the case when fractional derivatives are different. From Figure 1, we can observe that exact and proposed solution are very close, which supports tabulated errors of proposed solution. In Figure 2, the absolute error of exact and proposed solution for ψ_1 in case of $\alpha = 1.8,\ \beta = 1.8$ is plotted.

Table 1. Absolute error when N=10.

		$\alpha = 1.8, \beta = 1.8$			$\alpha = 1.5, \beta = 1.2$	
x, t	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
0.2	6.25e-08	2.69e-08	3.60e-08	2.90e-06	4.17e-07	6.90e-07
0.4	5.28e-07	7.30e-07	6.97e-07	2.92e-05	1.01e-05	1.20e-05
0.6	1.13e-06	4.87e-06	3.83e-06	7.30e-05	5.26e-05	5.27e-05
0.8	9.20e-07	1.39e-05	1.02e-05	7.08e-05	1.03e-04	9.78e-05

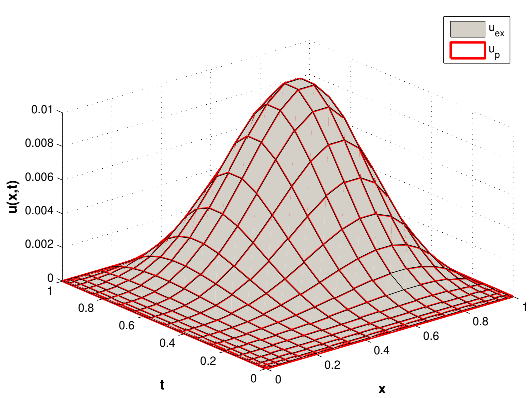

Figure 1. Exact and proposed solution for ψ_1 at $\alpha = \beta = 1.8$.

Case 2: Consider fractional partial differential equation (1.1) with initial and boundary conditions $u(x,0) = u(0,t) = 0$, $u(x,1) = \psi(x)$ and $u(1,t) = \psi(t)$. The coefficients $\eta(x,t) = 5$, $\mu(x,t) = \Gamma(1.5)(\psi(x))^2$ and $0 < \alpha, \beta \leq 2$. The exact

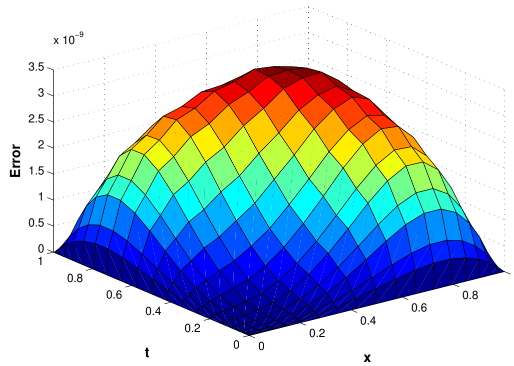

Figure 2. Error for ψ_1 at $\alpha = \beta = 1.8$.

solution is $u_{ex} = \psi(x)\ \psi(t)$. The function $\mathfrak{g}(x,t)$ is

$$\mathfrak{g}(x,t) = \frac{5}{\Gamma(2-\alpha)}\psi(x)(\psi(t))^{(1-\alpha)} + \frac{\Gamma(1.5)}{\Gamma(2-\beta)}(\psi(x))^{(3-\beta)}\psi(t).$$

In this case, we are taking non-zero boundary conditions but the chosen function ψ are same as in case 1. Table 2 shows the absolute error for multiple choices of α and β. From Table 2, it can be seen that the error in case of ψ_1 is considerably low, also the proposed solution is very close to exact solution for other choices of ψ. From Figure 3, we observe that the exact and proposed solution are again almost close to each other. Also, the error between exact and proposed method for ψ_2 when $\alpha = 1.8, \beta = 1.8$ and ψ_3 when $\alpha = 1.5, \beta = 1.2$ is plotted in Figure 4.

Table 2. Absolute error when N=10.

		$\alpha = 1.8, \beta = 1.8$			$\alpha = 1.5, \beta = 1.2$	
x, t	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
0.2	6.94e-18	5.06e-08	2.13e-07	6.53e-05	7.28e-06	1.26e-05
0.4	2.78e-17	2.66e-06	5.32e-06	1.17e-03	2.71e-04	3.43e-04
0.6	0.00	4.19e-05	4.99e-05	4.29e-04	2.25e-03	2.35e-04
0.8	1.11e-16	1.45e-04	1.54e-04	4.78e-03	5.47e-03	5.46e-03

In Example 7.1, we have discussed two different cases of fractional differential equation for zero and non-zero boundary conditions. The use of different chosen functions ψ shows the applicability and validity of presented numerical method to solve variety of fractional differential equations.

Example 7.2. Consider the fractional partial differential Eq. (1.1) with $\eta(x,t) = \mu(x,t) = 1$, initial-boundary conditions $u(x,0) = 0$ and $u(0,t) = 0$, $u(1,t) = \psi(t)$ where $\alpha = \beta = 1/4$ and $\mathfrak{g}(x,t) = \frac{4}{3\ \Gamma(3/4)}\left[(\psi(x))^{3/4}\psi(t) + \psi(t)(\psi(x))^{3/4}\right]$. The exact solution is given as $u(x,t) = \psi(t)\psi(x)$.

In order to compare the results with existing techniques, we will take $\psi(x) = x$ and $\psi(t) = t$. Migxu Yi et al. [30] solved this problem by block pulse operational matrix method and then recently H. Singh and C.S. Singh [25] solved this problem using Legendre scaling function operational matrices. Legendre scaling function

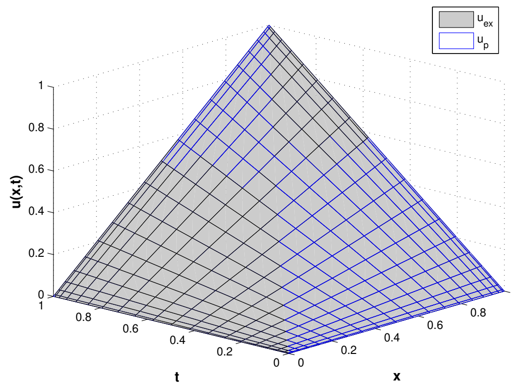

Figure 3. Exact and proposed solution for ψ_1 at $\alpha = \beta = 1.8$.

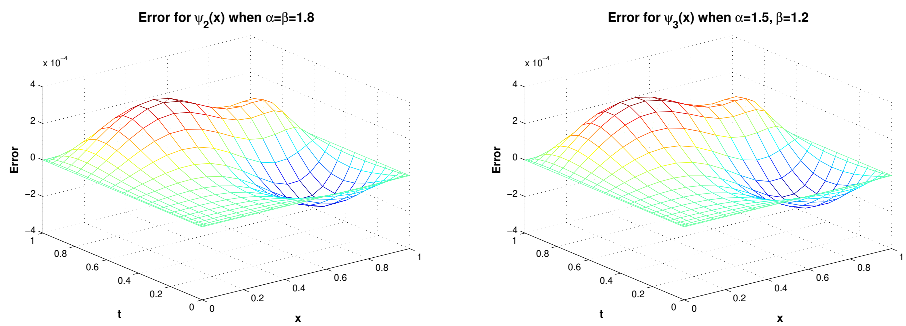

Figure 4. Error for different ψ's taking multiple values of α and β.

technique gives good results as compared to block pulse technique, so, we will compare our results with Legendre scaling function technique. Table 3 shows the comparison of exact solution with the solution by proposed method and Legendre scaling technique. It can be seen that our results are close to exact solution.

Table 3. Comparison of exact, proposed, other method when $\alpha = \beta = \frac{1}{4}$ and $N = 8$.

(x,t)	Exact	Method in [25]	Purposed
(1/8,1/8)	0.0156	0.0155	0.0156
(2/8,2/8)	0.0625	0.0625	0.0625
(3/8,3/8)	0.1406	0.1406	0.1406
(4/8,4/8)	0.2500	0.2500	0.2500
(5/8,5/8)	0.3906	0.3906	0.3906
(6/8,6/8)	0.5625	0.5625	0.5625
(7/8,7/8)	0.7656	0.7656	0.7656

Example 7.3. Again, take the fractional differential Equation (1.1) with $\mu(x,t) = \eta(x,t) = 1$, initial and boundary conditions $u(x,0) = 0$ and $u(0,t) = 0,\ u(1,t) = \psi(t)\sin(1)$ where $0 < \alpha, \beta \leq 1$ and $\mathfrak{g}(x,t) = \frac{(\psi(t))^{(1-\alpha)}}{\Gamma(2-\alpha)} \sin(\psi(x)) + \psi(t)\cos(\psi(x))$.

The exact solution is $u(x,t) = \psi(t)\sin(\psi(x))$.

In order to compare the results with existing techniques, we will take $\psi(x) = x$ and $\psi(t) = t$. M. Javidi and B. Ahmed [15] solved this problem by using numerical Laplace inversion technique. Table 4 shows the comparison of absolute error of proposed solution with numerical Laplace inversion scheme.

Table 4. Absolute error for $\alpha = 0.95$, $\beta = 1$ when N=10.

(x,t)	Method in [15]	Proposed
(0.2,0.2)	1.98e-07	2.98e-08
(0.4,0.4)	2.67e-07	4.64e-07
(0.6,0.6)	9.65e-07	4.64e-07
(0.8,0.8)	8.54e-06	2.98e-07
(1.0,1.0)	4.19e-05	3.07e-06

Example 7.4. Consider Equation (1.1) with $\eta(x,t) = \mu(x,t) = 1$, $0 < \alpha \leq 2, \beta = 2$, initial-boundary conditions $u(x,0) = (\psi(x))^{(\alpha-1)}$, $u(0,t) = 0$ and $u(1,t) = e^{(\psi(t))}$ where $\mathfrak{g}(x,t) = (\psi(x))^{(\alpha-1)}\ e^{\psi(t)}$ with exact solution $u(x,t) = (\psi(x))^{(\alpha-1)}\ e^{\psi(t)}$.

For $\psi(x) = x$ and $\psi(t) = t$, the problem reduces to fractional poisson equation having certain initial and boundary conditions and it is investigated by M. Rehman and R. A. Khan in [24]. In Figure 5, we can see that exact and proposed solution are very close. In Figure 6, the exact and proposed solutions are plotted for multiple values of α taking $t = 0.5$ and $t = 1$. From Figure 6, we observe that the solution for fractional order derivative converges to solution for integer order derivative in both cases. Absolute error for multiple values of α is tabulated in Table 5. From the Table 5, it is evident that the error decreases gradually with the increase in the value of α and it becomes very small for integral value of α.

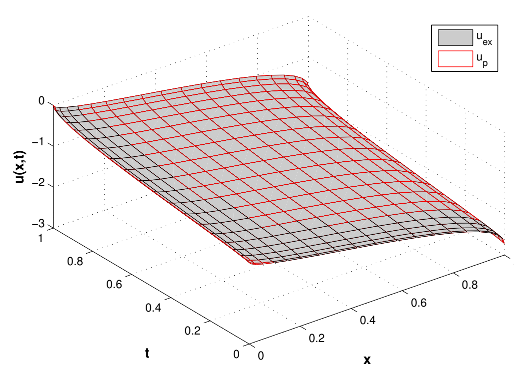

Figure 5. Exact and proposed solution at $\alpha = 1.5$

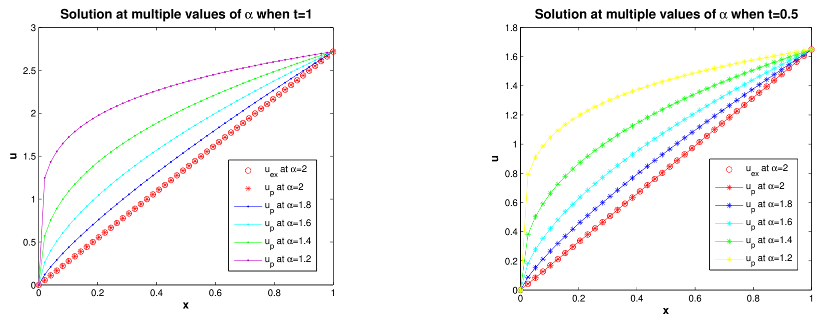

Figure 6. Exact and proposed solution for multiple values of α at $t = 1$ and $t = 0.5$.

Table 5. Absolute error for multiple values of α when N=10.

(x,t)	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
(0.1,0.1)	3.65e-04	3.23e-04	2.42e-04	1.29e-05	1.11e-16
(0.3,0.3)	9.98e-04	8.97e-04	6.80e-04	3.67e-05	2.22e-16
(0.5,0.5)	9.85e-04	9.75e-04	7.37e-04	3.96e-05	2.22e-16
(0.7,0.7)	9.82e-04	8.85e-04	6.72e-04	3.62e-05	2.22e-16
(0.9,0.9)	9.23e-04	8.27e-04	6.23e-04	3.34e-05	4.44e-16

8. Conclusion

In this paper, ψ-shifted Chebyshev polynomials are introduced to solve a class of fractional partial differential equations. ψ-shifted Chebyshev polynomials are used to obtain the operational matrices to reduce the problem under consideration into an algebraic system which is then solved numerically. The obtained numerical results are compared with other published results to illustrate the effectiveness and validity of proposed numerical scheme. In our paper, we have compared the results of our method with methods in [25] and [15]. In summary, we observe that the method works well for different choices of ψ and both homogeneous and inhomogeneous boundary conditions. Also, presented method is applicable for fractional derivatives of different orders.

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