SOLITARY AND LUMP WAVES INTERACTION IN VARIABLE-COEFFICIENT NONLINEAR EVOLUTION EQUATION BY A MODIFIED ANSÄTZ WITH VARIABLE COEFFICIENTS

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Abstract In this work, we examine variable-coefficient nonlinear evolution equations that often describe complex physical models more than constant coefficient models. A modified ansätz with variable coefficients is used for studying the solitary and lump waves interaction in these variable-coefficient nonlinear evolution equations. We discuss the variable-coefficient Kadomtsev-Petviashvili equation to achieve this goal. We present lump wave and interaction solutions between solitary and lump waves for this model. By choosing appropriate values of the variable coefficients, 3d plots and corresponding contour plots are drawn to illustrate the dynamical behaviors of the obtained solutions.

Keywords Kadomtsev-Petviashvili equation, lump wave, interaction solutions, dynamical behaviors.

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1. Introduction

Many complex physical phenomena, such as fiber optics, fluid dynamics, plasma physics, quantum mechanics, etc., arise in various scientific and engineering fields can be simulated in the form of nonlinear evolution equation (NLEE) [56]. To study these models, many effective methods are proposed for the determination of the analytical solutions of NLEE [1,11,18,52]. Recently, lump wave and interaction solutions between lump wave and solitary wave have attracted the attention of many scholars [2,8,19,24,44–46] aiming to make more progress in this field. Lump wave can be often found in oceanography, nonlinear fiber optics and biophysics [20,36,42,47]. Although the current research works focus on the constant-coefficient NLEE, but the variable-coefficient NLEE often describe more complex physical models and anticipate some new physical phenomena. Due to the computational

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complexity and lack of effective methods, relatively few works have studied the lump wave of variable-coefficient NLEE.

The smooth propagation of KdV solitons can exist in plane 1d geometry. However, 1d geometry may not be a reality in laboratory equipment and space and can't explain all observations in the aurora region and higher polar altitudes [7]. Transverse perturbations often occur in higher-dimensional systems, and the wave structure is modified by them. Based on the weakly transverse perturbations in planar geometry, a constant-coefficient Kadomtsev-Petviashvili (KP) equation is proposed [21]. However, the dust-acoustic waves (DAW) tend to appear by a nonplanar geometry. So, it is of great significance to study DAW in the non-planar geometry. Thus, a variable-coefficient KP equation is proposed for describing the non-planar geometry model by using the reductive perturbation method for unmagnified, collisionless and two-temperature ions in dusty plasma [48].

In Ref. [43], and in some of the references therein [9, 10, 53], the significant features of the time dependent coefficients was furnished. In what follows we summarize the important issues examined there. Interest in variable-coefficient nonlinear equations has grown steadily in recent years. It is well known that nonlinear wave equations with variable coefficients are more realistic in various physical situations than their constant coefficients counterparts. It should be pointed out that the existence of the inhomogeneities in the media influences the accompanied physical effects giving rise to spatial or temporal dispersion and nonlinearity variations [43]. For example, in realistic fibre transmission lines, no fibre is homogeneous due to long distance communication and manufacturing problems . When the media are inhomogeneous or the boundaries are nonuniform, variable-coefficient nonlinear evolution equation may arise.

In this work, a modified ansätz with variable coefficients is presented for finding the lump wave and interaction solutions between lump wave and solitary wave of NLEE. The proposed ansätz will be applied to the following variable-coefficient KP equation [57]

$$\tau(t)u_x^2 + \tau(t)u_{xx} + \delta(t)u_{xxxx} - \phi(t)u_{yy} + u_{xt} = 0, \qquad (1.1)$$

where u = u(x, y, t) is the amplitude of the long wave of two-dimensional fluid domain on varying topography or in turbulent over a sloping bottom. The coefficients $\tau(t)$, $\delta(t)$ and $\phi(t)$ represent nonlinearity, dispersion, and disturbed wave velocity along the y direction, respectively [15, 57]. Eq. (1.1) represents the many physical models containing the propagation of the two-dimensional dust-acoustic wave in the dusty plasma consisting of cold dust particles, isothermal electrons and surface waves through shallow seas and marines straits of varying width and depth with nonvanishing vorticity and so on [25]. Wang [42] presented the solitonic solution of Eq. (1.1). Yao [49] obtained the Wronskian and Gramian solutions of Eq. (1.1). The Bäcklund transformation was given by Wu in Ref. [15]. The lump and interactions solutions were derived by using Hirota's bilinear method [37], which was not very suitable for variable-coefficient NLEE. In this work, the lump and interactions solutions will be discussed by a modified ansätz with variable coefficients, which will become our main task. Some special cases of Eq. (1.1) have been presented as follows

(1) When $\phi(t) = 0, \tau(t) = -6, \delta(t) = 1$, and integrate once with respect to x, Eq.

(1.1) becomes a KdV equation [50, 51]

$$u_{xxx} + u_t - 6uu_x = 0, (1.2)$$

which describes the motion of long waves and one-dimensional nonlinear lattice in shallow water under the action of gravity.

(2) When $\phi(t) = \pm 1, \tau(t) = -6, \delta(t) = -1$, Eq. (1.1) becomes a constant-coefficient Kadomtsev-Petviashvili (KP) equation [5]

$$\pm u_{yy} + u_{xt} - u_{xxxx} - 6u_x^2 - 6uu_{xx} = 0.$$
(1.3)

The KP equation has been extensively investigated mathematically and physically in scientific phenomena, such as plasma physics, solid state physics, fiber optics, propagation of waves, chemical physics, and in other fields. The KP equation plays a fundamental role in the theory of propagation of waves and integrable systems. Moreover, it models shallow water waves with weakly nonlinear restoring forces.

The organization of this paper is as follows. Section 2 presents the lump wave and interaction solutions between lump wave and solitary wave based on a modified ansätz with variable coefficients. 3d plots and corresponding contour plots are drawn to show their dynamical behaviors by choosing different values of the variable coefficients; Section 3 gives a conclusion.

2. Lump wave and interaction solutions between lump wave and solitary wave

Under the transformation $\tau(t) = \frac{6\delta(t)}{\Psi_0}$ and $u = 2 \Psi_0 [ln\Im(x, y, t)]_{xx}$ into Eq. (1.1), the bilinear form of Eq. (1.1) can be written as

$$[\delta(t)D_x^4 - \phi(t)D_y^2 + D_t D_x]\Im \cdot \Im = 0,$$
(2.1)

where Ψ_0 is arbitrary constant, D is the bilinear derivative operators defined by

$$D_x^i D_y^j D_z^k D_t^m \Im \cdot \Im = \left(\frac{\partial}{\partial_x} - \frac{\partial}{\partial_{x'}}\right)^i \left(\frac{\partial}{\partial_y} - \frac{\partial}{\partial_{y'}}\right)^j \left(\frac{\partial}{\partial_z} - \frac{\partial}{\partial_{z'}}\right)^k \\ \left(\frac{\partial}{\partial_t} - \frac{\partial}{\partial_{t'}}\right)^m \Im(x, y, z, t) \Im(x', y', z', t')|_{x = x', y = y', z = z', t = t'}.$$

This is equivalent to

$$\Im[\delta(t)\Im_{xxxx} - \phi(t)\Im_{yy} + \Im_{xt}] + 3\delta(t)\Im_{xx}^{2} -4\delta(t)\Im_{x}\Im_{xxx} + \phi(t)\Im_{y}^{2} - \Im_{t}\Im_{x} = 0.$$
(2.2)

To seek the lump wave and interaction solutions between lump wave and solitary wave, a modified ansätz with variable coefficients is proposed as follows

$$\Im = [\Psi_3(t) + \Psi_1(t)x + \Psi_2(t)y]^2 + [\Psi_6(t) + \Psi_4(t)x + \Psi_5(t)y]^2 + \Psi_7(t) + \sigma_1(t)e^{\varsigma_3(t) + \varsigma_1(t)x + \varsigma_2(t)y} + \sigma_2(t)e^{-\varsigma_3(t) - \varsigma_1(t)x - \varsigma_2(t)y},$$
(2.3)

where $\Psi_i(t)$, $\varsigma_i(t)$ and $\sigma_i(t)(i = 1, 2, 3)$ are unkown functions. In previous work [5, 15, 25, 37, 49-51], $\Psi_i(t)$, $\varsigma_i(t)$ and $\sigma_i(t)(i = 1, 2, 3)$ are assumed to be constants, which is not very suitable for variable-coefficient NLEE. Eq. (2.3) has not yet been applied in Eq. (1.1) in other works.

2.1. Lump wave



Figure 1. Lump wave with $\phi(t) = \mu_1 = \Psi_0 = \Psi_5(t) = 1$, $\Psi_1(t) = 2$, y = 0, $\Psi_2(t) = \Psi_4(t) = -1$, (a) 3d plot; (b) corresponding contour plot.

It has become a very interesting topic for exploring lump solutions which are rationally localized solutions in all directions in space. Lump is generally localized for space and time variables, and has a bigger amplitude compared to its surrounding waves. In what follows, we examine specific cases of the given parameters.

When $\sigma_1(t) = \sigma_2(t) = 0$, Eq. (2.3) represents a lump wave. Substituting Eq. (2.3) into Eq. (2.2), the lump wave can be found as follows

$$u = \frac{2\Psi_0 \left(2\Psi_1(t)^2 + 2\Psi_4(t)^2\right)}{\Psi_7(t) + (\Psi_3(t) + x\Psi_1(t) + y\Psi_2(t))^2 + (\Psi_6(t) + x\Psi_4(t) + y\Psi_5(t))^2} - 2\Psi_0$$
$$\frac{\left[2\Psi_1(t)[\Psi_3(t) + x\Psi_1(t) + y\Psi_2(t)] + 2\Psi_4(t)[\Psi_6(t) + x\Psi_4(t) + y\Psi_5(t)]\right]^2}{\left[\Psi_7(t) + [\Psi_3(t) + x\Psi_1(t) + y\Psi_2(t)]^2 + [\Psi_6(t) + x\Psi_4(t) + y\Psi_5(t)]^2\right]^2}. (2.4)$$

All parameters have been interpreted in Appendix A. For a fixed t in Eq. (2.4), by solving the system $\{u_x = 0, u_y = 0\}$, three critical points can be obtained as

$$\begin{split} C_1 &= (\frac{\Psi_2(t)\Psi_6(t) - \Psi_3(t)\Psi_5(t)}{\Psi_1(t)\Psi_5(t) - \Psi_2(t)\Psi_4(t)}, \frac{\Psi_1(t)\Psi_6(t) - \Psi_3(t)\Psi_4(t)}{\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)}), \\ C_2, C_3 &= (\frac{\Psi_3(t)\Psi_5(t)}{\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)} + \frac{\Psi_2(t)\Psi_6(t)}{\Psi_1(t)\Psi_5(t) - \Psi_2(t)\Psi_4(t)} \\ &\pm \frac{\sqrt{3}\Psi_7(t)}{\sqrt{(\Psi_1(t)^2 + \Psi_4(t)^2)\Psi_7(t)}}, \frac{\Psi_1(t)\Psi_6(t) - \Psi_3(t)\Psi_4(t)}{\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)}). \end{split}$$

The corresponding amplitudes are

$$A_1 = \frac{4\Psi_0 \left(\Psi_1(t)^2 + \Psi_4(t)^2\right)}{\Psi_7(t)}, \ A_2 = A_3 = -\frac{\Psi_0 \left(\Psi_1(t)^2 + \Psi_4(t)^2\right)}{2\Psi_7(t)},$$

respectively. We see that the amplitudes count on the parameters Ψ_0 , $\Psi_1(t)$ and $\Psi_2(t)$, not on $\phi(t)$. This means that the amplitude keeps invariant when $\phi(t)$ chooses different functions in solution (2.4). Figures 1-3 describe the influence of disturbed wave velocity $\phi(t)$ on the lump wave in Eq. (2.4). Fig. 1(a) shows 3d plot of lump

wave with $\phi(t) = 1$, and Fig. 1(b) gives the corresponding contour plot. Fig. 2(a) shows 3d plot of lump wave with $\phi(t) = t$, and Fig. 2(b) presents the corresponding contour plot. Fig. 3(a) shows 3d plot of lump wave with $\phi(t) = 1 + \sin t$, and Fig. 3(b) gives the corresponding contour plot.



Figure 2. Lump wave with $\phi(t) = t$, $\mu_1 = \Psi_0 = \Psi_5(t) = 1$, $\Psi_1(t) = 2$, y = 0, $\Psi_2(t) = \Psi_4(t) = -1$, (a) 3d plot; (b) corresponding contour plot.



Figure 3. Lump wave with $\phi(t) = 1 + \sin t$, $\mu_1 = \Psi_0 = \Psi_5(t) = 1$, $\Psi_1(t) = 2$, y = 0, $\Psi_2(t) = \Psi_4(t) = -1$, (a) 3d plot; (b) corresponding contour plot.

2.2. Interaction solutions between lump wave and one solitary wave

When $\sigma_2(t) = 0$, Eq. (2.3) represents the interaction solutions between lump wave and one solitary wave. Substituting Eq. (2.3) into Eq. (2.2), the interaction solutions between lump wave and one solitary wave can be presented as follows

$$u = \left[2\Psi_0 \left(2\Psi_1(t)^2 + 2\Psi_4(t)^2 + \mu_3^2 \sigma_1(t) e^{\varsigma_3(t) + \mu_3 x + \mu_4 y}\right)\right] / \left[\frac{\Psi_1(t)^2 + \Psi_4(t)^2}{\mu_3^2} + \sigma_1(t) e^{\varsigma_3(t) + \mu_3 x + \mu_4 y} + \left(\Psi_3(t) + x\Psi_1(t) + y\Psi_2(t)\right)^2\right]$$

$$+ (\Psi_{6}(t) + x\Psi_{4}(t) + y\Psi_{5}(t))^{2}] - [2\Psi_{0}[\mu_{3}\sigma_{1}(t)e^{\varsigma_{3}(t) + \mu_{3}x + \mu_{4}y} + 2\Psi_{1}(t)[\Psi_{3}(t) + x\Psi_{1}(t) + y\Psi_{2}(t)] + 2\Psi_{4}(t)[\Psi_{6}(t) + x\Psi_{4}(t) + y\Psi_{5}(t)]]^{2}] /[[\frac{\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}}{\mu_{3}^{2}} + \sigma_{1}(t)e^{\varsigma_{3}(t) + \mu_{3}x + \mu_{4}y} + (\Psi_{3}(t) + x\Psi_{1}(t) + y\Psi_{2}(t))^{2} + (\Psi_{6}(t) + x\Psi_{4}(t) + y\Psi_{5}(t))^{2}]^{2}].$$
(2.5)



Figure 4. Interaction solutions between lump wave and one solitary wave with $\phi(t) = \mu_4 = \Psi_0 = 1$, $\Psi_1(t) = 2$, $\Psi_2(t) = \mu_3 = \mu_2 = -1$, $\varsigma_3(t) = t$, when y = -5 in (a) (d), y = 0 in (b) (e) and y = 5 in (c) (f).

All parameters have been interpreted in Appendix B. Figure 4 displays the fission process that one solitary wave splits into one solitary wave and one lump wave conversely at y = -5; 0; 5. When y = -5, a lump rises from the solitary wave can be seen in Fig. 4(a) and then separates in Fig. 4(b) at y = 0. When y = 5, the lump and solitary wave spread ahead respectively in Fig. 4(c). When $\phi(t) = \mu_3 = \Psi_0 = 1$, $\Psi_1(t) = 2$, $\Psi_2(t) = \mu_4 = \mu_2 = -1$, $\varsigma_3(t) = t$, the velocity of the solitary wave is 1 in solution (2.5). So, the amplitude of solitary wave keeps invariant before and after collisions, and then the asymptotic behavior of the solution (2.5) is

$$\lim_{t \to -\infty} \lim_{x \to -\infty} u(x, y, t) = 0,$$
$$\lim_{t \to +\infty} \lim_{x \to -\infty} u(x, y, t) = 0.$$

When $\phi(t) = \cos(2t) + 1$, we can find that the soliton presents a periodic structure in Fig. 5. Due to the influence of $\phi(t)$, the amplitude of solitary and lump wave has become smaller.



Figure 5. Interaction solutions between lump wave and one solitary wave with $\phi(t) = \cos(2t) + 1$, $\mu_4 = \Psi_0 = 1$, $\Psi_1(t) = 2$, $\Psi_2(t) = \mu_3 = \mu_2 = -1$, $\varsigma_3(t) = t$, when y = -5 in (a) (d), y = 0 in (b) (e) and y = 5 in (c) (f).

2.3. Interaction solutions between lump wave and two solitary waves

When $\sigma_1(t) \neq 0$ and $\sigma_2(t) \neq 0$, Eq. (2.3) represents the interaction solutions between lump wave and two solitary waves. Substituting Eq. (2.3) into Eq. (2.2), the interaction solutions between lump wave and two solitary waves can be presented as follows

$$\begin{split} u &= [2\Psi_0[2\Psi_1(t)^2 + \mu_5^2\sigma_1(t)e^{\varsigma_3(t)+\mu_5x+\mu_6y} + \mu_5^2\sigma_2(t)e^{-\varsigma_3(t)-\mu_5x-\mu_6y} \\ &+ 2\Psi_4(t)^2]]/[\Psi_7(t) + \sigma_1(t)e^{\varsigma_3(t)+\mu_5x+\mu_6y} + \sigma_2(t)e^{-\varsigma_3(t)-\mu_5x-\mu_6y} \\ &+ (\Psi_3(t) + x\Psi_1(t) + y\Psi_2(t))^2 + (\Psi_6(t) + x\Psi_4(t) + y\Psi_5(t))^2] \\ &- [2\Psi_0[\mu_5\sigma_1(t)e^{\varsigma_3(t)+\mu_5x+\mu_6y} - \mu_5\sigma_2(t)e^{-\varsigma_3(t)-\mu_5x-\mu_6y} \\ &+ 2\Psi_1(t) (\Psi_3(t) + x\Psi_1(t) + y\Psi_2(t)) + 2\Psi_4(t) (\Psi_6(t) + x\Psi_4(t) + y\Psi_5(t))]^2] \\/[[\Psi_7(t) + \sigma_1(t)e^{\varsigma_3(t)+\mu_5x+\mu_6y} + \sigma_2(t)e^{-\varsigma_3(t)-\mu_5x-\mu_6y} \\ &+ (\Psi_3(t) + x\Psi_1(t) + y\Psi_2(t))^2 + (\Psi_6(t) + x\Psi_4(t) + y\Psi_5(t))^2]^2]. \end{split}$$
(2.6)

All parameters have been interpreted in Appendix C. The fusion phenomenon between the lump wave and two solitary waves can be found in Fig. 6. It can be seen that the lump wave splits from one solitary and merges into the other one, and that the two solitary waves exchange the amplitudes through the energy transfer by the lump wave at y = -5; 0; 5. When $\phi(t) = \mu_6 = \Psi_0 = 1$, $\Psi_1(t) = 2$, $\Psi_2(t) = \mu_5 = -1$, $\varsigma_3(t) = t$, the velocities of two solitary waves are all constants, $v_1 = v_2 = -\frac{2}{3}$ in solution (2.6). So, the amplitudes of two solitary wave keep invariant before and after collisions, and then the asymptotic behavior of the solution (2.6) is

$$\lim_{t \to -\infty} \lim_{x \to -\infty} u(x, y, t) = 0,$$
$$\lim_{t \to -\infty} \lim_{x \to +\infty} u(x, y, t) = 0.$$

When $\phi(t) = \cos(2t) + 1$, we can see that two solitary waves present the periodic structure in Fig. 7. The amplitude of solitary and lump wave has become smaller.



Figure 6. Interaction solutions between lump wave and two solitary waves with $\phi(t) = \mu_6 = \mu_5 = \Psi_0 = 1$, $\Psi_1(t) = 2$, $\Psi_2(t) = -1$, $\varsigma_3(t) = t$ when y = -5 in (a) (d), y = 0 in (b) (e) and y = 5 in (c) (f).



Figure 7. Interaction solutions between lump wave and two solitary waves with $\phi(t) = \cos(2t) + 1$, $\mu_6 = \mu_5 = \Psi_0 = 1$, $\Psi_1(t) = 2$, $\Psi_2(t) = -1$, $\varsigma_3(t) = t$, when y = -5 in (a) (d), y = 0 in (b) (e) and y = 5 in (c) (f).

3. Conclusion

In this paper, a modified ansätz with variable coefficients is presented for conducting research on the solitary and lump waves interaction in variable-coefficient NLEE. Compared to the previous work [3, 12, 22, 26, 27, 54, 55], Eq. (2.3) contains more arbitrary functions and is more suitable for handling variable coefficients models. Applying the modified ansätz with variable coefficients into the (2+1)-dimensional variable-coefficient KP equation, lump wave and interaction solutions between lump wave and solitary wave are obtained. All calculation results have been verified by Mathematica.

By choosing appropriate values of the variable coefficients, Figures 1-3 displays the influence of disturbed wave velocity $\phi(t)$ on the lump wave in Eq. (2.4). Figure 4 and Figure 5 show the fission process that one solitary wave splits into one solitary

wave and one lump wave conversely in Eq. (2.5). Figure 6 and Figure 7 demonstrate the fusion phenomenon between the lump wave and two solitary waves in Eq. (2.6).

In Appendix A, Eq. (3.9) can be replaced by the following equation

$$\Psi_{2}(t) = \left[\phi(t)\Psi_{1}(t)\Psi_{4}(t)\Psi_{5}(t)\Psi_{7}(t) \pm \sqrt{3}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right) \\ *\sqrt{\delta(t)\phi(t)\Psi_{4}(t)^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\Psi_{7}(t)}\right] / \left[\phi(t)\Psi_{4}(t)^{2}\Psi_{7}(t)\right], \quad (3.1)$$

or

$$3\delta(t)[\Psi_1(t)^2 + \Psi_4(t)^2]^2 - \frac{\phi(t)[\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)]^2\Psi_7(t)}{\Psi_1(t)^2 + \Psi_4(t)^2} = 0$$

In this way, $\phi(t)$ and $\delta(t)$ become free parameters. Then, the corresponding plots can be given by choosing different values of $\delta(t)$, such as

$$\phi(t) = \delta(t) = \mu_1 = \Psi_0 = \Psi_5(t) = 1, \Psi_1(t) = 2, \Psi_2(t) = \Psi_4(t) = -1.$$
(3.2)

Then, Eq. (2.4) is reduced to

$$u = \frac{20}{\left(\frac{2t}{5} + 2x - y\right)^2 + \left(-\frac{4t}{5} - x + y\right)^2 + 375} - \frac{2\left[4\left(\frac{2t}{5} + 2x - y\right) - 2\left(-\frac{4t}{5} - x + y\right)\right]^2}{\left[\left(\frac{2t}{5} + 2x - y\right)^2 + \left(-\frac{4t}{5} - x + y\right)^2 + 375\right]^2}.$$
(3.3)

The corresponding figure of Eq. (3.3) is shown in Fig. 8. Let $\phi(t) = 1 - 12 \sin t, 1 + 4 \sin t$ and $1 - 28 \sin t$ in Eq. (3.2), respectively, the corresponding figure can be seen in Fig. 9. Obviously, the change of A in $\phi(t) = 1 + A \sin t$ did not change the amplitude of the lump wave. Appendix B and Appendix C can be treated in the same way.



Figure 8. lump wave (3.3) when x = 0 in (a), y = 0 in (b) and t = 0 in (c).

As a result, we have derived lump solutions for the considered KP Equation with variable coefficients based on the symbolic computation [4,6,13,14,16,17,23,28–35, 38–41]. The dynamical structure of the acquired lump solution has been studied via presenting variety of plots with some specific choices of the included free parameters to show the localizations of the solutions. Lump is generally localized for space and time variables, and has a bigger amplitude compared to its surrounding waves. It is worth stating that the modified ansätz with variable coefficients is a promising and



Figure 9. lump wave (2.4) with y = 0 when $\phi(t) = 1 - 12 \sin t$ in (a), $\phi(t) = 1 + 4 \sin t$ in (b) and $\phi(t) = 1 - 28 \sin t$ in (c).

robust mathematical tool to examine nonlinear identical models. In the future, we will discuss the application of this method in (3+1) nonlinear integrable equations with variable coefficients.

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Appendix A

$$\begin{split} \Psi_4'(t) &= \frac{\left[-\Psi_1(t)\Psi_2(t) - \Psi_4(t)\Psi_5(t)\right]\Psi_1'(t) + \left[\Psi_1(t)^2 + \Psi_4(t)^2\right]\Psi_2'(t)}{\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)}, \quad (3.4) \\ \Psi_5'(t) &= \frac{\left[-\Psi_2(t)^2 - \Psi_5(t)^2\right]\Psi_1'(t) + \left[\Psi_1(t)\Psi_2(t) + \Psi_4(t)\Psi_5(t)\right]\Psi_2'(t)}{\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)}, \quad (3.5) \\ \Psi_6'(t) &= \left[\phi(t)\left[\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)\right]\left[\Psi_4(t)\Psi_2(t)^2 - 2\Psi_1(t)\Psi_5(t)\Psi_2(t) - \Psi_4(t)\Psi_5(t)^2\right] - \left[\Psi_1(t)^2 + \Psi_4(t)^2\right]\left[\left(-\Psi_2(t)\Psi_3(t) - \Psi_5(t)\Psi_6(t)\right)\Psi_1'(t) + \left[\Psi_1(t)\Psi_3(t) + \Psi_4(t)\Psi_6(t)\right]\Psi_2'(t)\right]\right] / \left[\left(\Psi_1(t)^2 + \Psi_4(t)^2\right)\left[\Psi_1(t)\Psi_5(t) - \Psi_2(t)\Psi_4(t)\right]\left[2\Psi_2(t)\Psi_4(t)\Psi_5(t) + \Psi_1(t)\left(\Psi_2(t)^2 - \Psi_5(t)^2\right)\right] + \left(\Psi_1(t)^2 + \Psi_4(t)^2\right)\left[\left[\Psi_3(t)\Psi_5(t) - \Psi_4(t)\Psi_4(t)\right]\right] \\ &= \left[\Psi_1(t)\Psi_4(t)\right]\Psi_1'(t) + \left(\Psi_1(t)\Psi_4(t) - \Psi_4(t)\Psi_4(t)\right)\left[\left(\Psi_3(t)\Psi_5(t) - \Psi_4(t)\Psi_4(t)\right)\right] + \left(\Psi_4(t)\Psi_4(t)\Psi_4(t)\Psi_4(t)\right) + \left(\Psi_4(t)\Psi_4(t)\Psi_4(t)\Psi_4(t)\right) \right] \end{split}$$

$$-\Psi_{2}(t)\Psi_{6}(t)]\Psi_{1}(t) + (\Psi_{1}(t)\Psi_{6}(t) - \Psi_{3}(t)\Psi_{4}(t))\Psi_{2}(t)]]$$

/[($\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}$) ($\Psi_{1}(t)\Psi_{5}(t) - \Psi_{2}(t)\Psi_{4}(t)$)], (3.7)

$$\Psi_{7}(t) = \mu_{1} \exp\left(2\int \frac{\Psi_{4}(t)\Psi_{2}(t) - \Psi_{5}(t)\Psi_{1}(t)}{\Psi_{2}(t)\Psi_{4}(t) - \Psi_{1}(t)\Psi_{5}(t)}dt\right),$$
(3.8)

$$\delta(t) = \frac{\phi(t) \left(\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)\right){}^2\Psi_7(t)}{3 \left(\Psi_1(t)^2 + \Psi_4(t)^2\right){}^3},\tag{3.9}$$

with $\Psi_1(t)^2 + \Psi_4(t)^2 \neq 0$, $\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t) \neq 0$. μ_1 is integral constant.

Appendix B

$$\Psi_{4}(t) = \mu_{2} \exp\left(\int_{1}^{t} \frac{\mu_{4}\Psi_{1}'(t) - \mu_{3}\Psi_{2}'(t)}{\mu_{4}\Psi_{1}(t) - \mu_{3}\Psi_{2}(t)} dt\right), \varsigma_{1}(t) = \mu_{3}, \varsigma_{2}(t) = \mu_{4}, \quad (3.10)$$

$$\Psi_{5}(t) = \frac{\mu_{4} \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right) - \mu_{3}\Psi_{1}(t)\Psi_{2}(t)}{\mu_{3}\Psi_{4}(t)}, \Psi_{7}(t) = \frac{\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}}{\mu_{3}^{2}}, \quad (3.11)$$

$$\Psi_{6}'(t) = \left[\mu_{3}^{3}[\Psi_{4}(t)][\Psi_{2}(t)][\Psi_{1}(t)\Psi_{6}(t) - \Psi_{3}(t)\Psi_{4}(t)]\Psi_{1}'(t) + \Psi_{4}(t)[\Psi_{1}(t)\Psi_{3}(t)\right]$$

$$+ \Psi_{4}(t)\Psi_{6}(t)]\Psi_{2}'(t)] - \phi(t)\Psi_{2}(t)^{3} \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)] + \mu_{4}\mu_{3}^{2} \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right) \left(\phi(t)\Psi_{1}(t)\Psi_{2}(t)^{2} - \Psi_{4}(t)\Psi_{6}(t)\Psi_{1}'(t)\right) + \mu_{4}^{2}\mu_{3}\phi(t)\Psi_{2}(t) \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)^{2} - \mu_{4}^{3}\phi(t)\Psi_{1}(t) \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)^{2}] / [\mu_{3}^{2}\Psi_{4}(t) \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right) (\mu_{3}\Psi_{2}(t) - \mu_{4}\Psi_{1}(t))],$$
(3.12)

$$\begin{split} \Psi_{3}'(t) &= \left[\mu_{3}^{3}[\Psi_{4}(t)^{2}[\Psi_{2}(t) (\Psi_{1}(t)\Psi_{3}(t) + \Psi_{4}(t)\Psi_{6}(t)) \Psi_{1}'(t) + \Psi_{4}(t)[\Psi_{3}(t)\Psi_{4}(t) - \Psi_{1}(t)\Psi_{6}(t)]\Psi_{2}'(t)] - \phi(t)\Psi_{1}(t)\Psi_{2}(t)^{3}[\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}]\right] \\ &+ \mu_{4}\mu_{3}^{2} \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right) \left[\phi(t)\Psi_{2}(t)^{2} \left(3\Psi_{1}(t)^{2} + 2\Psi_{4}(t)^{2}\right) - \Psi_{3}(t)\Psi_{4}(t)^{2}\Psi_{1}'(t)\right] - 3\mu_{4}^{2}\mu_{3}\phi(t)\Psi_{1}(t)\Psi_{2}(t) \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)^{2} \\ &+ \mu_{4}^{3}\phi(t)\Psi_{1}(t)^{2} \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)^{2}\right] \\ &/ \left[\mu_{3}^{2}\Psi_{4}(t)^{2} \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right) \left(\mu_{3}\Psi_{2}(t) - \mu_{4}\Psi_{1}(t)\right)\right], \end{split}$$
(3.13)
$$\sigma_{1}'(t) &= \left[\sigma_{1}(t)[3\mu_{3}\mu_{4} \left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right) \left[\mu_{3}\phi(t)\Psi_{1}(t)\Psi_{2}(t)^{2} + \Psi_{4}(t)^{2}[\Psi_{1}(t)\varsigma_{3}'(t) - 2\Psi_{4}'(t)]\right] + \mu_{2}^{2}[\Psi_{2}(t)]\left[-\Psi_{1}(t)^{2} - \Psi_{4}(t)^{2}\right) \left(\mu_{2}\phi(t)\Psi_{2}(t)^{2} + 3\Psi_{4}(t)^{2}\varsigma_{4}'(t)\right) \right]$$

$$-2\Psi_{1}(t)]] + \mu_{3}^{2}[\Psi_{2}(t)[(-\Psi_{1}(t)^{2} - \Psi_{4}(t)^{2})(\mu_{3}\phi(t)\Psi_{2}(t)^{2} + 3\Psi_{4}(t)^{2}\varsigma_{3}(t)) + 6\Psi_{1}(t)\Psi_{4}(t)^{2}\Psi_{1}'(t)] + 6\Psi_{4}(t)^{4}\Psi_{2}'(t)] + \mu_{4}^{3}\phi(t)\Psi_{1}(t)(\Psi_{1}(t)^{2} - 3\Psi_{4}(t)^{2}) * (\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}) + 3\mu_{3}\mu_{4}^{2}\phi(t)\Psi_{2}(t)(\Psi_{4}(t)^{4} - \Psi_{1}(t)^{4})]] /[3\mu_{3}\Psi_{4}(t)^{2}(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2})(\mu_{3}\Psi_{2}(t) - \mu_{4}\Psi_{1}(t))],$$
(3.14)

$$\delta(t) = \frac{\phi(t) \left(\Psi_2(t)\Psi_4(t) - \Psi_1(t)\Psi_5(t)\right)^2 \Psi_7(t)}{3 \left(\Psi_1(t)^2 + \Psi_4(t)^2\right)^3},\tag{3.15}$$

with $\Psi_1(t)^2 + \Psi_4(t)^2 \neq 0$, $\mu_4 \Psi_1(t) - \mu_3 \Psi_2(t) \neq 0$, $\mu_3 \Psi_4(t) \neq 0$. $\mu_i(i = 2, 3, 4)$ is integral constant.

Appendix C

$$\Psi_4'(t) = \frac{\Psi_4(t)\left(\mu_6\Psi_1'(t) - \mu_5\Psi_2'(t)\right)}{\mu_6\Psi_1(t) - \mu_5\Psi_2(t)}, \varsigma_1(t) = \mu_5, \varsigma_2(t) = \mu_6, \tag{3.16}$$

$$\Psi_5(t) = \frac{\mu_6 \left(\Psi_1(t)^2 + \Psi_4(t)^2\right) - \mu_5 \Psi_1(t) \Psi_2(t)}{\mu_5 \Psi_4(t)},\tag{3.17}$$

$$\begin{split} \Psi_{7}(t) &= \frac{\frac{\mu_{5}^{4}\sigma_{1}(t)\sigma_{2}(t)}{\mu_{5}^{2}} + \Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}}{\mu_{5}^{2}}, \end{split} \tag{3.18} \\ \Psi_{6}'(t) &= \left[\mu_{5}^{3}\left[\Psi_{4}(t)\left|\Psi_{2}(t)\left(\Psi_{1}(t)\Psi_{6}(t) - \Psi_{3}(t)\Psi_{4}(t)\right)\Psi_{1}'(t) + \Psi_{4}(t)\left|\Psi_{1}(t)\Psi_{3}(t)\right.\right. \\ &+ \Psi_{4}(t)\Psi_{6}(t)\right]\Psi_{2}'(t)\right] - \phi(t)\Psi_{2}(t)^{3}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\right] \\ &+ \mu_{6}\mu_{5}^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\phi(t)\Psi_{1}(t)\Psi_{2}(t)^{2} - \Psi_{4}(t)\Psi_{6}(t)\Psi_{1}'(t)\right) \\ &+ \mu_{6}^{2}\mu_{5}(t)\Psi_{2}(t)\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)^{2} - \mu_{6}^{3}\phi(t)\Psi_{1}(t)\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)^{2}\right] \\ &/\left[\mu_{5}^{2}\Psi_{4}(t)\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\mu_{5}\Psi_{2}(t) - \mu_{6}\Psi_{1}(t)\right)\right], \end{aligned} \tag{3.19} \\ \Psi_{3}'(t) &= \left[\mu_{3}^{3}\left[\Psi_{4}(t)^{2}\left[\Psi_{2}(t)\left(\Psi_{1}(t)\Psi_{3}(t) + \Psi_{4}(t)\Psi_{6}(t)\Psi_{1}'(t)^{2} + \Psi_{4}(t)^{2}\right)\right] \\ &+ \mu_{6}\mu_{5}^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left[\phi(t)\Psi_{2}(t)^{2}\left(3\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\right] \\ &+ \mu_{6}\mu_{5}^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left[\phi(t)\Psi_{2}(t)^{2}\left(3\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)^{2} + \mu_{6}^{3}\phi(t)\Psi_{1}(t)^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\mu_{5}\Psi_{2}(t) - \mu_{6}\Psi_{1}(t)\right)\right], \end{aligned} \tag{3.20} \\ \sigma_{1}'(t) &= \left[\sigma_{1}(t)\left[3\mu_{5}\mu_{6}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\mu_{5}\psi_{2}(t) - \mu_{6}\Psi_{1}(t)\right)\right], \end{aligned} \tag{3.20} \\ \sigma_{1}'(t) &= \left[\sigma_{1}(t)\left[3\mu_{5}\mu_{6}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right\right)\left[\mu_{5}\psi_{1}(t)\Psi_{1}(t)\Psi_{2}(t)^{2} + \Psi_{4}(t)^{2}\right) \\ &+ \left\{\Psi_{4}(t)^{2}\left(\Psi_{1}(t)\Psi_{2}(t)\right] + \mu_{6}^{3}\psi_{6}(t)\Psi_{1}(t)\left(\Psi_{1}(t)^{2} - \Psi_{4}(t)^{2}\right)\right] \\ &+ \left\{\Phi_{4}(t)^{4}\Psi_{2}'(t)\right] + \mu_{6}^{3}\psi_{6}(t)\Psi_{1}(t)\left(\Psi_{1}(t)^{2} - \Psi_{4}(t)^{2}\right) \\ &+ \left\{\Psi_{4}(t)^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\mu_{5}\psi_{2}(t) - \mu_{6}\Psi_{1}(t)\right)\right], \end{aligned} \end{aligned} \tag{3.21} \\ \sigma_{2}'(t) &= \left[\sigma_{2}(t)\left[-3\mu_{5}\mu_{6}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\mu_{5}\psi_{2}(t)-\mu_{6}\Psi_{1}(t)\right)\right] \\ &+ \left\{\Phi_{4}(t)^{4}\Psi_{2}'(t)\right] + \mu_{6}^{3}(-\phi(t))\Psi_{1}(t)\left(\Psi_{1}(t)^{2} - \Psi_{4}(t)^{2}\right) \\ &+ \left\{\Psi_{4}(t)^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\mu_{5}\psi_{2}(t) - \mu_{6}\Psi_{1}(t)^{2}\right) \\ &+ \left\{\Psi_{4}(t)^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left(\mu_{5}\Psi_{2}(t) - \mu_{6}\Psi_{1}(t)^{2}\right)\right] \\ &+ \left\{\Psi_{4}(t)^{2}\left(\Psi_{1}(t)^{2} + \Psi_{4}(t)^{2}\right)\left($$

with $\Psi_1(t)^2 + \Psi_4(t)^2 \neq 0$, $\mu_5 \Psi_2(t) - \mu_6 \Psi_1(t) \neq 0$, $\mu_5 \Psi_4(t) \neq 0$. $\mu_i(i = 5, 6)$ is integral constant.

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