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SOLVING CONVOLUTION SINGULAR INTEGRAL EQUATIONS WITH REFLECTION AND TRANSLATION SHIFTS UTILIZING RIEMANN-HILBERT APPROACH

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Abstract In this paper, method of solution for some kinds of convolution singular integral equations with reflection will be discussed in class $\{0\}$. By means of the theory of Fourier analysis and the theory of boundary value problems of analytic functions, such equations can be transformed into Riemann boundary value problems (i.e., Riemann-Hilbert problems) with nodes and reflection, or a system of linear algebraic equations. In spite of the classical method for solution, we are to give a new method, by which analytic solutions and conditions of Noether solvability are obtained respectively. At the end of this paper, we propose two kinds of convolution singular integral equations with reflections and a finite set of translation shifts.

Keywords Singular integral equations of convolution type, Riemann-Hilbert problems, Noether theory, reflection, translation shifts.

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1. Introduction

Riemann-Hilbert problems and singular integral equations are the powerful tools widely used in scientific fields, such as mathematics, physics, biology and chemistry. Singular integral equations and integral equations of convolution type have systematically been researched, among which a series of valuable achievements were obtained, see [1, 2, 6–8, 16–19, 43] for references and therein. Singular integral equations with a shift have been studied for a long time, and many mathematicians devote themselves to research singular integral equations with Carleman shift by Fredholm operator theory [3–5, 10–12, 20, 21, 39, 40]. Recently, Li and Ren [22–33] dealt with the theory of Noether solvability, the general solutions of some kinds of singular integral equations with convolution kernels and constant coefficients in class $\{0\}$. In fact, integral equations of convolution type and singular integral equations, mathematically, belong to an interesting subject in the theory of integral equations and Riemann-Hilbert problems.

It is well known that equations of convolution type are closely related to singular integral equations and Riemann-Hilbert problems. When there occurs reflection in equations of convolution type, in the related Riemann-Hilbert problems and singular integral equations, the reflection will also occur. The main aim of this paper is to

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solve the following several kinds of singular integral equations of convolution type with reflection.

(1) Singular integral equations with reflection and one pair of kernels

$$
\begin{aligned}
&A_1 f(t) + A_2 f(-t) + \frac{B_1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(\tau)}{\tau - t}d\tau + \frac{B_2}{\pi i}\int_{-\infty}^{+\infty}\frac{f(-\tau)}{\tau - t}d\tau \\
&+ \frac{C_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} k_1(t-\tau)f(\tau)d\tau + \frac{C_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} k_2(t-\tau)f(-\tau)d\tau \\
=&\, g(t), \quad -\infty < t < +\infty.
\end{aligned} \tag{1.1}
$$

(2) Singular integral equations of dual type with reflection

$$
\begin{cases}
A_1 f(t) + B_1 f(-t) + \frac{C_1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(\tau)}{\tau-t}d\tau + \frac{D_1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(-\tau)}{\tau-t}d\tau \\
+\frac{E_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} k_1(t-\tau)f(\tau)d\tau + \frac{F_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} h_1(t-\tau)f(-\tau)d\tau = g(t),\ 0 < t < +\infty; \\
A_2 f(t) + B_2 f(-t) + \frac{C_2}{\pi i}\int_{-\infty}^{+\infty}\frac{f(\tau)}{\tau-t}d\tau + \frac{D_2}{\pi i}\int_{-\infty}^{+\infty}\frac{f(-\tau)}{\tau-t}d\tau \\
+\frac{E_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} k_2(t-\tau)f(\tau)d\tau + \frac{F_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} h_2(t-\tau)f(-\tau)d\tau = g(t), \quad -\infty < t < 0.
\end{cases} \tag{1.2}
$$

(3) Singular integral equations of Wiener-Hopf type with reflection

$$
\begin{aligned}
&Af(t) + \frac{C_1}{\pi i}\int_0^{+\infty}\frac{f(\tau)}{\tau - t}d\tau + \frac{C_2}{\pi i}\int_{-\infty}^{0}\frac{f(-\tau)}{\tau - t}d\tau + \frac{D_1}{\sqrt{2\pi}}\int_0^{+\infty} k(t-\tau)f(\tau)d\tau \\
&+ \frac{D_2}{\sqrt{2\pi}}\int_{-\infty}^{0} h(t-\tau)f(-\tau)d\tau = g(t), \quad 0 < t < +\infty.
\end{aligned} \tag{1.3}
$$

(4) Singular integral equations with reflection and two pairs of kernels

$$
\begin{aligned}
&A_1 f(t) + A_2 f(-t) + \frac{B_1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(\tau)}{\tau - t}d\tau + \frac{B_2}{\pi i}\int_{-\infty}^{+\infty}\frac{f(-\tau)}{\tau - t}d\tau \\
&+ \frac{C_1}{\sqrt{2\pi}}\int_0^{+\infty} k_1(t-\tau)f(\tau)d\tau + \frac{C_2}{\sqrt{2\pi}}\int_0^{+\infty} h_1(t-\tau)f(-\tau)d\tau \\
&+ \frac{D_1}{\sqrt{2\pi}}\int_{-\infty}^{0} k_2(t-\tau)f(\tau)d\tau + \frac{D_2}{\sqrt{2\pi}}\int_{-\infty}^{0} h_2(t-\tau)f(-\tau)d\tau \\
=&\, g(t), \quad -\infty < t < +\infty.
\end{aligned} \tag{1.4}
$$

In Eqs. (1.1)-(1.4), $A, A_j, B_j, C_j, D_j, E_j, F_j (j = 1, 2)$ are real constants, the given functions $k(t), h(t), g(t)$, $k_j(t)$, $h_j(t)(j = 1, 2)$, and $f(t) \in \{0\}$ is an unknown function. In such equations, the reflection occurs, that is, besides the unknown $f(t)$, $f(-t)$ is also appeared. These four kinds of equations have been widely applied in physics, engineering and technology, engineering mechanics, fracture mechanics and other fields. They can not be solved directly by the classical methods of Fredholm integral equations.

In this paper, we shall apply the theory of Fourier analysis and the principle of analytic continuation to solve Eqs. (1.1)-(1.4). Here, our approach of solving equations (1.1)-(1.4) is novel and effective, different from the ones in classical cases, that is to say, we firstly transform Eqs. (1.1)-(1.4) into a system of function equations by Fourier integral transform, and then we again transform the obtained equations into Riemann-Hilbert problems with reflection and nodes. By the method of boundary value of analytic functions and the theory of complex analysis, we can solve Eqs. (1.1)-(1.4) with analytic solutions and conditions of solvability in class $\{0\}$. Moreover, it is mentioned that the methods of solution for two of them are still effective when translation shifts, i.e., $f(t+a_j)$ and $f(-t-b_j)$, occur in addition. Thus, the results in this paper improve ones of [8, 16, 20, 39].

The variable of functions that appear in this paper is taken on the real axis X and their function values are complex.

2. Definitions and lemmas

In this section, some necessary background is provided. Now we present some definitions and lemmas, and mainly introduce the concepts of classes $\{0\}$ and $\{\{0\}\}$.

Let H_* be a set of all functions $F(x)$ which satisfies γ-Hölder condition in each compact subset of $\mathbb{R}$ (including a neighbourhood of ∞), where $0 < \gamma \leq 1$.

Definition 2.1. If $F(x) \in H_*$ and $F(x) \in L^2(\mathbb{R})$, we say that $F(x) \in \{\{0\}\}$.

Obviously $\{\{0\}\} \in L^2(\mathbb{R}) \cap H$, where H is the class of Hölder continuous functions (for the notation H, see [20, 39]. It is easy to verify that the function class $\{\{0\}\}$ is closedness under pointwise multiplication.

Definition 2.2. The Fourier transform $\mathcal{F}$ of $f \in L^1(\mathbb{R})$ will be denoted by $F(x)$:

$$F(x) = \mathcal{F}[f(t)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{ixt} dt \tag{2.1}$$

and the inverse transform operator $\mathcal{F}^{-1}$ of $F(x)$ is defined by $f(t)$:

$$f(t) = \mathcal{F}^{-1}[F(x)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(x) e^{-ixt} dx. \tag{2.2}$$

From [21,28,39], we know that $\mathcal{F}$ can be extended to $L^2(\mathbb{R})$ and $\mathcal{F}$ is an isometric operator in L^2. Moreover, we also have

$$\mathcal{F}^{-1} f(t) = \mathcal{F} f(-t) = F(-x).$$

Definition 2.3. If $F(x) \in \{\{0\}\}$, then we call that $f(t) \in \{0\}$, where $f(t) = \mathcal{F}^{-1}[F(x)]$.

From Definition 2.3, we know that $\{0\} \subset L^2(\mathbb{R})$ and $\mathcal{F} : \{0\} \longrightarrow \{\{0\}\}$.

Definition 2.4. The operators N, S, and T are defined as follows

$$(\mathrm{N}f)(t) = f(-t),\ (\mathrm{S}f)(t) = f(t)\mathrm{sgn}t,\ (\mathrm{T}f)(t) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t} d\tau,\ t \in (-\infty, +\infty). \tag{2.3}$$

It is clear that

$$\mathrm{SN} = -\mathrm{NS}, \quad \mathrm{T}^2 = \mathrm{N}^2 = \mathrm{S}^2 = \mathrm{I}, \tag{2.4}$$

where I is a unit operator. We can verify that T is an isometric operator in $L^2(\mathbb{R})$ and T: $\{\{0\}\} \to \{\{0\}\}$.

Definition 2.5. The convolution of $f(t)$ and $g(t)$ is given by

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t-s)g(s)ds. \tag{2.5}$$

According to the convolution theorem [8], we have

$$\mathcal{F}(f * g(t)) = \mathcal{F}f(t) \cdot \mathcal{F}g(t) = F(x)G(x), \tag{2.6}$$

where F, G are the Fourier transforms of f, g respectively. We know that $f, g \in \{0\}$ implies $f * g \in \{0\}$.

Lemma 2.1 plays an important role in this paper.

Lemma 2.1. *If $f(t) \in \{0\}$, then*

$$\mathcal{F}[\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t} d\tau] = -F(s)sgns, \ \textit{i.e.}, \ \mathcal{F}(Tf) = -SF. \tag{2.7}$$

Proof. Owing to

$$\begin{aligned} \mathcal{F}(Tf) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t} d\tau] e^{ist} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\tau) [\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{e^{ist}}{\tau - t} dt] d\tau, \end{aligned} \tag{2.8}$$

using the extended residue theorem [20, 39], we get

$$\int_{-\infty}^{+\infty} \frac{e^{ist}}{\tau - t} dt = -\pi i e^{is\tau} sgns. \tag{2.9}$$

Putting (2.9) into (2.8), we have

$$\mathcal{F}(Tf) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\tau) e^{is\tau} (-sgns) d\tau = -sgnsF(s). \tag{2.10}$$

□

Evidently, we have

$$\mathcal{F}[T(f(-t))] = -F(-s)sgns. \tag{2.11}$$

Lemma 2.2. *If $f \in \{0\}$ and $\mathcal{F}f(0) = 0$, then $Tf \in \{0\}$.*

Proof. Since $f \in \{0\}$, we have $\mathcal{F}f \in \{\{0\}\}$. From 2.1 it follows that (2.10) holds. Note that $\mathcal{F}f(\infty) = \mathcal{F}f(0) = 0$, we can get $\mathcal{F}(Tf) \in \{\{0\}\}$. Therefore, $Tf \in \{0\}$. □

In order to transform Eqs. (1.1)-(1.4) into Riemann-Hilbert problems, we investigate the relation between Fourier integral and Cauchy type integral. Let $f(t) \in \{0\}$, we define Cauchy type integral as follows

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{t - z} dt, \quad \mathrm{Im} z \neq 0. \tag{2.12}$$

Then, $F(z)$ is the sectionally holomorphic function on $\{\mathrm{Im} z > 0\} \cup \{\mathrm{Im} z < 0\}$. From [8, 19, 39], we can obtain

$$\frac{1}{2\pi i} \int_{0}^{+\infty} \frac{f(t)}{t - s} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_{+}(t) e^{ist} dt = F^{+}(s), \tag{2.13}$$

and

$$\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{f(t)}{t-s}dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f_{-}(t)e^{ist}dt=F^{-}(s), \tag{2.14}$$

where $F^{\pm}(s)$ are the Fourier transforms of $f_{\pm}(t)$ respectively. In fact, $F^{\pm}(s)$ also are the boundary values of $F(z)$ in the upper half planes $\mathbb{C}^{+}$ and the lower half planes $\mathbb{C}^{-}$, respectively. In (2.13) and (2.14), we have put

$$f_{\pm}(t)=\frac{1}{2}f(t)(\mathrm{sgn}t\pm 1). \tag{2.15}$$

In Sections 3-5, we shall study theory of Noether solvability and method of solution for some kinds of singular integral equations of convolution type with reflection. Indeed, the problem of finding their solutions is very important in practical applications.

3. Singular integral equations with one pair of kernels

Let us solve Eq. (1.1). To do this, we can represent Eq. (1.1) in the following form

$$\begin{aligned}&A_1f(t)+A_2f(-t)+B_1Tf(t)+B_2Tf(-t)+C_1k_1*f(t)+C_2k_2*f(-t)\\ =&g(t),\quad -\infty<t<+\infty,\end{aligned} \tag{3.1}$$

where A_j, B_j, C_j $(j=1,2)$ are constants with B_j not all equal to zero simultaneously. $k_1(t), k_2(t)$, $g(t)$ are the given functions, and $k_j(t), g(t)\in\{0\}$ $(j=1,2)$. The unknown function $f(t)$ is required to be in $\{0\}$ too.

We apply Fourier transforms to both sides of (3.1), then (3.1) is reduced to

$$[A_1-B_1sgns+C_1K_1(s)]F(s)+[A_2-B_2sgns+C_2K_2(s)]F(-s)=G(s), \tag{3.2}$$

where

$$K_1(s)=\mathcal{F}k_1(t),\ K_2(s)=\mathcal{F}k_2(t),\ G(s)=\mathcal{F}g(t),\ F(s)=\mathcal{F}f(t).$$

From $k_1(t), k_2(t), g(t), Tf(t)\in\{0\}$, we have $K_1(s), K_2(s), G(s), F(s)\in\{\{0\}\}$, and we know that these functions are continuous. Now we take the limits to (3.2) as $s\to 0$ and note that $\lim_{s\to\pm 0} sgns=\pm 1$, therefore, we have $G(0)=0$. From our previous discussions, we can get the necessary condition of the existence of solution for Eq. (3.1) is

$$(\mathcal{F}g)(0)=0,\ i.e.,\ G(0)=0.$$

Restricting ourselves to the normal type, i.e.,

$$A_j-B_jsgns+C_jK_j(s)\neq 0,\ s\in\mathbb{R},\ j=1,2.$$

Replacing s by $-s$ in (3.2), we obtain

$$[A_2+B_2sgns+C_2K_2(-s)]F(s)+[A_1+B_1sgns+C_1K_1(s)]F(-s)=G(-s). \tag{3.3}$$

Denote

$$\Delta_j(\pm s)=A_j\mp B_jsgns+C_jK_j(\pm s),\quad \forall\ j=1,2.$$

Then we may write (3.2),(3.3) as the following matrix equation:

$$M(s)X(s) = H(s), \tag{3.4}$$

where

$$M(s) = \begin{bmatrix} \Delta_1(s) & \Delta_2(s) \\ \Delta_2(-s) & \Delta_1(-s) \end{bmatrix}, \ X(s) = (F(s), F(-s))^T, \ H(s) = (G(s), G(-s))^T.$$

Thus, under condition $\mathcal{F}g(0) = 0$, Eq. (3.1) has a solution in $\{0\}$ if and only if Eq. (3.4) has a solution in $\{\{0\}\}$. Using the solvability theory of a system of linear algebraic equations, we would solve Eq. (3.4) with the unknown functions $F(s)$ and $F(-s)$. Denote

$$N(s) = \begin{bmatrix} \Delta_1(s) & \Delta_2(s) & G(s) \\ \Delta_2(-s) & \Delta_1(-s) & G(-s) \end{bmatrix}, \ \Delta(s) = \det M(s).$$

Thus we have

$$\begin{aligned} \Delta(s) =& A_1^2 - A_2^2 + \delta(B_2^2 - B_1^2) + A_1C_1(K_1(s) + K_1(-s)) - A_2C_2(K_2(s) \\ &+ K_2(-s)) + B_1C_1 sgns(K_1(s) - K_1(-s)) - B_2C_2 sgns(K_2(s) - K_2(-s)) \\ &+ C_1^2K_1(s)K_1(-s) - C_2^2K_2(s)K_2(-s). \end{aligned} \tag{3.5}$$

Since $K_j(\pm s) \in \{\{0\}\}$, we have $K_j(\pm s) \to 0$ and $\Delta(s) \to A_1^2 - A_2^2 + \delta(B_2^2 - B_1^2)$ when $s \to \infty$.

For convenience, we also assume that $A_1^2 - A_2^2 + \delta(B_2^2 - B_1^2) \neq 0$, where

$$\delta = \begin{cases} 0, & s = 0, \\ 1, & s \neq 0. \end{cases}$$

Then there exists an $X > 0$ such that $\Delta(s) \neq 0$ when $|s| > X$. Therefore, Eq. (3.3) has only solution given by the formula:

$$F(s) = \frac{1}{\Delta(s)}[G(s)\Delta_1(-s) - G(-s)\Delta_2(s)], \tag{3.6}$$

where $\Delta(s) \neq 0$ for any $|s| > X$. When $|s| \leq X$, we consider the following three cases.

1) If $\Delta(s) \neq 0$, then the system of equation (3.4) has only solution $M^{-1}(s)H(s)$.

2) If there exist $s_1, s_2, \cdots, s_n \in [-X, X]$ such that $\Delta(s_i) = 0$ and $\partial M(s_i) = \partial N(s_i)$ $(1 \leq i \leq n)$, then (3.4) has infinite solutions $(F(s), F(-s))$, where ∂M is rank of the matrix M.

3) If $\Delta(s_i') = 0$ and $\partial M(s_i') \neq \partial N(s_i')$ $(1 \leq i \leq n_1), s_i' \in [-X, X]$, then the conditions of solvability are

$$G(\pm s_i') = 0, \quad \forall\, i \in \{1, 2, \cdots, n_1\}. \tag{3.7}$$

These conditions can be written as follows

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(t)\cos(s_i't)dt = 0; \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(t)\sin(s_i't)dt = 0, \quad 1 \le i \le n_1. \tag{3.8}$$

For homogeneous case, i.e., $g(t) \equiv 0$, Eq. (3.1) has linearity independent solutions: $e^{is_1t}, e^{is_2t}, \cdots, e^{is_nt}$.

Now we can state the main results with respect to solutions of Eq. (3.1) in the following form.

Theorem 3.1. *For Eq. (3.1), under condition $G(0) = 0$.*

(1) If $\Delta(s) \neq 0$ $(-\infty < s < +\infty)$, Eq. (3.1) has a unique solution in class $\{0\}$, and its solution is given by $f(t) = \mathcal{F}^{-1}F(s)$, where $F(s)$ is given by (3.6).

(2) If $\Delta(s) = 0$ and $\partial M(s) = \partial N(s)$ for $s = s_1, s_2, \cdots, s_n$, Eq. (3.1) has infinite solutions.

(3) If $\Delta(s) = 0$ and $\partial M(s) \neq \partial N(s)$ for $s = s_1', s_2', \cdots, s_m'$, the conditions of solvability (3.8) must be augmented, and Eq. (3.1) has solutions given by the formula

$$f(t) = \mathcal{F}^{-1}F(s) + \Sigma_{j=1}^{m} c_j e^{is_j't}, \tag{3.9}$$

where c_j $(1 \le i \le m)$ are arbitrary constants.

Remak 3.1. Since $\frac{1}{\Delta(s)}$ is bounded on the whole real axis X, it follows from (3.6) that $F(s) \in H_* \cap L^2(-\infty, +\infty)$ (i.e., $F(s) \in \{\{0\}\}$), hence $f(t) \in \{0\}$.

At the end of this section, we give an example to illustrate the application of the solving method. In Eq. (3.1), we assume that

$$A_1 = 1,\ A_2 = 0,\ B_1 = B_2 = 1,\ C_1 = 1,\ C_2 = 0,\ k_1(t) = g(t) = \sqrt{\frac{\pi}{2}}\exp(-tsgnt). \tag{3.10}$$

By taking Fourier transforms to $k_1(t), g(t)$, we get

$$K_1(s) = G(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\sqrt{\frac{\pi}{2}}\exp(-tsgnt)e^{ist}dt = \frac{1}{1+s^2}. \tag{3.11}$$

Obviously, when $k_1(t), g(t) \in \{0\}$, then $K_1(s), G(s) \in \{\{0\}\}$. Via a simple calculation, we get

$$\Delta(s) = (1 + \frac{1}{1+s^2})^2,\ \Delta_1(-s) = 1 - sgns + \frac{1}{1+s^2},\ \Delta_2(s) = -sgns. \tag{3.12}$$

Thus, from (3.6) we have

$$F(s) = \frac{1}{\Delta(s)}[G(s)\Delta_1(-s) - G(-s)\Delta_2(s)] = \frac{1}{2+s^2}. \tag{3.13}$$

Taking the inverse Fourier transform to $F(s)$, we obtain

$$f(t) = \mathcal{F}^{-1}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(s)e^{-ist}ds = \frac{1}{2}\sqrt{\pi}\exp(-\sqrt{2}tsgnt), \tag{3.14}$$

therefore, we have

$$f(t) = \begin{cases} \frac{1}{2}\sqrt{\pi}\exp(-\sqrt{2}t), & t \ge 0, \\ \frac{1}{2}\sqrt{\pi}\exp(\sqrt{2}t), & t < 0. \end{cases} \tag{3.15}$$

It is clear that (3.15) is the solution of Eq. (3.1).

4. Singular integral equations of dual type

In this section, we study the solvability of dual singular integral equations with reflection, and we simplify Eq. (1.2) to the following form:

$$\begin{cases} A_1 f(t) + B_1 f(-t) + C_1 Tf(t) + D_1 Tf(-t) \\ +E_1 k_1 * f(t) + F_1 h_1 * f(-t) = g(t),\ 0 < t < +\infty, \\ A_2 f(t) + B_2 f(-t) + C_2 Tf(t) + D_2 Tf(-t) \\ +E_2 k_2 * f(t) + F_2 h_2 * f(-t) = g(t),\ \ -\infty < t < 0, \end{cases} \tag{4.1}$$

where $A_j, B_j, C_j, D_j, E_j, F_j$ $(j = 1, 2)$ are constants and all the functions appeared in (4.1) belong to $\{0\}$.

In the first equation of (4.1), by extending t to $-\infty < t < +\infty$, we get an equation

$$\begin{aligned} &A_1 f(t) + B_1 f(-t) + C_1 Tf(t) + D_1 Tf(-t) + E_1 k_1 * f(t) + F_1 h_1 * f(-t) \\ =&g(t) + \psi_-(t), \quad\ -\infty < t < +\infty, \end{aligned} \tag{4.2}$$

where $\psi_-(t)$ is an unknown function in class $\{0\}$ with $\psi_-(t) = 0$ when $0 < t < +\infty$.

Similarly, in the second one of (4.1), by extending t to $-\infty < t < +\infty$, we have

$$\begin{aligned} &A_2 f(t) + B_2 f(-t) + C_2 Tf(t) + D_2 Tf(-t) + E_2 k_2 * f(t) + F_2 h_2 * f(-t) \\ =&g(t) + \psi_+(t), \quad\ -\infty < t < +\infty, \end{aligned} \tag{4.3}$$

where $\psi_+(t)$ is an unknown function in $\{0\}$ with $\psi_+(t) = 0$ when $-\infty < t < 0$.

Now we take Fourier transforms to both (4.2) and (4.3), by the method of complex analysis and the theory of boundary value problems of analytic functions, we get the following matrix equations:

$$M(s)X(s) = \tilde{\Psi}(s) + \tilde{G}(s), \tag{4.4}$$

where

$$M(s) = \begin{bmatrix} A_1 - C_1 sgns + E_1 K_1(s) & B_1 - D_1 sgns + F_1 H_1(s) \\ A_2 - C_2 sgns + E_2 K_2(s) & B_2 - D_2 sgns + F_2 H_2(s) \end{bmatrix},$$

and

$$X(s) = (F(s), F(-s))^T,\ \tilde{G}(s) = (G(s), G(s))^T,\ \tilde{\Psi}(s) = (\Psi^+(s), \Psi^-(s))^T,$$
$$\Psi^{\pm}(s) = \mathcal{F}\psi_{\pm}(t).$$

In order that $F(s)$ is continuous at $s = 0$, it is necessary that $\Psi^{\pm}(s)$ are continuous at $s = 0$ and

$$\Psi^+(0) + G(0) = 0, \quad \Psi^-(0) + G(0) = 0.$$

And it is well known that $\Psi^{\pm}(s)$ are the one-sided Fourier transforms of $\psi(t)$, that is,

$$\Psi^+(s) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \psi(t) e^{ist} dt, \quad \Psi^-(s) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \psi(t) e^{ist} dt.$$

It is evident that $\Psi(s) = \Psi^+(s) - \Psi^-(s)$. In fact, by [28, 29, 31] and the extended residue theorem, we know that $\Psi^\pm(s)$ also are the boundary values of the Cauchy type integral:

$$\Psi(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Psi(s)}{s-z} ds, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^-. \tag{4.5}$$

To solve (4.4), we replace s by $-s$ in it, then we have

$$M(-s)X(-s) = \tilde{\Psi}(-s) + \tilde{G}(-s). \tag{4.6}$$

Thus by eliminating $F(\pm s)$ in (4.4) and (4.6), it gives rise to the following Riemann-Hilbert problem with reflection and node:

$$\begin{aligned} &\vartheta_1(s)\Delta(-s)\Psi^+(s) + \varrho_1(s)\Delta(s)\Psi^+(-s) - \vartheta_2(s)\Delta(-s)\Psi^-(s) - \varrho_2(s)\Delta(s)\Psi^-(-s) \\ =&[\vartheta_1(s) - \vartheta_2(s)]\Delta(-s)G(s) - [\varrho_1(s) - \varrho_2(s)]\Delta(-s)G(-s), \end{aligned} \tag{4.7}$$

where $\Delta(s) = \det M(s)$, that is, $\Delta(s)$ is the coefficient determinant of (4.4), and

$$\vartheta_j(s) = A_j - C_j sgns + E_j K_j(s), \quad \varrho_j(s) = B_j + D_j sgns + F_j H_j(-s), \quad j = 1, 2.$$

Since $\Psi(s) \in \{\{0\}\}$, we get $\Psi(\infty) = 0$. By using Sokhotski-Plemelj formula [20, 39] to $\Psi(z)$ in (4.5), we have

$$\Psi^\pm(s) = \pm\frac{1}{2}\Psi(s) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Psi(t)}{t-s} dt. \tag{4.8}$$

Set

$$\begin{aligned} &\alpha(s) = [\vartheta_1(s) + \vartheta_2(s)]\Delta(-s), \quad \beta(s) = [\varrho_1(-s) + \varrho_2(-s)]\Delta(s), \\ &\xi(s) = [\vartheta_1(s) - \vartheta_2(s)]\Delta(-s), \quad \zeta(s) = [\varrho_1(-s) - \varrho_2(-s)]\Delta(s), \\ &P(s) = \xi(s)G(s) - \zeta(s)G(-s). \end{aligned}$$

From (4.8), we know that (4.7) may be also transformed into a singular integral equation with reflection:

$$\frac{1}{2}\alpha(s)\Psi(s) + \frac{1}{2}\beta(s)\Psi(-s) + \xi(s)\cdot\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Psi(t)}{t-s}dt + \zeta(s)\cdot\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Psi(-t)}{t-s}dt = P(s). \tag{4.9}$$

We again replace s by $-s$ in (4.9), and let $Y(s) = (\Psi(s), \Psi(-s))^T$, thus we can obtain a system of singular integral equations of two-dimension in class $\{\{0\}\}$:

$$A(s)Y(s) + B(s)\cdot\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{Y(t)}{t-s}dt = E(s), \tag{4.10}$$

where

$$A(s) = \begin{bmatrix} \frac{1}{2}\alpha(s) & \frac{1}{2}\beta(s) \\ \frac{1}{2}\beta(-s) & \frac{1}{2}\alpha(-s) \end{bmatrix}, \; B(s) = \begin{bmatrix} \frac{1}{2}\xi(s) & -\frac{1}{2}\zeta(s) \\ \frac{1}{2}\zeta(-s) & -\frac{1}{2}\xi(-s) \end{bmatrix}, \; E(s) = \begin{bmatrix} P(s) \\ P(-s) \end{bmatrix}.$$

Next we will solve the matrix equation (4.10). To do this, we define

$$W(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{Y(t)}{t-z}dt, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^-. \tag{4.11}$$

According to Sokhotski-Plemelj formula, we get

$$W^{\pm}(s) = \pm\frac{1}{2}Y(s) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{Y(t)}{t-s}dt. \tag{4.12}$$

Therefore, from (4.12) we have

$$Y(s) = W^{+}(s) - W^{-}(s).$$

Putting (4.12) into (4.10), we may obtain

$$(A(s) + B(s))W^{+}(s) + (A(s) - B(s))W^{-}(s) = E(s), \tag{4.13}$$

where $W(s)$ is a 2-dimensional unknown vector function. For convenience, we let

$$U(s) = A(s) + B(s), \quad V(s) = A(s) - B(s). \tag{4.14}$$

It is easy to see that (4.10) has a solution if and only if (4.13) has a solution. Thus, we should only study (4.13) in place of (4.10). Here, we consider only the case of normal type:

$$\det U(s)V(s) \neq 0,\ i.e.,\ \det(A^2(s) - B^2(s))) \neq 0 \tag{4.15}$$

holds on the whole real axis $\bar{X} = X \cup \{\infty\}$, therefore $U(s)$ and $V(s)$ are reversible matrixes. We let

$$a(s) = U^{-1}(s)V(s), \quad b(s) = U^{-1}(s)E(s). \tag{4.16}$$

Then (4.13) is equivalent to the following system of Riemann-Hilbert problem of two-dimension in class $\{\{0\}\}$:

$$W^{+}(s) + a(s)W^{-}(s) = b(s), \quad s \in (-\infty, +\infty). \tag{4.17}$$

From $\det a(s) \neq 0$, we know that (4.17) is a Riemann-Hilbert problem of normal type, which satisfies the following conditions:

$$W(\infty) = 0,\ i.e.,\ W^{\pm}(\infty) = 0. \tag{4.18}$$

In the following, we shall introduce a kind of transformation to reduce (4.17) to a class of generalized linear Riemann-Hilbert problems of function group. To do this, we make a linear transform χ as follows

$$\chi : \eta = \frac{z}{iz-1}, \quad z \in \mathbb{C}^{+} \cup \mathbb{C}^{-}, \tag{4.19}$$

obviously,

$$t = \frac{s}{is-1}, \quad s \in X,$$

and the inverse transform χ^{-1} of (4.19) have the same form with χ. The transform (4.19) maps the real axis X on the complex plane $\mathbb{C}$ onto a circle $\Gamma : \Gamma = \{\eta \in \mathbb{C} || \eta + \frac{i}{2}| = \frac{1}{2}\}$ on the complex plane η. We denote by D^{+} and D^{-} an interior region and an exterior region of Γ, respectively, that is,

$$D^{+} = \{\eta \in \mathbb{C} || \eta + \frac{i}{2}| < \frac{1}{2}\}, \quad D^{-} = \{\eta \in \mathbb{C} || \eta + \frac{i}{2}| > \frac{1}{2}\}.$$

Then, χ maps the upper half plane $\mathbb{C}^+$ and the lower half plane $\mathbb{C}^-$ onto the D^+ and D^- respectively. (4.17) is readily reduced to the following system of Riemann-Hilbert problem on the complex plane $\mathbb{C}$ by (4.19),

$$\Theta^+(t) + R(t)\Theta^-(t) = \varrho(t), \quad t \in \Gamma, \tag{4.20}$$

where

$$W(s) = W(\frac{t}{it-1}) = \Theta(t),\ a(s) = a(\frac{t}{it-1}) = R(t),\ b(s) = b(\frac{t}{it-1}) = \varrho(t).$$

It is easy to see that $R(t)$ and $\varrho(t)$ belong to H on Γ and $\varrho(-i) = 0$, hence (4.20) must fulfill the following additional conditions

$$\Theta(-i) = 0,\ i.e.,\ \Theta^+(-i) = \Theta^-(-i) = 0. \tag{4.21}$$

Thus, we have

Theorem 4.1. *$\Theta(\eta)$ is a bounded solution of (4.20) that satisfies (4.21), if and only if $W(z) = \Theta(\frac{\eta}{i\eta-1})$ is a bounded solution of (4.17) that satisfies (4.18).*

Next we would solve the Riemann-Hilbert problem (4.20) under the complementary condition (4.21). Assume that $\sigma(\eta)$ is a canonical solution matrix of (4.20), and its canonical solutions are $\sigma^j(\eta)(j = 1, 2)$. If we denote by $\kappa_j(j = 1, 2)$ the partial index of $\sigma(\eta)$, then the whole index κ of (4.20) is defined as follows

$$\kappa = \sum_{j=1}^{2} \kappa_j. \tag{4.22}$$

Similar to the methods in [9, 34, 35, 41], we may obtain the following boundedly analytic solution for Riemann-Hilbert problem (4.20) under conditions (4.21) by using the principle of analytic continuation [20, 39]:

$$\Theta(\eta) = \sum_{j=1}^{2} \sigma^j(\eta)\frac{p_{\kappa_j}(\eta)}{(\eta+i)^{\kappa_j}} - \frac{1}{2\pi i}\sum_{j=1}^{2} \sigma^j(\eta)\int_\Gamma \frac{\psi_j(s)}{s-\eta}ds, \quad \eta \in D^+ \cup D^-, \tag{4.23}$$

where $\psi(s) = [\sigma(s)]^{-1}M(s)$, $\psi_j(s)(j = 1, 2)$ are two components of $\psi(s)$, and $p_{\kappa_j}(\eta)$ is a polynomial of the degree κ_j.

On the solvability of Riemann-Hilbert problem (4.20), we have the following conclusions.

Theorem 4.2. *Under conditions (4.21), the solution of (4.20) can be expressed by (4.23). If $\kappa_j \geq 0$, then $\Theta(\eta)$ contains κ arbitrary constants. If $\kappa_j < 0$, then $p_{\kappa_j}(\eta) \equiv 0$ $(j = 1, 2)$, and the conditions of solvability become*

$$\int_\Gamma \frac{\psi_j(s)}{(s+i)^l}ds = 0, \quad \forall\ l = 0, -1, \ldots, \kappa_j + 1, \quad j = 1, 2, \tag{4.24}$$

and there are exactly $-(\kappa_1 + \kappa_2)$ conditions of solvability. If $\kappa_1\kappa_2 < 0$, without loss of generality, we assume that $\kappa_2 < 0 < \kappa_1$, then $\Theta(\eta)$ contains κ_1 arbitrary constants, and the following conditions of solvability are also fulfilled

$$\int_\Gamma \frac{\psi_2(s)}{(s+i)^l}ds = 0, \quad \forall\ l = 0, -1, \ldots, \kappa_2 + 1, \tag{4.25}$$

therefore there are $-\kappa_2$ *conditions of solvability for Eq. (4.20), and* $p_{\kappa_2}(\eta) \equiv 0$.

After that, by using the inverse transform of (4.19), we may obtain a solution of (4.17) under conditions (4.18) as follows

$$W(z) = \sum_{j=1}^{2} X^j(z) \frac{p_{\kappa_j}(z)}{(z+i)^{\kappa_j}} - \frac{1}{2\pi i} \sum_{j=1}^{2} X^j(z) \int_{-\infty}^{+\infty} \frac{\zeta_j(s)}{s-z} ds, \quad \forall\, z \in \mathbb{C}^+ \cup \mathbb{C}^-, \tag{4.26}$$

where $X(z) = \sigma(\frac{\eta}{i\eta - 1})$, $\zeta(s) = [X(s)]^{-1} M(s)$, $X^j(z)(j = 1, 2)$ are two solution components of $X(z)$, and $\zeta_j(s)(j = 1, 2)$ are two components of $\zeta(s)$.

Using Sokhotski-Plemelj formula to $W(z)$ in (4.26), we may obtain $W^{\pm}(s)$, and then we substitute $W^{\pm}(s)$ into (4.12), thus we can get $Y(s)$. In fact, if we assume that $Y(s) = (Y_1(s), Y_2(s))^T$ is a solution of (4.10), then $Y_*(s) = (Y_2(-s), Y_1(-s))^T$ is also a solution of it and so does $\frac{1}{2}(Y(s) + Y_*(s))$. Hence, it is easily verified that

$$\Psi(s) = \frac{1}{2}(Y_1(s) + Y_2(-s)) \tag{4.27}$$

is a solution of (4.9). Putting $\Psi(s)$ into (4.4), similariy to the discussion in Section 3, we may obtain the solution $F(s)$ of (4.4), thus

$$f(t) = \mathcal{F}^{-1} F(s) \tag{4.28}$$

is the solution of (4.1), and $f(t) \in \{0\}$. The detailed discussion will be omitted here.

Now we can state our main result about the solution of Eq. (4.1).

Theorem 4.3. *Assume that (4.18) and (4.21) are fulfilled, then Eq. (4.1) has a solution in class* $\{0\}$*, and its general solution is given by (4.28), where* $F(s)$ *is determined by (4.4).*

Note that the method of solution used here may be solved the above-mentioned equations (that is, Eqs. (1.1)-(1.4)) in the case of non-normal, that is,

$$\det(U(s)V(s)) = 0. \tag{4.29}$$

5. Singular integral equations of Wiener-Hopf type

Method mentioned in Sections 3 and 4 may be also used to solve Wiener-Hopf equation with reflection. Thus we can rewrite Eq. (1.3) in the form

$$\begin{aligned} &Af(t) + C_1 T f_+(t) + C_2 T f_+(-t) + D_1 k * f_+(t) + D_2 h * f_+(-t) \\ =&g(t),\ 0 < t < +\infty, \end{aligned} \tag{5.1}$$

where A, C_j, D_j $(j = 1, 2)$ are real constants, $k(t), h(t), g(t) \in \{0\}$ and their one-sided Fourier transforms belong to $\{\{0\}\}$, and an unknown function $f(t)$ also belong to $\{0\}$.

In (5.1), we extend t to $t \in (-\infty, 0)$ and denote $f(t) = f_+(t), g(t) = g_+(t)$, then (5.1) may be written as

$$\begin{aligned} &Af_+(t) + C_1 T f_+(t) + C_2 T f_+(-t) + D_1 k * f_+(t) + D_2 h * f_+(-t) \\ =&g_+(t) + f_-(t), \ -\infty < t < +\infty, \end{aligned} \tag{5.2}$$

where

$$f_+(t)=\begin{cases}f(t), & t\geq 0,\\ 0, & t<0;\end{cases}\qquad f_-(t)=\begin{cases}0, & t\geq 0,\\ -f(t), & t<0.\end{cases}$$

We apply the Fourier transform in both sides of (5.2) and obtain

$$\begin{aligned}&AF^+(s)-C_1\text{sgn}sF^+(s)-C_2\text{sgn}sF^+(-s)+D_1K(s)F^+(s)+D_2H(s)F^+(-s)\\ =&G^+(s)-F^-(s),\end{aligned}\tag{5.3}$$

namely,

$$[A-C_1\text{sgn}s+D_1K(s)]F^+(s)+F^-(s)-[C_2\text{sgn}s-D_2H(s)]F^+(-s)=G^+(s),\tag{5.4}$$

which is a Riemann-Hilbert problem with reflection and node.

Similar to the discussion in Section 4, we replace s by $-s$ in (5.4) and get

$$[C_2\text{sgn}s+D_2H(-s)]F^+(s)+[A+C_1\text{sgn}s+D_1K(-s)]F^+(-s)+F^-(-s)=G^+(-s).\tag{5.5}$$

Via (5.4) and (5.5), we have

$$\begin{cases}[A-C_1\text{sgn}s+D_1K(s)]F^+(s)+F^-(s)-[C_2\text{sgn}s-D_2H(s)]F^+(-s)=G^+(s);\\ [C_2\text{sgn}s+D_2H(-s)]F^+(s)+[A+C_1\text{sgn}s+D_1K(-s)]F^+(-s)+F^-(-s)=G^+(-s).\end{cases}\tag{5.6}$$

Since $F(s)\in\{\{0\}\}$, we have $F(\infty)=0$. Again using the following Sokhotski-Plemelj formula

$$F^+(s)+F^-(s)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{F(t)}{t-s}dt,\quad F^+(s)-F^-(s)=F(s),\tag{5.7}$$

we can transform (5.6) to the following matrix equations with a singular integral in class $\{\{0\}\}$:

$$\frac{1}{2}\Theta(s)X(s)+\frac{\Upsilon(s)}{2\pi i}\int_{-\infty}^{+\infty}\frac{X(t)}{t-s}dt=N(s),\tag{5.8}$$

where $X(s)=(F(s),F(-s))^T$, $N(s)=(G^+(s),G^+(-s))^T$, and

$$\Theta(s)=\begin{pmatrix}p(s) & q(s)\\ q(-s) & p(-s)\end{pmatrix},\quad \Upsilon(s)=\begin{pmatrix}u(s) & -v(s)\\ v(-s) & -u(-s)\end{pmatrix},\tag{5.9}$$

in which we have put

$$\begin{aligned}p(s)&=A-1-C_1\text{sgn}s+D_1K(s);\quad q(s)=C_2\text{sgn}s+D_2H(s);\\ u(s)&=A+1-C_1\text{sgn}s+D_1K(s);\quad v(s)=C_2\text{sgn}s-D_2H(s).\end{aligned}\tag{5.10}$$

(5.8) is the characteristic system of equations of dimension 2 with singular integral in class $\{\{0\}\}$. For (5.8), its solvability is similar to the discussion in Sections 3 and 4. Here, we also only consider the case of normal type, that is,

$$\det(\Theta(s)\pm\Upsilon(s))\neq 0,\quad \forall\ s\in X.\tag{5.11}$$

About Eq. (5.8), there is no essential difference for the methods of solution with Eq. (4.10), and we will not elaborate. Now we can formulate the main results with respect to the solutions of Eq. (5.1) as follows.

Theorem 5.1. *Under conditions (5.11), Eq. (5.1) has a solution in class* $\{0\}$*, and its solutions and conditions of solvability are similar to those in Theorem 4.2.*

Remark 5.1. In Eq. (5.1), if $k(t), h(t), g(t) \in L^2[0, +\infty)$, then their one-sided Fourier transforms belong to the Hölder continuous class on $[0, +\infty)$, and the unknown function $f(t)$ is required to be in $\{0\}$, in this case, we can solve Eq. (5.1) similarly.

6. Singular integral equations with reflections and translation shifts

Methods of solution used in Sections 3 and 4 are also in effect for equations similar to (1.1) and (1.2) with both reflection and a finite set of translation shifts $f(t+a_j)$ and $f(-t-b_j)$, namely,

$$\begin{cases} \sum_{j=1}^{n}[A_j^{(1)}f(t+a_j) + B_j^{(1)}Tf(t+a_j) + k_j^{(1)} * f(t+a_j) + C_j^{(1)}f(-t-b_j) \\ +D_j^{(1)}Tf(-t-b_j) + h_j^{(1)} * f(-t-b_j)] = g(t), \quad 0 < t < +\infty; \\ \sum_{j=1}^{n}[A_j^{(2)}f(t+a_j) + B_j^{(2)}Tf(t+a_j) + k_j^{(2)} * f(t+a_j) + C_j^{(2)}f(-t-b_j) \\ +D_j^{(2)}Tf(-t-b_j) + h_j^{(2)} * f(-t-b_j)] = g(t), \quad -\infty < t < 0, \end{cases} \tag{6.1}$$

and

$$\begin{aligned} &\sum_{j=1}^{n}[A_j f(t+a_j) + B_j Tf(t+a_j) + k_j * f(t+a_j) + C_j f(-t-b_j) \\ &+ D_j Tf(-t-b_j) + h_j * f(-t-b_j)] = g(t), \quad -\infty < t < +\infty, \end{aligned} \tag{6.2}$$

where $a_j, b_j, A_j, B_j, C_j, D_j (1 \le j \le n)$ are constants, and all the functions appeared in Eqs. (6.1) and (6.2) belong to $\{0\}$. Using the Fourier transforms, Eqs. (6.1) and (6.2) may be transformed into Riemann-Hilbert problems with reflections and nodes. The solving method is similar to Eqs. (3.1) and (4.1) mentioned above. Further discussion will be omitted here.

But for equations similar to (1.3) or (1.4) of the same type, the above described method is not effective.

7. Conclusions

In this paper, we dealt with the existence of solutions for some classes of convolution singular integral equations with reflection and translation shifts. By means of the theory of Fourier analysis and of a system of linear algebraic equations, we transformed such equations into the Riemann-Hilbert problems. The exact solution, denoted by integrals, of equations and the conditions of solvability are obtained in class $\{0\}$. Here, our method of solving equations is different from those of the classical Riemann-Hilbert problems, and it is novel and effective. Other integral equations with reflection can be solved by using our method. Especially, the method used here is also effective for certain kinds of such equations with translation shifts and reflection. Indeed, we can also research the similar problem in the setting of Clifford analysis (see [13–15, 35–38, 42]). Thus, this paper generalizes the classical theory of Riemann-Hilbert problems and integral equations.

Finally, as for the solvability of Eq. (1.4) (i.e., equations with two pairs of convolution kernels), by using Fourier transformation, we can transform Eq. (1.4) into a Riemann-Hilbert problem similar to (4.9), which may be discussed by using the same method as shown in Section 4.

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