Journal of Applied Analysis and Computation
Volume 12, Number 2, April 2022, 622–661

Website:http://www.jaac-online.com
DOI:10.11948/20210258

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THE FRACTIONAL DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN IN $\mathbb{H}_P^{\nu,\eta;\psi}$

J. Vanterler da C. Sousa[1,†]

Abstract This present paper is dedicated to investigate the existence, uniqueness and minimization properties of weak solutions for a fractional differential equation in the sense of the ψ-Hilfer fractional operator, with p-Laplacian in the ψ-fractional space $\mathbb{H}_p^{\nu,\eta;\psi}$. To obtain such results, we use a variational structure for the main operator of the problem and the Harnack inequality.

Keywords ψ-Hilfer fractional derivative, fractional Differential Equations, p-Laplacian, existence, uniqueness.

MSC(2010) 26A33, 34A12, 35R11, 35A15, 35J66.

1. Introduction

In this present paper, we are interested in the existence, uniqueness and minimization properties of weak solutions for the following fractional differential equation with p-Laplacian given by

$$\begin{cases} {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)=\dfrac{f(x)}{u^{\theta}(x)}, \\ \mathbf{I}_{0+}^{\eta(\eta-1);\psi}u(0)=\mathbf{I}_{T}^{\eta(\eta-1);\psi}u(T)=0 \end{cases} \tag{1.1}$$

where ${}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}(\cdot)$, ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(\cdot)$ are the ψ-Hilfer fractional derivatives of order ν ($\frac{1}{p}<\nu\leq 1$) and type η ($0\leq\eta\leq 1$) and where $\Omega=[0,T]$ is a bounded domain in $\mathbb{R}$, $1<p<\infty, 0<\theta\leq 1, f\in L^{\frac{1}{\theta}}(\Omega), f\geq 0$ and $f\neq 0$.

We recall that the ψ-Riemann–Liouville fractional integrals and ψ-Hilfer fractional derivatives.

Let $I=(0,T)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(0,T]$, having a continuous derivative $\psi'(x)$ on I. The left-sided and right-sided ψ-Riemann-Liouville fractional integrals of a function f with respect to another function ψ on $J=[0,T]$ are defined by [35–38]

$$\mathbf{I}_{0+}^{\alpha;\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\psi'(t)\left(\psi(x)-\psi(t)\right)^{\alpha-1}f(t)\,dt \tag{1.2}$$

[†]Email: jose.vanterler@ufabc.edu.br (J. Vanterler da C. Sousa)
[1]Center for Mathematics, Computing and Cognition, Federal University of ABC, Avenida dos Estados, 5001, Bairro Bangu, 09.210-580, Santo André, SP-Brazil

and

$$\mathbf{I}_T^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t)\, dt. \tag{1.3}$$

On the other hand, let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, $J = [0, T]$ and $f, \psi \in \mathcal{C}^n(J, \mathbb{R})$ are two functions such that ψ is increasing and $\psi(x) \neq 0$, for all $x \in J$. The ψ-Hilfer fractional derivative left-sided and right-sided, denoted by ${}^{\mathbf{H}}\mathbf{D}_{a^+}^{\alpha,\beta;\psi}(\cdot)$ and ${}^{\mathbf{H}}\mathbf{D}_{b^-}^{\alpha,\beta;\psi}(\cdot)$ of a function f of order α and type $0 \leq \beta \leq 1$, is defined by [35–38]

$${}^{\mathbf{H}}\mathbf{D}_{a^+}^{\alpha,\beta;\psi} f(x) = \mathbf{I}_{a^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathbf{I}_{a^+}^{(1-\beta)(n-\alpha);\psi} f(x) \tag{1.4}$$

and

$${}^{\mathbf{H}}\mathbf{D}_{b^-}^{\alpha,\beta;\psi} f(x) = \mathbf{I}_{b^-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathbf{I}_{b^-}^{(1-\beta)(n-\alpha);\psi} f(x) \tag{1.5}$$

where $\mathbf{I}_{a^+}^{\alpha;\psi}(\cdot)$ and $\mathbf{I}_{b^-}^{\alpha;\psi}(\cdot)$ by defined in Eq. (1.2) and Eq. (1.3) respectively.

Optimization problems appear extensively throughout the history of Mathematics, especially the Brachistochrone problem, solved by Newton and Leibniz. Such problems gained rigor from the 17th century on wards, on the study of differential equations. A common operator for such equations is the Laplacian, denoted by Δ, which appears naturally in the mathematical modeling of many physical phenomena, such as the wave equation, heat flux equation, vibrating membrane equation, etc. Its uses in math problems are numerous. In particular, in Differential Geometry, we are interested in the possible relationships of the Δ spectrum between two manifolds, considering some hypotheses as conditions of curvature of these manifolds. A natural extension of Laplacian is p-Laplacian, which in turn is generalized by fractional p-Laplacian. Problems involving fractional p-Laplacian, throughout the decade, have been gaining prominence, both theoretically and in the context of applications [4, 7, 8, 12, 14, 15, 21, 26].

Problems about the existence, non-existence, regularity and multiplicity of weak solutions for the fractional p-Laplacian, have been the subject of increasing research over the years. Once the theory of differential equations, in particular, involving p-Laplacian problems, started to consider the fractional aspect, new results with a great impact on mathematics were emerging, and started to attract the attention of many researchers. Some results can be checked on [1, 5, 17, 27, 45].

In 2018, Li and Wei [22] investigated the existence and multiplicity of nontrivial solutions of fractional p-Laplacian equations of the form

$$\begin{cases} (-\Delta)_p^s u = \lambda f(x, u), x \in \Omega, \\ u(0) = 0,\ x \in \mathbb{R}^n/\Omega \end{cases}$$

$\lambda \in (0, \infty)$, $0 < s < \lambda < p < \infty$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with smooth boundary.

Giacomoni et al. [16], elaborate an interesting work on positive solutions to the following singular and non local elliptic problem with $\Omega \subset \mathbb{R}^N$ (smooth boundary)

$N > 2s$ $(0 < s < 1)$ given by

$$\begin{cases} (-\Delta)_p^s u = \lambda[k(x)u^{-\delta} + f(u)], \ x \in \Omega, \\ u = 0, \ \ x \in \mathbb{R}^n/\Omega, \\ u > 0, \in \Omega \end{cases}$$

where $\delta > 0$, $\lambda > 0$, $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive C^2 function and $K : \Omega \to \mathbb{R}^+$ is Holder continuous function in Ω which behave as $dis(x, \partial\Omega)^{-\beta}$ near the boundary with $0 \leq \beta < 2s$.

In 2021, Arora et al. [3], investigated the existence, uniqueness, non-existence and regularity of weak solutions of the nonlinear fractional elliptic problem given

$$\begin{cases} 2(-\Delta)_p^s u = \dfrac{K_\delta u}{u^\gamma}, \quad u > 0, \quad x \in \Omega, \\ u(0) = 0, \quad \text{in } \mathbb{R}^n/\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary $s \in (0, 1)$, $p \in (1, +\infty)$, $\gamma > 0$ and K_δ satisfies the asymptotic boundary behavior, for any $x \in \Omega$

$$\frac{c_1}{d^\delta(x)} \leq K_\delta(x) \leq \frac{c_2}{d^\delta(x)}$$

for some $\delta \in [0, sp]$, where for any $x \in \Omega$, $d(x) = dist(x, \partial\Omega) = inf_{y\in\partial\Omega}|x - y|$. The operator $(-\Delta)_p^s$ is known as fractional p-Laplacian.

Fractional derivatives and integrals are proved to be more useful in the modeling of different physical and natural phenomena. The p-Laplacian fractional boundary value problems related to nonlocal conditions have many applications in various fields: non-Newtonian mechanics, nonlinear elasticity, combustion theory, population biology, and other [2, 9, 20, 24, 25, 28, 29]. Fractional differential systems with p-Laplacian operators have also tremendous attention. Once it made discussions about fractional differential equations with p-Laplacian interesting and important, there was an exponential growth of works published in the literature. What is noticeable is that, over the years, the fractional calculus has consolidated itself in several areas, and has proved to be very important to explore problems in other areas [6, 18, 19, 39–44]. There are many papers concerning fractional differential equations with the p-Laplacian operator that address the existence, uniqueness, multiplicity of weak solutions. Here we will highlight a new theory that has been built using the ψ-Hilfer fractional derivative to attack variational problems [31–34].

In 2021 Sousa et al. [33], investigated the existence and non-existence of weak solutions to the nonlinear problem with a fractional p-Laplacian given by

$$\begin{cases} {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\xi(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\xi(x)\right) = \lambda|\xi(x)|^{p-2}\xi(x) + b(x)|\xi(x)|^{q-1}\xi(x), \\ \mathbf{I}_{0+}^{\eta(\eta-1);\psi}\xi(0) = \mathbf{I}_T^{\eta(\eta-1);\psi}\xi(T) = 0 \end{cases} \tag{1.6}$$

where ${}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}(\cdot)$, ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(\cdot)$ are the ψ-Hilfer fractional derivatives left-sided and right-sided of order $\frac{1}{p} < \nu < 1$ and type η $(0 \leq \eta \leq 1)$, $1 < q < p - 1 < \infty$, $b \in L^\infty(\Omega)$ and $\mathbf{I}_{0+}^{\eta(\eta-1);\psi}(\cdot)$, $\mathbf{I}_T^{\eta(\eta-1);\psi}(\cdot)$ are ψ-Riemann-Liouville fractional integrals of order $\eta(\eta-1)$ $(0 \leq \eta \leq 1)$, for all $x \in \Omega := [0,T]$. In the same year, Sousa [32] discussed necessary and sufficient conditions for Eq. (1.6) and investigated the bifurcation of solutions through the technique of the variety of Nehari and Fibering maps.

Before stating precisely our main results, let's introduce the energy functional and the definition of weak solutions.

Let $\psi(\cdot)$ be an increasing and positive monotone function on $[0,T]$, having a continuous derivative $\psi'(\cdot) \neq 0$ on $(0,T)$. If $0 < \nu \leq 1$ and $0 \leq \eta \leqslant 1$, then

$$\int_0^T \left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\xi(t)\right)\theta(t)dt = \int_0^T \xi(t)\psi'(t)\,{}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\frac{\theta(t)}{\psi'(t)}\right)dt \tag{1.7}$$

for any $\xi \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $\xi(0) = \xi(T) = 0$.

From Eq. (1.1), yields

$$\int_0^T {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)dx = \int_0^T \frac{f(x)}{u^{\theta}(x)}dx.$$

From $\phi \in C_0^{\infty}([0,T],\mathbb{R})$, yields

$$\int_0^T {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)\phi(x)dx = \int_0^T \frac{f(x)}{u^{\theta}(x)}\phi(x)dx. \tag{1.8}$$

Using Eq. (1.7), we get

$$\begin{aligned}&\int_0^T {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)\phi(x)dx\\ &= \int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\,\psi'(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(\frac{\phi(x)}{\psi'(x)}\right)dx.\end{aligned} \tag{1.9}$$

If ${}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\frac{\phi(x)}{\psi'(x)}\right) = \frac{1}{\psi'(x)}\,{}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\phi(x)$, for all $x \in [0,T]$, then Eq. (1.8), can be rewritten as

$$\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)dx = \int_0^T \frac{f(x)}{u^{\theta}(x)}\phi(x)dx.$$

Consider $\phi = u$, yields

$$\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p}dx = \int_0^T u^{1-\theta}(x)f(x)\,dx. \tag{1.10}$$

So, from Eq. (1.10), we have the functional associated to Eq. (1.1), $\mathbf{E}_{\nu,\eta}^{\theta} : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{R}$, given by

$$\mathbf{E}_{\nu,\eta}^{\theta}(u) := \frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p}dx - \frac{1}{1-\theta}\int_0^T u^{1-\theta}(x)f(x)\,dx.$$

Definition 1.1. A function $u \in \mathbb{H}_p^{\nu,\eta;\psi}$ it is called weak problem solving Eq. (1.1) if $u > 0$ in $[0,T]$ and the following identity is valid

$$\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx = \int_0^T \frac{f(x)}{u^{\theta}(x)}dx \tag{1.11}$$

$\forall \varphi \in \mathbb{H}_p^{\nu,\eta;\psi}$.

Although highlighted above some papers on fractional differential equations with p-Laplacian, there are still open questions. In particular, it is worth noting that fractional differential equation problems involving p-Laplacian, discussed over a variational structure with the ψ-Hilfer fractional derivative, is very restricted. In this sense, one of the reasons for the elaboration of this paper is to expand and contribute to the growth of the area, in particular, to the theory of fractional differential equations and variational problems.

Inspired by the above papers and open questions, we will now highlight the main contributions to be discussed in this paper. The contributions obtained in this paper are divided into two stages.

In the first step, we investigate the existence and uniqueness of weak solutions for the fractional differential equation with p-Laplacian (see Eq. (1.1)), in other words, we will investigate the following results:

Lemma 1.1. *Let $u \in \mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})$ non-negative, satisfying*

$$\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x)\, dx = \int_0^T \frac{f(x)}{u_1(x)^{\widetilde{\nu}}} \varphi(x)\, dx$$

$\forall \varphi(x) \in C_0^\infty([0,T],\mathbb{R})$. *So u is a weak solution to* Eq. (1.1)*.*

Theorem 1.1. *Suppose f is a non-negative function on $L^{\frac{1}{\nu}}([0,T],\mathbb{R})$ and $0 < \widetilde{\nu} \leq 1$. Then, the* Eq. (1.1) *has a unique solution $u \in \mathbb{H}_p^{\nu,\eta;\psi}$.*

In the second step of this paper, we investigate that the energy functional associated with Eq. (1.1) has a unique minimizer and that this minimizer is the weak solution u of Eq. (1.1) in other words, let's investigate the following results, namely:

Lemma 1.2. *Let $0 < \theta \leq 1$. The functional $\mathbf{E}_{\nu,\eta}^\theta : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{R}$ has a unique minimizer, which is nonnegative.*

Lemma 1.3. *The solution u_n found in* Lemma 3.1 *is the only positive minimizer of the functional*

$$\mathbf{H}_n(v) = \frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v\right\|_p^p - \int_0^T \mathbf{U}_n(v(x)) f_n(x) dx$$

on what

$$\begin{aligned} \mathbf{U}_n(t) &= \int_0^t \left(s^+ + \frac{1}{n}\right)^{-\theta} ds \\ &= \begin{cases} \dfrac{1}{1-\theta}\left(t+\dfrac{1}{n}\right)^{1-\theta} - \dfrac{1}{1-\theta}\left(\dfrac{1}{n}\right)^{1-\theta}, & \text{if } t \geq 0, \\ \left(\dfrac{1}{n}\right)^{-\theta} t & \text{if } t < 0. \end{cases} \end{aligned}$$

Theorem 1.2. *The u solution found in* Theorem 3.1 *minimize $\mathbf{E}_{\nu,\eta}^\theta$ with $0 < \theta \leq 1$.*

Theorem 1.3. *Let $0 < \nu \leq 1$ and $0 \leq \eta \leq 1$. We have*

$$\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_\theta\right\|_p^p = \min\left\{\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v\right\|_p^p : v \in \mathcal{M}\right\}$$

with $0 < \theta \leq 1$.

A natural and important consequence of the results investigated here is that they are valid for a wide class of particular cases, that is, from the choice of $\psi(\cdot)$ and the limits $\beta \to 1$ and $\beta \to 0$, we have a wide class of particular cases. A special case is for $\nu = 1$ and $\psi(t) = t$ (integer case), that is, the following problem given by

$$\begin{cases} \left(|u'(x)|^{p-2}\ u'(x)\right)' = \dfrac{f(x)}{u^{\theta}(x)}, \\ u(0) = u(T) = 0, \end{cases}$$

where u' is the classical derivative.

In the rest, the paper is organized as follows: In section 2, we present some variational results that are of paramount importance to obtain the main results of this paper. In section 3, we investigate our first main result, that is, the existence and uniqueness of weak solutions for the fractional differential equation with p-Laplacian. In this sense, we investigate minimization properties for the functional energy $\mathbf{E}_{\nu,\eta,\theta}$ referring to Eq. (1.1), closes section 4.

2. Preliminaries framework

In this section, we present definitions and results involving fractional operators and variational structure, essential to investigate the main results of this paper.

Definition 2.1 ([32, 33]). Let $0 < \nu \leq 1$, $0 \leq \eta \leq 1$ and $1 < p < \infty$. The ψ-fractional derivative space $\mathbb{H}_p^{\nu,\eta;\psi} := \mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})$ is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R})$, and is given by

$$\begin{aligned} &\mathbb{H}_p^{\nu,\eta;\psi} \\ &= \left\{ u \in L^p([0,T],\mathbb{R});\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \in L^p([0,T],\mathbb{R}),\ \mathbf{I}_{0+}^{\eta(\eta-1)} u(0) = \mathbf{I}_{T}^{\eta(\eta-1)} u(T) = 0 \right\} \\ &= \overline{C_0^{\infty}([0,T],\mathbb{R})} \end{aligned} \tag{2.1}$$

with the following norm

$$\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}} = \left(\|u\|_{L^p}^p + \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \right\|_{L^p}^p \right)^{1/p}, \tag{2.2}$$

where ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative with $0 < \nu \leq 1$ and $0 \leq \eta \leq 1$.

Choosing $p = 2$, in definition Eq. (2.1), we have the ψ-fractional derivative space $\mathbb{H}_2^{\nu,\eta;\psi}$ is defined on $\overline{C_0^{\infty}([0,T],\mathbb{R})}$ with respect to the norm [32, 33]

$$\|u\|_{\mathbb{H}_2^{\nu,\eta;\psi}} = \left(\int_0^T |u(x)|^2\, dx + \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^2 dx \right)^{1/2}.$$

The space $\mathbb{H}_2^{\nu,t;\psi}$ is a Hilbert space with the norm [32, 33]

$$\|u\|_{\mathbb{H}_2^{\nu,\eta;\psi}} = \left(\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(t) \right|^2 \mathrm{dt} \right)^{1/2}$$

with $0 < \nu \leq 1$ and $0 \leq \eta \leq 1$.

Proposition 2.1 ([32,33]). *Let $0 < \nu \leq 1$, $0 \leq \eta \leq 1$ and $1 < p < \infty$. Assume that $\nu > 1/p$ and the sequence $\{u_k\}$ converges weakly to u in $\mathbb{H}_p^{\nu,\eta;\psi}$ i.e., $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R})$, i.e., $\|u-u_k\|_\infty \to 0$ as $k \to \infty$.*

Proposition 2.2 ([32,33]). *The space $\mathbb{H}_p^{\nu,\eta;\psi}$ is compactly embedded in $C([0,T],\mathbb{R})$.*

Proposition 2.3 ([32,33]). *Let $0 < \nu \leq 1$, $0 \leq \eta \leq 1$ and $1 < p < \infty$. The fractional derivative space $\mathbb{H}_p^{\nu,\eta;\psi}$ is a reflexive and separable Banach space.*

Theorem 2.1. *The space $\left(\mathbb{H}_p^{\nu,\eta,\psi}, \|\cdot\|_{\mathbb{H}_p^{\nu,\eta,\psi}}\right)$ is uniformly convex.*

Proof. Indeed, let $p \in [2,\infty)$. Then for each $z,w \in \mathbb{R}$, it holds

$$\left|\frac{z+w}{2}\right|^p + \left|\frac{z-w}{2}\right|^p \leqslant \frac{1}{2}(|z|^p + |w|^p).$$

Let $\xi,\zeta \in \mathbb{H}_p^{\nu,\eta,\psi}$ satisfy $\|\xi\|_{\mathbb{H}_p^{\nu,\eta,\psi}} = \|\zeta\|_{\mathbb{H}_p^{\nu,\eta,\psi}} = 1, and\, \|\xi-\zeta\|_{\mathbb{H}_p^{\nu,\eta,\psi}} \geqslant \varepsilon \in (0,2]$. Then, we have

$$\begin{aligned}
\left\|\frac{\xi+\zeta}{2}\right\|^p_{\mathbb{H}_p^{\nu,\eta,\psi}} + \left\|\frac{\xi-\zeta}{2}\right\|^p_{\mathbb{H}_p^{\nu,\eta,\psi}} &= \int_0^T \left(\left|\frac{{}^H\mathbf{D}_{0+}^{\nu,\eta,\psi}\xi(x) + {}^\mathbf{H}\,\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta(x)^p}{2}\right|\right)dx \\
&\quad + \int_0^T \left(\left|\frac{{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\xi(x) + {}^\mathbf{H}\,\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta(x)^p}{2}\right|\right)dx \\
&\leqslant \int_0^T \frac{1}{2}\left(|{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\xi(x)|^p + |{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta(x)|^p\right)dx \\
&= \frac{1}{2}\left(\|\xi\|_{\mathbb{H}_p^{\nu,\eta,\psi}} + \|\zeta\|_{\mathbb{H}_p^{\nu,\eta,\psi}}\right) = 1
\end{aligned}$$

which yields

$$\left\|\frac{\xi+\zeta}{2}\right\|^p_{\mathbb{H}_p^{\nu,\eta,\psi}} \leqslant 1 - \left(\frac{\varepsilon}{2}\right)^p. \tag{2.3}$$

On the other hand, if $p \in (1,2)$ then for each $z,w \in \mathbb{R}$ it holds

$$\left|\frac{z+w}{2}\right|^{p'} + \left|\frac{z-w}{2}\right|^{p'} \leqslant \left(\frac{1}{2}(|z|^p + |w|^p)\right)^{\frac{1}{p-1}}. \tag{2.4}$$

A straight forward computation proves that if $v \in \mathbb{H}_p^{\nu,\eta,\psi}$ then $\left\||{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta|^p\right\|_{\mathbb{H}_{p-1}^{\nu,\eta,\psi}} = \|\zeta\|^{p'}_{\mathbb{H}_p^{\nu,\eta,\psi}}$.

Let $\zeta_1,\zeta_2 \in \mathbb{H}_p^{\nu,\eta,\psi}$ then $\left|{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_1\right|^{p'}, \left|{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_2\right|^{p'} \in L^{p-1}([0,T])$ with $0 < p-1 < 1$ and according to

$$\left\||{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_1|^{p'} + |{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_2|^{p'}\right\|_{\mathbb{H}_{p-1}^{\nu,\eta,\psi}} \geqslant \left\||{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_1|^{p'}\right\|_{\mathbb{H}_{p-1}^{\nu,\eta,\psi}} + \left\||{}^\mathbf{H}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_2|^{p'}\right\|_{\mathbb{H}_{p-1}^{\nu,\eta,\psi}} \tag{2.5}$$

consequently

$$\left\|\frac{\zeta_1+\zeta_2}{2}\right\|^p_{\mathbb{H}_p^{\nu,\eta,\psi}} + \left\|\frac{\zeta_1-\zeta_2}{2}\right\|^p_{\mathbb{H}_p^{\nu,\eta,\psi}}$$

$$= \left\| \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|^{p'} \right\|_{\mathbb{H}_{p-1}^{\nu,\eta,\psi}} + \left\| \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\left(\frac{\zeta_1-\zeta_2}{2}\right)\right|^{p'} \right\|_{\mathbb{H}_{p-1}^{\nu,\eta,\psi}} \tag{2.6}$$

$$\leqslant \left\| \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|^{p'} + \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\left(\frac{\zeta_1-\zeta_2}{2}\right)\right|^{p'} \right\|_{\mathbb{H}_{p-1}^{\nu,\eta,\psi}} \tag{2.7}$$

$$= \left[\int_0^T \left(\left| \frac{{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_1 + {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_2}{2}\right|^{p'} + \left| \frac{{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_1 - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_2}{2}\right|^{p'} \right)^{p-1} dx \right]^{\frac{1}{p-1}} \tag{2.8}$$

$$\leqslant \left[\frac{1}{2}\int_0^T \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_1\right|^p + \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta,\psi}\zeta_2\right|^p \right) dx \right]^{\frac{1}{p-1}} \tag{2.9}$$

$$= \left(\frac{1}{2}\|\zeta_1\|^p_{\mathbb{H}_p^{\nu,\eta,\psi}} + \frac{1}{2}\|\zeta_2\|^p_{\mathbb{H}_p^{\nu,\eta,\psi}}\right)^{\frac{1}{p-1}}. \tag{2.10}$$

For $\xi,\zeta \in \mathbb{H}_p^{\nu,\eta,\psi}$ with $\|\xi\|_{\mathbb{H}_p^{\nu,\eta,\psi}} = \|\zeta\|_{\mathbb{H}_p^{\nu,\eta,\psi}} = 1$ and $\|\xi-\zeta\|_{\mathbb{H}_p^{\nu,\eta,\psi}} \geqslant \varepsilon \in (0,2]$, we have

$$\left\|\frac{\xi+\zeta}{2}\right\|^{p'} \leqslant 1 - \left(\frac{\varepsilon}{2}\right)^{p'}. \tag{2.11}$$

From (2.3) and (2.11) in either case there exists $\delta(\varepsilon) > 0$ such that $\|\xi+\zeta\|_{\mathbb{H}_p^{\nu,\eta,\psi}} \leqslant 2(1-\delta(\varepsilon))$. □

Next we present the Harnack's inequality in the fractional sense with respect to another function.

Theorem 2.2 ([30]). *Let $t_* \geq 0, 0 < \sigma_1 < \sigma_2 < \sigma_3$ and $\rho > 0$. Let further $\nu \in (0,1)$, $0 \leq \eta \leq 1$, $\psi(0) = 0$ and $u_0 \geq 0$. Then for any function $u \in Z(t_*, t_* + \sigma_3\rho)$ and that satisfies*

$$\partial_t^{\nu,\eta;\psi}(u-u_0)(t) = 0, \;\; a.a.\, t \in (t_*, t_*+\sigma_3\rho) \tag{2.12}$$

there holds the inequality

$$\sup_{W-} u \leq \sigma_3\sigma_1 \inf_{W+} u \tag{2.13}$$

where $W- = (t_*+\sigma_1\rho, t_*+\sigma_2\rho)$ e $W+ = (t_*+\sigma_2\rho, t_*+\sigma_3\rho)$.

Theorem 2.3 ([13], Schaefer's Theorem). *Let X a real Banach space and $\Lambda : X \to X$ a continuous and compact application. Suppose the set*

$$\{u \in X, u = \lambda A(u),\; for\ some\ 0 \leq \lambda \leq 1\} \tag{2.14}$$

be bounded. So A has a fixed point.

Proposition 2.4 ([11]). *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies the following growth condition:*

$$|f(x,s)| \leq C|s|^{q-1} + b(x), \quad x \in \Omega, \quad s \in \mathbb{R}$$

wherein $C \geq 0$ is a constant, $q > 1$, $b \in L^{q'}(\Omega)$. Let $\mathbf{F} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\mathbf{F}(x,s) = \int_0^s f(x,\tau)d\tau.$$

Then:
1) $\mathbf{F}$ *is a Caratheodory function and there are* $c_1 \geq 0$ *and* $c \in L^{q'}(\Omega)$ *such that*

$$|\mathbf{F}(x,s)| \leq c_1|s|^q + c(x),\; x \in \Omega,\; s \in \mathbb{R}.$$

2) The functional $\Phi : L^q(\Omega) \to \mathbb{R}$ *defined by*

$$\Phi(u) = \int_\Omega \mathbf{F}(x, u(x))dx$$

is of class C^1 *and the Frechet derivative of* Φ *in* u *is the functional defined by*

$$\langle \Phi'(u), v\rangle = \int_0^T f(x, u(x)v(x))dx,\; u \in L^q(\Omega),\; v \in L^{q'}(\Omega).$$

Lemma 2.1 ([10])**.** *Let* x, y *vectors in* $\mathbb{R}^n$*. So there are positive constants* c_p *and* $\widetilde{c}_p$ *that only depend on* p*, such that*

$$\left|\widetilde{\Psi}_p(x) - \widetilde{\Psi}_p(y)\right| \leq c_p \begin{cases} |x-y|^{p-1}, & \text{if } 1 < p \leq 2, \\ (|x| + |y|)^{p-2}\,|x-y|\,, & \text{if } p \geq 2 \end{cases} \tag{2.15}$$

and

$$\left(\widetilde{\Psi}_p(x) - \widetilde{\Psi}_p(y)\right)(x-y) \geq \widetilde{c}_p \begin{cases} \dfrac{|x-y|^2}{(|x|+|y|)^{p-2}}, & \text{if } 1 < p \leq 2, \\ |x-y|^p\,, & \text{if } p \geq 2 \end{cases} \tag{2.16}$$

for all $x, y \in \mathbb{R}^n - \{0\}$.

Theorem 2.4. *Let* $u_1, u_2 \in \mathbb{H}_p^{\alpha,\beta;\psi}$ *such that*

$$\begin{aligned} &\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx \\ \leq &\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx \end{aligned} \tag{2.17}$$

for all $\varphi \in \mathbb{H}_p^{\alpha,\beta;\psi}$*,* $\varphi \leq 0$*. Hence,* $u_1 \leq u_2$ *a.e in* $[0, T]$*.*

Proof. Consider $(u_1 - u_2)^+ = max\,\{u_1 - u_2, 0\}$. So $(u_1 - u_2)^+ \in \mathbb{H}_p^{\alpha,\beta;\psi}$, because $u_1, u_2 \in \mathbb{H}_p^{\alpha,\beta;\psi}$. We also have,

$${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_1 - u_2)^+ = \begin{cases} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_1 - u_2)^+, & \text{if } u_1 > u_2, \\ 0, & \text{if } u_1 \leq u_2. \end{cases} \tag{2.18}$$

Using the hypothesis for $\varphi = (u_1 - u_2)^+$ and $u_1 > u_2$, yields

$$\begin{aligned} 0 \geq \int_0^T &\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x)\right|^{p-2} u_1(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\right|^{p-2} u_2(x)\right) \\ &\times {}^{\mathbf{H}}\,\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_1(x) - u_2(x))^+ dx \end{aligned}$$

$$= \int_0^T \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x) \right|^{p-2} u_1(x) - \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x) \right|^{p-2} u_2(x) \right)$$
$$\times {}^{\mathbf{H}}\, \mathbf{D}_{0+}^{\nu,\eta;\psi} (u_1(x) - u_2(x)) dx.$$

But by Eq. (2.16), we obtain

$$\left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x) \right|^{p-2} u_1(x) - \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x) \right|^{p-2} u_2(x) \right) {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (u_1(x) - u_2(x)) \geq 0.$$

Therefore, $\Omega_0 := \{x \in [0,T], u_1(x) > u_2(x)\}$ has null measure or

$$\left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x) \right|^{p-2} u_1(x) - \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x) \right|^{p-2} u_2(x) \right) {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (u_1(x) - u_2(x)) = 0.$$

a.e in ω_0.

The last condition cannot occur. In fact, otherwise we would have from Eq. (2.16) that $u_1 - u_2 = 0$ a.e in ω_0. Therefore, Ω_0 has null measure. □

Corollary 2.1. *Suppose that $u \in \mathbb{H}_p^{\alpha,\beta;\psi}$ is such that*

$$\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx \geq 0 \tag{2.19}$$

for all $\varphi \in \mathbb{H}_p^{\alpha,\beta;\psi}$, $\varphi \geq 0$ and $u \geq 0$. Hence, $u \geq 0$ in $[0,T]$.

Theorem 2.5. *Let $u_1, u_2 \in \mathbb{H}_p^{\alpha,\beta;\psi}(\Omega) \cap C^0(\Omega)$ such that*

$$\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx$$
$$\leq \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx \tag{2.20}$$

for all $\varphi \in \mathbb{H}_p^{\alpha,\beta;\psi}$, $\varphi \geq 0$ and $u_1 \leq u_2$ in $[0,T]$. So, exactly one of the following possibilities occurs: $u_1 = u_2$ in $[0,T]$ or $u_1 < u_2$ in $[0,T]$.

Proof. Suppose the existence of a certain $x_0 \in \Omega$ such that $u(x_0) = 0$, define the following set

$$\mathcal{A} = \{x \in \Omega; u(x) = 0\}.$$

We have $\mathcal{A} \neq 0$ and since u is a continuous function it follows that $\mathcal{A}$ is a closed interval in Ω. If Ω is open, there exist $\delta > 0$ such that $(x_0 - 5\delta, x_0 + 5\delta \subset \Omega)$. By the Harnack's inequality (Theorem 2.2) with respect to ψ, there are $c, s > 0$ such that

$$||v|| \leq cs \inf_{(x_0-\delta, x_0+\delta)} u.$$

As $u \geq 0$ in Ω and $u(x_0) = 0$, follow that $\inf_{(x_0-\delta,x_0+\delta)} u = 0$, so

$$\int_{(x_0-\delta,x_0+\delta)} |u(s)|^2 ds = 0.$$

With $u \geq 0$ and continuous it follows that $u = 0$ in $(x_0 - \delta, x_0 + \delta)$. Therefore, $\mathcal{A}$ is an open interval. Since Ω is connected, we must have $\mathcal{A} = \Omega$. Therefore, $u = 0$ in Ω or $u > 0$ in Ω. □

Corollary 2.2. *Suppose that Ω in open domain. If $u \in \mathbb{H}_p^{\alpha,\beta;\psi}(\Omega)$ such that*

$$\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx \geq 0 \tag{2.21}$$

for all $\varphi \in \mathbb{H}_p^{\alpha,\beta;\psi}(\Omega)$, $\varphi \geq 0$ and $u \geq 0$ in Ω. So, exactly one of the following possibilities occurs: $u = 0$ in Ω or $u > 0$ in Ω.

Before attacking the main results of this paper, consider the following problem

$$\begin{cases} -\Delta_p u = f(x), \text{ in } \Omega, \\ u(0) = 0, \qquad \partial\Omega, \end{cases} \tag{2.22}$$

where Ω is the ball of radius R centered on the origin and $f \in L^\infty(\Omega)$ is radial, that is, $f(x) = f(r)$ where $r = |x|$.

Theorem 2.6 ([8]). *Suppose that in* Eq. (2.22) *have $\Omega = B_R(0)$ and $f(x) = f(r)$ where $r = |x|$. So the only solution of* Eq. (2.22) *is given by*

$$u(r) = \int_r^R \psi_{p'} \left(\int_0^\theta \left(\frac{s}{\theta} \right)^{N-1} f(s) ds \right) d\theta$$

where $\psi_{p'}$ is the inverse of $\psi_p(t) = |t|^{p-2} t$.

Consider the following fractional problem given by

$$\begin{cases} {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi} \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right) = g(x), \\ \mathbf{I}_{0+}^{\eta(\eta-1);\psi} u(0) = \mathbf{I}_{T}^{\eta(\eta-1);\psi} u(T) = 0 \end{cases} \tag{2.23}$$

$g \in L^{p'}([0,T], \mathbb{R})$, $0 < p < 1$ and $p' = \dfrac{p}{p-1}$. The condition of bounded will be extended with $u \in \mathbb{H}_p^{\nu,\eta;\psi}$.

Definition 2.2. Let $0 < \nu \leq 1$ and $0 \leq \eta \leq 1$. A function $u \in \mathbb{H}_p^{\nu,\eta;\psi}$ is called weak solution of the Eq. (2.23) if the following identity holds

$$\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx = \int_0^T g(x)\, dx \tag{2.24}$$

$\forall \varphi \in \mathbb{H}_p^{\nu,\eta;\psi}$ and $\forall x \in [0,T]$.

Lemma 2.2. *The functional $\mathbf{E}_{\nu,\eta} : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{R}$ given by*

$$\mathbf{E}_{\nu,\eta}^{\theta}(u) = \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^p - \int_0^T g(x) u(x)\, dx \tag{2.25}$$

$u \in \mathbb{H}_p^{\nu,\eta;\psi}$ is of class C^1 and

$$\left\langle \mathbf{E}'_{\nu,\eta}(u), \phi \right\rangle = \int_0^T \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \phi(x) - g(x) u(x) \right) dx \tag{2.26}$$

$\forall x \in \mathbb{H}_p^{\nu,\eta;\psi}$.

Proposition 2.5. *Let u be a weak solution to the* Eq. (2.23) *with $f \in L^\infty([0,T],\mathbb{R})$. Then, $u \in L^\infty([0,T],\mathbb{R})$ and*

$$|u| \leq ||f||_\infty^{\frac{1}{p-1}} \phi$$

on what ϕ is a torsion p-function of $[0,T]$.

Proof. Let ϕ a torsion p-function of Ω, that is, the solution of

$$\begin{cases} {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\right) = 1, \text{ in } \Omega, \\ \phi = 0. \end{cases} \tag{2.27}$$

Since Ω is bounded, there exists a interval $B = [0,R]$ such that $\Omega \subset B$. Let ϕ be the torsion p-function of B. As the function 1 is radial it follows from Theorem 2.6 that Φ is radial and using the functions $\xi_p(t) = |t|^{p-2}t$ and $\xi_{p'}(t) = |t|^{p'-2}t$, we have

$$\Phi(t) = \int_r^R \xi_{p'}\left(\int_0^\theta \frac{s}{\theta}\right)^{N-1} d\theta = \int_r^R \left(\frac{\theta}{N}\right)^{p'-1} d\theta = \frac{N^{1-p'}}{p'}\left(R^{p'} - r^{p'}\right)$$

that is,

$$\Phi(x) = \frac{N^{1-p'}}{p'}(R^{p'} - |x|^{p'}),\ x \in B.$$

Extending ϕ as zero out of Ω and noticing that $\Phi \geq 0$ into B, yields

$$\xi := (\phi - \Phi)^+ \in \mathbb{H}_p^{\alpha,\beta,\psi}(\Omega) \cap \mathbb{H}_p^{\alpha,\beta,\psi}(B).$$

Thus,

$$\begin{aligned} &\int_\Omega \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\xi(x)dx \\ &= \int_\Omega \xi(x)dx \\ &= \int_B \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\xi(x)dx \\ &= \int_\Omega \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\xi(x)dx, \end{aligned}$$

that is,

$$\begin{aligned} &\int_\Omega \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right) \\ &\times\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\xi(x)dx = 0. \end{aligned}$$

Soon,

$$\begin{aligned} &\int_\Omega \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right) \\ &\times\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(\phi - \Phi)^+(x)dx = 0 \end{aligned}$$

implies that

$$\int_{\phi\geq\Phi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)-\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi(x)\right)$$
$$\times\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(\phi-\Phi)(x)dx=0.$$

So, follow from Eq. (2.16),

$${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi-\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\Phi=0,\ q.t.p$$

in $\{x\in\Omega;\phi(x)\geq\Phi(x)\}$. So, ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(\phi-\Phi)^{+}=0$, q.t.p in Ω and then $(\phi-\Phi)^{+}=0$ in $\mathbb{H}_{p}^{\alpha,\beta;\psi}(\Omega)$. Therefore, $\phi\leq\Phi$ q.t.p in Ω. Furthermore, it follows from the Corollary 2.1 that $\phi\geq 0$. Thus, $0\leq\phi\leq\Phi$ and, as Φ is bounded, we have $\phi\in L^{\infty}(\Omega)$. In Ω we also have

$$\begin{aligned}
&{}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)\\
&=f\leq||f||_{\infty}\\
&=\ {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(||f||_{\infty}^{1/p'-1}\phi(x)\right)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(||f||_{\infty}^{1/p'-1}\phi(x)\right)\right),
\end{aligned}$$

in the weak sense. Similarly,

$$\begin{aligned}
&{}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(-u(x))\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(-u(x))\right)\\
&=|-1|^{p-2}(-1)\,{}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)\\
&\leq -f\\
&=\ {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(||f||_{\infty}^{1/p'-1}\phi(x)\right)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(||f||_{\infty}^{1/p'-1}\phi(x)\right)\right)
\end{aligned}$$

in the weak sense. Thus, by Theorem 2.4, yields

$$|u|\leq||f||_{\infty}^{1/p-1}\phi$$

and as, $f,\phi\in L^{\infty}(\Omega)$, we concluded that $u\in L^{\infty}(\Omega)$. □

Theorem 2.7. *Suppose $\Omega=[0,T]$ is a bounded domain and $g\in L^{p'}([0,T],\mathbb{R})$. So, the* Eq. (2.23) *has a unique solution $u\in\mathbb{H}_{p}^{\nu,\eta;\psi}$ in the weakly sense.*

Proof. The proof of this result will be discussed in two steps.

Step 1: Existence.

Note that the functional

$$\mathbf{E}_{\nu,\eta}(u):=\frac{1}{p}\int_{0}^{T}\ \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p}-\int_{0}^{T}g(x)u\left(x\right)dx$$

with $u\in\mathbb{H}_{p}^{\nu,\eta;\psi}$ is well defined, because by the inequalities of Holder and Poincare, we obtain

$$|\mathbf{E}_{\nu,\eta}(u)|\leq\frac{1}{p}\ \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u\right\|_{p}^{p}+\left\|g\right\|_{p'}\left\|u\right\|_{p}$$

$$\leq \frac{1}{p}\ \|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})} + c\,\|g\|_{p'}\,\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}}\,.$$

Thus, it follows from the Holder and Poincare inequalities that

$$\begin{aligned}\mathbf{E}_{\nu,\eta}(u) &\geq \frac{1}{p}\ \|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})} - \|g\|_{p'}\,\|u\|_p \\ &\geq \frac{1}{p}\ \|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})} - c\,\|g\|_{p'}\,\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}}\,.\end{aligned} \tag{2.28}$$

So, like the function $p(t) = \dfrac{1}{p}t^p - c\,\|g\|_{p'}\,t$ is lower bound, it follows that

$$\mu := \inf_{u\in\mathbb{H}_p^{\nu,\eta;\psi}} \mathbf{E}_{\nu,\eta}(u) > -\infty. \tag{2.29}$$

We have that there exists a sequence $\{u_k\} \in \mathbb{H}_p^{\nu,\eta;\psi}$ such that

$$\lim_{k\to\infty} \mathbf{E}_{\nu,\eta}(u_k) = \mu. \tag{2.30}$$

Thus, the sequence $(\mathbf{E}_{\nu,\eta}(u_k)_{k\in\mathbb{N}})$ is bounded in $\mathbb{R}$ and then it follows from Eq. (2.28) that there exists is $M > 0$ such that

$$\frac{1}{p}\,\|u_k\|^p_{\mathbb{H}_p^{\nu,\eta;\psi}} - c\,\|f\|_{p'}\,\|u_k\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq M,\ \forall k\in\mathbb{N}. \tag{2.31}$$

Therefore, $\{u_k\}$ is a bounded sequence in $\mathbb{H}_p^{\nu,\eta;\psi}$, because

$$\lim_{t\to+\infty} p(t) = +\infty.$$

As $\mathbb{H}_p^{\nu,\eta;\psi}$ is reflexive, without loss of generality taking a convergent subsequence $\{u_k\}_{k\in\mathbb{N}}$, there exists $u \in \mathbb{H}_p^{\nu,\eta;\psi}$ such that $u_k \rightharpoonup u$ weakly in $\mathbb{H}_p^{\nu,\eta;\psi}$. Consider the function

$$\mathbf{F}(w) = \int_0^T g(x)w(x)dx,\ w\in\mathbb{H}_p^{\nu,\eta;\psi}$$

follows from the Holder inequality, follows that

$$|\mathbf{F}(w)| \leq \|g\|_{p'}\,\|w\|_p$$

and then $\mathbf{F} \in \mathbb{H}_{-p'}^{\nu,\eta;\psi}$. So, $\mathbf{F}(u_k) \to \mathbf{F}(u)$, that is,

$$\lim_{k\to+\infty}\int_0^T g(x)u_k(x)dx = \int_0^T f(x)u(x)dx. \tag{2.32}$$

Observe that

$$\begin{aligned}\mathbf{E}_{\nu,\eta}(u) &= \frac{1}{p}\,\|u\|^p_{\mathbb{H}_p^{\nu,\eta;\psi}} - \int_0^T g(x)u(x)dx \\ &\leq \frac{1}{p}\left(\lim_{k\to+\infty}\inf\|u_k\|_{\mathbb{H}_p^{\nu,\eta;\psi}}\right)^p - \int_0^T g(x)u(x)dx \\ &= \lim_{k\to\infty}\inf\left(\frac{1}{p}\,\|u_k\|_{\mathbb{H}_p^{\nu,\eta;\psi}} - \int_0^T g(x)u_k(x)dx\right)^p\end{aligned}$$

$$
\begin{aligned}
&+\lim_{k\to\infty}\left(\int_0^T g(x)u_k(x)dx-\int_0^T g(x)u(x)dx\right)\\
&=\lim_{k\to\infty}\inf \mathbf{E}_{\nu,\eta}(u_k)+\lim_{k\to\infty}\left(\int_0^T g(x)u_k(x)dx-\int_0^T g(x)u(x)dx\right). \quad (2.33)
\end{aligned}
$$

Taking $k\to\infty$ on both sides of inequality (2.33), follows from Eq. (2.30) and Eq. (2.32) that $\mathbf{E}_{\nu,\eta}(u)\le\mu$.

On the other hand,

$$
\mu=\inf_{w\in\mathbb{H}_p^{\nu,\eta;\psi}}\le \mathbf{E}_{\nu,\eta}(u).
$$

So $\mathbf{E}_{\nu,\eta}(u)=\mu$ then u minimizes $\mathbf{E}_{\nu,\eta,n}$. Therefore, it follows from Lemma 2.2 that u is a weak solution to the Eq. (2.23).

Step 2: Uniqueness.

Let $u_1,u_2\in\mathbb{H}_p^{\nu,\eta;\psi}$ weak solutions of the Eq. (2.23) for $g=g_1$ and $g=g_2$, respectively. Thus

$$
\begin{aligned}
&\left\langle \begin{array}{l} {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right)\\ -\ {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right)\end{array}, u_1-u_2\right\rangle\\
&=\langle g_1-g_2,u_1-u_2\rangle
\end{aligned}
$$

and then follows from the inequality (2.16) that

$$
\begin{aligned}
&\left\langle \begin{array}{l} {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right)\\ -\ {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right)\end{array}, u_1-u_2\right\rangle\\
&=\int_0^T\left\langle\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right.\\
&\quad-\left.\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x),{}^{\mathbf{H}}\,\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_1-u_2\right)\right\rangle dx\\
&\ge c_p\begin{cases}\displaystyle\int_0^T\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_1-u_2\right)(x)\right|^{p-2}dx, & \text{if }\ p>2,\\ \displaystyle\int_0^T\frac{\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_1-u_2\right)(x)\right|^{2}}{\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|\right)^{2-p}}dx, & \text{if }\ 1<p\le 2.\end{cases} \quad (2.34)
\end{aligned}
$$

So for $p>2$, yields

$$
\begin{aligned}
\int_0^T\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_1-u_2\right)(x)\right|^{p}dx&\le\frac{1}{c_p}\langle g_1-g_2,u_1-u_2\rangle\\
&\le\frac{1}{c_p}\left\|g_1-g_2\right\|_{p'}\left\|u_1-u_2\right\|_p\\
&\le\frac{S_p}{c_p}\left\|g_1-g_2\right\|_{p'}\left\|u_1-u_2\right\|_{\mathbb{H}_p^{\nu,\eta;\psi}}
\end{aligned}
$$

that implies

$$\|u_1-u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq \left(\frac{S_p}{c_p}\right)^{\frac{1}{p-1}} \|g_1-g_2\|_{p'}^{\frac{1}{p-1}}. \tag{2.35}$$

For $1<p<2$, we get

$$\begin{aligned}
&\int_0^T \frac{\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_1-u_2)(x)\right|^2}{\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|\right)^{2-p}}dx\\
\leq&\frac{1}{c_p}\langle g_1-g_2,u_1-u_2\rangle \leq \frac{1}{c_p}\|g_1-g_2\|_{p'}\|u_1-u_2\|_p\\
\leq&\frac{S_p}{c_p}\|g_1-g_2\|_{p'}\|u_1-u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}}.
\end{aligned}$$

By the Holder inequality, we have

$$\begin{aligned}
&\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_1-u_2)(x)\right|^p dx\\
\leq&\left\|\frac{\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_1-u_2)(x)\right|^p}{\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|\right)^{\frac{p(2-p)}{2}}}\right\|_{\frac{2}{p}}\\
&\times\left\|\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|\right)^{\frac{p(2-p)}{2}}\right\|_{\frac{2}{2-p}}\\
\leq&\frac{S_p}{c_p}\|g_1-g_2\|_{p'}\|u_1-u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})}.
\end{aligned}$$

Combining these last two inequalities, we obtain

$$\begin{aligned}
&\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_1-u_2)(x)\right|^p dx\\
\leq&\left(\frac{S_p}{c_p}\|g_1-g_2\|_{p'}\|u_1-u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})}\right)^{\frac{p}{2}}\\
&\left(\int_0^T\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|^p+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2(x)\right|\right)^p dx\right)^{\frac{2-p}{2}}\\
\leq&\left(\frac{S_p}{c_p}\|g_1-g_2\|_{p'}\|u_1-u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})}\right)^{\frac{p}{2}}\\
&\left(\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1\right\|_p+\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_2\right\|_p\right)^{\frac{p(2-p)}{2}}
\end{aligned}$$

and then

$$\frac{\|u_1-u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^{\frac{p}{2}}}{\left(\|u_1\|_{\mathbb{H}_p^{\nu,\eta;\psi}}+\|u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}}\right)^{\frac{p(p-2)}{2}}} \leq \left(\frac{S_p}{c_p}\|g_1-g_2\|_{p'}\right)^{\frac{p}{2}}. \tag{2.36}$$

From Eq. (2.35) and Eq. (2.36), follows that $g_1=g_2$ then $u_1\neq u_2$. $\square$

Remark 2.1. For each $u \in \mathbb{H}_p^{\nu,\eta;\psi}$, we get

$$\begin{aligned}
&\left|\left\langle {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right),\phi\right\rangle\right| \\
&= \left|\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)dx\right| \\
&\leq \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u\right\|_p^{p-1}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi\right\|_p
\end{aligned}$$

and then ${}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right) u \in \mathbb{H}_p^{\nu,\eta;\psi}$. So we have ${}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right) : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{H}_{p'}^{\nu,\eta;\psi}$. Note that for $g \in \mathbb{H}_{p'}^{\nu,\eta;\psi}$,

$$|g(u)| \leq \|g\|_{\mathbb{H}_{p'}^{\nu,\eta;\psi}}\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}}$$

$\forall u \in \mathbb{H}_p^{\nu,\eta;\psi}$ and, we conclude that the Theorem 2.7 remains valid. Therefore, the operator ${}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right) : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{H}_{p'}^{\nu,\eta;\psi}$ is bijective.

Theorem 2.8. *Let $\Omega = [0,T] \subset \mathbb{R}$ a bounded domain. Then:*

(1) ${}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right) : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{H}_{p'}^{\nu,\eta;\psi}$ *is uniformly continuous in bounded sets for* $0 < \nu \leq 1$ *and* $0 \leq \eta \leq 1$.

(2) $\left({}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)\right)^{-1} : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{H}_{p'}^{\nu,\eta;\psi}$ *is continuous for* $0 < \nu \leq 1$ *and* $0 \leq \eta \leq 1$.

(3) *The operator* $\left({}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)\right)^{-1} : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{H}_{p'}^{\nu,\eta;\psi} \hookrightarrow L^q([0,T],\mathbb{R})$ *is compact if* $1 \leq q \leq p^*$, $0 < \nu \leq 1$ *and* $0 \leq \eta \leq 1$, *in*

$$p^* = \begin{cases} \dfrac{p}{1-\nu p}, & \textit{if } p < 1, \\ \infty, & \textit{if } p \geq 1. \end{cases} \tag{2.37}$$

Proof. (1) Consider $\mathcal{C} \subset \mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})$ a bounded set, that is, there exists $\exists M > 0$ such that

$$\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq M,\ \forall u \in \mathcal{C}.$$

We prove that $\Delta_p^{\nu,\eta;\psi}u(x) := {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right)$ is uniformly continuous in $\mathcal{C}$.

Indeed, let $u, v \in \mathcal{C}$. Thus,

$$\begin{aligned}
&\left\|\Delta_p^{\nu,\eta;\psi}u(x) - \Delta_p^{\nu,\eta;\psi}u(x)\right\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \\
&= \sup_{\|\phi\|_{\mathbb{H}_p^{\nu,\eta;\psi}}}\int_0^T \left\langle \Delta_p^{\nu,\eta;\psi}u - \Delta_p^{\nu,\eta;\psi}v, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi\right\rangle dx
\end{aligned}$$

$$\leq \sup_{\|\phi\|_{\mathbb{H}_p^{\nu,\eta;\psi}}=1} \int_0^T \left| \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) - \right.$$

$$\left. \times \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x) \right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x) \right| \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \phi(x) \right| dx.$$

So if $1 < p \leq 2$ then, we obtain

$$\begin{aligned}
& \left\| \Delta_p^{\nu,\eta;\psi} u - \Delta_p^{\nu,\eta;\psi} u \right\|_{\mathbb{H}_{p'}^{\nu,\eta;\psi}} \\
& \leq c_p \sup_{\|\phi\|_{\mathbb{H}_p^{\nu,\eta;\psi}}=1} \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x) \right|^{p-2} \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \phi(x) \right| dx \\
& \leq c_p \sup_{\|\phi\|_{\mathbb{H}_p^{\nu,\eta;\psi}}=1} \left\| \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right|^{p-1} \right\|_{p'} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \phi \right\|_p \\
& = c_p \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^{p-1}. \qquad (2.38)
\end{aligned}$$

Also, for $p > 2$, yields

$$\begin{aligned}
\left\| \Delta_p^{\nu,\eta;\psi} u - \Delta_p^{\nu,\eta;\psi} u \right\|_{\mathbb{H}_{p'}^{\nu,\eta;\psi}} & \leq c_p \sup_{\|\phi\|_{\mathbb{H}_p^{\nu,\eta;\psi}}=1} \left(\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}} + \|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \right)^{p-2} \\
& \quad \|u-v\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \|\phi\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \\
& = c_p \left(\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}} + \|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \right)^{p-2} \|u-v\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \\
& \leq c_p (2M)^{p-2} \|u-v\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^{\frac{1}{p}}. \qquad (2.39)
\end{aligned}$$

Therefore, $\Delta_p^{\nu,\eta;\psi}$ is uniformly continuous in C.

(2) From Eq. (2.35) and Eq. (2.36) follows that $\left(\Delta_p^{\nu,\eta;\psi} u\right)^{-1}$ is continuous.

(3) As $\mathbb{H}_p^{\nu,\eta;\psi} \hookrightarrow L^q([0,T],\mathbb{R})$ is compact and the composition of a continuous operator with a compact operator is compact, follows as an immediate consequence of (2). □

3. Existence and uniqueness

In this section, we investigate the main results of this paper, that is, through the results presented in the preliminary section, we investigate the existence and uniqueness of solutions for the Eq. (1.1).

Proposition 3.1. *If $u_n \to u$ in $\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})$ then*

$$\begin{aligned}
& \lim_{n\to\infty} \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx \\
& = \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx \qquad (3.1)
\end{aligned}$$

$\forall \varphi \in \mathbb{H}_p^{\nu,\eta;\psi}$.

Proof. Using Holder inequality, we have

$$\left|\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx \right.$$
$$\left. -\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx\right|$$
$$\leq \left\|\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right\|_{p'}$$
$$\left\|\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)\right\|_p.$$

Thus, if $1 < p \leq 2$ then it follows from the inequality (2.15),

$$\left\|\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right\|_{p'}$$
$$\leq c_p \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right\|^{p-1}$$

implies that

$$\left|\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx \right.$$
$$\left. -\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx\right|$$
$$\leq c_p \left(\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\ dx\right)^{\frac{1}{p'}} \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi\right\|_p$$
$$= c_p \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_n-u)\right\|_p^{p-1} \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi\right\|_p$$
$$= c_p \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(u_n-u)\right\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^{p-1}. \tag{3.2}$$

Therefore,

$$\lim_{n\to\infty}\left|\begin{array}{l}\displaystyle\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx \\ \displaystyle\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx\end{array}\right| = 0.$$

On the other hand, for $p > 2$, using the inequalities (2.15) and the Holder inequality, follows that

$$\left|\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx \right.$$
$$\left. -\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx\right|$$

$$\leq c_p \left(\int_0^T \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right| + \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right| \right)^{p'(p-2)} \right.$$

$$\left. \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p'} dx \right)^{\frac{1}{p'}} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi \right\|_p$$

$$\leq c_p \left\| \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right| + \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right| \right)^{p'(p-2)} \right\|_{\frac{p-1}{p-2}}^{\frac{1}{p'}}$$

$$\left\| \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{\frac{1}{p'}} \right\|_{p-1} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi \right\|_p$$

$$\leq c_p \left(\int_0^T \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right| + \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right| \right)^{p} dx \right)^{\frac{p-2}{p}}$$

$$\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \right\|_p \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi \right\|_p$$

$$\leq c_p \left(\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n \right\|_p + \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \right\|_p \right)^{p-2}$$

$$\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \right\|_p \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi \right\|_p$$

that is,

$$\left| \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx \right.$$

$$\left. - \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx \right|$$

$$\leq c_p \left(\|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}} + \|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \right)^{p-2} \|u_n - u\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \|\varphi\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \tag{3.3}$$

for all $x \in [0,T]$.

Therefore, we concluded that

$$\lim_{n\to\infty} \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx$$

$$= \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx.$$

□

Consider the following auxiliary fractional problem given by

$$\begin{cases} {}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi} \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right) = \dfrac{f_n(x)}{\left(u_n(x) + \frac{1}{n} \right)^{\nu}}, \\ \mathbf{I}_{0+}^{\eta(\eta-1);\psi} u_n(0) = \mathbf{I}_{T}^{\eta(\eta-1);\psi} u_n(T) = 0, \end{cases} \tag{3.4}$$

on what $f_n(x) = \min\{f, n\}$.

Lemma 3.1. *Let $f \in L^1([0,T])$ and $\nu \geq 0$. So, for each $n \in \mathbb{N}^*$, the* Eq. (3.4) *has only one weak non-negative solution $u_n \in \mathbb{H}_p^{\nu,\eta;\psi} \cap L^\infty([0,T], \mathbb{R})$, that is,*

$$\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) dx$$

$$= \int_0^T \frac{f_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^\nu}\varphi(x)\,dx,\ \ \forall \varphi \in \mathbb{H}_p^{\nu,\eta;\psi}.$$

Proof. The proof of this result will be investigated in two steps.

Step 1: Existence.

For each $w \in L^p([0,T],\mathbb{R})$, we get

$$\left|\frac{f_n(x)}{\left(|w|+\frac{1}{n}\right)^\nu}\right| \leq \frac{n}{\left(\frac{1}{n}\right)^\nu} = n^{\nu+1}$$

and so,

$$\frac{f_n(x)}{\left(|w|+\frac{1}{n}\right)^\nu} \in L^\infty([0,T],\mathbb{R}) \subset L^{p'}([0,T],\mathbb{R}).$$

Hence, by Theorem 2.7, the following problem has only one weak solution $v \in \mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})$

$$\begin{cases} {}^{\mathbf{H}}\mathbf{D}_T^{\nu,\eta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v(x)\right) = \dfrac{f_n(x)}{\left(|w|+\frac{1}{n}\right)^\nu}, \\ \mathbf{I}_{0+}^{\eta(\eta-1);\psi}v(0) = \mathbf{I}_T^{\eta(\eta-1);\psi}v(T). \end{cases} \tag{3.5}$$

So we can define the map $\Gamma : L^p([0,T],\mathbb{R}) \to L^p([0,T],\mathbb{R})$ with $\Gamma(w) = v$. Therefore,

$$\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi(x)dx = \int_0^T \frac{f_n(x)}{\left(|w|+\frac{1}{n}\right)^\nu}\varphi(x)\,dx, \tag{3.6}$$

$\forall \varphi \in \mathbb{H}_p^{\nu,\eta;\psi}$.

So,

$$\begin{aligned} \int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v(x)\right|^p dx &= \int_0^T \frac{f_n(x)}{\left(|w|+\frac{1}{n}\right)^\nu}v(x)dx \\ &\leq \frac{n}{\left(\frac{1}{n}\right)^\nu}\int_0^T |v(x)|dx \\ &= n^{\nu+1}\int_0^T |v(x)|dx. \end{aligned}$$

As immersion $\mathbb{H}_p^{\nu,\eta;\psi} \hookrightarrow L^1([0,T],\mathbb{R})$ is continuous, yields

$$\begin{aligned} \|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}} &\leq n^{\nu+1}\|v\|_{L^1([0,T],\mathbb{R})} \\ &\leq Cn^{\nu+1}\|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \end{aligned} \tag{3.7}$$

that implies

$$\|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq \left(Cn^{\nu+1}\right)^{\frac{1}{p-1}} \tag{3.8}$$

that is,

$$\|\Gamma(w)\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq \left(Cn^{\nu+1}\right)^{\frac{1}{p-1}}. \tag{3.9}$$

As immersion $\mathbb{H}_p^{\nu,\eta;\psi} \hookrightarrow L^p([0,T],\mathbb{R})$ is compact, follows from the previous inequality that $(w_n) \subset L^p(\Omega)$ it is a bounded sequence so $(\Gamma(w_n))$ has a convergent subsequence in $L^p([0,T],\mathbb{R})$. Therefore, $\Gamma : L^p([0,T],\mathbb{R}) \to L^p([0,T],\mathbb{R})$ is a compact operator. Also, if $u = \Gamma(u)\lambda$ for some $0 \leq \lambda \leq 1$, then follows that

$$\begin{aligned} \|u\|_p &= \|\Gamma(u)\lambda\|_p \\ &\leq c_1 \|\Gamma(u)\lambda\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \\ &\leq c_1 |\lambda| \left(Cn^{\nu+1}\right)^{\frac{1}{p-1}}. \end{aligned}$$

Thus, the set $\{u \in L^p(\Omega) : u = \lambda\Gamma(u) \text{ for some } 0 \leq \lambda \leq 1\}$ is bounded. Therefore, by Scharefer's fixed point theorem (see Theorem 2.3), there exists $u_n \in \mathbb{H}_p^{\nu,\eta;\psi}$ such that $u_n = \Gamma(u_n)$. Note that to prove u_n is a weak solution to Eq. (3.4), it is enough to prove that $u_n \geq 0$. Observe from Eq. (3.6), with $v = u_n$ and $w = u_n$

$$\begin{aligned} &\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \phi(x)dx \\ &= \int_0^T \frac{f_n(x)}{\left(|u_n| + \frac{1}{n}\right)^\nu} \phi(x)\, dx, \ \forall \phi \in \mathbb{H}_p^{\nu,\eta;\psi}. \end{aligned} \tag{3.10}$$

So, using Corollary 2.1, we have $u_n \geq 0$. Therefore, u_n is a weak solution to the Eq. (3.4). Also, as $\dfrac{f_n(x)}{\left(|u_n| + \frac{1}{n}\right)^\nu} \in L^\infty(\Omega)$, from the Proposition 2.5, we have $u_n \in \mathbb{H}_p^{\nu,\eta;\psi} \cap L^\infty([0,T],\mathbb{R})$.

Step 2: Uniqueness.

Let $u_n, v_n \in \mathbb{H}_p^{\nu,\eta;\psi}$ weak solutions of the Eq. (3.4). Choosing $\varphi = u_n - v_n$ as a test function we have

$$\begin{aligned} &\int_0^T \left\{ \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v_n(x) \right\} \\ &\times {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (u_n(x) - v_n(x))\, dx \\ &= \int_0^T f_n(x) \left(\frac{1}{\left(|u_n| + \frac{1}{n}\right)^\nu} - \frac{1}{\left(|v_n| + \frac{1}{n}\right)^\nu} \right) (u_n(x) - v_n(x))\, dx \leq 0 \end{aligned}$$

$\forall \phi \in \mathbb{H}_p^{\nu,\eta;\psi}$.

On the other hand, from inequality (2.16), yields

$$\begin{aligned} &\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v_n(x) \right) \\ &\times {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (u_n(x) - v_n(x)) \geq 0. \end{aligned} \tag{3.11}$$

Therefore,

$$\begin{aligned} &\int_0^T \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v_n(x) \right) \\ &\times {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (u_n(x) - v_n(x))\, dx = 0. \end{aligned}$$

Then, using Eq. (3.11) again, we obtain

$$\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)-\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v_n(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v_n(x)\right)$$
$$\times\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_n(x)-v_n(x)\right)=0$$

a.e in $[0,T]$. □

Lemma 3.2. *Let $f\in L^1([0,T],\mathbb{R})$ and $\widetilde{\nu}>0$.*

(1) *A sequence $\{u_n\}$ is increasing in relation to n.*

(2) *If $[0,T]'\subset\subset[0,T]$ then $u_n>0$ in $[0,T]'$ and there exists a positive constant $C_{[0,T]'}$ (independent n) such that for all $n\in\mathbb{N}^*$*

$$u_n\geq C_{[0,T]'}>0,\ \forall x\in[0,T]'. \tag{3.12}$$

Proof. (1) Note that, if $u_n(x)-u_{n+1}(x)\geq 0$ then

$$u_{n+1}(x)+\frac{1}{n+1}\leq u_n(x)+\frac{1}{n}$$

and so,

$$\left(u_{n+1}(x)+\frac{1}{n+1}\right)^{\nu}-\left(u_n(x)+\frac{1}{n}\right)^{\nu}\leq 0.$$

We also have

$$0\leq f_n=\min\{f,n\}\leq\min\{f,n+1\}=f_{n+1}.$$

Therefore, choosing $(u_n-u_{n+1})^+=\max\{u_n-u_{n+1},0\}$ as a test function, yields

$$\int_0^T\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)\right|^{p-2}{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)-\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right|^{p-2}{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right)$$
$${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_n(x)-u_{n+1}(x)\right)^+dx$$
$$\leq\int_0^T f_{n+1}(x)\left(\frac{1}{\left(u_n+\frac{1}{n}\right)^{\nu}}-\frac{1}{\left(u_{n+1}+\frac{1}{n+1}\right)^{\nu}}\right)\left(u_n(x)-u_{n+1}(x)\right)dx\leq 0.$$

On the other hand, from the inequality (2.16), it follows that

$$\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)-\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right)$$
$$\times\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_n(x)-u_{n+1}(x)\right)^+\geq 0. \tag{3.13}$$

Therefore,

$$\int_0^T\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)-\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right)$$
$$\times\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_n(x)-u_{n+1}(x)\right)^+dx=0.$$

Then using Eq. (3.13) again, yields

$$\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_n(x)-\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_{n+1}(x)\right)$$
$$\times\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_n(x)-u_{n+1}(x)\right)^+=0$$

a.e in $[0,T]$, that implies from inequality (2.16), that ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\left(u_n(x)-u_{n+1}(x)\right)^+=0$, a.e in Ω, $\forall x\in[0,T]$, i.e,

$$\left\|\left(u_n(x)-u_{n+1}(x)\right)^+\right\|_{\mathbb{H}_p^{\nu,\eta;\psi}}=0.$$

So, $\left(u_n(x)-u_{n+1}(x)\right)^+=0$ in $\mathbb{H}_p^{\nu,\eta;\psi}$, that implies $u_n(x)-u_{n+1}(x)\leq 0$ a.e in $[0,T]$, i.e, $u_n(x)\leq u_{n+1}(x)$ a.e in $[0,T]$.

(2) Since the sequence u_n is increasing with respect to n, we only need to prove that u_1 satisfies Eq. (3.12).

From Proposition 2.5, there exists c_1 (dependent only on $[0,T],N,p$) such that

$$\|u\|_{L^\infty([0,T],\mathbb{R})}\leq c_1\,\|f_1\|_\infty^{\frac{1}{p-1}}=C. \tag{3.14}$$

Thus, for all $\varphi\in\mathbb{H}_p^{\nu,\eta;\psi}$ with $\varphi\geq 0$, yields

$$\begin{aligned}\int_0^T\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u_1(x)\,{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi\left(x\right)dx&=\int_0^T\frac{f_1}{\left(u_1+1\right)^{\widetilde{\nu}}}\varphi\left(x\right)dx\\&\geq\int_0^T\frac{f_1}{\left(C+1\right)^{\widetilde{\nu}}}\varphi\left(x\right)dx\end{aligned}$$

because $u_1\leq C$, that implies $(u_1+1)^{\widetilde{\nu}}\leq(C+1)^{\widetilde{\nu}}$ and then, as $f_1\geq 0$, we obtain

$$\frac{f_1}{\left(u_1+1\right)^{\widetilde{\nu}}}\geq\frac{f_1}{\left(C+1\right)^{\widetilde{\nu}}}$$

we assume, by hypothesis, $f_1\neq 0$. Thus, $\dfrac{f_1}{\left(u_1+1\right)^{\widetilde{\nu}}}\neq 0$ and then $u_1\neq 0$. As $u_1\in L^\infty\left(\Omega\right)$, we have by [23] that $u_1\in C^{1,\eta}\left([0,T]\right)$, for an appropriate constant. In particular, $u_1\in C^0\left([0,T],\mathbb{R}\right)$. Therefore, by Theorem 2.5, implies that $u_1>0$. As u_1 is continuous, we concluded that the Eq. (3.12) is valid. $\square$

Lemma 3.3. *Let* $u\in\mathbb{H}_p^{\nu,\eta;\psi}$ *not negative, satisfying*

$$\int_0^T\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\right|^{p-2}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}u(x)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\varphi\left(x\right)dx=\int_0^T\frac{f(x)}{u_1(x)^{\widetilde{\nu}}}\varphi\left(x\right)dx$$

$\forall\varphi\left(x\right)\in C_0^\infty\left([0,T],\mathbb{R}\right)$. *So* u *is a weak solution to* Eq. (1.1).

Proof. Let w an arbitrary function in $\mathbb{H}_p^{\nu,\eta;\psi}$ and taking $\{\xi_n\}\subset C_0^\infty\left([0,T],\mathbb{R}\right)$ such that $\xi_n\to|w|$ in $\mathbb{H}_p^{\nu,\eta;\psi}$. Hence, $\xi_n\to|w|$ in $L^p\left([0,T],\mathbb{R}\right)$ and passing to a subsequence, if necessary, we also have the convergence a.e in $[0,T]$. As $u,f\geq 0$, $\xi_n\in C_0^\infty\left([0,T],\mathbb{R}\right)$. Using Fatou's Lemma and Holder's inequality, we have

$$\left|\int_0^T\frac{f(x)w(x)}{u(x)^{\widetilde{\nu}}}dx\right|\leq\lim_{n\to\infty}\inf\int_0^T\frac{f(x)\xi_n(x)}{u(x)^{\widetilde{\nu}}}dx$$

$$\begin{aligned}
&= \lim_{n\to\infty} \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \xi_n(x) dx \\
&\leq \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \right\|_p^{p-1} \lim_{n\to\infty} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \xi_n \right\|_p \\
&= \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \right\|_p^{p-1} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w \right\|_p . \qquad (3.15)
\end{aligned}$$

Now, let $\varphi \in \mathbb{H}_p^{\nu,\eta;\psi}$. So there is a sequence $(\varphi_n) \subset C_0^\infty([0,T],\mathbb{R})$ such that $\varphi_n \to \varphi$ in $\mathbb{H}_p^{\nu,\eta;\psi}$. Therefore, using Eq. (3.15) for $w = \varphi_n - \varphi$, yields

$$\lim_{n\to\infty} \left| \int_0^T \frac{f(x)}{u(x)^{\widetilde{\nu}}} (\varphi_n(x) - \varphi(x)) dx \right| \leq \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u \right\|_p^{p-1} \lim_{n\to\infty} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (\varphi_n - \varphi) \right\| = 0. \qquad (3.16)$$

Therefore,

$$\lim_{n\to\infty} \int_0^T \frac{f(x)}{u(x)^{\widetilde{\nu}}} \varphi_n(x) dx = \int_0^T \frac{f(x)}{u(x)^{\widetilde{\nu}}} \varphi_n(x) dx. \qquad (3.17)$$

On the other hand, we have $\varphi_n \in C_0^\infty([0,T],\mathbb{R})$ and by the Holder inequality, we have

$$\begin{aligned}
&\left| \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (\varphi_n(x) - \varphi(x)) \, dx \right| \\
&\leq \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right\|_p^{p-1} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} (\varphi_n(x) - \varphi(x)) \right\| .
\end{aligned}$$

So, it follows from the hypothesis that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \frac{f(x)}{u(x)^{\widetilde{\nu}}} \varphi_n(x) dx \\
&= \lim_{n\to\infty} \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi_n(x) \, dx \\
&= \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) \, dx. \qquad (3.18)
\end{aligned}$$

Combining the Eq. (3.17) and Eq. (3.18), we obtain

$$\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \varphi(x) \, dx = \int_0^T \frac{f(x)}{u(x)^{\widetilde{\nu}}} \varphi_n(x) dx.$$

Therefore, u is a weak solution to Eq. (1.1). □

Theorem 3.1. *Suppose f is a non-negative function on $L^{\frac{1}{\nu}}([0,T],\mathbb{R})$ and $0 < \widetilde{\nu} \leq 1$. Then, the* Eq. (1.1) *has a unique solution $u \in \mathbb{H}_p^{\nu,\eta;\psi}$.*

Proof. Let's prove this result in two steps.

Step 1: Existence

Note that u_n is a solution of

$$\begin{cases}
{}^{\mathbf{H}}\mathbf{D}_{T}^{\nu,\eta;\psi} \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w(x) \right|^{p-2} \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w(x) \right) = g \text{ in } [0,T], \\
\mathbf{I}_{0+}^{\eta(\eta-1);\psi} w(0) = \mathbf{I}_{T}^{\eta(\eta-1);\psi} w(T) = 0,
\end{cases}$$

wherein $g = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\widetilde{\nu}}}$. Therefore, by proving the Theorem 2.7, we have that u_n minimizes the functional

$$\mathbf{E}_{\nu,\eta,n}(w) = \frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w(x)\right|^p dx - \int_0^T \frac{f_n(x)}{\left(u_n + \frac{1}{n}\right)^{\widetilde{\nu}}} w(x)dx.$$

From Lemma 3.1, $\{u_n\}$ is growing. Hence,

$$0 \leq u_n \leq u(x) := \lim u_n(x) \leq \infty.$$

Furthermore, by doing $\varphi = u_n$ in Eq. (3.10), we get

$$\begin{aligned}
\|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^p &= \int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^p dx \\
&= \int_0^T \frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^{\widetilde{\nu}}} w_n(x)dx \\
&\leq \int_0^T f(x) u_n(x)^{1-\widetilde{\nu}} dx \\
&\leq \|f\|_{\frac{1}{\widetilde{\nu}}} \left\|u_n^{1-\widetilde{\nu}}\right\|_{\frac{1}{1-\widetilde{\nu}}} \\
&\leq \|f\|_{\frac{1}{\widetilde{\nu}}} \left(C \|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^p\right)^{1-\widetilde{\nu}} \\
&= C^{1-\widetilde{\nu}} \|f\|_{\frac{1}{\widetilde{\nu}}} \|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^{1-\widetilde{\nu}}
\end{aligned}$$

that is,

$$\|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^p \leq C^{\frac{1-\widetilde{\nu}}{p-(1-\widetilde{\nu})}} \|f\|_{\frac{1}{\widetilde{\nu}}}^{\frac{1-\widetilde{\nu}}{p-(1-\widetilde{\nu})}}.$$

Hence, $\{u_n\}$ is bounded in $\mathbb{H}_p^{\nu,\eta;\psi}$. So, as $\mathbb{H}_p^{\nu,\eta;\psi}$ is reflexive, without loss of generality taking a convergent subsequence $\{u_k\}$, there exists $\overline{u} \in \mathbb{H}_p^{\nu,\eta;\psi}$ such that

$$u_n \to \overline{u} \text{ weakly in } \mathbb{H}_p^{\nu,\eta;\psi}. \tag{3.19}$$

So, going to a subsequence, if necessary we have $u_n \to u$ in $L^1([0,T],\mathbb{R})$ and then, going again to a subsequence, we have $u_n \to u$ a.e in $[0,T]$. Therefore, $u = \overline{u} \in \mathbb{H}_p^{\nu,\eta;\psi}$ also follows from Eq. (3.19) that

$$\|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq \lim_{n\to\infty} \inf \|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}}. \tag{3.20}$$

Since u_n minimizes the functional $\mathbf{E}_{\nu,\eta,n}$, yields

$$\mathbf{E}_{\nu,\eta,n}(u_n) \leq \mathbf{E}_{\nu,\eta,n}(u)$$

that is,

$$\begin{aligned}
&\frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^p dx - \int_0^T \frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^{\widetilde{\nu}}} u_n(x)dx \\
\leq &\frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^p dx - \int_0^T \frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^{\widetilde{\nu}}} u(x)dx
\end{aligned}$$

that implies

$$\begin{aligned}
&\frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^p dx \\
&\leq \frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^p dx + \int_0^T \frac{(u_n(x)-u(x))}{\left(u_n(x)+\frac{1}{n}\right)^{\widetilde{\nu}}} f_n(x)dx \\
&\leq \frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^p dx
\end{aligned}$$

once $u_n \leq u$ a.e in $[0,T]$. Hence,

$$\lim_{n\to\infty} \inf \|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq \|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}}. \tag{3.21}$$

Therefore, from Eq. (3.20) and Eq. (3.21) follows that

$$\lim_{n\to\infty} \|u_n\|_{\mathbb{H}_p^{\nu,\eta;\psi}} = \|u\|_{\mathbb{H}_p^{\nu,\eta;\psi}}. \tag{3.22}$$

As $\mathbb{H}_p^{\nu,\eta;\psi}$ is uniformly convex, follows from Eq(3.19) and Eq. (3.22) that $u_n \to u$ in $\mathbb{H}_p^{\nu,\eta;\psi}$. Therefore, using the Proposition 3.1, we conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\, dx \\
&= \int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\, dx
\end{aligned} \tag{3.23}$$

$\forall \phi \in \mathbb{H}_p^{\nu,\eta;\psi}$.

On the other hand, for each $\phi \in C_0^\infty([0,T],\mathbb{R})$ we obtain from Eq. (3.12) that

$$0 \leq \left|\frac{f_n\phi}{\left(u+\frac{1}{n}\right)^{\widetilde{\nu}}}\right| \leq \frac{\|\phi\|_{L^\infty}}{(C_{\Omega'})^{\widetilde{\nu}}}$$

on what $\Omega' = \{x \in \Omega, \phi \neq 0\}$. Besides that, $\{f_n\}$ converge for f a.e and how the immersion $\mathbb{H}_p^{\nu,\eta;\psi} \hookrightarrow L^p([0,T],\mathbb{R})$ is compact, possibly going to a subsequence, we have $u_n \to u$ strong in $L^p([0,T],\mathbb{R})$ and a.e in $[0,T]$. Thus, applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^T \frac{f_n(x)\phi(x)}{\left(u(x)+\frac{1}{n}\right)^{\widetilde{\nu}}}dx = \int_0^T \frac{f(x)\phi(x)}{u(x)^{\widetilde{\nu}}}dx,\ \forall\phi \in C_0^\infty([0,T],\mathbb{R}). \tag{3.24}$$

We also have that u not is a weak solution of Eq. (3.4) and so,

$$\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\, dx = \int_0^T \frac{f_n(x)\phi(x)}{\left(u(x)+\frac{1}{n}\right)^{\widetilde{\nu}}}dx \tag{3.25}$$

$\forall \phi \in \mathbb{H}_p^{\nu,\eta;\psi}$.

Therefore, follow from Eq. (3.23), Eq. (3.24) and Eq. (3.25), that

$$\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)\, dx = \int_0^T \frac{f(x)\phi(x)}{u(x)^{\widetilde{\nu}}}dx$$

$\forall \phi \in C_0^\infty([0,T],\mathbb{R})$. So, by Lemma 3.3 u is a weak solution of Eq. (1.1).

Step 2: Uniqueness.

Let $u_1, u_2 \in \mathbb{H}_p^{\nu,\eta;\psi}$ weak solutions of Eq. (1.1). Considering $\varphi = u_1 - u_2$, we have

$$\begin{aligned}&\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x) \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \left(u_1(x) - u_2(x)\right) dx \\ &= \int_0^T \frac{f(x)}{u_1(x)^{\widetilde{\nu}}} \left(u_1(x) - u_2(x)\right) dx\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x) \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \left(u_1(x) - u_2(x)\right) dx \\ &= \int_0^T \frac{f(x)}{u_2(x)^{\widetilde{\nu}}} \left(u_1(x) - u_2(x)\right) dx.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^T \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\right) \\ &\times {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \left(u_1(x) - u_2(x)\right) dx \\ &= \int_0^T f(x) \left(u_1(x) - u_2(x)\right) \left(\frac{1}{u_1(x)^{\widetilde{\nu}}} - \frac{1}{u_2(x)^{\widetilde{\nu}}}\right) dx.\end{aligned}$$

It follows from the inequality (2.16) that the left side of this equality is then negative. We also have that the right side is less than or equal to 0. Therefore,

$$\begin{aligned}&\int_0^T \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_1(x) - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_2(x)\right) \\ &\times {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \left(u_1(x) - u_2(x)\right) dx = 0.\end{aligned}$$

So, it follows from inequality (2.16) that ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \left(u_1(x) - u_2(x)\right) = 0$, a.e. So, $\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} \left(u_1 - u_2\right)\right\|_p = 0$, i.e, $\|u_1 - u_2\|_{\mathbb{H}_p^{\nu,\eta;\psi}} = 0$. Therefore, $u_1 = u_2$. □

4. Minimization of functional energy

Before starting our main purpose of this section, that is, to investigate the minimization of the functional energy related to Eq. (1.1), let's present the following essential remark for the realization of this section.

Remark 4.1. For $1 < q < p^*$, the solutions of

$$\begin{cases} {}^{H}\mathbf{D}_{T}^{\nu,\eta;\psi} \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right|^{p-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x)\right) = f(x,u), \\ \mathbf{I}_{0+}^{\eta(\eta-1);\psi} u(0) = \mathbf{I}_{T}^{\eta(\eta-1);\psi} u(T) = 0 \end{cases} \tag{4.1}$$

are the critical points of a class functional C^1 in $\mathbb{H}_p^{\nu,\eta;\psi}$.

Indeed, by definition $u \in \mathbb{H}_p^{\nu,\eta;\psi}$ is weakly solution of Eq. (4.1) if

$$\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x) dx$$
$$= \int_0^T f(x, u(x)) v(x) dx, \; \forall v \in \mathbb{H}_p^{\nu,\eta;\psi}.$$

In addition, it follows the functional $\varpi(u) := \dfrac{1}{p} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right\|_p^p$ is class C^1 in $\mathbb{H}_p^{\nu,\eta;\psi}$ and

$$\langle \varpi'(u), v \rangle := \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x) dx.$$

On the other hand, like immersion $\mathbb{H}_p^{\nu,\eta;\psi} \hookrightarrow L^q([0,T], \mathbb{R})$ is continuous, follows from Proposition 2.4 that the functional energy associated with the Eq. (4.1), defined by

$$\mathbf{E}_{\nu,\eta}(u) = \frac{1}{p} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right\|_p^p - \int_0^T f(x, u(x)) dx$$

is of class C^1 in $\mathbb{H}_p^{\nu,\eta;\psi}$ and its derivative is given by the expression

$$\left\langle \mathbf{E}'_{\nu,\eta}(u), v \right\rangle$$
$$= \int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \right|^{p-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u(x) \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x) dx - \int_0^T f(x, u(x)) v(x) dx$$

$\forall u, v \in \mathbb{H}_p^{\nu,\eta;\psi}$. Thus, the solutions of the problem Eq. (4.1) are the critical points of $\mathbf{E}_{\nu,\eta}$.

Initially, we prove that the weak solution of Eq. (1.1) minimizes the functional $\mathbf{E}_{\nu,\eta}^{\theta} : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{R}$ define by

$$\mathbf{E}_{\nu,\eta}^{\theta}(v) = \frac{1}{p} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p - \frac{1}{1-\theta} \int_0^T v(x)^{1-\theta} f(x) dx$$

is known as the energy functional associated with Eq. (1.1). Since $0 < \theta < 1$, this functional is not derivable. Later, we prove that the u solution minimizes the quotient

$$\frac{\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p}{\left(\displaystyle\int_0^T |v(x)|^{1-\theta} f(x) dx \right)^{\frac{p}{1-\theta}}}, \; v \in \mathbb{H}_p^{\nu,\eta;\psi} \setminus \{0\}. \tag{4.2}$$

Remark 4.2. Let $0 < \widetilde{\eta} < 1$, $\gamma > 1$ and $a, b \geq 0$ such that $a + b = 1$. The the function $t \to t^{\widetilde{\eta}}$is strictly concave and the function $t \to t^{\gamma}$ is strictly convex, that is,

$$(ax + by)^{\widetilde{\eta}} \geq a x^{\widetilde{\eta}} + b y^{\widetilde{\eta}}, \; \forall x, y \in \mathbb{R},$$
$$(ax + by)^{\gamma} \geq a x^{\gamma} + b y^{\gamma}, \; \forall x, y \in \mathbb{R}$$

and these inequalities are restricted where $x \neq y$.

As $0<1-\theta<1$ and $p>1$, for $w_1, w_2 \in \mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})$, we have

$$\begin{aligned}
&(aw_1+bw_2)^{1-\theta} \geq aw_1^{1-\theta}+bw_2^{1-\theta},\\
&\left(a\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}w_1\right|+b\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}w_2\right|\right)^p \leq a\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}w_1\right|^p+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}w_2\right|^p
\end{aligned}$$

is the first inequality is strict, at the points where $w_1 \neq w_2$.

Lemma 4.1. *Let $0<\theta\leq 1$. The functional $\mathbf{E}_{\nu,\eta}^{\theta}:\mathbb{H}_p^{\nu,\eta;\psi}\to\mathbb{R}$ has a unique minimizer, which is nonnegative.*

Proof. The proof of this result will be presented in 3 steps.

Step 1: Existence

Note that the functional $\mathbf{E}_{\nu,\eta}^{\theta}$ is well defined, that is, by the Holder inequality and $\mathbb{H}_p^{\nu,\eta;\psi}\hookrightarrow L^1[0,T]$ we have

$$\begin{aligned}
\left|\mathbf{E}_{\nu,\eta}^{\theta}(v)\right| &\leq \frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p^p+\|f\|_{\frac{1}{\theta}}\left\|\left(v^+\right)^{1-\theta}\right\|_{\frac{1}{1-\theta}}\\
&=\frac{1}{p}\|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^p+\|f\|_{\frac{1}{\theta}}\left\|\left(v^+\right)^{1-\theta}\right\|_1^{1-\theta}\\
&\leq\frac{1}{p}\|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})}^p+c\|f\|_{\frac{1}{\theta}}\|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^{1-\theta}. \qquad (4.3)
\end{aligned}$$

Also, similarly we have

$$\left|\mathbf{E}_{\nu,\eta}^{\theta}(v)\right| \geq \frac{1}{p}\|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^p-c\|f\|_{\frac{1}{\theta}}\|v\|_{\mathbb{H}_p^{\nu,\eta;\psi}}^{1-\theta}. \qquad (4.4)$$

So, as $0<1-\theta<1<p$, it follows that the function $p(t)=\dfrac{1}{p}(t)^p-c\|f\|_{\frac{1}{\theta}}t^{1-\theta}$ is bounded inferiorly in $\mathbb{R}_+$ and then

$$\mu:=\inf_{v\in\mathbb{H}_p^{\nu,\eta;\psi}}\mathbf{E}_{\nu,\eta}^{\theta}(v)>-\infty. \qquad (4.5)$$

It is interesting to note that $\mu<0$. In fact, fixing $v\in\mathbb{H}_p^{\nu,\eta;\psi}$ such that

$$\int_0^T(v^+(x))^{1-\theta}f(x)dx\geq 0$$

we have

$$\mathbf{E}_{\nu,\eta}^{\theta}(tv)=t^{1-\theta}\left(\frac{t^{p-(1-\theta)}}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p^p-\frac{1}{1-\theta}\int_0^T(v^{+1-\theta})f(x)dx\right)$$

for all $t\geq 0$. How the function $t\to\mathbf{E}_{\nu,\eta}^{\theta}(tv)$ is negative for values of t between its two roots, we can conclude that the functional $\mathbf{E}_{\nu,\eta}^{\theta}$ takes negative values.

Now, let's take a minimizing sequence corresponding to μ, that is, a sequence $\{w_k\}\subset\mathbb{H}_p^{\nu,\eta;\psi}$ such that

$$\lim_{k\to\infty}\mathbf{E}_{\nu,\eta}^{\theta}(w_k)=\mu. \qquad (4.6)$$

So, the sequence $\left(\mathbf{E}^{\theta}_{\nu,\eta}(w_k)\right)_{k\in\mathbb{N}}$ is bounded to $\mathbb{R}$ and then follows from Eq. (4.4) that there exists $M>0$ such that

$$\frac{1}{p}\left\|w_k\right\|^p_{\mathbb{H}_p^{\nu,\eta;\psi}} - c\left\|f\right\|_{\frac{1}{\theta}}\left\|w_k\right\|^{1-\theta}_{\mathbb{H}_p^{\nu,\eta;\psi}} \leq M$$

$\forall k\in\mathbb{N}$. Therefore, $\{w_k\}$ is a bounded sequence in $\mathbb{H}_p^{\nu,\eta;\psi}$ because $\lim\limits_{t\to\infty} p(t)=+\infty$.

As $\mathbb{H}_p^{\nu,\eta;\psi}$ is reflexive, (w_k) has a subsequence that we continue to denote by (w_k), weakly converged on $\mathbb{H}_p^{\nu,\eta;\psi}$. Moving possibly to a subsequence, we have that there $w\in L^1([0,T],\mathbb{R})$ such that $w_k\to w$ in $L^1([0,T],\mathbb{R})$. Note that

$$w_k^+ = \frac{1}{2}\left(|w_k|+w_k\right) \to \frac{1}{2}(|w|+w) = w^+ \text{ in } L^1([0,T],\mathbb{R}) \tag{4.7}$$

when $k\to\infty$. As

$$\left|a^{\widetilde{\eta}} - b^{\widetilde{\eta}}\right| \leq |a-b|^{\widetilde{\eta}},\ \forall a,b\geq 0,\ 0<\widetilde{\eta}\leq 1 \tag{4.8}$$

by Holder inequality, we have

$$\begin{aligned}\left|\int_0^T \left(\left(w_k^+(x)\right)^{1-\theta} - \left(w^+(x)\right)^{1-\theta}\right) f(x)dx\right| &\leq \int_0^T \left|w_k^+(x)-w^+(x)\right|^{1-\theta}|f(x)|dx\\ &\leq \left\|w_k^+ - w^+\right\|_1^{1-\theta}\left\|f\right\|_{\frac{1}{\theta}}.\end{aligned}$$

Therefore, it follows from Eq. (4.7) that

$$\lim_{k\to\infty}\int_0^T (w_k^+(x))^{1-\theta} f(x)dx = \int_0^T (w^+(x))^{1-\theta} f(x)dx. \tag{4.9}$$

Observe that

$$\begin{aligned}\mathbf{E}^{\theta}_{\nu,\eta}(w) &= \frac{1}{p}\left\|w\right\|^p_{\mathbb{H}_p^{\nu,\eta;\psi}} - \frac{1}{1-\theta}\int_0^T (w^+(x))^{1-\theta} f(x)dx\\ &\leq \frac{1}{p}\lim_{k\to\infty}\inf\left\|w\right\|^p_{\mathbb{H}_p^{\nu,\eta;\psi}} - \frac{1}{1-\theta}\int_0^T (w^+(x))^{1-\theta} f(x)dx\\ &= \lim_{k\to\infty}\inf \mathbf{E}^{\theta}_{\nu,\eta}(w_k)\\ &\quad + \frac{1}{1-\theta}\lim_{k\to\infty}\left(\int_0^T (w_k^+(x))^{1-\theta} f(x)dx - \int_0^T (w^+(x))^{1-\theta} f(x)dx\right)\\ &= \mu\end{aligned}$$

that is, $\mathbf{E}^{\theta}_{\nu,\eta}(w)\leq\mu$. On the other hand

$$\mu = \inf_{v\in\mathbb{H}_p^{\nu,\eta;\psi}} \mathbf{E}^{\theta}_{\nu,\eta}(v) \leq \mathbf{E}^{\theta}_{\nu,\eta}(w). \tag{4.10}$$

So, $\mathbf{E}^{\theta}_{\nu,\eta}(w)=\mu$ and then w minimize $\mathbf{E}^{\theta}_{\nu,\eta}$.

Step 2: w is not negative.

We know that

$${}^{\mathbf{H}}\mathbf{D}^{\nu,\eta;\psi}_{0+}(w)^+ = \begin{cases} {}^{\mathbf{H}}\mathbf{D}^{\nu,\eta;\psi}_{0+}(w), & \text{if } w\geq 0,\\ 0, & \inf w<0\end{cases}$$

and

$$
{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(w)^- = \begin{cases} 0, & \text{if} w \geq 0, \\ -\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(w), & \text{if } w < 0. \end{cases}
$$

If ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}(w)^- \neq 0$, then

$$
\begin{aligned}
\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w \right\|_p &= \left(\int_0^T \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w^+ \right\|^p dx \right)^{\frac{1}{p}} + \left(\int_0^T \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w^- \right\|^p dx \right)^{\frac{1}{p}} \\
&> \left(\int_0^T \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w^+ \right\|^p dx \right)^{\frac{1}{p}} \\
&= \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w^+ \right\|_p .
\end{aligned}
$$

Thus,

$$
\mathbf{E}_{\nu,\eta}^{\theta}\left(w^+\right) < \mathbf{E}_{\nu,\eta}^{\theta}\left(w\right) \tag{4.11}
$$

which contradicts the fact that w is a minimizer of $\mathbf{E}_{\nu,\eta}^{\theta}$. Therefore,

$$
w^- = 0. \tag{4.12}
$$

Thereby, $w \geq 0$.

Step 3: Uniqueness

Let w_1 and w_2 minimizers of $\mathbf{E}_{\nu,\eta}^{\theta}$ and suppose that

$$
D = \{x \in [0,T]/w_1(x) \neq w_2(x)\}
$$

has positive measure. As we have already prove that w_1 and w_2 are non-negative. Let $a, b \geq 0$ be such that $a + b = 1$. Therefore, due to the triangular inequality and Remark 4.2, we have

$$
\begin{aligned}
\mu &\leq \mathbf{E}_{\nu,\eta}^{\theta}(aw_1 + bw_2) \\
&= \frac{1}{p} \int_0^T \left| a\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w_1 + b\ {}^{\mathbf{H}}\mathbb{D}_{a+}^{\nu,\eta;\psi} w_2 \right|^p dx - \frac{1}{1-\theta} \int_0^T (aw_1 + w_2)^{1-\theta} f dx \\
&\leq \frac{1}{p} \int_0^T \left(a \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w_1^p \right| + b \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w_2 \right| \right)^p dx - \frac{1}{1-\theta} \int_0^T (aw_1 + bw_2)^{1-\theta} f dx \\
&< \frac{1}{p} \int_0^T \left(a \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w_1^p \right| + b \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} w_2 \right| \right)^p dx - \frac{1}{1-\theta} \int_0^T (aw_1^{1-\theta} + bw_2^{1-\theta} f dx \\
&= a\mathbf{E}_{\nu,\eta}^{\theta}\left(w_1\right) + b\mathbf{E}_{\nu,\eta}^{\theta}\left(w_2\right) \\
&= (a+b)\mu.
\end{aligned}
$$

So we arrive at the absurd $\mu < \mu$. Hence $|D| = 0$ and $w_1 = w_2$ a.e in $[0,T]$. □

Lemma 4.2. *The solution u_n found in* Lemma 3.1 *is the only positive minimizer of the functional*

$$
\mathbf{H}_n(v) = \frac{1}{p} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p - \int_0^T \mathbf{U}_n(v(x)) f_n(x) dx
$$

on what

$$\begin{aligned}\mathbf{U}_n(t) &= \int_0^t \left(s^+ + \frac{1}{n}\right)^{-\theta} ds\\ &= \begin{cases} \dfrac{1}{1-\theta}\left(t+\dfrac{1}{n}\right)^{1-\theta} - \dfrac{1}{1-\theta}\left(\dfrac{1}{n}\right)^{1-\theta}, & \text{if } t \geq 0,\\ \left(\dfrac{1}{n}\right)^{-\theta} t, & \text{if } t < 0.\end{cases}\end{aligned}$$

Proof. First, let's prove that $\mathbf{H}_n$ is of class C^1.

Define
$$g(x,s) = \frac{f_n(x)}{\left(s^+ + \frac{1}{n}\right)^{\theta}}. \tag{4.13}$$

Note that, for all $s \in \mathbb{R}$, the function $x \to g(x,s)$ is Lebesgue measurable, because $f_n \in L^1([0,T],\mathbb{R})$. We also have to stop almost everything $x \in [0,T]$ the function $s \to g(x,s)$ is continuous in $\mathbb{R}$. Therefore, g define in Eq. (4.13) it is a function of Caratheodory.

We have that g satisfies the growth condition

$$|g(x,s)| \leq C|s|^{q-1} + b(x),\ \ x \in [0,T], s \in \mathbb{R} \tag{4.14}$$

where $C \geq 0$ is a constant, $1 < q < p^*$ and $b \in L^{q'}([0,T],\mathbb{R})$, because

$$\begin{aligned}|g(x,s)| &= \frac{f_n(x)}{\left(s^+ + \frac{1}{n}\right)^{\theta}}\\ &\leq \frac{n}{\left(\frac{1}{n}\right)^{\theta}}\\ &= n^{\theta+1}.\end{aligned}$$

Therefore, it follows from Remark 4.1 that the functional $\mathbf{H}_n : \mathbb{H}_p^{\nu,\eta;\psi} \to \mathbb{R}$ class C^1 and

$$\langle \mathbf{H}'_n(v), \phi\rangle = \int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x)\right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v(x)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}\phi(x)dx - \int_0^T g(x,v)\phi(x)dx.$$

Let's prove that $\mathbf{H}_n$ has a minimizer. For this, first, note that

$$\mathbf{U}_n(v(x)) = \frac{1}{1-\theta}\left(v(x)+\frac{1}{n}\right)^{1-\theta} - \frac{1}{1-\theta}\left(\frac{1}{n}\right)^{1-\theta} \leq \frac{1}{1-\theta}\left(v^+(x)+\frac{1}{n}\right)^{1-\theta},$$

for $v(x) \geq 0$ and

$$\mathbf{U}_n(v(x)) = v(x)\left(\frac{1}{n}\right)^{-\theta} \leq 0 \leq \frac{1}{1-\theta}\left(\frac{1}{n}\right)^{1-\theta} = \frac{1}{1-\theta}\left(v^+(x)+\frac{1}{n}\right)^{1-\theta},$$

for $v(x) \leq 0$.

Thus,
$$\mathbf{U}_n(v) \leq \frac{1}{1-\theta}\left(v^+ + \frac{1}{n}\right)^{1-\theta},$$

which, by Holder's inequality and Sobolev's immersion, implies

$$\begin{aligned}\int_0^T \mathbf{U}_n(v(x))f_n(x)dx &\leq \int_0^T \frac{1}{1-\theta}\left(v^+(x)+\frac{1}{n}\right)^{1-\theta} f(x)dx \\ &\leq \frac{1}{1-\theta}\left\|f\right\|_{\frac{1}{\theta}}\left\|v+1\right\|_1^{1-\theta} \\ &\leq \frac{1}{1-\theta}\left\|f\right\|_{\frac{1}{\theta}}\left(c\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p+|\Omega|\right)^{1-\theta} \\ &= c_1\left(c\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p+c_2\right)^{1-\theta}. \end{aligned} \tag{4.15}$$

Thus,

$$\begin{aligned}\mathbf{H}_n(v) &= \frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p^p - \int_0^T \mathbf{U}_n\left(v(x)\right)f_n(x)dx \\ &\geq \frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p^p - c_1\left(c\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p+c_2\right)^{1-\theta}.\end{aligned}$$

Since the function $t\in[0,+\infty)\to\dfrac{t^p}{p}-c_1(t+c_2)^{1-\theta}$ is inferiorly bounded, we have

$$\lambda := \inf_{v\in\mathbb{H}_p^{\nu,\eta;\psi}}\mathbf{H}_n(v) > -\infty.$$

Therefore, exist a sequence $\{w_k\}\subset\mathbb{H}_p^{\nu,\eta;\psi}$ such that

$$\mathbf{H}_n(w_k)\to\lambda \text{ when } k\to\infty. \tag{4.16}$$

It follows from Eq. (4.16) that w_k is bounded to $\mathbb{H}_p^{\nu,\eta;\psi}$, because

$$\lim_{t\to+\infty}\left[\frac{t^p}{p}-c_1(t+c_2)^{1-\theta}\right]=+\infty.$$

As $\mathbb{H}_p^{\nu,\eta;\psi}$ is reflexive, , without loss of generality taking a subsequence $\{w_k\}$ is weakly convergent on $\mathbb{H}_p^{\nu,\eta;\psi}$. Note also, there exists $w\in L^1([0,T],\mathbb{R})$ such that

$$w_k\to w \text{ in } L^1([0,T],\mathbb{R}), \text{ when } k\to\infty. \tag{4.17}$$

Note that

$$\mathbf{U}_n'(t)=\left(t^++\frac{1}{n}\right)^{-\theta}\leq\left(\frac{1}{n}\right)^{-\theta}=n^\theta,\ \forall t\in\mathbb{R}. \tag{4.18}$$

Thus, $\mathbf{U}_n$ is Lipschitzian and using the fact that $|f_n|\leq n$ is Eq. (4.18), then

$$\begin{aligned}\left|\int_0^T \mathbf{U}_n\left(w_k\right)f_n dx-\int_0^T \mathbf{U}_n\left(w\right)f_n dx\right| &\leq n\int_0^T\left|\mathbf{U}_n\left(w_k\right)-\mathbf{U}_n\left(w\right)\right|dx \\ &\leq n^{\theta+1}\int_0^T\left|w_k-w\right|dx \\ &\leq n^{\theta+1}c\left\|w_k-w\right\|_{\mathbb{H}_p^{\nu,\eta;\psi}}.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\int_0^T \mathbf{U}_n\left(w_k\right) f_n dx=\int_0^T \mathbf{U}_n\left(w\right) f_n dx$$

that implies

$$\begin{aligned}\mathbf{H}_n(w) &\leq \frac{1}{p}\lim_{k\to\infty}\inf\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p^p-\int_0^T \mathbf{U}_n(w) f_n dx\\ &=\lim_{k\to\infty}\inf\left(\frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}v\right\|_p^p-\int_0^T \mathbf{U}_n(w_k) f_n dx\right)\\ &=\lim_{k\to\infty}\inf\mathbf{H}_n(w_k)\\ &=\lambda. \end{aligned}\tag{4.19}$$

So w minimizes $\mathbf{H}_n$.

Once $\mathbf{U}_n\left(w\right)\leq \mathbf{U}_n(w^+)$, we can conclude that ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi}w^-=0$, that is, $w\geq 0$. As $\mathbf{H}_n$ is of class C^1 and w is a minimizer, we conclude that w is a critical point of $\mathbf{H}_n$. Therefore, w is a solution of Eq. (3.4) implying that $w=u_n$ and then u_n minimizes $\mathbf{H}_n$.

To finish the proof, let's get the uniqueness of $\mathbf{H}_n$. We have already prove that $\mathbf{H}_n$ is of class C^1, and therefore every minimizer of $\mathbf{H}_n$ is a critical point. We also prove that every critical point of $\mathbf{H}_n$ is a solution of Eq. (3.4). Hence, every minimizer of $\mathbf{H}_n$ is a solution of Eq. (3.4). Furthermore, it follows from Lemma 3.1 that the Eq. (3.4) has only one solution. Therefore, $\mathbf{H}_n$ has a unique minimizer, which is u_n. □

Theorem 4.1. *The u solution found in* Theorem 3.1 *minimize $\mathbf{E}_{\nu,\eta}^{\theta}$ with $0<\theta\leq 1$.*

Proof. Note that

$$\begin{aligned}\lim_{n\to\infty}\mathbf{U}_n(t) &= \begin{cases}\dfrac{1}{1-\theta}t^{1-\theta} & \text{, if } t\geq 0\\ 0 & \text{, if } t<0\end{cases}\\ &=\frac{\left(t^+\right)^{1-\theta}}{1-\theta}.\end{aligned}$$

So,

$$\begin{aligned}\lim_{n\to\infty} f_n(x)\mathbf{U}_n(u_n(x)) &\leq f(x)\left|\mathbf{U}_n(u_n(x))\right|\\ &\leq \frac{f(x)}{1-\theta}\left(u(x)+\frac{1}{n}\right)^{1-\theta}\\ &\leq \frac{f(x)}{1-\theta}\left(u(x)+1\right)^{1-\theta}\end{aligned}$$

and

$$\left\|(u+1)^{1-\theta}f\right\|_1\leq \left(\|u\|_1+|\Omega|\right)^{1-\theta}\|f\|_{\frac{1}{\theta}}<\infty.$$

Therefore, by the Lebesgue dominated convergence theorem, yields

$$\lim_{n\to\infty}\int_0^T f_n(x)\mathbf{U}_n(u_n(x))dx=\frac{1}{1-\theta}\int_0^T f(x)u^{1-\theta}dx. \tag{4.20}$$

Analogously, if $v \in \mathbb{H}_p^{\nu,\eta;\psi}$, we get

$$\begin{aligned} |f_n(x)\mathbf{U}_n(v(x))| &\leq f(x) \left|\mathbf{U}_n(v^+(x))\right| \\ &\leq \frac{f(x)}{1-\theta}\left(v^+(x) + \frac{1}{n}\right) \in L^1([0,T],\mathbb{R}) \end{aligned}$$

and

$$\lim_{n\to\infty} \int_0^T f_n(x)\mathbf{U}_n(v(x))dx = \frac{1}{1-\theta}\int_0^T f(x)\left(v^+(x)\right)^{1-\theta} dx. \tag{4.21}$$

Since $u_n \geq 0$ and u_n is a minimizer of $\mathbf{H}_n$, we have

$$\frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_n\right\|_p^p - \int_0^T f_n(x)\mathbf{U}_n(u_n(x))dx \leq \frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v\right\|_p^p - \int_0^T f_n(x)\mathbf{U}_n(v(x))dx.$$

Hence, how do we know that $u_n \to u$ in $\mathbb{H}_p^{\nu,\eta;\psi}([0,T],\mathbb{R})$ (see Proof Theorem 3.1), from Eq. (4.20) and Eq. (4.21), we have

$$\begin{aligned} &\frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v\right\|_p^p - \frac{1}{1-\theta}\int_0^T f(x)u^{1-\theta}dx \\ &\leq \frac{1}{p}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v\right\|_p^p - \frac{1}{1-\theta}\int_0^T f(x)\left(v^+(x)\right)^{1-\theta} dx \end{aligned}$$

that is

$$\mathbf{E}_{\nu,\eta}^{\theta}(u) \leq \mathbf{E}_{\nu,\eta}^{\theta}(v), \ \forall v \in \mathbb{H}_p^{\nu,\eta;\psi}.$$

Therefore, we conclude the prove. □

Now we will prove that u minimizes the quotient

$$\frac{\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v\right\|_p^p}{\left(\displaystyle\int_0^T |v|^{1-\theta} f dx\right)}, \ v \in \mathbb{H}_p^{\nu,\eta;\psi} \setminus \{0\}.$$

Note that this is equivalent to proving the following theorem that

$$u_\theta := \frac{u}{\left(\displaystyle\int_0^T |u|^{1-\theta} f dx\right)^{\frac{1}{1-\theta}}}$$

and

$$\mathcal{M} := \left\{ v \in \mathbb{H}_p^{\nu,\eta;\psi} : \int_0^T |v|^{1-\theta} f dx = 1 \right\}.$$

Theorem 4.2. *Let $0 < \nu \leq 1$ and $0 \leq \eta \leq 1$. We have*

$$\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_\theta\right\|_p^p = \min\left\{ \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v\right\|_p^p : v \in \mathcal{M} \right\}$$

with $0 < \theta \leq 1$.

Proof. Using the Eq. (1.11) for $\varphi = u$, we obtain

$$\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p = \int_0^T u(x)^{1-\theta} f(x)dx. \tag{4.22}$$

Therefore,

$$\mathbf{E}_{\nu,\eta}^{\theta}(u) = \left(\frac{1}{p} - \frac{1}{1-\theta}\right) \int_0^T u(x)^{1-\theta} f(x)dx.$$

Set $v \in \mathcal{M}$. For every $t > 0$ we have

$$\mathbf{E}_{\nu,\eta}^{\theta}(u) \leq \mathbf{E}_{\nu,\eta}^{\theta}(t\,|v|) = \frac{t^p}{p} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p - \frac{t^{1-\theta}}{1-\theta},$$

and, by Eq. (4.22), this inequality is equivalent to

$$t^{1-\theta} \left(\frac{1}{1-\theta} - \frac{t^{p-(1-\theta)} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p}{p} \right) \leq \left(\frac{1}{1-\theta} - \frac{1}{p}\right) \int_0^T u(x)^{1-\theta} f(x)dx. \tag{4.23}$$

For $t = \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^{-\frac{p}{p-(1-\theta)}}$, we get $t^{1-\theta} \leq \int_0^T u(x)^{1-\theta} f(x)dx$, that is,

$$\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^{-\frac{p(1-\theta)}{p-(1-\theta)}} \leq \int_0^T u(x)^{1-\theta} f(x)dx$$

or yet

$$\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p \geq \left(\int_0^T u(x)^{1-\theta} f(x)dx \right)^{1-\frac{p}{1-\theta}}.$$

Therefore, it follows from Eq. (4.22) that

$$\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} u_\theta \right\|_p^p = \left(\int_0^T u(x)^{1-\theta} f(x)dx \right)^{1-\frac{p}{1-\theta}} \leq \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\nu,\eta;\psi} v \right\|_p^p$$

what ends the prove, since $u_\theta \in M$. □

Acknowledgment: The authors warmly thank the anonymous referee for her/his useful and nice comments on the paper.

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