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THE EXISTENCE OF SOLUTION AND DEPENDENCE ON FUNCTIONAL PARAMETER FOR BVP OF FRACTIONAL DIFFERENTIAL EQUATION

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Abstract In this paper, fractional differential equations of p-q-Laplacian with instantaneous and non-instantaneous impulses are considered. The existence result is obtained by using the variational approach. Furthermore, we establish the dependence on functional parameters for classical solutions of the boundary value problem with L^1 right hand side. The interesting points are p-q-Laplace operator and dependence on functional parameters.

Keywords Variational approach, p-q-Laplace operator, instantaneous impulse, non-instantaneous impulse, continuous dependence on functional parameters.

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1. Introduction

In recent years, the study of the fractional differential equations has attached much attraction, please refer to [4, 5, 7, 8, 10–12, 15–17, 20, 24, 25, 31].

More recently, impulsive differential equations have attracted much attentions since they can consider the impact of sudden change on the state and can reflect the changing law of things. Impulsive differential equation includes the differential equation with instantaneous and non-instantaneous impulses. Instantaneous impulse appears in the situation that the state is affected by external uncertainty suddenly, please refer to the monographs of Afrouzi et al. [3], Bai et al. [7], Liang et al. [20], Li et al. [21], Qiao et al. [23], Yang et al. [30], Zhang et al. [32]. Non-instantaneous impulse appears in the situation that state is affected by external environment lasting for a period of time, please refer to the monographs of Agarwal et al. [1], Agarwal et al. [2], Bai et al. [6], Bai et al. [9], Khaliq et al. [18], Li et al. [21], Qiao et al. [23], Zhang et al. [32]. The two impulses often appear in the same differential system. In [28], Tian and Zhang considered the existence of solutions for second-order differential equations with instantaneous and non-instantaneous impulses by using Ekeland¡¯s Variational Principle. In [32], Zhang and Liu discussed the existence of solutions to the fractional Dirichlet boundary value problem with instantaneous and non-instantaneous impulses based on the variational method. In [21], Li and Chen studied the multiplicity of solutions for a

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class of p-Laplacian type fractional differential equations with both instantaneous and non-instantaneous impulses by using the critical points theorem of B. Ricceri and obtained some interesting results.

In [35], Zhou and Deng studied the existence of solutions for fractional differential equations of p-Laplacian with instantaneous and non-instantaneous impulses

$$\begin{cases} {}_tD_T^{\alpha}\,\phi_p({}_0^CD_t^{\alpha}\,u)(t)+g(t)|u(t)|^{p-2}u(t)=f_i(t,u(t)), t\in(s_i,t_{i+1}], i=0,1,2,\ldots,n,\\ \Delta({}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u))(t_i)=I_i(u(t_i)), i=1,2,\ldots,n,\\ {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(t)={}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(t_i^+), t\in(t_i,s_i], i=1,2,\ldots,n,\\ {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(s_i^-)={}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(s_i^+), i=1,2,\ldots,n,\\ u(0)=u(T)=0,\ \text{where } p\geq 2, \alpha\in(\frac{1}{p},1], \end{cases} \tag{1.1}$$

For the problem (1.1), the main result is as follows

Theorem 1.1 ([35]). *Assume that there exist a_i, $b_i>0$, and $\sigma_i\in[0,1)$ such that $|f_i(t,u)|\leq a_i+b_i|u|^{\sigma_i}$, for any $(t,u)\in[0,T]\times R$, where $i=1,2,\ldots,n$ and there exist c_i, $d_i>0$, $\delta_i\in[0,1)(i=1,2,\ldots,n)$ such that $|I_i(u)|\leq c_i+d_i|u|^{\delta_i}$, for any $u\in R$. Then problem (1.1) has at least one classical solution.*

In [13], the p-q-Laplacian equation appears in a lot of applications such as plasma physics, chemical reaction design and biophysics. On the other hand, at present, people have only studied the kind of fractional differential equations of p-Laplacian with instantaneous and non-instantaneous impulses, and less on fractional differential equations of p-q-Laplacian with instantaneous and non-instantaneous impulses. At the same time, the dependence on functional parameters is considered.

One of our main goals in this paper is to consider the existence of a nontrivial solution of the following fractional differential equations with instantaneous and non-instantaneous impulses

$$\begin{cases} {}_tD_T^{\alpha}\,\phi_p({}_0^CD_t^{\alpha}\,u)(t)+{}_tD_T^{\alpha}\,\phi_q({}_0^CD_t^{\alpha}\,u)(t)+\phi_p(u(t))+\phi_q(u(t))=f_i(t,u(t))\\ +g_i(t,u(t))x(t), t\in(s_i,t_{i+1}], i=0,1,2,\ldots,n,\\ \Delta({}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u))(t_i)+\Delta({}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^{\alpha}\,u))(t_i)=I_i(u(t_i)), i=1,2,\ldots,n,\\ {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(t)={}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(t_i^+), t\in(t_i,s_i], i=1,2,\ldots,n,\\ {}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^{\alpha}\,u)(t)={}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^{\alpha}\,u)(t_i^+), t\in(t_i,s_i], i=1,2,\ldots,n,\\ {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(s_i^-)+{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^{\alpha}\,u)(s_i^-)={}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^{\alpha}\,u)(s_i^+)\\ +{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^{\alpha}\,u)(s_i^+), i=1,2,\ldots,n,\\ u(0)=u(T)=0, \end{cases} \tag{1.2}$$

where $\alpha\in(\frac{1}{q},1]$, $1<q\leq p<+\infty$, $\phi_r(s):=|s|^{r-2}s$, $s\neq 0$, $\phi_r(0)=0$ for $r=p$ or q, ${}_0^CD_t^{\alpha}$, ${}_tD_T^{\alpha}$ are the left Caputo fractional derivative and the right Riemann-Liouville fractional derivative, $0=s_0<t_1<s_1<t_2<s_2<\cdots<t_n<s_n<t_{n+1}=T$, $I_i\in C(R,R)$ and $f_i\in C((s_i,t_{i+1}]\times R,R)$, $F_i(t,u)=\int_0^u f_i(t,s)ds$, $G_i(t,u)=\int_0^u g_i(t,s)ds$, $i=1,2,\ldots,n$. Let $m>0$ be a fixed number.

The functional parameter x belongs to the set $L_m = \{v : [0,T] \to R | v$ is measurable and $\int_0^T v^2(t)dt \leq m\}$.

The instantaneous impulses are defined as follows

$$\Delta({}_tD_T^{\alpha-1}\phi_r({}_0^CD_t^\alpha u))(t_i) = {}_tD_T^{\alpha-1}\phi_r({}_0^CD_t^\alpha)u(t_i^+) - {}_tD_T^{\alpha-1}\phi_r({}_0^CD_t^\alpha)u(t_i^-),$$
$$r = p, q.$$

The instantaneous impulses start abruptly at the points t_i and the non-instantaneous impulses continue on the interval $(t_i, s_i]$.

Now we state our results. First, we need the following assumptions

(H_1) There exist a_i, $b_i > 0$ and $\rho \in (0, q-1)$ such that

$$|f_i(t,u)| \leq a_i + b_i|u|^\rho, \ \forall (t,u) \in [0,T] \times R, \quad \text{where} \quad i = 1, 2, \dots, n.$$

(H_2) There exist c_i, $d_i > 0$ and $l \in (0, q)$ such that

$$\int_0^t I_i(s)ds \leq c_i + d_i|t|^l, \ \forall (t,u) \in [0,T] \times R, \quad \text{where} \quad i = 1, 2, \dots, n.$$

(H_3) G_i, $g_i : [0,T] \times R \to R$ are Carathéodory functions and there exists function $g \in L^2[0,T]$, such that $|G_i(t,u)| \leq g(t)$, $|g_i(t,u)| \leq g(t)$ for a.e. $t \in [0,T]$ and all $u \in R$, $i = 1, 2, \dots, n$.

Theorem 1.2 is the existence result for (1.2).

Theorem 1.2. *Assume that the conditions* (H_1)-(H_3) *are satisfied. Then problem* (1.2) *has at least one classical solution.*

Remark 1.1. Let $q = p$ and $x(t) \equiv 0$, problem (1.2) is reduced to problem (1.1), the condition (H_2) become $\int_0^t I_i(s)ds \leq c_i + d_i|t|^l$, $l \in (0, p)$ which improve the condition $|I_i(u)| \leq c_i + d_i|u|^{\delta_i}$, $\delta \in [0, 1)$ in [35].

Theorem 1.3 is about the dependence on functional parameters for classical solutions to the problem (1.2).

Theorem 1.3. *Assume that conditions* (H_1)-(H_3) *hold. Let* $\{x_k\}_{k=1}^\infty \subset L_m$ *satisfy* $x_k \rightharpoonup x_0$ *in* $L^p[0,T]$*. Then, for any sequence* $\{u_k\}_{k=1}^\infty$ *of solutions to problem* (1.2) *corresponding to* x_k*, there exists a subsequence* $\{u_{k_n}\}_{n=1}^\infty \subset X$ *and an element* $u_0 \in X$ *such that* $u_{k_n} \rightharpoonup u_0$ *in* X *and that* u_0 *is a classical solution to problem* (1.2) *corresponding to* x_0*. In addition,* $u_{k_n} \to u_0$ *(strongly) in* X*.*

Corollary 1.1. *Assume that the functions* f_i*,* G_i *and the impulsive functions* I_i *are bounded, then problem* (1.2) *has at least one classical solution.*

This paper has two novelties. Firstly, comparing with problem (1.1), we extend the p-Laplace operator to the p-q-Laplace operator and prove the existence of the solution for problem (1.2) in a more general case $1 < q \leq p < +\infty$. If $q = p$ and the functional parameter $x(t) \equiv 0$, problem (1.2) is reduced to problem (1.1). Theorem 1.2 is a generalization of Theorem 1.1 since $p > 1$. Secondly, the functional is dependent on the functional parameter x, which has not been considered before.

The organization of this article is as follows. In section 2, we introduce some fundamental knowledge, lemmas and theorems. In section 3, we prove the main results that the problem (1.2) has at least one nontrivial classical solution and the dependence on functional parameters for the classical solution.

2. Preliminaries

In this part, we recall some definitions, lemmas and theorems which are useful to the main results.

Definition 2.1. Let $\alpha \in (\frac{1}{q}, 1], 1 < q \le p < +\infty$. The fractional space

$$\begin{aligned} W_0^{\alpha,p}[0,T] =& \{u : [0,T] \to R | u \in L^p([0,T],R),\ {}_0^C D_t^\alpha u \in L^p([0,T],R), \\ & u(0) = u(T) = 0\} \end{aligned}$$

is defined by the closure of $C_0^\infty([0,T],R)$ with the norm

$$\|u\|_{W_0^{\alpha,p}} = (\int_0^T |u|^p dt + \int_0^T |{}_0^c D_t^\alpha u|^p dt)^{\frac{1}{p}}.$$

Proposition 2.1 ([26]). *Let $\alpha \in (0,1]$, $p > 1$ and the space $W_0^{\alpha,p}(\Omega)$ is reflexive.*

Proposition 2.2 ([33, 34]). *Let $\alpha \in (0,1]$ and $p \in (1,+\infty)$. For any $u \in W_0^{\alpha,p}[0,T]$, we have*

$$\|{}_0^C D_t^\alpha u\|_{L^p} \ge \frac{\Gamma(1+\alpha)}{T^\alpha} \|u\|_{L^p}.$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|{}_0^C D_t^\alpha u\|_{L^p} \ge \frac{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}{T^{\alpha-\frac{1}{p}}} \|u\|_\infty.$$

Proposition 2.3 ([33,34]). *Let $\alpha \in (0,1]$ and u, $v \in L^p([a,b],R)$, $p > 1$. Then*

$$\int_a^b v(t)\, {}_t D_b^\alpha u(t) dt = \int_a^b u(t)\, {}_a D_t^\alpha v(t) dt,$$
$$\int_a^b v(t)\, {}_t D_b^{\alpha-1} u(t) dt = \int_a^b u(t)\, {}_a D_t^{\alpha-1} v(t) dt.$$

Proposition 2.4 ([33,34]). *Let $\alpha \in (0,1]$ and the space $W_0^{\alpha,p}[0,T]$ and $W_0^{\alpha,q}[0,T]$ are compactly embedded in $C([0,T],R)$.*

Proposition 2.5 ([29]). *Let $s \in (0,1)$ and $N \ge 1$. Suppose that Ω is bounded domain in R^N with Lipschitz boundary $\partial\Omega$. Then the embedding $W_0^{s,N/s}(\Omega) \hookrightarrow W^{s,N/s}(\Omega) \hookrightarrow\hookrightarrow L^v(\Omega)$ is compact for all $v \in [1,+\infty)$.*

Remark 2.1. For $W_0^{\alpha,p}[0,T]$ and $W_0^{\alpha,q}[0,T]$, $1 < q \le p < +\infty$. It is easy to see $W_0^{\alpha,p}[0,T] \hookrightarrow W_0^{\alpha,1/\alpha}[0,T]$ for $\alpha \in (\frac{1}{q}, 1]$. By Proposition 2.5, $W_0^{\alpha,p}[0,T] \hookrightarrow\hookrightarrow L^v[0,T]$, $v \in [1,+\infty)$.

Proposition 2.6 ([19,33]). *Let $n-1 < \alpha < n$, $n \in N$ and $u \in AC([0,T])$. The left and right Caputo fractional derivatives denoted by ${}_0^C D_t^\alpha u(t)$ and ${}_t^C D_T^\alpha u(t)$ are respectively defined by*

$$\begin{aligned} {}_0^C D_t^\alpha u(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^n(\tau) d\tau, \\ {}_t^C D_T^\alpha u(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T (\tau-t)^{n-\alpha-1} u^n(\tau) d\tau. \end{aligned}$$

$$ {}_a^C D_t^\alpha u(t) = {}_a D_t^\alpha u(t) - \frac{u(a)}{\Gamma(1-a)}(t-a)^{-\alpha}, $$
$$ {}_t^C D_b^\alpha u(t) = {}_t D_b^\alpha u(t) - \frac{u(b)}{\Gamma(1-a)}(b-t)^{-\alpha}. $$

Proposition 2.7 ([19, 33]). *Let $n-1 < \alpha < n$, $n \in N$ and $t \in [0,T]$. Denote the left and right Riemann-Liouville fractional derivatives of $u(t)$ by ${}_0D_t^\alpha u(t)$ and ${}_tD_T^\alpha u(t)$, respectively, with definitions as follows*

$$ {}_0D_t^\alpha u(t) = \frac{d^n}{dt^n}\, {}_0D_t^{\alpha-n} u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}u(\tau)d\tau, $$
$$ {}_tD_T^\alpha u(t) = (-1)^n \frac{d^n}{dt^n}\, {}_tD_T^{\alpha-n} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^T (\tau-t)^{n-\alpha-1}u(\tau)d\tau. $$

In particular, for $u \in W_0^{\alpha,p}[0,T]$, it follows

$$ {}_0^C D_t^\alpha u(t) = {}_0D_t^\alpha u(t), \ \ {}_t^C D_T^\alpha u(t) = {}_tD_T^\alpha u(t). $$

In this paper, we shall use the space X as follows.

Define the space $X = W_0^{\alpha,p} \cap W_0^{\alpha,q}$ with the norm $\|u\|_X = \|u\|_{W_0^{\alpha,p}} + \|u\|_{W_0^{\alpha,q}}$. We shall state the relationship between $(W_0^{\alpha,p}, \|\cdot\|_{W_0^{\alpha,p}})$ and $(X, \|\cdot\|_{W_0^{\alpha,p}})$.

Lemma 2.1. *For the space X, the norms $\|\cdot\|_{W_0^{\alpha,p}}$ and $\|\cdot\|_X$ are equivalent.*
Proof. *On one hand, we have*

$$ \|u\|_{W_0^{\alpha,p}} \le \|u\|_{W_0^{\alpha,p}} + \|u\|_{W_0^{\alpha,q}}, $$
$$ \|u\|^q_{W_0^{\alpha,p}} = (\int_0^T |u|^p dt + \int_0^T |{}_0^C D_t^\alpha u|^p dt)^{\frac{q}{p}}. $$

On the other hand, by Hölder's inequality, we can also get

$$ \begin{aligned} \|u\|^q_{L^q} + \|{}_0^C D_t^\alpha u\|^q_{L^q} &\le T^{(1-\frac{q}{p})}(\int_0^T |u|^p dt)^{\frac{q}{p}} + T^{(1-\frac{q}{p})}(\int_0^T |{}_0^C D_t^\alpha u|^p dt)^{\frac{q}{p}} \\ &= T^{(1-\frac{q}{p})}(\|u\|_{L^p})^q + T^{(1-\frac{q}{p})}(\|{}_0^C D_t^\alpha u\|_{L^p})^q. \end{aligned} $$

And, for $1 < q \le p < +\infty$, there exists a c large enough, such that

$$ T^{(1-\frac{q}{p})}(\int_0^T |u|^p dt)^{\frac{q}{p}} + T^{(1-\frac{q}{p})}(\int_0^T |{}_0^c D_t^\alpha u|^p dt)^{\frac{q}{p}} \le c(\int_0^T |u|^p dt + \int_0^T |{}_0^C D_t^\alpha u|^p dt)^{\frac{q}{p}}. $$

Therefore, these two norms are equivalent. □

Lemma 2.2. *$(W_0^{\alpha,p}, \|\cdot\|_{W_0^{\alpha,p}})$ and $(X, \|\cdot\|_{W_0^{\alpha,p}})$ are the same.*

Proof. We only need to show $W_0^{\alpha,p} \subset W_0^{\alpha,q}$, $\forall u \in W_0^{\alpha,p}[0,T]$, we have $u \in L^p([0,T],R)$, and so u is measurable and $\int_T |u|^p dx < \infty$. Since $1 < q \le p < +\infty$, we have $\int_T |u|^q dx < \infty$, therefore we can get $u \in L^q([0,T],R)$. Similarly, we can also get ${}_0^C D_t^\alpha \in L^q([0,T],R)$. Therefore $u \in W_0^{\alpha,q}[0,T]$ and $u(0) = u(T) = 0$. □

Remark 2.2. By Proposition 2.1, we have $(W_0^{\alpha,p}, \|\cdot\|_{W_0^{\alpha,p}})$ is reflexive Banach space, so $(X, \|\cdot\|_{W_0^{\alpha,p}})$ is a reflexive Banach space.

Define the functional $\varphi_x : X \to R$ by

$$\begin{aligned}\varphi_x(u) =&\frac{1}{p}\|u\|^p_{W_0^{\alpha,p}} + \frac{1}{q}\|u\|^q_{W_0^{\alpha,q}} + \sum_{i=1}^{n}\int_0^{u(t_i)} I_i(s)ds - \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t,u(t))dt \\ &- \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} G_i(t,u(t))x(t)dt.\end{aligned} \tag{2.1}$$

By using the continuity of f_i and I_i and assumption (H_3), we obtain that φ is Fréchet differentiable on X and

$$\begin{aligned}\langle\varphi_x'(u);v\rangle =& \int_0^T \phi_p({}_0^C D_t^\alpha\, u(t))({}_0^C D_t^\alpha\, v(t))dt + \int_0^T \phi_q({}_0^C D_t^\alpha\, u(t))({}_0^C D_t^\alpha\, v(t))dt \\ &+ \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}[|u(t)|^{p-2}u(t) + |u(t)|^{q-2}u(t)]v(t)dt + \sum_{i=1}^{n} I_i(u(t_i))v(t_i) \\ &- \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}[f_i(t,u(t)) + g_i(t,u(t))x(t)]v(t)dt.\end{aligned} \tag{2.2}$$

Definition 2.2. A function $u \in X$ is a weak solution of problem (1.2), if u satisfies $\langle\varphi_x'(u);v\rangle = 0$ for all $v \in X$.

Definition 2.3. A function u satisfying ${}_tD_T^\alpha\, \phi_p({}_0^C D_t^\alpha\, u)(t) + {}_tD_T^\alpha\, \phi_q({}_0^C D_t^\alpha\, u)(t) \in L^1(s_i,t_{i+1}]$, $i = 0,1,\ldots,n$ is a classical solution of problem (1.2), if u satisfies the equation of problem (1.2), impulsive conditions and the boundary condition $u(0) = u(T) = 0$.

Theorem 2.1 will be used in the proof of existence of solutions for (1.2).

Theorem 2.1 ([22]). *Let M be a complete metric space and let $J : M \to (-\infty,\infty]$ be a lower semicontinuous functional, bounded from below, and not identically equal to ∞. Let $\varepsilon > 0$ be given and $z \in M$ be such that $J(z) \leq inf_M J + \varepsilon$.*

Then, there exists $v \in M$ such that $J(v) \leq J(z) \leq inf_M J + \varepsilon$, $d(z,v) \leq 1$, and for any $u \neq v$ in M, $J(v) < J(u) + \varepsilon d(v,u)$, where $d(\cdot,\cdot)$ denotes the distance between two elements in M.

3. Main results

Lemma 3.1. *If $u \in X$ is a weak solution of the problem* (1.2)*, u is a classical solution of the problem* (1.2).

Proof. Let $u \in X$ be a weak solution of the problem (1.2), then $u(0) = (T) = 0$ and Definition 2.2 hold.

Step 1. We'll show the first equation in (1.2).

For this, we first show that the first equation in (1.2) in the interval $(s_n,T]$.

We can select a test function $v \in C_0^\infty[0,T]$ satisfying $v(t) \equiv 0$ for $t \in [0,s_n]$.

Substituting $v(t)$ into (2.2) and by Definition 2.2, we have

$$\begin{aligned}&\int_{s_n}^{T}\phi_p({}_0^C D_t^\alpha u(t))({}_{s_n}^C D_t^\alpha v(t))dt+\int_{s_n}^{T}\phi_q({}_{s_n}^C D_t^\alpha u(t))({}_0^C D_t^\alpha v(t))dt\\&=\int_{s_n}^{T}[f_n(t,u(t))+g_n(t,u(t))x(t)]v(t)dt-\int_{s_n}^{T}[|u(t)|^{p-2}u(t)+|u(t)|^{q-2}u(t)]v(t)dt.\end{aligned}\tag{3.1}$$

Using Proposition 2.3 and Proposition 2.6, we have

$$\int_{s_n}^{T}\phi_p({}_0^C D_t^\alpha u(t))({}_{s_n}^C D_t^\alpha v(t))dt=\int_{s_n}^{T}v(t)\,{}_tD_T^\alpha(\phi_p({}_0^C D_t^\alpha u(t)))dt.\tag{3.2}$$

Substituting (3.2) into (3.1), and by dubois-Reymond theorem, we have

$$\begin{aligned}&{}_tD_T^\alpha\,\phi_p({}_0^C D_t^\alpha u)(t)+{}_tD_T^\alpha\,\phi_q({}_0^C D_t^\alpha u)(t)+|u(t)|^{p-2}u(t)+|u(t)|^{q-2}u(t)\\&=f_n(t,u(t))+g_n(t,u(t))x(t),t\in(s_n,T],\end{aligned}\tag{3.3}$$

which means the first equations in (1.2) is satisfied in $t\in(s_n,T]$.

Next we'll show that the first equations in (1.2) is satisfied in $t\in(s_i,t_{i+1}]$, $i=0,1,\ldots,n-1$.

We can select a test function $v\in C_0^\infty[0,T]$ satisfying $v(t)\equiv 0$ for $t\in[0,s_i]\cup[t_{i+1},T]$. Substituting $v(t)$ into (2.2) and by Definition 2.2, we have

$$\begin{aligned}&\int_{s_i}^{T}\phi_p({}_0^C D_t^\alpha u(t))({}_{s_i}^C D_t^\alpha v(t))dt+\int_{s_i}^{T}\phi_q({}_{s_n}^C D_t^\alpha u(t))({}_{s_i}^C D_t^\alpha v(t))dt\\&=\int_{s_i}^{t_{i+1}}[f_i(t,u(t))+g_i(t,u(t))x(t)]v(t)-[|u(t)|^{p-2}u(t)+|u(t)|^{q-2}u(t)]v(t)dt.\end{aligned}\tag{3.4}$$

Using Proposition 2.3, Proposition 2.6 and $v(t)\equiv 0$ for $t\in[0,s_i]\cup[t_{i+1},T]$, we have

$$\begin{aligned}\int_{s_i}^{T}\phi_p({}_0^C D_t^\alpha u(t))({}_{s_i}^C D_t^\alpha v(t))dt&=\int_{s_i}^{T}v(t)\,{}_tD_T^\alpha(\phi_p({}_0^C D_t^\alpha u(t)))dt\\&=\int_{s_i}^{t_{i+1}}v(t)\,{}_tD_T^\alpha(\phi_p({}_0^C D_t^\alpha u(t)))dt.\end{aligned}\tag{3.5}$$

Substituting (3.5) into (3.4), and by dubois-Reymond theorem, we have

$$\begin{aligned}&{}_tD_T^\alpha\,\phi_p({}_0^C D_t^\alpha u)(t)+{}_tD_T^\alpha\,\phi_q({}_0^C D_t^\alpha u)(t)+|u(t)|^{p-2}u(t)+|u(t)|^{q-2}u(t)\\&=f_i(t,u(t))+g_i(t,u(t))x(t),t\in(s_i,t_{i+1}],\end{aligned}\tag{3.6}$$

which means the first equations in (1.2) is satisfied in $t\in(s_i,t_{i+1}]$.

Since $u\in X\subset C([0,T])$ and $f_i\in C((s_i,t_{i+1}]\times R,R)$ and assumption (H_3), one has

$${}_tD_T^\alpha\,\phi_p({}_0^C D_t^\alpha u)(t)+{}_tD_T^\alpha\,\phi_q({}_0^C D_t^\alpha u)(t)\in L^1(s_i,t_{i+1}].$$

Step 2. We'll show the third and fourth equation in (1.2).

By Proposition 2.6 we have ${}_tD_T^\alpha\,u(t)=-\frac{d}{dt}\,{}_tD_T^{\alpha-1}\,u(t)$, so we get

$${}_tD_T^\alpha\,\phi_p({}_0^C D_t^\alpha u(t))=-\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_p({}_0^C D_t^\alpha u(t)))\in L^p(s_i,t_{i+1}],\tag{3.7}$$

$$ {}_tD_T^\alpha\,\phi_q({}_0^CD_t^\alpha\,u(t)) = -\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\,u(t))) \in L^p(s_i,t_{i+1}], \tag{3.8} $$

Hence, we have ${}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u(t))$, ${}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\,u(t)) \in AC([s_i,t_{i+1}])$.

Substituting (3.3) into (2.2) and by Definition 2.2, we have

$$ \begin{aligned} &\int_0^T \phi_p({}_0^CD_t^\alpha\,u(t))({}_0^CD_t^\alpha\,v(t))dt + \int_0^T \phi_q({}_0^CD_t^\alpha\,u(t))({}_0^CD_t^\alpha\,v(t))dt \\ &+\sum_{i=1}^n I_i(u(t_i))v(t_i) \\ =&\sum_{i=1}^n\int_{s_i}^{t_{i+1}} v(t)({}_tD_T^\alpha\,\phi_p({}_0^CD_t^\alpha\,u)(t))dt + \sum_{i=1}^n\int_{s_i}^{t_{i+1}} v(t)({}_tD_T^\alpha\,\phi_q({}_0^CD_t^\alpha\,u)(t))dt. \end{aligned} $$

Then using (3.7) and (3.8), we can get

$$ \begin{aligned} &\int_0^T \phi_p({}_0^CD_t^\alpha\,u(t))({}_0^CD_t^\alpha\,v(t))dt+\int_0^T \phi_q({}_0^CD_t^\alpha\,u(t))({}_0^CD_t^\alpha\,v(t))dt+\sum_{i=1}^n I_i(u(t_i))v(t_i) \\ =&-\sum_{i=1}^n\int_{s_i}^{t_{i+1}} v(t)\frac{d}{dt}\left({}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t) + {}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\,u)(t)\right)dt. \end{aligned} \tag{3.9} $$

By integration by parts, one has

$$ \begin{aligned} &-\sum_{i=1}^n\int_{s_i}^{t_{i+1}} v(t)\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t))dt \\ =&-\sum_{i=0}^n v(t)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t)|_{s_i}^{t_{i+1}} + \sum_{i=1}^n\int_{s_i}^{t_{i+1}} {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t)v'(t)dt. \end{aligned} \tag{3.10} $$

By Proposition 2.3 and (3.10), one has

$$ \begin{aligned} &-\sum_{i=0}^n v(t)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t)|_{s_i}^{t_{i+1}} + \sum_{i=1}^n\int_{s_i}^{t_{i+1}} {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t)v'(t)dt \\ =&-\sum_{i=0}^n v(t)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t)|_{s_i}^{t_{i+1}} + \sum_{i=1}^n\int_{s_i}^{t_{i+1}} \phi_p({}_0^CD_t^\alpha\,u)(t)\,{}_0^CD_t^{\alpha-1}\,v'(t)dt. \end{aligned} \tag{3.11} $$

By Proposition 2.6 and (3.11), one has

$$ \begin{aligned} &-\sum_{i=0}^n v(t)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t)|_{s_i}^{t_{i+1}} + \sum_{i=1}^n\int_{s_i}^{t_{i+1}} \phi_p({}_0^CD_t^\alpha\,u)(t)\,{}_0^CD_t^{\alpha-1}\,v'(t)dt \\ =&\sum_{i=0}^n v(s_i^+)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(s_i^+) - \sum_{i=0}^n v(t_{i+1}^-)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\,u)(t_{i+1}^-) \\ &+\sum_{i=1}^n\int_{s_i}^{t_{i+1}} \phi_p({}_0^CD_t^\alpha\,u(t))({}_0^CD_t^\alpha\,v(t))dt. \end{aligned} \tag{3.12} $$

Similarly, one has

$$
\begin{aligned}
&-\sum_{i=1}^{n}\int_{s_i}^{t_{i+1}} v(t)\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t))dt\\
=&\sum_{i=0}^{n} v(s_i^+)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(s_i^+) - \sum_{i=0}^{n} v(t_{i+1}^-)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t_{i+1}^-)\\
&+\sum_{i=1}^{n}\int_{s_i}^{t_{i+1}} \phi_q({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^\alpha\, v(t))dt.
\end{aligned}
\tag{3.13}
$$

Substituting (3.10)-(3.13) into (3.9), one has

$$
\begin{aligned}
&\sum_{i=1}^{n}\int_{t_i}^{s_i} \phi_p({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^\alpha\, v(t))dt + \sum_{i=1}^{n}\int_{t_i}^{s_i} \phi_q({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^\alpha\, v(t))dt\\
&+\sum_{i=1}^{n} I_i(u(t_i))v(t_i)\\
=&\sum_{i=0}^{n} v(s_i^+)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(s_i^+) - \sum_{i=0}^{n} v(t_{i+1}^-)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t_{i+1}^-)\\
&+\sum_{i=0}^{n} v(s_i^+)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(s_i^+) - \sum_{i=0}^{n} v(t_{i+1}^-)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t_{i+1}^-).
\end{aligned}
\tag{3.14}
$$

By Proposition 2.3 and ${}_tD_T^\alpha\, u(t) = -\frac{d}{dt}\,{}_tD_T^{\alpha-1}\, u(t)$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\int_{t_i}^{s_i} \phi_p({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^\alpha\, v(t))dt + \sum_{i=1}^{n}\int_{t_i}^{s_i} \phi_q({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^\alpha\, v(t))dt\\
=&\sum_{i=1}^{n}\int_{t_i}^{s_i} v(t)\,{}_tD_T^\alpha(\phi_p({}_0^CD_t^\alpha\, u(t)))dt + \sum_{i=1}^{n}\int_{t_i}^{s_i} v(t)\,{}_tD_T^\alpha(\phi_q({}_0^CD_t^\alpha\, u(t)))dt\\
=&-\sum_{i=1}^{n}\int_{t_i}^{s_i} v(t)\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t)) + v(t)\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t))dt.
\end{aligned}
\tag{3.15}
$$

Without loss of generality, we take the test function $v \in C_0^\infty(t_i, s_i]$ satisfying $v(t) \equiv 0$ for $t \in [0, t_i] \cup (s_i, T]$, $i = 1, 2, \ldots, n$. Substituting $v(t)$ into (3.14), (3.15) into (3.14), we have

$$
\sum_{i=1}^{n}\int_{t_i}^{s_i} v(t)\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t))dt + \sum_{i=1}^{n}\int_{t_i}^{s_i} v(t)\frac{d}{dt}({}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t))dt
$$
$$=0.$$

By dubois-Reymond theorem, we have ${}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t))$, ${}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u(t))$ are constants, $t \in (t_i, s_i]$, $i = 1, 2, \ldots, n$. That is

$$
{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t_i^+)) = {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(s_i^-)) = {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t)),
$$

$$ {}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u(t_i^+)) = {}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u(s_i^-)) = {}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u(t)), $$

which means the third and fourth conditions in (1.2) are satisfied.
Step 3. We'll show the second and fifth conditions in (1.2).

By Proposition 2.6, one has

$$ \sum_{i=1}^n \int_{t_i}^{s_i} \phi_p({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^\alpha\, v(t))dt = \sum_{i=1}^n \int_{t_i}^{s_i} \phi_p({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^{\alpha-1}\, v'(t))dt. \tag{3.16} $$

By Proposition 2.3 and (3.16), one has

$$ \sum_{i=1}^n \int_{t_i}^{s_i} \phi_p({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^{\alpha-1}\, v'(t))dt = \sum_{i=1}^n \int_{t_i}^{s_i} {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t))v'(t)dt. \tag{3.17} $$

By ${}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t))$ is constant and (3.17), one has

$$ \begin{aligned} &\sum_{i=1}^n \int_{t_i}^{s_i} {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t))v'(t)dt \\ =&\sum_{i=1}^n v(s_i)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t_i^+)) - \sum_{i=1}^n v(t_i)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t_i^+)). \end{aligned} \tag{3.18} $$

Similarly, one has

$$ \begin{aligned} &\sum_{i=1}^n \int_{t_i}^{s_i} \phi_q({}_0^CD_t^\alpha\, u(t))({}_0^CD_t^{\alpha-1}\, v'(t))dt \\ =&\sum_{i=1}^n v(s_i)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u(t_i^+)) - \sum_{i=1}^n v(t_i)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u(t_i^+)). \end{aligned} \tag{3.19} $$

Substituting (3.16)-(3.19) into (3.14), one has

$$ \begin{aligned} &\sum_{i=1}^n v(s_i)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t_i^+) - \sum_{i=1}^n v(t_i)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t_i^+) \\ &+\sum_{i=1}^n v(s_i)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t_i^+) - \sum_{i=1}^n v(t_i)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t_i^+) \\ &+\sum_{i=1}^n I_i(u(t_i))v(t_i) \\ =&\sum_{i=0}^n v(s_i^+)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(s_i^+) - \sum_{i=0}^n v(t_{i+1}^-)\,{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t_{i+1}^-) \\ &+\sum_{i=0}^n v(s_i^+)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(s_i^+) - \sum_{i=0}^n v(t_{i+1}^-)\,{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t_{i+1}^-). \end{aligned} \tag{3.20} $$

Without loss of generality, assume $v(t_k) \neq 0$, $n \le k < n+1$, $v(s_i) = 0$, $i = 0, \dots, n$,

$v(T)=0$ for all $v\in W_0^{1,p}[0,T]$, substituting it into (3.20), one has

$$\begin{aligned}&\sum_{i=1}^{n} {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t_i^+)-\sum_{i=1}^{n} {}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(t_i^-)\\&+\sum_{i=1}^{n} {}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t_i^+)-\sum_{i=1}^{n} {}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(t_i^-)\\=&I_i(u(t_i)),\end{aligned}$$

which means the second condition in (1.2) is satisfied.

Let $v(t_k)=0$, $n\le k<n+1$, $v(s_i)\neq 0$, $i=0,\dots,n$. Substituting $v\in W_0^{1,p}[0,T]$ into (3.20), since ${}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u(t))$, ${}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u(t))$ are constants, one has

$$\begin{aligned}&{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(s_i^-)+{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(s_i^-)\\=&{}_tD_T^{\alpha-1}\,\phi_p({}_0^CD_t^\alpha\, u)(s_i^+)+{}_tD_T^{\alpha-1}\,\phi_q({}_0^CD_t^\alpha\, u)(s_i^+)\end{aligned}$$

which means the fifth condition in (1.2) is satisfied.

From above, u satisfies equation, boundary conditions and impulsive conditions of problem (1.2), this means u is a classical solution of problem (1.2). □

Lemma 3.2. *Suppose that (H_1)-(H_3) are satisfied. Then there exists $\gamma>0$ such that $\varphi_x(u)>0$ for $u\in X$ with $\|u\|_X=\gamma$.*

Proof. We firstly illustrate some conclusions under assumptions (H_1) and (H_3).
(h_1) From (H_1), we infer that $|F_i(t,u)|\le a_i|u|+\frac{b_i}{\rho+1}|u|^{\rho+1}$.
(h_2) From (H_3) and Hölder's inequality, we have

$$\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} G_i(t,u(t))x(t)dt\le\sqrt{m}\sqrt{\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} g^2(t)dt}.$$

Combining with (H_2) and Proposition 2.2, there exists constants C_i, $i=1,2,\cdots,6$, such that

$$\begin{aligned}\varphi_x(u)=&\frac1p\|u\|^p_{W_0^{\alpha,p}}+\frac1q\|u\|^q_{W_0^{\alpha,q}}+\sum_{i=1}^{n}\int_0^{u(t_i)} I_i(s)ds-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t,u(t))dt\\&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} G_i(t,u(t))x(t)dt\\\ge&\frac1p\|u\|^p_{W_0^{\alpha,p}}+\frac1q\|u\|^q_{W_0^{\alpha,q}}-T\sum_{i=0}^{n}(a_i\|u\|_\infty+\frac{b_i}{\rho+1}\|u\|_\infty^{\rho+1})\\&-\sum_{i=0}^{n}(c_i+d_i\|u\|_\infty^l)-\sqrt{m}\sqrt{\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} g^2(t)dt}\\\ge&(\frac1p\|u\|^p_{W_0^{\alpha,p}}-C_1\|u\|_{W_0^{\alpha,p}}-C_2\|u\|^{\rho+1}_{W_0^{\alpha,p}}-C_3\|u\|^l_{W_0^{\alpha,p}})\\&+(\frac1q\|u\|^q_{W_0^{\alpha,q}}-C_4\|u\|_{W_0^{\alpha,q}}-C_5\|u\|^{\rho+1}_{W_0^{\alpha,q}}-C_6\|u\|^l_{W_0^{\alpha,q}})\\&-\sqrt{m}\sqrt{\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} g^2(t)dt}-\sum_{i=0}^{n}c_i,\end{aligned}\tag{3.21}$$

which implies there exists $\gamma > 0$ such that $\varphi_x(u) > 0$ for $u \in X$ with $\|u\|_X = \gamma$. $\square$

Proof of Theorem 1.2. We'll apply Theorem 2.1 to prove the theorem. Let γ be defined in Lemma 3.2, $M = \overline{B_\gamma(0)} \subset X$. By (2.2), φ_x is continuous, then φ_x is lower semicontinuous. From (3.21), we have φ_x is bounded from below.
Step 1. We'll prove there exists $v \in M$ such that

$$c - \varepsilon < \varphi_x(v) \le c + \varepsilon, \ c = \inf_{u \in M} \varphi_x(u). \tag{3.22}$$

From Lemma 3.2, one has $\inf_{u \in \partial B_\gamma(0)} \varphi_x(u) > 0$. By (2.1), $\varphi_x(0) = 0$. Thus $\inf_{u \in B_\gamma(0)} \varphi_x(u) \le 0$ and $\inf_{u\in M} \varphi_x(u) = \inf_{u \in B_\gamma(0)} \varphi_x(u)$. Let $0 < \varepsilon < \inf_{u \in \partial B_\gamma(0)} \varphi_x(u) - c$, it is clear that there exists $z \in M$ such that $\varphi_x(z) \le \varepsilon + \inf_{u\in M} \varphi_x(u)$. By Theorem 2.1, there exists $v \in M$ such that

$$\inf_{u\in M} \varphi_x(u) - \varepsilon < \varphi_x(v) \le \varphi_x(z) \le \varepsilon + \inf_{u \in M} \varphi_x(u),$$

which means (3.22) is proved.

Besides, from Theorem 2.1 for any $u \ne v$ in M, we have

$$\varphi_x(v) < \varphi_x(u) + \varepsilon \|u - v\|_X. \tag{3.23}$$

Step 2. We'll prove

$$\|\varphi_x'(v)\|_{X^*} \le \varepsilon. \tag{3.24}$$

Define functional

$$I(u) = \varphi_x(u) + \varepsilon\|u - v\|_X. \tag{3.25}$$

By (3.23) and (3.25), one has $I(v) = \varphi_x(v) < \varphi_x(u) + \varepsilon\|u - v\|_X = I(u)$ for all $u \ne v$. So v is the minimum point of (3.25). Therefore $I(v + ty) - I(v) \ge 0$ for all $y \in B_\gamma(0)$, and

$$\begin{aligned}\lim_{t\to 0^+} \frac{I(v+ty) - I(v)}{t} &= \lim_{t \to 0^+} \frac{\varphi_x(v+ty) + \varepsilon\|v + ty - v\|_X - \varphi_x(v)}{t} \\ &= \langle \varphi_x'(v), y\rangle + \varepsilon\|y\|_X \ge 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0^-} \frac{I(v+ty) - I(v)}{t} &= \lim_{t \to 0^-} \frac{\varphi_x(v+ty) + \varepsilon\|v + ty - v\|_X - \varphi_x(v)}{t} \\ &= \langle \varphi_x'(v), y\rangle - \varepsilon\|y\|_X \le 0,\end{aligned}$$

which means (3.24) is proved.
Step 3. We'll prove the existence of classical solution u_0 of problem (1.2).

By (3.22) and (3.24), there exists sequence $\{u_n\} \subset B_\gamma(0)$ such that

$$\varphi_x(u_n) \to c, \ \varphi_x'(u_n) \to 0.$$

By (3.21), $\{u_n\}$ is bounded. Because of the reflexivity of X, the sequence $\{u_n\}$ weakly converges to u_0 in X. Next we'll prove that $\{u_n\}$ strongly converges to u_0

in X. By (2.2) and Definition 2.2, one has

$$
\begin{aligned}
0 &\leftarrow \langle \varphi'_x(u_n) - \varphi'_x(u_0), u_n - u_0\rangle \\
&= \langle \varphi'_x(u_n), u_n - u_0\rangle - \langle \varphi'_x(u_0), u_n - u_0\rangle \\
&= \int_0^T [\phi_p({}_0^C D_t^\alpha u_n) - \phi_p({}_0^C D_t^\alpha u_0) + \phi_q({}_0^C D_t^\alpha u_n) - \phi_q({}_0^C D_t^\alpha u_0)]\, {}_0^C D_t^\alpha (u_n - u_0) dt \\
&\quad + \sum_{i=0}^n \int_{s_i}^{t_{i+1}} [|u_n|^{p-2}u_n - |u_0|^{p-2}u_0 + |u_n|^{q-2}u_n - |u_0|^{q-2}u_0](u_n - u_0)dt \\
&\quad - \sum_{i=0}^n \int_{s_i}^{t_{i+1}} [f_i(t,u_n) - f_i(t,u_0) + g_i(t,u_n)x - g_i(t,u_0)x](u_n - u_0)dt \\
&\quad + \sum_{i=1}^n [I_i(u_n(t_i) - I_i(u_0(t_i))](u_n - u_0).
\end{aligned}
\tag{3.26}
$$

Since $u_n \rightharpoonup u_0$ in X, we obtain $\{u_n\}$ uniformly converges to u_0 in $C[0,T]$. Let $r = p, q$, one has

$$
\begin{aligned}
&\sum_{i=0}^n \int_{s_i}^{t_{i+1}} |u_n|^{r-2} u_{k_n}(u_n - u_0)dt \to \sum_{i=0}^n \int_{s_i}^{t_{i+1}} |u_0|^{r-2} u_0(u_n - u_0)dt \text{ as } n \to \infty, \\
&\sum_{i=1}^n I_i(u_n(t_i))(u_n - u_0) \to \sum_{i=1}^n I_i(u_0(t_i))(u_n - u_0) \text{ as } n \to \infty.
\end{aligned}
\tag{3.27}
$$

The Lebesgue's dominated convergence theorem implies that for any $u_n - u_0 \in X$,

$$
\begin{aligned}
&\sum_{i=0}^n \int_{s_i}^{t_{i+1}} f_i(t,u_n)(u_n - u_0)dt \to \sum_{i=0}^n \int_{s_i}^{t_{i+1}} f_i(t,u_0)(u_n - u_0)dt, \\
&\sum_{i=0}^n \int_{s_i}^{t_{i+1}} g_i(t,u_n)x(u_n - u_0)dt \to \sum_{i=0}^n \int_{s_i}^{t_{i+1}} g_i(t,u_0)x(u_n - u_0)dt.
\end{aligned}
\tag{3.28}
$$

Substituting (3.27)-(3.28) into (3.26), one has

$$
\int_0^T [\phi_p({}_0^C D_t^\alpha u_n) - \phi_p({}_0^C D_t^\alpha u_0) + \phi_q({}_0^C D_t^\alpha u_n) - \phi_q({}_0^C D_t^\alpha u_0)]\, {}_0^C D_t^\alpha (u_n - u_0) dt \to 0.
$$

Next we'll show $\{u_n\}$ strongly converges to u_0 in X. By Equation (2.2) in [27], we can get the following two results.

If $2 \le q \le p < +\infty$, there exists $c_p > 0$ such that

$$
\begin{aligned}
&\int_0^T [\phi_p({}_0^C D_t^\alpha u_n) - \phi_p({}_0^C D_t^\alpha u_0) + \phi_q({}_0^C D_t^\alpha u_n) - \phi_q({}_0^C D_t^\alpha u_0)]\, {}_0^C D_t^\alpha (u_n - u_0) dt \\
\ge& c_p \int_0^T [|{}_0^C D_t^\alpha (u_n - u_0)|^p + |{}_0^C D_t^\alpha (u_n - u_0)|^q] dt \\
\ge& c_p (\|{}_0^C D_t^\alpha (u_n - u_0)\|^p + \|{}_0^C D_t^\alpha (u_n - u_0)\|^q).
\end{aligned}
$$

If $1 < q \le p < 2$, by Hölder's inequality, similar to the proof of Tian Ge, there exists $d_p > 0$ such that

$$\begin{aligned}
&\int_0^T [\phi_p({}_0^C D_t^\alpha u_n) - \phi_p({}_0^C D_t^\alpha u_0) + \phi_q({}_0^C D_t^\alpha u_n) - \phi_q({}_0^C D_t^\alpha u_0)]\, {}_0^C D_t^\alpha (u_n - u_0) dt \\
\ge & d_p \int_0^T \frac{|{}_0^C D_t^\alpha (u_n - u_0)|^2}{(|{}_0^C D_t^\alpha (u_n)| + |{}_0^C D_t^\alpha (u_0)|)^{2-p}} + \frac{|{}_0^C D_t^\alpha (u_n - u_0)|^2}{(|{}_0^C D_t^\alpha (u_n)| + |{}_0^C D_t^\alpha (u_0)|)^{2-q}} dt \\
\ge & \frac{d_p}{2^{\frac{(p-1)(2-p)}{p}} (\|u_n\| + \|u_0\|)^{2-p}} (\int_0^T |{}_0^C D_t^\alpha (u_n - u_0)|^p dt)^{\frac{2}{p}} \\
& + \frac{d_p}{2^{\frac{(q-1)(2-q)}{q}} (\|u_n\| + \|u_0\|)^{2-q}} (\int_0^T |{}_0^C D_t^\alpha (u_n - u_0)|^q dt)^{\frac{2}{q}} \\
\ge & [\frac{d_p}{2^{\frac{(p-1)(2-p)}{p}} (\|u_n\| + \|u_0\|)^{2-p}} + \frac{d_p}{2^{\frac{(q-1)(2-q)}{q}} (\|u_n\| + \|u_0\|)^{2-q}}] \| {}_0^C D_t^\alpha (u_n - u_0)\|^2.
\end{aligned}$$

By the above, if $1 < q \le p < \infty$, we have $\|u_n - u_0\|_X \to 0$ as $n \to \infty$. Then

$$\varphi_x(u_0) = c,\ \varphi_x'(u_0) = 0.$$

Therefore, u_0 is a weak solution of problem (1.2). By Lemma 3.1, the problem (1.2) has a nontrivial classical solution. □

Proof of Theorem 1.3. As a result of $\int_{s_i}^{t_{i+1}} F_i(t,0)dt = \int_{s_i}^{t_{i+1}} G_i(t,0)x_k(t)dt = 0$ by definition of F, G. Thus, $\varphi_{x_k}(0) = 0$. By Theorem 1.2, for any fixed x_k, problem (1.2) has a non-trivial classical solution u_k such that $\varphi_{x_k}(u_k) = \inf_{u \in B_\gamma(0)} \varphi_{x_k}(u) \le \varphi_{x_k}(0)$.

Step 1. We'll prove that there exists a subsequence $\{u_{k_n}\}_{n=1}^\infty$ and $u_{k_n} \rightharpoonup u_0$ in X. By (3.21), we see that

$$\begin{aligned}
&(\frac{1}{p}\|u_k\|^p_{W_0^{\alpha,p}} - C_1\|u_k\|_{W^{\alpha,p}} - C_2\|u_k\|^{\rho+1}_{W_0^{\alpha,p}} - C_3\|u_k\|^l_{W_0^{\alpha,p}}) \\
&+ (\frac{1}{q}\|u_k\|^q_{W_0^{\alpha,q}} - C_4\|u_k\|_{W_0^{\alpha,q}} - C_5\|u_k\|^{\rho+1}_{W_0^{\alpha,q}} - C_6\|u_k\|^l_{W_0^{\alpha,q}}) \\
&- \sqrt{m}\sqrt{\sum_{i=0}^n \int_{s_i}^{t_{i+1}} g^2(t)dt} - \sum_{i=0}^n c_i \\
\le & \varphi_{x_k}(u_k) = \inf \varphi_{x_k}(u) \le \varphi_{x_k}(0) = 0,
\end{aligned}$$

which means that $\{u_{k_n}\}$ is bounded in X. Hence, the sequence $\{u_k\}_{k=1}^\infty$ has a subsequence $\{u_{k_n}\}_{k=1}^\infty$ which is weakly convergent in X to some $u_0 \in X$.

Step 2. We'll prove that $\{u_{k_n}\}$ strongly converges to u_0 in X.

The proof is similar to Step 3 in Theorem 1.2.

The difference is that $\int_{s_i}^{t_{i+1}} [g_i(t,u_{k_n})x_{k_n} - g_i(t,u_0)x_0]\,(u_{k_n} - u_0)dt \to 0$ for $i = 0, 1, \cdots, n$.

In fact, since $x_{k_n} \rightharpoonup x_0$ in X one has $x_{k_n} \to x_0$ in $C[0,T]$. By Hölder's inequality and the Lebesgue's dominated convergence theorem, one has

$$
\begin{aligned}
&\int_{s_i}^{t_{i+1}} [g_i(t,u_{k_n})x_{k_n} - g_i(t,u_0)x_0]\,(u_{k_n}-u_0)dt \\
=&\int_{s_i}^{t_{i+1}} [g_i(t,u_{k_n})(x_{k_n}-x_0) + (g_i(t,u_{k_n}) - g_i(t,u_0))x_0]\,(u_{k_n}-u_0)dt \\
=&\int_{s_i}^{t_{i+1}} |g_i(t,u_{k_n}) - g_i(t,u_0)|x_0(u_{k_n}-u_0)dt \\
&+\int_{s_i}^{t_{i+1}} g_i(t,u_{k_n})(x_{k_n}-x_0)(u_{k_n}-u_0)dt \\
\le& 2\int_{s_i}^{t_{i+1}} g(t)x_0(u_{k_n}-u_0)dt + \int_{s_i}^{t_{i+1}} g_i(t,u_{k_n})(x_{k_n}-x_0)(u_{k_n}-u_0)dt \\
\to& 0 \quad \text{as} \quad n\to\infty.
\end{aligned}
\tag{3.29}
$$

The proof of Step 2 is complete.
Step 3. We'll prove that u_0 is a classical solution to problem (1.2) corresponding to x_0.

Since $\{u_{k_n}\}$ is the solution sequence of (1.2) with x_{k_n}, we have

$$
\begin{aligned}
0 =&\langle \varphi'_{x_{k_n}}(u_{k_n}), v\rangle \\
=&\int_0^T \phi_p({}_0^C D_t^\alpha u_{k_n})({}_0^C D_t^\alpha v(t))dt + \int_0^T \phi_q({}_0^C D_t^\alpha u_{k_n})({}_0^C D_t^\alpha v(t))dt \\
&+\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} [|u_{k_n}|^{p-2}u_{k_n} + |u_{k_n}|^{q-2}u_{k_n}]v(t)dt + \sum_{i=1}^{n} I_i(u_{k_n}(t_i)v(t) \\
&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} [f_i(t,u_{k_n}) + g_i(t,u_{k_n})x_{k_n}]v(t)dt
\end{aligned}
\tag{3.30}
$$

for any $v\in X$.

Similar to the proof of (3.29), we have

$$
\int_{s_i}^{t_{i+1}} g_i(t,u_{k_n})x_{k_n}dt \to \int_{s_i}^{t_{i+1}} g_i(t,u_0)x_0dt \quad \text{as} \quad n\to\infty. \tag{3.31}
$$

Since $\{u_{k_n}\}$ strongly converges to u_0 in X, it remains to show

$$
\lim_{n\to\infty}\int_0^T \phi_p({}_0^C D_t^\alpha u_{k_n})({}_0^C D_t^\alpha v(t))dt = \int_0^T \phi_p({}_0^C D_t^\alpha u_0)({}_0^C D_t^\alpha v(t))dt.
$$

By Lagrange's mean value theorem and Hölder's inequality, there exist a $\xi(t)$ that is between ${}_0^C D_t^\alpha u_{k_n}(t)$ and ${}_0^C D_t^\alpha u_0(t)$, such that

$$
\begin{aligned}
&\int_0^T |\phi_p({}_0^C D_t^\alpha u_{k_n}(t)) - \phi_p({}_0^C D_t^\alpha u_0(t))|({}_0^C D_t^\alpha v(t))dt \\
\le& \int_0^T (p-1)|\xi(t)|^{p-2}|{}_0^C D_t^\alpha u_{k_n} - {}_0^C D_t^\alpha u_0|({}_0^C D_t^\alpha v(t))dt \\
\le& (p-1)(\int_0^T |{}_0^C D_t^\alpha u_{k_n} - {}_0^C D_t^\alpha u_0|^p dt)^{\frac{1}{p}}(\int_0^T |\xi(t)|^{(p-2)q}|{}_0^C D_t^\alpha v(t)|^q dt)^{\frac{1}{q}}
\end{aligned}
$$

$$\leq (p-1)\|{}_0^C D_t^\alpha (u_{k_n}-u_0)\|_{L^p}\left(\int_0^T |\xi(t)|^p\right)^{\frac{(p-2)}{p}}\left(\int_0^T |{}_0^C D_t^\alpha v(t)|^p\right)^{\frac{1}{p}}.$$

Because of ${}_0^C D_t^\alpha u_{k_n} \in L^p[0,T]$, we have $\xi(t)\in L^p[0,T]$, ${}_0^C D_t^\alpha v(t)\in L^p[0,T]$, we can get $(\int_0^T |\xi(t)|^p)^{\frac{(p-2)}{p}}$ and $(\int_0^T |{}_0^C D_t^\alpha v(t)|^p)^{\frac{1}{p}}$ is bounded in X. By $\|u_{k_n}-u_0\|_X \to 0$, we have

$$\lim_{n\to\infty}\int_0^T \phi_p({}_0^C D_t^\alpha u_{k_n})({}_0^C D_t^\alpha v(t))dt = \int_0^T \phi_p({}_0^C D_t^\alpha u_0)({}_0^C D_t^\alpha v(t))dt. \tag{3.32}$$

Substituting (3.31)-(3.32) into (3.30), let $n\to\infty$, we have

$$\begin{aligned}&\int_0^T \phi_p({}_0^C D_t^\alpha u_0(t))({}_0^C D_t^\alpha v(t))dt + \int_0^T \phi_q({}_0^C D_t^\alpha u_0(t))({}_0^C D_t^\alpha v(t))dt\\ &= -\sum_{i=0}^n \int_{s_i}^{t_{i+1}} [|u_0(t)|^{p-2}u_0(t)+|u_0(t)|^{q-2}u_0(t)]v(t)dt - \sum_{i=1}^n I_i(u_0(t_i))v(t_i)\\ &\quad+\sum_{i=0}^n \int_{s_i}^{t_{i+1}} [f_i(t,u_0(t))+g_i(t,u_0(t))x_0(t)]v(t)dt.\end{aligned}$$

Hence, u_0 is a weak solution to the problem (1.2) corresponding to $x=x_0$. And, by Lemma 3.1, we get that u_0 is a classical solution to the problem (1.2) corresponding to $x=x_0$. □

Example 3.1. Let $\alpha=\frac{5}{6}$, $q=2$, $p=3$, $T>0$, $f_i(t,u)=\frac{1}{3}u^{\frac{1}{3}}$, $g_i(t,u)=u^{\frac{1}{2}}$, $I_i(u)=\frac{1}{2}u^{\frac{1}{4}}$, $i=1,2,\ldots,n$ in (1.2). And let $a_i=1$, $b_i=2$, $c_i=d_i=1(i=1,2,\ldots,n)$, $l=\frac{4}{5}$, $\rho=\frac{1}{3}$, $x(t)=\frac{1}{1+t}$, $m=1$ in (1.2). It's easy to check that (H_1), (H_2) and (H_3) hold. Then, by Theorem 1.2., problem (1.2) has a classical solution. And it's easy to check that Theorem 1.3. hold.

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