Journal of Applied Analysis and Computation
Volume 12, Number 2, April 2022, 609–621

Website:http://www.jaac-online.com
DOI:10.11948/20210250

CONTROL DESIGN FOR A CLASS OF GENERAL NONLINEAR REACTION DIFFUSION EQUATIONS

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Abstract We consider a class of nonlinear parabolic equation with general source function $f(u)$, conduction function $g(u)$ and conduction coefficient $\rho(|\nabla u|^2)$ in multi-dimensional space. We establish new control conditions to guarantee that the positive solution exists globally. At the same time, under suitable control conditions, by means of the Sobolev inequality in multi-dimensional space, we obtain upper and lower bounds of the blow-up time t^* in $\mathbb{R}^n$ $(n \geqslant 2)$. Our work generalize the models, improve the method and remove the constraint of spatial dimension in many literatures.

Keywords Control design, reaction diffusion equation, global existence, blow up, multi-dimensional space.

MSC(2010) 35B35, 35G05, 35G16, 35L30.

1. Introduction

In this paper, we deal with the initial-boundary value problem

$$\begin{cases} u_t - \operatorname{div}\left(\rho(|\nabla u|^2)\nabla u\right) = f(u), \quad (\boldsymbol{x}, t) \in \Omega \times (0, t^*), \\ \rho(|\nabla u|^2)\dfrac{\partial u}{\partial \boldsymbol{\nu}} = g(u), \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, t^*), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geqslant 0, \quad \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded star-shaped domain in $\mathbb{R}^n$ $(n \geqslant 2)$ with smooth boundary $\partial\Omega$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega$, ∇u denotes the gradient of u, $\frac{\partial u}{\partial \boldsymbol{\nu}}$ is the outward normal derivative on the boundary $\partial\Omega$. $f(u)$, $g(u)$ and $\rho(|\nabla u|^2)$ are source function, conduction function and conduction coefficient respectively, $u_0(\boldsymbol{x}) \geqslant 0$ and $u_0 \not\equiv 0$, t^* is the blow-up time if blow-up occurs.

The dynamical property of reaction diffusion equations have received considerable attentions in the past decades. Payne et al. [15, 16] studied the following nonlinear reaction diffusion problems

$$u_t - \operatorname{div}(\rho(u)\nabla u) = f(u), \ (\boldsymbol{x}, t) \in \Omega \times (0, t^*),$$

and

$$u_t - \operatorname{div}\left(\rho(|\nabla u|^2)\nabla u\right) = f(u), \ (\boldsymbol{x}, t) \in \Omega \times (0, t^*),$$

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with *homogeneous Dirichlet* boundary condition and nonnegative initial conditions. They obtained the lower and upper bounds of blow-up time t^* in *three-dimensional* bounded domain $\Omega \subset \mathbb{R}^3$. Payne et al. [18] and Li [6] studied semilinear reaction diffusion equations, respectively

$$u_t - \operatorname{div}\left(|\nabla u|^{2p}\nabla u\right) = 0,\ (\boldsymbol{x}, t) \in \Omega \times (0, t^*),$$

and

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - f(u),\ (\boldsymbol{x}, t) \in \Omega \times (0, t^*),$$

with *inhomogeneous Neumann* boundary conditions. They established respectively the conditions on the nonlinearities to guarantee that the solution $u(\boldsymbol{x}, t)$ exists globally or blows up at some finite time. And obtained the upper and lower bounds of the blow-up time if blow-up occurs in a *three-dimensional* bounded star-shaped domain $\Omega \subset \mathbb{R}^3$. For more literatures on reaction diffusion equation, we refer readers to [1–5, 9, 13, 14, 17, 19–22] and the references cited therein.

Motivated by the above works and on the basis of our researches [5–12, 21, 22], we intend to study the global existence and the blow-up phenomena for the nonlinear parabolic problem (1.1) in *multi-dimensional* space. It's known that the behaviour of the solution depends on the functions ρ, f, g, the domain Ω, and the initial data u_0. From the physical standpoint, ρ is the conduction coefficient, f is the source function and g is the conduction function transmitting into interior of Ω from the boundary of Ω. To our knowledge, this is the first work to study problem (1.1) and to estimate the lower bound of blow-up time in $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$.

The main contributions of this paper are: (a) we establish some conditions to ensure that the positive solution exists globally, which are weaker than the conditions given in [4–6]; (b) we naturally derive blow-up conditions by calculations; (c) under the conditions that ensure the occurrence of blow-up phenomena, we give the upper and lower bounds estimate of blow-up time; (d) the lower bound of blow-up time is estimated by using general Sobolev inequality in *multi-dimensional* space.

The present work is organized as follows. In Sec. 2, we establish some new sufficient conditions on f, ρ and g to ensure that $u(\boldsymbol{x}, t)$ exists globally. In Sec. 3, we give sufficient conditions to ensure that the solution blows up at finite time and derive an explicit upper bound for t^*. In Sec. 4, under the conditions on ρ, f and g that ensure the occurrence of blow-up phenomena, we give the lower bound of t^* by means of general Sobolev inequality and differential inequality technique in *multi-dimensional* space.

2. Global existence conditions of positive solution

In this section, we give new sufficient conditions on f, ρ and g to ensure that $u(\boldsymbol{x}, t)$ exists globally. The global existence result is as follows.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ be a bounded star-shaped domain assumed to be convex in $n-1$ orthogonal directions and with smooth boundary $\partial\Omega$. Assume that $u_0(\boldsymbol{x}) \geqslant 0$, $u_0(\boldsymbol{x}) \not\equiv 0$ and the functions ρ, f, and g satisfy*

$$\rho(s) \geqslant \beta s^q,\ \ f(s) \begin{cases} \leqslant -\alpha s^p, & s > 0, \\ = 0, & s \leqslant 0, \end{cases} \qquad g(s) \begin{cases} \leqslant \gamma s^q, & s > 0, \\ \geqslant 0, & s \leqslant 0, \end{cases} \tag{2.1}$$

where $\alpha, \beta, \gamma > 0$ and $p > q$. Then the solution $u(\boldsymbol{x}, t)$ of problem (1.1) is positive and $u(\boldsymbol{x}, t)$ cannot blow up in the measure $\Phi(t) = \frac{1}{2}\int_\Omega u^2 dx$ at finite time.

In order to prove Theorem 2.1, we give the following general lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded star-shaped domain assumed to be convex in $n-1$ orthogonal directions. Then for any nonnegative increasing C^1 function $h(w)$, we have*

$$\int_{\partial\Omega} h(w)dS \leq \frac{n}{\rho_0}\int_\Omega h(w)d\boldsymbol{x} + \frac{d}{\rho_0}\int_\Omega h'(w)|\nabla w|d\boldsymbol{x},$$

where

$$\rho_0 := \min_{\boldsymbol{x}\in\partial\Omega}(\boldsymbol{x}\cdot\boldsymbol{\nu}), \quad d := \max_{\boldsymbol{x}\in\Omega}|\boldsymbol{x}|.$$

Proof. The proof can be found in [21]. □

The proof of Theorem 2.1. From the background of the model and assumptions, it is easy to know that the solution $u(\boldsymbol{x}, t)$ is positive. Next, we intend to show that the positive solution of problem (1.1) does not blow up.

Multiplying the equation of (1.1) by u and making use of the divergence theorem, we have

$$\begin{aligned} 0 &= \int_\Omega uu_t d\boldsymbol{x} - \int_\Omega u[\mathrm{div}(\rho(|\nabla u|^2)\nabla u) + f(u)]d\boldsymbol{x} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 d\boldsymbol{x} + \int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x} - \int_{\partial\Omega} u\rho(|\nabla u|^2)\frac{\partial u}{\partial\boldsymbol{\nu}}dS - \int_\Omega uf(u)d\boldsymbol{x} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 d\boldsymbol{x} + \int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x} - \int_{\partial\Omega} ug(u)dS - \int_\Omega uf(u)d\boldsymbol{x}. \end{aligned} \tag{2.2}$$

The above calculations inspire us to define the following functional

$$\Phi(t) := \frac{1}{2}\int_\Omega u^2 d\boldsymbol{x}. \tag{2.3}$$

By (2.1)-(2.3), we have

$$\begin{aligned} \Phi'(t) &= -\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x} + \int_{\partial\Omega} ug(u)dS + \int_\Omega uf(u)d\boldsymbol{x} \\ &\leqslant -\beta\int_\Omega |\nabla u|^{2(q+1)}d\boldsymbol{x} + \gamma\int_{\partial\Omega} u^{q+1}dS - \alpha\int_\Omega u^{p+1}d\boldsymbol{x}. \end{aligned} \tag{2.4}$$

Using Lemma 2.1, we have

$$\int_{\partial\Omega} u^{q+1}dS \leqslant \frac{n}{\rho_0}\int_\Omega u^{q+1}d\boldsymbol{x} + \frac{d(q+1)}{\rho_0}\int_\Omega u^q|\nabla u|d\boldsymbol{x}. \tag{2.5}$$

Substituting (2.5) into (2.4), we obtain

$$\Phi'(t) \leqslant -\beta\int_\Omega |\nabla u|^{2(q+1)}d\boldsymbol{x} + \frac{n\gamma}{\rho_0}\int_\Omega u^{q+1}d\boldsymbol{x} + \frac{d\gamma(q+1)}{\rho_0}\int_\Omega u^q|\nabla u|d\boldsymbol{x} - \alpha\int_\Omega u^{p+1}d\boldsymbol{x}. \tag{2.6}$$

Applying Hölder inequality and Young inequality yields

$$\begin{aligned}\int_\Omega u^q|\nabla u|d\boldsymbol{x} &\leqslant \left(\delta^{\frac{1}{2q+1}}\int_\Omega u^{q\frac{2(q+1)}{2q+1}}d\boldsymbol{x}\right)^{\frac{2q+1}{2(q+1)}}\left(\delta^{-1}\int_\Omega|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{1}{2(q+1)}}\\ &\leqslant \frac{2q+1}{2(q+1)}\delta^{\frac{1}{2q+1}}\int_\Omega u^{q\frac{2(q+1)}{2q+1}}d\boldsymbol{x}+\frac{1}{2(q+1)\delta}\int_\Omega|\nabla u|^{2(q+1)}d\boldsymbol{x},\ \forall\delta>0.\end{aligned}\tag{2.7}$$

Inserting (2.7) into (2.6), we have

$$\begin{aligned}\Phi'(t)\leqslant&\left(\frac{d\gamma}{2\rho_0\delta}-\beta\right)\int_\Omega|\nabla u|^{2(q+1)}d\boldsymbol{x}+\frac{n\gamma}{\rho_0}\int_\Omega u^{q+1}d\boldsymbol{x}\\ &+\frac{d\gamma(2q+1)}{2\rho_0}\delta^{\frac{1}{2q+1}}\int_\Omega u^{q\frac{2(q+1)}{2q+1}}d\boldsymbol{x}-\alpha\int_\Omega u^{p+1}d\boldsymbol{x}.\end{aligned}\tag{2.8}$$

Choosing $\delta=\frac{d\gamma}{2\rho_0\beta}$ in (2.8), we deduce

$$\begin{aligned}\Phi'(t)\leqslant&\frac{n\gamma}{\rho_0}\int_\Omega u^{q+1}d\boldsymbol{x}+(2q+1)\beta\left(\frac{d\gamma}{2\rho_0\beta}\right)^{1+\frac{1}{2q+1}}\int_\Omega u^{q\frac{2(q+1)}{2q+1}}d\boldsymbol{x}-\alpha\int_\Omega u^{p+1}d\boldsymbol{x}\\ =&\int_\Omega\left(\frac{n\gamma}{\rho_0}\frac{u^q}{u^p}-\frac{\alpha}{2}\right)u^{p+1}d\boldsymbol{x}\\ &+\int_\Omega\left((2q+1)\beta^{-\frac{1}{2q+1}}\left(\frac{d\gamma}{2\rho_0}\right)^{1+\frac{1}{2q+1}}\frac{u^{q\frac{2(q+1)}{2q+1}}}{u^{p+1}}-\frac{\alpha}{2}\right)u^{p+1}d\boldsymbol{x}.\end{aligned}\tag{2.9}$$

Since $p>q$ which implies that $q\frac{2(q+1)}{2q+1}=q+\frac{q}{2q+1}<p+1$ and $\alpha>0$, we conclude that $\Phi(t)$ remains bounded for all $t>0$.

In fact, if $u(\boldsymbol{x},t)$ blows up at finite time t^*, then

$$\lim_{t\to t^*}\Phi(t)=+\infty.\tag{2.10}$$

From (2.10), we know that there exists some $t_0>0$ such that

$$\Phi'(t)\leqslant 0,\quad t\in[t_0,t^*).\tag{2.11}$$

Then we can deduce that

$$\Phi(t)\leqslant\Phi(t_0),\ t\in[t_0,t^*),\tag{2.12}$$

which shows that $\Phi(t)$ is bounded in $[t_0,t^*)$, this is contradict with (2.10).

The proof of Theorem 2.1 is completed. □

Remark 2.1. In fact, from the above inequality (2.4), we know that if there is no source term and the boundary is adiabatic, that is, $f(u)\equiv 0$ and $g(u)\equiv 0$, then the solution of the problem (1.1) exists globally. The condition $q>1$ in [4–6] is not required in our result.

3. An upper bound estimate of blow-up t^*

In this section, we give sufficient conditions on $f(u)$, $g(u)$, $\rho(|\nabla u|^2)$ and $u_0(\boldsymbol{x})$ in problem (1.1) to guarantee that the solution $u(\boldsymbol{x},t)$ blows up at some finite time t^*, and under these conditions we show an explicit upper bound estimate of blow-up t^*.

Multiplying the equation of (1.1) by u and integrating on Ω, we obtain

$$\begin{aligned}\int_\Omega uu_t d\boldsymbol{x} &= \int_\Omega u\left[\operatorname{div}(\rho(|\nabla u|^2)\nabla u)+f(u)\right]d\boldsymbol{x}\\ &= -\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x}+\int_{\partial\Omega} u\rho(|\nabla u|^2)\frac{\partial u}{\partial \boldsymbol{\nu}}dS+\int_\Omega uf(u)d\boldsymbol{x}\\ &= -\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x}+\int_{\partial\Omega} ug(u)dS+\int_\Omega uf(u)d\boldsymbol{x},\end{aligned}$$

that is

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 d\boldsymbol{x} = -\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x}+\int_{\partial\Omega} ug(u)dS+\int_\Omega uf(u)d\boldsymbol{x}. \tag{3.1}$$

The above calculations inspire us to define the functional

$$\Phi(t) := \frac{1}{2}\int_\Omega u^2 d\boldsymbol{x}. \tag{3.2}$$

We assume that the nonnegative integrable functions f, g satisfy

$$sg(s) \geqslant (\lambda+1)G(s),\ \ sf(s) \geqslant (\lambda+1)F(s),\ \ s\geqslant 0, \tag{3.3}$$

with

$$G(\xi)=\int_0^\xi g(s)ds,\ F(\xi)=\int_0^\xi f(s)ds,\ \lambda>1,$$

and $\rho(\cdot)$ is a positive C^1 function that satisfies

$$(2\alpha-1)\rho(s)+2\alpha s\rho'(s)=0,\ \ s\geqslant 0,\ \ \alpha(\lambda+1)\geqslant 1. \tag{3.4}$$

By (3.1)-(3.4), using the divergence theorem and the equation of (1.1), we obtain

$$\begin{aligned}\Phi'(t) &= \int_\Omega uu_t d\boldsymbol{x}\\ &= -\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x}+\int_{\partial\Omega} ug(u)dS+\int_\Omega uf(u)d\boldsymbol{x}\\ &\geqslant -\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x}+(\lambda+1)\int_{\partial\Omega} G(u)dS+(\lambda+1)\int_\Omega F(u)d\boldsymbol{x}\\ &\geqslant (\lambda+1)\left(-\alpha\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x}+\int_{\partial\Omega} G(u)dS+\int_\Omega F(u)d\boldsymbol{x}\right)\\ &:= \Psi(t),\end{aligned}$$

that is

$$\Phi'(t) \geqslant \Psi(t). \tag{3.5}$$

Differentiating $\Psi(t)$ with respect to t, we have

$$\frac{1}{\lambda+1}\Psi'(t) = -\alpha\int_\Omega 2|\nabla u|^2\rho'(|\nabla u|^2)\nabla u\cdot\nabla u_t d\boldsymbol{x}-\alpha\int_\Omega 2\rho(|\nabla u|^2)\nabla u\cdot\nabla u_t d\boldsymbol{x}$$

$$
\begin{aligned}
&+\int_{\partial\Omega} g(u)u_t dS+\int_\Omega f(u)u_t d\boldsymbol{x}\\
=&-\int_\Omega \left[2\alpha|\nabla u|^2\rho'(|\nabla u|^2)+(2\alpha-1)\rho(|\nabla u|^2)\right]\nabla u\cdot\nabla u_t d\boldsymbol{x}\\
&-\int_\Omega \rho(|\nabla u|^2)\nabla u\cdot\nabla u_t d\boldsymbol{x}+\int_{\partial\Omega} g(u)u_t dS+\int_\Omega f(u)u_t d\boldsymbol{x}\\
=&-\int_\Omega \rho(|\nabla u|^2)\nabla u\cdot\nabla u_t d\boldsymbol{x}+\int_{\partial\Omega} g(u)u_t dS+\int_\Omega f(u)u_t d\boldsymbol{x}\\
=&-\int_\Omega \rho(|\nabla u|^2)\nabla u\cdot\nabla u_t d\boldsymbol{x}+\int_{\partial\Omega} \rho(|\nabla u|^2)\frac{\partial u}{\partial \boldsymbol{\nu}}u_t dS+\int_\Omega f(u)u_t d\boldsymbol{x}\\
=&\int_\Omega u_t \mathrm{div}\left(\rho(|\nabla u|^2)\nabla u\right)d\boldsymbol{x}+\int_\Omega f(u)u_t d\boldsymbol{x}\\
=&\int_\Omega \left[\mathrm{div}\left(\rho(|\nabla u|^2)\nabla u\right)+f(u)\right]u_t d\boldsymbol{x}=\int_\Omega u_t^2 d\boldsymbol{x}\geqslant 0,
\end{aligned}
\tag{3.6}
$$

where we use (3.1), (3.4), the divergence theorem and the equation of (1.1). If $\Psi(0)>0$ and (3.6), we have $\Psi(t)>0$ for arbitrary $t\in(0,t^*)$. According to (3.2), (3.5), (3.6) and making use of Schwartz inequality, we get

$$
\Phi'(t)\Psi(t)\leqslant[\Phi'(t)]^2=\left(\int_\Omega uu_t d\boldsymbol{x}\right)^2\leqslant\int_\Omega u^2 d\boldsymbol{x}\cdot\int_\Omega u_t^2 d\boldsymbol{x}\leqslant\frac{2}{\lambda+1}\Phi(t)\Psi'(t),
$$

which is equivalent to the inequality

$$
\left[\Psi(t)\Phi^{-\frac{\lambda+1}{2}}(t)\right]'\geqslant 0.
$$

Refering to corresponding calculation procedure in [5,21], we obtain

$$
\lim_{t\to t^*}\Phi(t)=+\infty,
$$

where

$$
t^*\leqslant T=\frac{2\Phi(0)}{(\lambda-1)\Psi(0)}.
$$

Synthesizing the above calculation process, we can formulate the following result.

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^n(n\geqslant 2)$ be a bounded star-shaped domain assumed to be convex in n-1 orthogonal directions. We assume $u_0(\boldsymbol{x})\geqslant 0$, $u_0(\boldsymbol{x})\not\equiv 0$, the nonnegative integrable functions f, g satisfy (3.3) for $p>1$, ρ is a positive C^1 function and satisfies (3.4). Moreover, we assume $\Psi(0)>0$ with*

$$
\Psi(0)=(\lambda+1)\left(-\int_\Omega \rho(|\nabla u_0|^2)|\nabla u_0|^2 d\boldsymbol{x}+\int_{\partial\Omega}G(u_0)dS+\int_\Omega F(u_0)d\boldsymbol{x}\right).
$$

Then we conclude that $u(\boldsymbol{x},t)$ of problem (1.1) blows up at some finite time $t^*\leqslant T$, with

$$
T=\frac{2\Phi(0)}{(\lambda-1)\Psi(0)}.
$$

Remark 3.1. If we choose $f(u) = u^\alpha$ or $f(u) \equiv 0$, $g(u) = u^\beta$, $(\alpha, \beta > 1)$, $u_0(x) =$ constant > 0, then all the conditions in the Theorem 3.1 are satisfied.

If we choose $\rho(s) = s^{\frac{p-2}{2}}$, $(p > 2)$, then our model is the equation in [6].

If we choose $\lambda = 2p - 3$, $(p > 2)$, from our upper bound of blow-up time, one can get the corresponding result in [6].

If we choose $\rho(s) = s^p$, $(p > 0)$ and $f(u) \equiv 0$, then our model is the equation in [18].

4. A lower bound estimate of blow-up t^*

In this section, in *multi-dimensional* space, we establish certain conditions on the data of problem (1.1) to guarantee that the solution blows up at finite time t^*, then we derive a lower bound for blow-up time t^*.

Assuming that the nonnegative integrable functions ρ, f, g and $u_0(\boldsymbol{x})$ satisfy the assumptions in Theorem 3.1. Moreover

$$0 \leqslant f(s) \leqslant a_1 s^{1+\frac{\sigma}{2n-3}},\ 0 < g(s) \leqslant a_2 s^{1+\frac{\sigma}{2n-2}},\ s > 0, \tag{4.1}$$

and ρ is a positive function that satisfies

$$\rho(s) \geqslant b_1 + b_2 s^q,\ s > 0, \tag{4.2}$$

where $\sigma \geqslant 1$ will be defined later, $q \geqslant 0$, $b_1 \geqslant 0$, a_1, a_2, b_2 are positive constants.

Multiplying the equation of (1.1) by $u^{2\sigma-1}$ and integrating on Ω, we obtain

$$\int_\Omega u^{2\sigma-1} u_t d\boldsymbol{x} = \int_\Omega u^{2\sigma-1}[\mathrm{div}(\rho(|\nabla u|^2)\nabla u) + f(u)]d\boldsymbol{x},$$

that is

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^{2\sigma} d\boldsymbol{x} =& 2\sigma \int_{\partial\Omega} u^{2\sigma-1} u_t d\boldsymbol{x} \\
=& 2\sigma \int_\Omega u^{2\sigma-1}[\mathrm{div}(\rho(|\nabla u|^2)\nabla u) + f(u)]d\boldsymbol{x} \\
=& 2\sigma\int_{\partial\Omega} u^{2\sigma-1}\rho(|\nabla u|^2)\frac{\partial u}{\partial \boldsymbol{\nu}}dS - 2\sigma(2\sigma-1)\int_\Omega u^{2(\sigma-1)}\rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x} \\
&+ 2\sigma\int_\Omega u^{2\sigma-1} f(u) d\boldsymbol{x},
\end{aligned} \tag{4.3}$$

where we use the divergence theorem and the equation of (1.1).

The above calculations inspire us to define the auxiliary functional

$$\varphi(t) := \int_\Omega u^{2\sigma} d\boldsymbol{x},\ \text{with } \sigma = (\mu - 1)(q + 1) + 1, \tag{4.4}$$

for some constant $\mu \geqslant 1$ to be choosed. From (1.1) and (4.1)- (4.4), we have

$$\begin{aligned}
\varphi'(t) = & 2\sigma \int_{\partial\Omega} u^{2\sigma-1} g(u) dS - 2\sigma(2\sigma-1)\int_\Omega u^{2(\sigma-1)}\rho(|\nabla u|^2)|\nabla u|^2 d\boldsymbol{x} \\
&+ 2\sigma \int_\Omega u^{2\sigma-1} f(u) d\boldsymbol{x}
\end{aligned}$$

$$\leqslant 2\sigma a_2\int_{\partial\Omega}u^{\sigma(2+\frac{1}{2^{n-2}})}dS-2\sigma(2\sigma-1)b_2\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}$$
$$+2\sigma a_1\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x}. \tag{4.5}$$

Using Lemma 2.2 in [21] and Cauchy inequality, we have

$$\int_{\partial\Omega}u^{\sigma(2+\frac{1}{2^{n-2}})}dS\leqslant\frac{n}{\rho_0}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-2}})}d\boldsymbol{x}+\frac{\sigma(2+\frac{1}{2^{n-2}})d}{\rho_0}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-2}})-1}|\nabla u|d\boldsymbol{x},$$
$$\int_\Omega u^{\sigma(2+\frac{1}{2^{n-2}})}d\boldsymbol{x}=\int_\Omega u^{\sigma(1+\frac{1}{2^{n-2}})}u^\sigma d\boldsymbol{x}\leqslant\frac{1}{2}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x}+\frac{1}{2}\varphi(t).$$

Therefore,

$$\int_{\partial\Omega}u^{\sigma(2+\frac{1}{2^{n-2}})}dS\leqslant\frac{n}{2\rho_0}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x}+\frac{n}{2\rho_0}\varphi(t)$$
$$+\frac{\sigma(2+\frac{1}{2^{n-2}})d}{\rho_0}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-2}})-1}|\nabla u|d\boldsymbol{x}. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\varphi'(t)\leqslant\left(\frac{\sigma a_2 n}{\rho_0}+2\sigma a_1\right)\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x}+\frac{2\sigma^2a_2(2+\frac{1}{2^{n-2}})d}{\rho_0}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-2}})-1}|\nabla u|d\boldsymbol{x}$$
$$-2\sigma(2\sigma-1)b_2\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}+\frac{\sigma a_2 n}{\rho_0}\varphi(t). \tag{4.7}$$

Then by means of Hölder inequality, Young inequality and (4.4), we have

$$\begin{aligned}
&\int_\Omega u^{\sigma(2+\frac{1}{2^{n-2}})-1}|\nabla u|d\boldsymbol{x}\\
=&\int_\Omega u^{\sigma(1+\frac{1}{2^{n-2}})+q(\mu-1)}u^{\mu-1}|\nabla u|d\boldsymbol{x}\\
\leqslant&\left(\int_\Omega u^{[\sigma(1+\frac{1}{2^{n-2}})+q(\mu-1)]\frac{2(q+1)}{2q+1}}d\boldsymbol{x}\right)^{\frac{2q+1}{2(q+1)}}\left(\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{1}{2(q+1)}}\\
=&\left(\theta^{-\frac{1}{2q+1}}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})\frac{q+1}{2q+1}+2(\sigma-1)\frac{q}{2q+1}}d\boldsymbol{x}\right)^{\frac{2q+1}{2(q+1)}}\left(\theta\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{1}{2(q+1)}}\\
\leqslant&\frac{2q+1}{2(q+1)}\theta^{-\frac{1}{2q+1}}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})\frac{q+1}{2q+1}+2(\sigma-1)\frac{q}{2q+1}}d\boldsymbol{x}\\
&+\frac{1}{2(q+1)}\theta\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x},\ \forall\theta>0.
\end{aligned} \tag{4.8}$$

Using Hölder inequality and Young inequality again, we obtain

$$\begin{aligned}
&\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})\frac{q+1}{2q+1}+2(\sigma-1)\frac{q}{2q+1}}d\boldsymbol{x}\\
\leqslant&\left(\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x}\right)^{\frac{q+1}{2q+1}}\left(\int_\Omega u^{2(\sigma-1)}d\boldsymbol{x}\right)^{\frac{q}{2q+1}}\\
\leqslant&\frac{q+1}{2q+1}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x}+\frac{q}{2q+1}\int_\Omega u^{2(\sigma-1)}d\boldsymbol{x}
\end{aligned}$$

$$\leqslant \frac{q+1}{2q+1}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x} + \frac{q}{2q+1}|\Omega|^{\frac{1}{\sigma}}(\varphi(t))^{\frac{\sigma-1}{\sigma}}, \tag{4.9}$$

where $|\Omega|$ is the volume of the domain Ω. Combining (4.8) and (4.9), we have

$$\begin{aligned}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-2}})-1}|\nabla u|d\boldsymbol{x} \leqslant& \frac{1}{2}\theta^{-\frac{1}{2q+1}}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x} + \frac{q}{2(q+1)}\theta^{-\frac{1}{2q+1}}|\Omega|^{\frac{1}{\sigma}}(\varphi(t))^{\frac{\sigma-1}{\sigma}} \\ &+ \frac{1}{2(q+1)}\theta\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}.\end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.7), we get

$$\varphi'(t) \leqslant \widetilde{c_1}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x} + \widetilde{c_2}\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x} + \widetilde{c_3}(\varphi(t))^{\frac{\sigma-1}{\sigma}} + \widetilde{c_4}\varphi(t), \tag{4.11}$$

where

$$\begin{aligned}\widetilde{c_1} &= \frac{\sigma a_2 n}{\rho_0} + 2\sigma a_1 + \frac{\sigma^2 a_2(2+\frac{1}{2^{n-2}})d}{\rho_0}\theta^{-\frac{1}{2q+1}}, \\ \widetilde{c_2} &= \frac{\sigma^2 a_2(2+\frac{1}{2^{n-2}})d}{\rho_0(q+1)}\theta - 2\sigma(2\sigma-1)b_2, \\ \widetilde{c_3} &= \frac{\sigma^2 a_2(2+\frac{1}{2^{n-2}})dq}{\rho_0(q+1)}\theta^{-\frac{1}{2q+1}}|\Omega|^{\frac{1}{\sigma}}, \\ \widetilde{c_4} &= \frac{\sigma a_2 n}{\rho_0}.\end{aligned}$$

Next, we estimate the first term on the right side of (4.11). By Lemma 4.1 in [21], we have

$$\begin{aligned}\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x} &= \int_\Omega u^{2\sigma(1+\frac{1}{2^{n-2}})}d\boldsymbol{x} \\ &\leqslant (1+2d)^{n-3}\left(\frac{n}{2\rho_0}\varphi(t)+\sigma\left(1+\frac{d}{\rho_0}\right)\int_\Omega u^{2\sigma-1}|\nabla u|d\boldsymbol{x}\right)^{1+\frac{1}{2^{n-2}}}.\end{aligned} \tag{4.12}$$

Using Hölder inequality and (4.4), we have

$$\begin{aligned}&\int_\Omega u^{2\sigma-1}|\nabla u|d\boldsymbol{x} \\ =& \int_\Omega u^{\sigma}u^{\sigma-1}|\nabla u|d\boldsymbol{x} \\ =& \int_\Omega u^{\sigma+q(\mu-1)}u^{\mu-1}|\nabla u|d\boldsymbol{x} \\ \leqslant& \left(\int_\Omega u^{[\sigma+q(\mu-1)]\frac{2(q+1)}{2q+1}}d\boldsymbol{x}\right)^{\frac{2q+1}{2(q+1)}}\left(\int_\Omega u^{2(\mu-1)(q+1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{1}{2(q+1)}} \\ =& \left[\left(\int_\Omega u^{2\sigma\frac{q+1}{2q+1}+2(\sigma-1)\frac{q}{2q+1}}d\boldsymbol{x}\right)^{2q+1}\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right]^{\frac{1}{2(q+1)}},\end{aligned} \tag{4.13}$$

and

$$\int_\Omega u^{2\sigma\frac{q+1}{2q+1}+2(\sigma-1)\frac{q}{2q+1}}d\boldsymbol{x}=\int_\Omega u^{2\sigma-\frac{2q}{2q+1}}d\boldsymbol{x}\leqslant|\Omega|^{\frac{q}{\sigma(2q+1)}}(\varphi(t))^{\frac{\sigma(2q+1)-q}{\sigma(2q+1)}}. \tag{4.14}$$

Inserting (4.14) into (4.13), we have

$$\int_\Omega u^{2\sigma-1}|\nabla u|d\boldsymbol{x}\leqslant\left(|\Omega|^{\frac{q}{\sigma}}(\varphi(t))^{\frac{\sigma(2q+1)-q}{\sigma}}\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{1}{2(q+1)}}. \tag{4.15}$$

Substituting (4.15) into (4.12) and using Lemma 4.2 in [21] yields

$$\begin{aligned}
&\int_\Omega u^{\sigma(2+\frac{1}{2^{n-3}})}d\boldsymbol{x}\\
&\leqslant(1+2d)^{n-3}\Bigg[\frac{n}{2\rho_0}\varphi(t)+\sigma\left(1+\frac{d}{\rho_0}\right)\\
&\quad\times\left(\left(|\Omega|^{\frac{q}{\sigma}}\varphi(t)\right)^{\frac{\sigma(2q+1)-q}{\sigma}}\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{1}{2(q+1)}}\Bigg]^{1+\frac{1}{2^{n-2}}}\\
&\leqslant(1+2d)^{n-3}2^{\frac{1}{2^{n-2}}}\left(\frac{n}{2\rho_0}\right)^{1+\frac{1}{2^{n-2}}}(\varphi(t))^{1+\frac{1}{2^{n-2}}}\\
&\quad+(1+2d)^{n-3}2^{\frac{1}{2^{n-2}}}\left[\sigma\left(1+\frac{d}{\rho_0}\right)\right]^{1+\frac{1}{2^{n-2}}}\\
&\quad\times|\Omega|^{\frac{q}{\sigma}\frac{2^{n-2}+1}{(q+1)2^{n-1}}}(\varphi(t))^{\frac{\sigma(2q+1)-q}{\sigma}\frac{2^{n-2}+1}{(q+1)2^{n-1}}}\left(\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{2^{n-2}+1}{(q+1)2^{n-1}}}.
\end{aligned} \tag{4.16}$$

Then using Young inequality with ε to estimate the second term on right side of (4.16), we have

$$\begin{aligned}
&(\varphi(t))^{\frac{\sigma(2q+1)-q}{\sigma}\frac{2^{n-2}+1}{(q+1)2^{n-1}}}\left(\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{2^{n-2}+1}{(q+1)2^{n-1}}}\\
=&\left(\varepsilon^{-\frac{2^{n-2}+1}{(2q+1)2^{n-2}-1}}(\varphi(t))^{\frac{\sigma(2q+1)-q}{\sigma}\frac{2^{n-2}+1}{(2q+1)2^{n-2}-1}}\right)^{\frac{(2q+1)2^{n-2}-1}{(q+1)2^{n-1}}}\\
&\times\left(\varepsilon\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}\right)^{\frac{2^{n-2}+1}{(q+1)2^{n-1}}}\\
\leqslant&\frac{(2q+1)2^{n-2}-1}{(q+1)2^{n-1}}\varepsilon^{-\frac{2^{n-2}+1}{(2q+1)2^{n-2}-1}}(\varphi(t))^{\frac{\sigma(2q+1)-q}{\sigma}\frac{2^{n-2}+1}{(2q+1)2^{n-2}-1}}\\
&+\frac{2^{n-2}+1}{(q+1)2^{n-1}}\varepsilon\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x},\ \forall\varepsilon>0.
\end{aligned} \tag{4.17}$$

Inserting (4.17) into (4.16) and recalling (4.11), we get

$$\varphi'(t)\leqslant c_1(\varphi(t))^{\beta_1}+c_2(\varphi(t))^{\beta_2}+c_3(\varphi(t))^{\beta_3}+c_4\varphi(t)+c_5\int_\Omega u^{2(\sigma-1)}|\nabla u|^{2(q+1)}d\boldsymbol{x}, \tag{4.18}$$

where

$$\begin{cases} c_1 = \widetilde{c_1}(1+2d)^{n-3}2^{\frac{1}{2^{n-2}}}\left(\dfrac{n}{2\rho_0}\right)^{1+\frac{1}{2^{n-2}}}, \\ c_2 = \widetilde{c_1}(1+2d)^{n-3}2^{\frac{1}{2^{n-2}}}\left[\sigma\left(1+\dfrac{d}{\rho_0}\right)\right]^{1+\frac{1}{2^{n-2}}}|\Omega|^{\frac{q}{\sigma}\frac{2^{n-2}+1}{(q+1)2^{n-1}}} \\ \qquad \times\dfrac{(2q+1)2^{n-2}-1}{(q+1)2^{n-1}}\varepsilon^{-\frac{2^{n-2}+1}{(2q+1)2^{n-2}-1}}, \\ c_3 = \widetilde{c_3}, \\ c_4 = \widetilde{c_4}, \\ c_5 = \widetilde{c_1}(1+2d)^{n-3}2^{\frac{1}{2^{n-2}}}\left[\sigma\left(1+\dfrac{d}{\rho_0}\right)\right]^{1+\frac{1}{2^{n-2}}}|\Omega|^{\frac{q}{\sigma}\frac{2^{n-2}+1}{(q+1)2^{n-1}}}\dfrac{2^{n-2}+1}{(q+1)2^{n-1}}\varepsilon+\widetilde{c_2}, \end{cases} \tag{4.19}$$

and

$$\begin{cases} \beta_1 = 1+\dfrac{1}{2^{n-2}} > 1, \\ \beta_2 = \dfrac{\sigma(2q+1)-q}{\sigma}\dfrac{2^{n-2}+1}{(2q+1)2^{n-2}-1} > 0, \\ \beta_3 = \dfrac{\sigma-1}{\sigma} \geqslant 0. \end{cases} \tag{4.20}$$

We now select θ, ε such that $c_5 = 0$. For instance

$$\theta = \frac{\sigma(2\sigma-1)b_2\rho_0(q+1)}{\sigma^2 a_2(2+\frac{1}{2^{n-2}})d},\ \varepsilon = \frac{\sigma(2\sigma-1)b_2}{\widetilde{c_1}k},$$

where

$$k = (1+2d)^{n-3}2^{\frac{1}{2^{n-2}}}\left[\sigma\left(1+\frac{d}{\rho_0}\right)\right]^{1+\frac{1}{2^{n-2}}}|\Omega|^{\frac{q}{\sigma}\frac{2^{n-2}+1}{(q+1)2^{n-1}}}\frac{2^{n-2}+1}{(q+1)2^{n-1}}.$$

From (4.18), we obtain the differential inequality

$$\varphi'(t) \leqslant c_1(\varphi(t))^{\beta_1} + c_2(\varphi(t))^{\beta_2} + c_3(\varphi(t))^{\beta_3} + c_4\varphi(t). \tag{4.21}$$

Inequality (4.21) can be rewritten as

$$\frac{d\varphi}{c_1(\varphi(t))^{\beta_1} + c_2(\varphi(t))^{\beta_2} + c_3(\varphi(t))^{\beta_3} + c_4\varphi(t)} \leqslant dt. \tag{4.22}$$

Integrating (4.22) over $[0,t]$ and taking the limit as $t \to t^*$ we obtain

$$\int_{\varphi(0)}^{+\infty}\frac{d\eta}{c_1\eta^{\beta_1}+c_2\eta^{\beta_2}+c_3\eta^{\beta_3}+c_4\eta} \leqslant t^*. \tag{4.23}$$

Remark 4.1. Since $\beta_1 > 1$, the infinite integral on the left hand side of (4.23) converges.

From the above analysis and Theorem 3.1, we can summarize the following theorem on lower bound estimate of blow-up time t^*.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n (n \geqslant 3)$ be a bounded star-shaped domain assumed to be convex in $n-1$ orthogonal directions, the nonnegative functions f, g, ρ, and $u_0(\boldsymbol{x})$ satisfy the assumptions in Theorem 3.1 and (4.1), (4.2). Then the nonnegative solution $u(\boldsymbol{x}, t)$ to (1.1) blows up at finite time in the measure φ defined in (4.4), and the blow-up time t^* is bounded from below by (4.23).*

Remark 4.2. (1) Fixing σ, we find that β_3 remains stationary, β_1 and β_2 are deceasing functions with respect to n, so the blow-up phenomena occurs later with n increasing.

(2) Fixing the space dimension n, β_1 remains stationary, β_2, β_3 are increasing functions with respect to σ, so the blow-up phenomena occurs earlier with σ increasing.

From the above, we can know that lower bound of blow-up time t^* depends closely on σ, q and the space dimension n.

Remark 4.3. From Theorem 4.1, we can derive the lower bound of blow-up time with only heat source or only heat conduction, and derive that the solution will not blow up (the energy functional $\varphi(t)$ is decreasing) if $f(u) \equiv 0$ and $g(u) \equiv 0$.

Remark 4.4. In the past work such as [4–6,15,17,18], the lower bound of the blow-up time t^* have been indicated in three-dimensional bounded star-shaped domain. We obtain the lower bound of the blow-up time in n-dimensional ($n \geqslant 3$) bounded star-shaped domain.

Acknowledgements

The authors thank the editor and anonymous referees for their valuable suggestions and comments, which improved the presentation of this paper.

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