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ENTIRE FUNCTIONS THAT SHARE A SET WITH THEIR DIFFERENCES*

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Abstract In this paper, we study the uniqueness of entire functions concerning deficient value and exponent of convergence, and have mainly proved the following theorem: Let $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$, where $\omega^n = 1$, $n \geq 1$ is an integer, let k be a positive integer, and let f be a nonconstant entire function such that $\lambda(f) < \rho(f) < \infty$. If $f(z)$ and $\Delta_\eta^k f(z)$ share S IM, where η is a nonzero complex number, then $f(z) = e^{az+b}$, where $a(\neq 0)$ and b are two finite complex numbers. The results obtained in this paper improve some results due to Li ([15]).

Keywords Entire functions, exponent of convergence, unicity, differences.

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1. Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory, see ([9, 13, 25, 26]):

$$T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \cdots.$$

We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$.

Let f be a nonconstant meromorphic function. Define

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r, f)}{\log r},$$

by the order of f.

Let α be a complex number, and let f be a transcendental meromorphic function of order $\rho(f)$. If

$$\overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f-\alpha}\right)}{\log r} < \rho(f)$$

for $\rho(f) > 0$; and $N\left(r, \frac{1}{f-\alpha}\right) = O(\log r)$ for $\rho(f) = 0$, then α is called a Borel exceptional value of f.

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The exponents of convergence of zeros and poles of f are defined by

$$\lambda(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ N(r, \frac{1}{f})}{\log r},$$

and

$$\lambda\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log^+ N(r, f)}{\log r}.$$

We define

$$E(a, f) = \{z : \ f(z) - a = 0\},$$

where each zero of $f(z) - a$ with multiplicity m is repeated m times in $E(a, f)$. Similarly, we define

$$\overline{E}(a, f) = \{z : \ f(z) - a = 0\},$$

where each zero of $f(z) - a$ with multiplicity m is repeated 1 time in $\overline{E}(a, f)$.

Let m be a positive integer, let $a_1, a_2, \cdots, a_m$ be distinct complex numbers, and let $S = \{a_1, a_2, \cdots, a_m\}$. We define

$$E(S, f) = \{z : \ f(z) \in S\}.$$

If $E(S, f) = E(S, g)$, then we say that f and g share the set S CM; if $\overline{E}(S, f) = \overline{E}(S, g)$, then we say that f and g share the set S IM. If $N\left(r, \frac{1}{f-\alpha}\right) + N\left(r, \frac{1}{g-\alpha}\right) - 2N(r, \alpha) \le S(r, f) + S(r, g)$, where $N(r, \alpha)$ is called the counting function of common zeros of both $f(z) - \alpha$ and $g(z) - \alpha$ with multiplicity been counted, then we call that f and g share α CM almost.

Let f and g be two nonconstant meromorphic functions and $\overline{E}(1, f) = \overline{E}(1, g)$. We denote by $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity being not counted. Similarly, we have the notation $\overline{N}_L\left(r, \frac{1}{g-1}\right)$. Especially, if $E(1, f) = E(1, g)$, then $\overline{N}_L\left(r, \frac{1}{f-1}\right) = \overline{N}_L\left(r, \frac{1}{g-1}\right) = 0$.

We denote by $N_{(k}(r, f)$ the counting function for poles of f with multiplicity$\ge k$, and by $\overline{N}_{(k}(r, f)$ the corresponding one for which multiplicity is not counted. Set $N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f) + \cdots + \overline{N}_{(k}(r, f)$.

For a nonzero complex constant $\eta \in \mathbb{C}$, we define the difference operators of f as $\triangle_\eta f(z) = f(z+\eta) - f(z)$ and $\triangle_\eta^k f(z) = \triangle_\eta(\triangle_\eta^{k-1} f(z)), k \in \mathbb{N}, k \ge 2$.

Uniqueness of meromorphic functions is an important topic of value distribution theory. In recent years, many articles have studied this aspect, see ([1, 5, 7, 16, 17, 20–22]).

In this paper, we consider uniqueness of entire functions sharing a set with their difference operators.

In ([15]), Li proved

Theorem 1.1. *Let f be a nonconstant entire function with $\lambda(f) < \rho(f) < \infty$ and $\rho(f) \ne 1$, and let a, b be two distinct entire functions such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If f and $\Delta_\eta f$ share $\{a, b\}$ CM, then $\Delta_\eta f(z) = f(z)$ for all $z \in \mathbb{C}$.*

Theorem 1.2. *Let $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$, where $\omega^n = 1$, $n \ge 2$ is an integer, let η be a nonzero complex number, and let f be a nonconstant entire function such that $\lambda(f) < \rho(f) < \infty$ and $\rho(f) \ne 1$. If $f(z)$ and $\Delta_\eta f(z)$ share S CM. Then $\Delta_\eta f(z) = tf(z)$ for all $z \in \mathbb{C}$, where $t \ne -1$ is a complex number satisfying $t^n = 1$.*

Theorem 1.3. *Let m and n be two distinct positive integers such that $n \geq 2m+3$, and n and $n-m$ are relatively prime, let a and b be two nonzero complex numbers such that $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots, and let f be a nonconstant entire function such that $\lambda(f) < \rho(f) < \infty$ and $\rho(f) \neq 1$. If $f(z)$ and $\Delta_\eta f(z)$ share S CM, where $S = \{\omega : \omega^n + a\omega^{n-m} + b = 0\}$, then $\Delta_\eta f(z) = f(z)$ for all $z \in \mathbb{C}$.*

In ([22]), Qi et al. got rid of the condition $\rho(f) \neq 1$ in Theorem A and proved

Theorem 1.4. *Let f be a nonconstant entire function with $\lambda(f) < \rho(f) < \infty$, and let a, b be two distinct entire functions such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If f and $\Delta_\eta f$ share $\{a, b\}$ CM, then $f(z) = Ae^{\mu z}$, where A, μ are two nonzero constants satisfying $e^{\mu z} = 2$. Furthermore, $\Delta_\eta f(z) = f(z)$ for all $z \in \mathbb{C}$.*

In ([20]), Niu et al. studied the case that f and $\Delta_\eta^n f$ $(n \geq 2)$ share {a, b} CM and proved

Theorem 1.5. *Let f be a nonconstant entire function with $\lambda(f) < \rho(f) < \infty$, and let a, b be two distinct entire functions such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If f and $\Delta_\eta^2 f$ share $\{a, b\}$ CM, then f must take one of the following conclusions:*

(i) $f(z) = Ae^{\mu z}$, where A, μ are two nonzero constants satisfying $e^{\mu z} = 2$. Furthermore, $\Delta_\eta f(z) = f(z)$ for all $z \in \mathbb{C}$;

(ii) $f(z) = H(z)e^{Az}$, for all $z \in \mathbb{C}$, where $H(z)$ is an entire function and $\lambda(f) = \rho(H) < 1$.

Theorem 1.6. *Let f be a nonconstant entire function with $\lambda(f) < \rho(f) < \infty$, and let a, b be two distinct entire functions such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If f and $\Delta_\eta^n f$ $(n \geq 3)$ share $\{a, b\}$ CM, then $f(z) = H(z)e^{Az}$, for all $z \in \mathbb{C}$, where $H(z)$ is an entire function and $\lambda(f) = \rho(H) < 1$.*

Naturally, we pose the following problem.

Problem 1.1. *In Theorem 1.2 and Theorem 1.3, whether $\rho(f) \neq 1$ can be deleted or not, whether f and $\Delta_\eta f(z)$ share S CM can be replaced by f and $\Delta_\eta f(z)$ share S IM or not, and whether $\Delta_\eta f(z)$ can be replaced by $\Delta_\eta^k f(z)$ for any positive integer k or not?*

In this paper, we give a positive answer to Problem 1.1 and have proved

Theorem 1.7. *Let $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$, where $\omega^n = 1$, $n \geq 1$ is an integer, let k be a positive integer, and let f be a nonconstant entire function of finite order such that $\delta(0, f) > \frac{1}{2}$. If $f(z)$ and $\Delta_\eta^k f(z)$ share S CM, where η is a nonzero complex number, then $\Delta_\eta^k f(z) = tf(z)$ for all $z \in \mathbb{C}$, where t is a complex number satisfying $t^n = 1$.*

Theorem 1.8. *Let $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$, where $\omega^n = 1$, $n \geq 1$ is an integer, let k be a positive integer, and let f be a nonconstant entire function of finite order such that $\delta(0, f) > \frac{4}{5}$. If $f(z)$ and $\Delta_\eta^k f(z)$ share S IM, where η is a nonzero complex number, then $\Delta_\eta^k f(z) = tf(z)$ for all $z \in \mathbb{C}$, where t is a complex number satisfying $t^n = 1$.*

Theorem 1.9. *Let $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$, where $\omega^n = 1$, $n \geq 1$ is an integer, let k be a positive integer, and let f be a nonconstant entire function such that $\lambda(f) < \rho(f) < \infty$. If $f(z)$ and $\Delta_\eta^k f(z)$ share S IM, where η is a nonzero complex* *number, then* $f(z) = e^{az+b}$ *for all* $z \in \mathbb{C}$*, where* $a(\neq 0)$ *and* b *are two finite complex numbers.*

Theorem 1.10. *Let* m *and* n *be two distinct positive integers such that* $n \geq 2m+1$*, and* n *and* $n-m$ *are relatively prime, let* k *be a positive integer, let* a *and* b *be two nonzero complex numbers such that* $\omega^n + a\omega^{n-m} + b = 0$ *has no multiple roots, and let* f *be a nonconstant entire function of finite order such that* $\delta(0, f) > \frac{3}{4}$*. If* $f(z)$ *and* $\Delta_\eta^k f(z)$ *share* S *CM, where* $S = \{\omega : \omega^n + a\omega^{n-m} + b = 0\}$*, then* $\Delta_\eta^k f(z) = f(z)$ *for all* $z \in \mathbb{C}$*.*

Theorem 1.11. *Let* m *and* n *be two distinct positive integers such that* $n \geq 5m+1$*, and* n *and* $n-m$ *are relatively prime, let* k *be a positive integer, let* a *and* b *be two nonzero complex numbers such that* $\omega^n + a\omega^{n-m} + b = 0$ *has no multiple roots, and let* f *be a nonconstant entire function of finite order such that* $\delta(0, f) > \frac{19}{20}$*. If* $f(z)$ *and* $\Delta_\eta^k f(z)$ *share* S *IM, where* $S = \{\omega : \omega^n + a\omega^{n-m} + b = 0\}$*, then* $\Delta_\eta^k f(z) = f(z)$ *for all* $z \in \mathbb{C}$*.*

Theorem 1.12. *Let* m *and* n *be two distinct positive integers such that* $n \geq 2m+1$*, and* n *and* $n-m$ *are relatively prime, let* k *be a positive integer, let* a *and* b *be two nonzero complex numbers such that* $\omega^n + a\omega^{n-m} + b = 0$ *has no multiple roots, and let* f *be a nonconstant entire function such that* $\lambda(f) < \rho(f) < \infty$*. If* $f(z)$ *and* $\Delta_\eta^k f(z)$ *share* S *IM, where* $S = \{\omega : \omega^n + a\omega^{n-m} + b = 0\}$*, then* $f(z) = e^{az+b}$ *for all* $z \in \mathbb{C}$*, where* $a(\neq 0)$ *and* b *are two finite complex numbers.*

By Theorem 1.9 and Theorem 1.12, we get the following results.

Proposition 1.1. *Let* $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$*, where* $\omega^n = 1$*,* $n \geq 1$ *is an integer, let* η *be a nonzero finite complex number, and let* f *be a nonconstant entire function such that* $\lambda(f) < \rho(f) < \infty$ *and* $\rho(f) \neq 1$*. Then* $f(z)$ *and* $\Delta_\eta^k f(z)$ *can not share the set* S *IM.*

Proposition 1.2. *Let* m *and* n *be two distinct positive integers such that* $n \geq 2m+1$*, and* n *and* $n-m$ *are relatively prime, let* a *and* b *be two nonzero complex numbers such that* $\omega^n + a\omega^{n-m} + b = 0$ *has no multiple roots, and let* f *be a nonconstant entire function such that* $\lambda(f) < \rho(f) < \infty$ *and* $\rho(f) \neq 1$*. Then* $f(z)$ *and* $\Delta_\eta^k f(z)$ *can not share the set* S *IM, where* $S = \{\omega : \omega^n + a\omega^{n-m} + b = 0\}$*.*

2. Some Lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1 ([3, 10, 12]). *Let* f *be a meromorphic function of finite order, and let* η *be a nonzero finite complex number. Then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = S(r, f).$$

Lemma 2.2 ([3, 10, 12]). *Let* f *be a nonconstant meromorphic function, and let* k *be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.3 ([19]). *Let f be a nonconstant meromorphic function, and $R(f) = \frac{P(f)}{Q(f)}$, where $P(f) = \sum_{k=0}^{p} a_k f^k$ and $Q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in f. If the coefficients $\{a_k(z)\}$, $\{b_j(z)\}$ are small functions of f and $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$, then*

$$T(r, R(f)) = max\{p, q\} \cdot T(r, f) + S(r, f).$$

Lemma 2.4 ([3, 10, 12]). *Let f and g be two nonconstant entire functions. If $E(1, f) = E(1, g)$ and*

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_2\left(r, \frac{1}{f}\right)}{T(r, f)} < \frac{1}{2}, \quad \varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_2\left(r, \frac{1}{g}\right)}{T(r, g)} < \frac{1}{2},$$

where E is a set with finite logarithmic measure, then either $f \equiv g$ or $fg \equiv 1$.

Lemma 2.5 ([6]). *Let f and g be two meromorphic functions. If $\overline{E}(1, f) = \overline{E}(1, g)$, then one of the following cases must occur:*

(i) $$T(r, f) + T(r, g) \le 2\left\{N_2(r, f) + N_2(r, g) + N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right)\right\} + 3\overline{N}_L\left(r, \frac{1}{f-1}\right) + 3\overline{N}_L\left(r, \frac{1}{g-1}\right) + S(r, f) + S(r, g);$$

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where $a(\neq 0)$ and b are two constants.*

Lemma 2.6 ([23]). *Let η be a nonconstant finite complex number, let n be a positive integer, and let f be a transcendental meromorphic function of finite order satisfying $\sum_{a\neq\infty} \delta(a, f) = 1$, $\delta(\infty, f) = 1$. If $\Delta_\eta^n f(z) \not\equiv 0$, then*

(i) $T(r, \Delta_\eta^n f) = T(r, f) + S(r, f)$;

(ii) $\delta(0, \Delta_\eta^n f) = \delta(\infty, \Delta_\eta^n f) = 1$.

Lemma 2.7 ([26]). *Let f be a nonconstant entire function of finite order, if α is a Borel exception value , then $\delta(\alpha, f) = 1$.*

Lemma 2.8 ([7]). *Let n be a positive integer, let f be a transcendental meromorphic function of finite order with two Borel exceptional values 0 and ∞, and let $\eta(\neq 0)$ be a constant such that $\Delta_\eta^n f \not\equiv 0$. If f and $\Delta_\eta^n f$ share $0, \infty$ CM, then $f(z) = e^{az+b}$, where $a(\neq 0)$, b are constants.*

3. Proof of Theorems

3.1. Proof of Theorem 1.7

Set

$$F(z) = f^n(z), \ G(z) = (\Delta_\eta^k f(z))^n. \tag{3.1}$$

Then F and G are two nonconstant entire functions. It follows from $E(S, f) = E(S, \Delta_\eta^k f)$ that $E(1, F) = E(1, G)$.

Since $\delta(0,f) > \frac{1}{2}$, then

$$\overline{\lim_{r\to\infty}} \frac{N\left(r,\frac{1}{f}\right)}{T(r,f)} < \frac{1}{2}. \tag{3.2}$$

So,

$$N\left(r,\frac{1}{f}\right) \le \frac{2c+1}{4} T(r,f) + S(r,f), \tag{3.3}$$

where $c = \overline{\lim\limits_{r\to\infty}} \frac{N\left(r,\frac{1}{f}\right)}{T(r,f)}$.

Thus,

$$N\left(r,\frac{1}{F}\right) \le \frac{2c+1}{4} T(r,F) + S(r,f). \tag{3.4}$$

Clearly,

$$N_2\left(r,\frac{1}{F}\right) \le N\left(r,\frac{1}{F}\right). \tag{3.5}$$

By (3.4) and (3.5), we get

$$\overline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{N_2\left(r,\frac{1}{F}\right)}{T(r,F)} < \frac{1}{2}. \tag{3.6}$$

Obviously,

$$\begin{aligned} m\left(r,\frac{1}{f}\right) \le & m\left(r,\frac{\Delta_\eta^k f}{f}\right) + m\left(r,\frac{1}{\Delta_\eta^k f}\right)\\ \le & m\left(r,\frac{1}{\Delta_\eta^k f}\right) + S(r,f). \end{aligned} \tag{3.7}$$

It follows from (3.7) and Lemma 2.6 that

$$\begin{aligned} \frac{1}{2} < \delta(0,f) \le & \underline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{m\left(r,\frac{1}{f}\right)}{T(r,f)}\\ \le & \underline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{m\left(r,\frac{1}{\Delta_\eta^k f}\right)}{T(r,\Delta_\eta^k f)} \cdot \overline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{T\left(r,\Delta_\eta^k f\right)}{T(r,f)} + \overline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{S(r,f)}{T(r,f)}\\ \le & \underline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{m\left(r,\frac{1}{\Delta_\eta^k f}\right)}{T(r,\Delta_\eta^k f)}. \end{aligned} \tag{3.8}$$

Likewise,

$$\overline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{N_2\left(r,\frac{1}{G}\right)}{T(r,G)} < \frac{1}{2}. \tag{3.9}$$

By Lemma 2.4, we know that either $F \equiv G$ or $FG \equiv 1$.

Suppose that $FG \equiv 1$, then $f^n(\Delta_\eta^k f)^n \equiv 1$. That is

$$\left(\frac{\Delta_\eta^k f}{f}\right)^n \equiv \frac{1}{f^{2n}}. \tag{3.10}$$

Since f is an entire function, it follows $f \neq 0$. By (3.10) and Lemma 2.1, we get

$$\begin{aligned} 2nT(r,f) =& 2nT\left(r,\frac{1}{f}\right)+O(1)=T\left(r,\frac{1}{f^{2n}}\right)+O(1) \\ =& m\left(r,\frac{1}{f^{2n}}\right)+O(1)=m\left(r,(\frac{\Delta_\eta^k f}{f})^n\right)+O(1) \\ =& n\cdot m\left(r,\frac{\Delta_\eta^k f}{f}\right)+O(1)=S(r,f). \end{aligned} \tag{3.11}$$

It gives $T(r,f)=S(r,f)$, a contradiction. So by Lemma 2.4, we know that $F\equiv G$. Hence $\Delta_\eta^k f\equiv tf$, where $t^n=1$. This completes the proof of Theorem 1.7.

3.2. Proof of Theorem 1.8

It follows from $\overline{E}(S,f)=\overline{E}(S,\Delta_\eta^k f)$ and (3.1) that $\overline{E}(1,F)=\overline{E}(1,G)$.

Since $\delta(0,f)>\frac{4}{5}$, then

$$N_2\left(r,\frac{1}{F}\right)\leq N\left(r,\frac{1}{F}\right)\leq \frac{5c+1}{10}T(r,F)+S(r,f), \tag{3.12}$$

where $c=\overline{\lim\limits_{r\to\infty}}\frac{N\left(r,\frac{1}{F}\right)}{T(r,F)}<\frac{1}{5}$.

By Lemma 2.2, we have

$$\begin{aligned} \overline{N}_L\left(r,\frac{1}{F-1}\right) \leq& \overline{N}\left(r,\frac{1}{F'}\right)\leq N\left(r,\frac{1}{F'}\right)-\left[N\left(r,\frac{1}{F'}\right)-\overline{N}\left(r,\frac{1}{F'}\right)\right] \\ \leq& N\left(r,\frac{1}{F}\right)+\overline{N}(r,F)-\left[N\left(r,\frac{1}{F'}\right)-\overline{N}\left(r,\frac{1}{F'}\right)\right] \\ \leq& N\left(r,\frac{1}{F}\right)-\left[N\left(r,\frac{1}{F'}\right)-\overline{N}\left(r,\frac{1}{F'}\right)\right]+S(r,F) \\ \leq& N_2\left(r,\frac{1}{F}\right)+S(r,F). \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we get

$$2N_2\left(r,\frac{1}{F}\right)+3\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq \frac{5c+1}{2}T(r,F)+S(r,F). \tag{3.14}$$

Similarly,

$$2N_2\left(r,\frac{1}{G}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\leq \frac{5c+1}{2}T(r,G)+S(r,G). \tag{3.15}$$

Suppose that

$$\begin{aligned} T(r,F)+T(r,G)\leq& 2N_2\left(r,\frac{1}{F}\right)+3\overline{N}_L\left(r,\frac{1}{F-1}\right)+S(r,F) \\ &+2N_2\left(r,\frac{1}{G}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,G). \end{aligned} \tag{3.16}$$

It follows from (3.14)-(3.16) that

$$\left(1-\frac{5c+1}{2}\right)\{T(r,F)+T(r,G)\}\le S(r,F)+S(r,G).$$

It follows from $c<\frac{1}{5}$ that $T(r,F)+T(r,G)\le S(r,F)+S(r,G)$, a contradiction. Thus, by Lemma 2.5, we obtain

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)}, \tag{3.17}$$

where $A(\neq 0)$ and B are two constants.

Clearly,

$$T(r,F)=T(r,G)+O(1). \tag{3.18}$$

Next we consider three cases:

Case 1. $B\neq 0,-1$. In the following, we consider two subcases.

Case 1.1. $A-B-1\neq 0$. From (3.17), we have

$$\overline{N}\left(r,\frac{1}{G+\frac{A-B-1}{B+1}}\right)=\overline{N}\left(r,\frac{1}{F}\right).$$

By the second fundamental theorem and (3.18), we get

$$\begin{aligned} T(r,G)\le&\overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}\left(r,\frac{1}{G+\frac{A-B-1}{B+1}}\right)+S(r,G)\\ \le&\overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}\left(r,\frac{1}{F}\right)+S(r,G)\\ <&\frac{2}{5}T(r,G)+S(r,G). \end{aligned}$$

It follows $T(r,G)=S(r,G)$, a contradiction.

Case 1.2. $A-B-1=0$. Then by (3.17), we have $\overline{N}\left(r,\frac{1}{G+\frac{1}{B}}\right)=\overline{N}(r,F)$.

By the second fundamental theorem, we get

$$\begin{aligned} T(r,G)\le&\overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}\left(r,\frac{1}{G+\frac{1}{B}}\right)+S(r,G)\\ \le&\overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,F)+S(r,G)\\ <&\frac{1}{5}T(r,G)+S(r,G). \end{aligned}$$

So we get $T(r,G)=S(r,G)$, a contradiction.

Case 2. $B=-1$. Then (3.17) becomes

$$F=\frac{A}{(A+1)-G}. \tag{3.19}$$

Next we consider two subcases.

Case 2.1. $A+1\neq 0$. By (3.19), we have $\overline{N}\left(r,\frac{1}{G-(A+1)}\right)=\overline{N}(r,F)$. Similarly, we deduce a contradiction as in Case 1.

Case 2.2. $A+1=0$. By (3.19), we have $FG \equiv 1$. Next, using the same argument as used in the proof of Theorem 1.7, we get a contradiction.

Case 3. $B=0$. Then (3.17) gives

$$F=\frac{G+(A-1)}{A}. \tag{3.20}$$

Now we consider two subcases.

Case 3.1. $A-1\neq 0$. By (3.20), we have $\overline{N}\left(r,\frac{1}{G+(A-1)}\right)=\overline{N}(r,F)$. Similarly, we deduce a contradiction as in Case 1.

Case 3.2. $A-1=0$. Then by (3.20), we get $F\equiv G$.

Thus, we have $\Delta_\eta^k f\equiv tf$, where $t^n=1$. This completes the proof of Theorem 1.8.

3.3. Proof of Theorem 1.9

Since f be a nonconstant entire function such that $\lambda(f)<\rho(f)<\infty$, then by Lemma 2.7, we have

$$\delta(0,f)=\delta(\infty,f)=1.$$

By Theorem 1.8, we get $\Delta_\eta^k f\equiv tf$, where $t^n=1$. Hence f and $\Delta_\eta^k f$ share 0, ∞ CM. So, it follows from Lemma 2.8 that $f(z)=e^{az+b}$ for all $z\in\mathbb{C}$, where $a(\neq 0)$ and b are two complex numbers. This completes the proof of Theorem 1.9.

3.4. Proof of Theorem 1.10

Set

$$F(z)=f^n(z)+af^{n-m}(z), \tag{3.21}$$
$$G(z)=(\Delta_\eta^k f(z))^n+a(\Delta_\eta^k f(z))^{n-m}. \tag{3.22}$$

Then F and G are two nonconstant entire functions. Since $E(S,f)=E(S,\Delta_\eta^k f)$, then $E(-b,F)=E(-b,G)$.

It follows from Lemma 2.3 that

$$T(r,F)=nT(r,f)+S(r,f), \tag{3.23}$$
$$T(r,G)=nT(r,\Delta_\eta^k f)+S(r,f). \tag{3.24}$$

Since $\delta(0,f)>\frac{3}{4}$, then

$$\varlimsup_{r\to\infty}\frac{N\left(r,\frac{1}{f}\right)}{T(r,f)}<\frac{1}{4}. \tag{3.25}$$

Clearly,

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right)\leq&2N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^m+a}\right)\\<&\left(\frac{1}{2}+m\right)T(r,f)+S(r,f).\end{aligned} \tag{3.26}$$

By (3.5), (3.23) and $n \geq 2m+1$, we get

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_2\left(r, \frac{1}{F}\right)}{T(r,F)} < \frac{1}{2}. \tag{3.27}$$

By the same argument as used in the proof of Theorem 1.7, we have

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{N\left(r, \frac{1}{\Delta_\eta^k f}\right)}{T(r,\Delta_\eta^k f)} < \frac{1}{4}.$$

Thus, we get

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_2\left(r, \frac{1}{G}\right)}{T(r,G)} < \frac{1}{2}. \tag{3.28}$$

By Lemma 2.4, (3.27), (3.28) and $E(-b,F) = E(-b,G)$, we get either $F \equiv G$ or $FG \equiv b^2$.

Suppose that $FG \equiv b^2$, that is

$$f^{n-m}(f^m + a) \cdot [(\Delta_\eta^k f)^n + a(\Delta_\eta^k f)^{n-m}] \equiv b^2. \tag{3.29}$$

It follows from (3.29) and that f is an entire function that $f \neq 0$, $f^m + a \neq 0$, $f \neq \infty$.

By the second fundamental theorem, we have

$$\begin{aligned} &mT(r,f) = T(r,f^m) \\ \leq &N(r,f^m) + N\left(r, \frac{1}{f^m}\right) + N\left(r, \frac{1}{f^m+a}\right) + S(r,f) \leq S(r,f). \end{aligned} \tag{3.30}$$

It gives $T(r,f) = S(r,f)$, a contradiction.

So by Lemma 2.4, we know that $F \equiv G$. That is

$$f^n + af^{n-m} = (\Delta_\eta^k f)^n + a(\Delta_\eta^k f)^{n-m}. \tag{3.31}$$

Set

$$h = \frac{\Delta_\eta^k f}{f}. \tag{3.32}$$

Thus, we get

$$(h^n - 1)f^m = -a(h^{n-m} - 1). \tag{3.33}$$

Since n, $n-m$ are relatively prime, then $h = 1$ is the only common root of $h^n = 1$ and $h^{n-m} = 1$. Let $\omega_1, \omega_2, \cdots, \omega_{n-1}$ be $n-1$ distinct simple roots of $h^n = 1$ such that $\omega_j \neq 1$ and $\omega_j^{n-m} \neq 1$ for $1 \leq j \leq n-1$. Next, we consider the following two cases.

Case 1. h is a constant. If $h^n \neq 1$, then by (3.33), we get f is a constant, a contradiction. Thus, $h^n = 1$. Then by (3.33) we know that $h^{n-m} = 1$. Obviously, we obtain $h = 1$. Hence, $\Delta_\eta^k f \equiv f$.

Case 2. h is not a constant. Then by (3.33), we get

$$f^m = -a\frac{h^{n-m-1} + h^{n-m-2} + \cdots + h + 1}{(h-\omega_1)(h-\omega_2)\cdots(h-\omega_{n-1})}. \tag{3.34}$$

It follows from (3.34) and the condition that f is a nonconstant entire function that $h \neq \omega_1, \omega_2, \cdots, \omega_{n-1}$.

In the following, we consider two subcases.

Case 2.1. $m \geq 2$. Since $n-1 \geq 2m \geq 4$, by Picard's Theorem, we get h is a constant, a contradiction.

Case 2.2. $m = 1$. Then $n \geq 2m+1 \geq 3$. Without loss of generality, we consider the case $n = 3$.

By (3.25) and (3.34), we get

$$\overline{N}(r,h) + \overline{N}\left(r, \frac{1}{h+1}\right) \leq N\left(r, \frac{1}{f}\right) < \frac{1}{4}T(r,f). \tag{3.35}$$

By the second fundamental theorem, (3.34), (3.35) and Lemma 2.3, we obtain

$$\begin{aligned} T(r,f) =& 2T(r,h) + S(r,h) \\ \leq& \overline{N}(r,h) + \overline{N}\left(r, \frac{1}{h+1}\right) + \overline{N}\left(r, \frac{1}{h-\omega_1}\right) + \overline{N}\left(r, \frac{1}{h-\omega_2}\right) + S(r,h) \\ \leq& N\left(r, \frac{1}{f}\right) + S(r,f) \\ <& \frac{1}{4}T(r,f) + S(r,f). \end{aligned}$$

It follows $T(r,f) = S(r,f)$, a contradiction.

By the above discuss, we deduce that $h = 1$, that is $\Delta_\eta^k f \equiv f$. This completes the proof of Theorem 1.10.

3.5. Proof of Theorem 1.11

It follows from $\overline{E}(S,f) = \overline{E}(S, \Delta_\eta^k f)$, (3.21) and (3.22) that $\overline{E}(-b,F) = \overline{E}(-b,G)$.

Since $\delta(0,f) > \frac{19}{20}$, then we have

$$\overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{F}\right)}{T(r,F)} < \frac{1}{20}. \tag{3.36}$$

So by (3.26), we get

$$N_2\left(r, \frac{1}{F}\right) < \left(\frac{1}{10} + m\right) T(r,f) + S(r,f). \tag{3.37}$$

By (3.13) and (3.37), we get

$$2N_2\left(r, \frac{1}{F}\right) + 3\overline{N}_L\left(r, \frac{1}{F-1}\right) < \left(\frac{1}{2} + 5m\right) T(r,f) + S(r,f). \tag{3.38}$$

Similarly, we have

$$2N_2\left(r, \frac{1}{G}\right) + 3\overline{N}_L\left(r, \frac{1}{G-1}\right) < \left(\frac{1}{2} + 5m\right) T(r, \Delta_\eta^k f) + S(r,f). \tag{3.39}$$

Suppose that

$$T(r,F) + T(r,G) \leq 2N_2\left(r, \frac{1}{F}\right) + 3\overline{N}_L\left(r, \frac{1}{F-1}\right) + S(r,F)$$

$$+ 2N_2\left(r, \frac{1}{G}\right) + 3\overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r, G). \tag{3.40}$$

It follows from (3.38)-(3.40) and $n \geq 5m+1$ that $T(r, f) \leq S(r, f)$, a contradiction.

Next, using the same argument as used in the proof of Theorem 1.8 and Theorem 1.10, we deduce that $\Delta_\eta^k f \equiv f$. This completes the proof of Theorem 1.11.

3.6. Proof of Theorem 1.12

Since f is a nonconstant entire function such that $\lambda(f) < \rho(f) < \infty$, then by Lemma 2.7, we have

$$\delta(0, f) = \delta(\infty, f) = 1.$$

By the second fundamental theorem, for any nonzero constant a, we have

$$\begin{aligned}
T(r, f) \leq & \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-a}\right) + S(r, f) \\
\leq & \overline{N}\left(r, \frac{1}{f-a}\right) + S(r, f) \\
\leq & N_{1)}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + S(r, f) \\
\leq & N_{1)}\left(r, \frac{1}{f-a}\right) + \frac{1}{2}N_{(2}\left(r, \frac{1}{f-a}\right) + S(r, f) \\
\leq & \frac{1}{2}N_{1)}\left(r, \frac{1}{f-a}\right) + \frac{1}{2}N\left(r, \frac{1}{f-a}\right) + S(r, f).
\end{aligned}$$

It follows that

$$T(r, f) \leq N_{1)}\left(r, \frac{1}{f-a}\right) + S(r, f).$$

Thus, we get

$$N_{(2}\left(r, \frac{1}{f-a}\right) = S(r, f).$$

So we deduce that f and $\Delta_\eta^k f$ share the set S CM almost. Next using the same argument as used in the proof of Theorem 1.10 we get $\Delta_\eta^k f \equiv f$. Hence, f and $\Delta_\eta^k f$ share 0, ∞ CM. It follows from Lemma 2.8 that $f(z) = e^{az+b}$ for all $z \in \mathbb{C}$, where $a(\neq 0)$ and b are two complex numbers. This completes the proof of Theorem 1.12.

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