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SOLVABILITY OF STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS FOR A CLASS OF FRACTIONAL ADVECTION-DISPERSION EQUATIONS THROUGH VARIATIONAL APPROACH*

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Abstract In this paper, we probe into the solvability of Sturm-Liouville problem for fractional advection-dispersion equations without traditional Ambrosetti-Rabinowitz conditions. Some existence results of infinitely many small negative energy and large energy solutions are obtained by employing variant fountain theorems. The nonlinearity f and l_i $(i = 1, 2, \dots, m)$ are considered under certain appropriate assumptions which are distinct from those assumed in previous articles. In addition, the main result is confirmed by an example which is provided.

Keywords Sturm-Liouville boundary conditions, variant fountain theorems, fractional advection-dispersion equations, variational method.

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1. Introduction

Consider the following Sturm-Liouville problem of fractional advection-dispersion equation (FADE for short) given by

$$\begin{cases} -\dfrac{d}{dx}\left[\dfrac{1}{2}\,{}_0D_x^{-\zeta}(v'(x)) + \dfrac{1}{2}\,{}_xD_T^{-\zeta}(v'(x))\right] + K(x)v(x) \\ = f(x, v(x)) + \displaystyle\sum_{i=1}^{m} l_i(x, v(x)),\ \ a.e.\ x \in [0, T], \\ \alpha v(0) - \beta\left[\dfrac{1}{2}\,{}_0D_x^{-\zeta}(v'(0)) + \dfrac{1}{2}\,{}_xD_T^{-\zeta}(v'(0))\right] = 0, \\ \gamma v(T) + \sigma\left[\dfrac{1}{2}\,{}_0D_x^{-\zeta}(v'(T)) + \dfrac{1}{2}\,{}_xD_T^{-\zeta}(v'(T))\right] = 0, \end{cases} \tag{1.1}$$

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where ${}_0D_x^{-\zeta}$ and ${}_xD_T^{-\zeta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \zeta < 1$ individually, $K(x) \in L^\infty([0,T], R^+)$ with $K^0 = ess\sup_{[0,T]} K(x)$, $K_0 = ess\inf_{[0,T]} K(x) > 0$, $\alpha, \gamma > 0, \beta, \sigma \ge 0$, and $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and $l_i \in C([0,T] \times \mathbb{R}, \mathbb{R})$ for every $i = 1, 2, \ldots, m$.

Motivated by increasing interest in the current literature concerning fractional advection-dispersion equations which can be used to simulate physical phenomena, such as exhibiting anomalous diffusion on certain conditions, and depicts nonsymmetric or symmetric transition and solute transportation and so on. In [4], Ervin and Roop studied the following form FADE

$$-\frac{d}{dx}(p\ {}_0D_x^{-\beta} + (1-p)_xD_T^{-\beta})v'(x) + b(x)v'(x) + c(x)v(x) = \nabla F(x, v(x)) \quad (1.2)$$

for $a.e.$ $x \in [0,T]$, where ${}_0D_x^{-\beta}$ and ${}_xD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integral operators individually, $0 \le \beta < 1$, $p \in [0,1]$ is a constant depicting the deflection of conveyance transversion, b, c, F meets certain proper conditions. If taking $p = \frac{1}{2}$ in (1.2), then the FADE (1.2) delineates symmetric mutations. Sun and Zhang in [23] probed into the FADE (1.2) with $b(x) = c(x) = 0, T = 1$, and the boundary conditions $v(0) = v(1) = 0$. For more physical background information and applications about FADE, see [1, 5, 7, 10, 12–15, 18, 19, 25, 28–30] and so on. Critical point theory and variational approach have become an valid tool to solve this type of FADE problem. In [8], Jiao and Zhou gained the existence of nontrivial solutions for following FADE by employing the usual Ambrosetti-Rabinowitz (A-R) condition ($i.e.$ there exist $\widetilde{\mu} > 2$, and $\widetilde{r} > 0$ such that for any $x \in [0,T], \xi \in \mathbb{R}, |\xi| \ge \widetilde{r}, 0 < \widetilde{\mu} F(x,\xi) \le \xi f(x,\xi)$),

$$\begin{cases} \dfrac{d}{dx}\left(\dfrac{1}{2}\ {}_0D_x^{-\beta}(v'(x)) + \dfrac{1}{2}\ {}_xD_T^{-\beta}(v'(x))\right) + \nabla F(x, v(x)) = 0, \ a.e.\ x \in [0,T], \\ v(0) = v(T) = 0, \end{cases} \quad (1.3)$$

where ${}_0D_x^{-\beta}$ and ${}_xD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integral operators individually, $0 \le \beta < 1$, and $\nabla F(x,y)$ is the gradient of F at y. Since that time, a lot of literatures concerning FADE have been written by employing the (A-R) condition. Its importance is as a result of the truth that it guarantees the boundedness of the Palais-Smale sequences for the energy functional related to the problem under consideration. The (A-R) condition is a superlinear growth assumption concerning the nonlinearity and proved that it can be expressed as $F(x,\xi) \ge \flat_1|\xi|^{\widetilde{\mu}} - \flat_2, \forall (x,\xi) \in [0,T] \times \mathbb{R}$, for some $\flat_1, \flat_2 > 0$. We notice that some functional such as $f(x,\xi) = \xi \log(1 + |\xi|)$ is superlinear at infinity, but does not satisfy the (A-R) condition. So, some new assumptions have been used by scholars to replace the (A-R) condition and overcome this restriction.

Recently, some FADE problems without the Ambrosetti-Rabinowitz condition have been researched by many scholars. For instance, in [6], the authors introduced the following assumption on f, there exists $\widehat{\sigma} \ge 1$ such that $\widehat{\sigma}\mathcal{F}(s,\xi) \ge \mathcal{F}(s,\xi)$ for any $(s,\xi) \in [0,T] \times \mathbb{R}^N, s \in [0,1]$, where $\mathcal{F}(s,\xi) = (\nabla F(s,\xi), \xi) - 2F(s,\xi)$. The authors in [3] obtained multiplicity results exist in the asymptotically quadratic case and subquadratic case to the above boundary value problem. In [2], the author obtained the existence of infinitely many small or high energy solutions to the above boundary value problem by applying the variant fountain theorems.

Very little research has been done on the Sturm-Liouville problem of FADE by taking advantage of the variant fountain theorems. In [24], the authors investigated the existence of solutions for the Sturm-Liouville problem of discontinuous fractional-order differential equation by critical-point theory

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{2}\,{}_0D_x^{-\beta}(v'(x))+\dfrac{1}{2}\,{}_xD_T^{-\beta}(v'(x))\right)=\lambda f(v(x)),\ a.e.\ x\in[0,T], \\ av(0)-b\left(\dfrac{1}{2}\,{}_0D_x^{-\beta}(v'(0))+\dfrac{1}{2}\,{}_xD_T^{-\beta}(v'(0))\right)=0, \\ cv(T)+d\left(\dfrac{1}{2}\,{}_0D_x^{-\beta}(v'(T))+\dfrac{1}{2}\,{}_xD_T^{-\beta}(v'(T))\right)=0, \end{cases} \tag{1.4}$$

where ${}_0D_x^{-\beta}$ and ${}_xD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0\le\beta<1$ individually, λ is a positive parameter, $a,c>0,b,d\ge 0$, and $f:\mathbb{R}\to\mathbb{R}$ is an almost everywhere continuous function. In [21], the authors studied the existence of weak solutions for the following damped-like fractional boundary value problem from the point of view of variational approach

$$\begin{cases} \quad\dfrac{d}{dx}\left(\dfrac{1}{2}\,{}_0D_x^{-\beta}(v'(x))+\dfrac{1}{2}\,{}_xD_T^{-\beta}(v'(x))\right) \\ \quad+p(x)\left(\dfrac{1}{2}\,{}_0D_x^{-\beta}(v'(x))+\dfrac{1}{2}\,{}_xD_T^{-\beta}(v'(x))\right)+q(x)v(x) \\ =f(x,v(x))+\displaystyle\sum_{j=1}^{n}g_j(x,v(x)),\quad a.e.\ x\in[0,T], \\ v(0)=v(T)=0, \end{cases} \tag{1.5}$$

where ${}_0D_x^{-\beta}$ and ${}_xD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integral operators individually, $0\le\beta<1$, and $p\in C([0,T]),q\in L^\infty([0,T]),\ f\in C([0,T]\times\mathbb{R},\mathbb{R})$ and $g_j\in C([0,T]\times\mathbb{R},\mathbb{R})$ for every $j=1,2,\dots,n$.

Here, we are interested in the existence of infinitely many small or high energy solutions for Sturm-Liouville problem of FADE. Our analysis will be on the account of variant fountain theorem which has been adopted for Dirichlet boundary problem in some literatures. Difficulties such as how to construct suitable function and how to prove the boundedness of the required sequences in the theorem need to overcome due to the weaker conditions and Sturm-Liouville boundary conditions taken into consideration. The innovation is twofold: For one thing, we study the Sturm-Liouville problem for fractional advection-dispersion equations under no Ambrosetti-Rabinowitz condition, it can free ourself from the (Palais-Smale)-type assumptions. The conditions we give are different from those assumed in Reference [2,3,6,21,24], although there is no Ambrosetti-Rabinowitz condition in reference [2,3,6,17,21,24,27], either. This is the essential difference between this paper and the previous paper. For another thing, boundary conditions in this paper are more general cases, which cover the Dirichlet boundary condition as special cases.

The framework of this article is as follows. Some fundamental preliminaries and lemmas are stated in the next section. The fundamental consequences of this article are given in the last section, as well as an application to FADE (1.1).

2. Preliminaries and lemmas

Definition 2.1 ([9,22]). Let y be a function defined on $[a,b]$. Then the left and right Riemann-Liouville fractional derivatives of order $0 \le \delta < 1$ for function y are represented by

$$ {}_aD_t^{\delta} y(t) = \frac{d}{dt}\, {}_aD_t^{\delta-1} y(t) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dt} \int_a^t (t-s)^{-\delta} y(s) ds,\ t \in [a,b], $$

and

$$ {}_tD_b^{\delta} y(t) = -\frac{d}{dt}\, {}_tD_b^{\delta-1} y(t) = -\frac{1}{\Gamma(1-\delta)} \frac{d}{dt} \int_t^b (s-t)^{-\delta} y(s) ds,\ t \in [a,b]. $$

Definition 2.2 ([9,22]). If $\delta \in (0,1)$ and $y \in AC([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order δ for function y denoted by ${}_a^cD_t^{\delta} y(t)$ and ${}_t^cD_b^{\delta} y(t)$, individually, exist a.e. on $[a,b]$. ${}_a^cD_t^{\delta} y(t)$ and ${}_t^cD_b^{\delta} y(t)$ are represented as

$$ {}_a^cD_t^{\delta} y(t) = {}_aD_t^{\delta-1} y'(t) = \frac{1}{\Gamma(1-\delta)} \int_a^t (t-s)^{-\delta} y'(s) ds,\ t \in [a,b], $$

and

$$ {}_t^cD_b^{\delta} y(t) = -{}_tD_b^{\delta-1} y'(t) = -\frac{1}{\Gamma(1-\delta)} \int_t^b (s-t)^{-\delta} y'(s) ds,\ t \in [a,b]. $$

If $\delta = 1$, then ${}_a^cD_t^{1} y(t) = y'(t)$ and ${}_t^cD_b^{1} y(t) = -y'(t)$, for every $t \in [a,b]$. Especially, ${}_a^cD_t^{0} y(t) = {}_t^cD_b^{0} y(t) = y(t)$ for every $t \in [a,b]$.

Proposition 2.1 ([9,22]). *The left and right Riemann-Liouville fractional integral operators have the property of semigroup, i.e.,*

$$ {}_aD_t^{-\alpha_1}({}_aD_t^{-\alpha_2} y(t)) = {}_aD_t^{-\alpha_1-\alpha_2} y(t), $$

and

$$ {}_tD_b^{-\alpha_1}({}_tD_b^{-\alpha_2} y(t)) = {}_tD_b^{-\alpha_1-\alpha_2} y(t),\ \forall \alpha_1, \alpha_2 > 0. $$

in any point $t \in [a,b]$ for a continuous function y and for almost every point in $[a,b]$ if the function $y \in L^1([a,b],\mathbb{R}^N)$.

Proposition 2.2 ([9,22]). *If $h \in L^p([a,b],\mathbb{R}^N)$, $g \in L^q([a,b],\mathbb{R}^N)$ and $p \ge 1, q \ge 1, \frac{1}{p} + \frac{1}{q} \le 1 + \delta$ or $p \ne 1, q \ne 1, \frac{1}{p} + \frac{1}{q} = 1 + \delta$, then*

$$ \int_a^b [{}_aD_t^{-\delta} h(t)] g(t) dt = \int_a^b [{}_tD_b^{-\delta} g(t)] h(t) dt,\ \delta > 0. $$

Definition 2.3. Let $\frac{1}{2} < \varrho \le 1$. We denote the fractional derivative space $E_\varrho = \{v : [0,T] \to \mathbb{R}^N : v$ is absolutely continous and ${}_0^cD_x^{\varrho} v(x) \in L^2([0,T],\mathbb{R}^N)\}$ as the closure of $C^\infty([0,T],\mathbb{R}^N)$ endued with the norm

$$ \|v\|_\varrho = \left(\int_0^T |{}_0^cD_x^{\varrho} v(x)|^2 dx + \int_0^T |v(x)|^2 dx \right)^{\frac{1}{2}},\ \forall v \in E_\varrho. \tag{2.1} $$

Remark 2.1. Following Definition 4.1 in [24], we are aware that the fractional derivative space E_ϱ is the space of functions $v \in L^2([0,T], \mathbb{R}^N)$, which has an ϱ-order Caputo fractional derivative ${}_0^cD_x^\varrho v(x) \in L^2([0,T], \mathbb{R}^N)$.

Proposition 2.3 ([24]). *Let $0 < \varrho \le 1$, the fractional derivative space E_ϱ is a reflexive and separable Banach space.*

Lemma 2.1 ([24]). *If $\frac{1}{2} < \varrho \le 1$, then for any $v \in E_\varrho$, we have*

$$
\begin{aligned}
-\cos(\pi\varrho)\int_0^T |{}_0^cD_x^\varrho v(x)|^2 dx &\le -\int_0^T ({}_0^cD_x^\varrho v(x),\ {}_x^cD_T^\varrho v(x))dx \\
&\le -\frac{1}{\cos(\pi\varrho)}\int_0^T |{}_0^cD_x^\varrho v(x)|^2 dx. \qquad (2.2)
\end{aligned}
$$

Lemma 2.2 ([16]). *Let $\frac{1}{2} < \varrho \le 1$, $v \in E_\varrho$, the norm $\|v\|_\varrho$ is isovalent to*

$$
\begin{aligned}
\|v\| = \Bigg(&-\int_0^T ({}_0^cD_x^\varrho v(x),\ {}_x^cD_T^\varrho v(x))dx + \int_0^T K(x)(v(x))^2 dx \\
&+\frac{\gamma}{\sigma}(v(T))^2 + \frac{\alpha}{\beta}(v(0))^2\Bigg)^{\frac{1}{2}}, \qquad (2.3)
\end{aligned}
$$

i.e. there exist $A_{1\varrho}, A_{2\varrho} > 0$ satisfying

$$\frac{1}{A_{2\varrho}}\|v\| \le \|v\|_\varrho \le A_{1\varrho}\|v\|, \qquad (2.4)$$

where

$$A_{1\varrho} = \left(\max\left\{2T\frac{\beta}{\alpha}, -\frac{2T^{2\varrho}}{(\Gamma(\varrho+1))^2\cos\pi\varrho}\right\} - \frac{1}{\cos\pi\varrho}\right)^{\frac{1}{2}}, \qquad (2.5)$$

$$
\begin{aligned}
A_{2\varrho} = \Bigg(&2\max\left\{-\frac{1}{\cos\pi\varrho}, K^0\right\} + 2\frac{\gamma}{\sigma}\max\left\{T^{-\frac{1}{2}}, \frac{-T^{\varrho-\frac{1}{2}}}{\Gamma(\varrho+1)\cos\pi\varrho}\right\}^2 \qquad (2.6) \\
&+2\frac{\alpha}{\beta}\max\left\{T^{-\frac{1}{2}}, \frac{T^{\varrho-\frac{1}{2}}}{\Gamma(\varrho+1)}\right\}^2\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Lemma 2.3 ([16]). *For $v \in E_\varrho$, there exists $A_{3\varrho} > 0$ such that $\|v\|_\infty \le A_{3\varrho}\|v\|$, where*

$$
\begin{aligned}
&\|v\|_\infty = \max_{x\in[0,T]}|v(x)|, \\
&A_{3\varrho} = \sqrt{2}A_{1\varrho}\max\left\{T^{-\frac{1}{2}}, \frac{T^{\varrho-\frac{1}{2}}}{\Gamma(\varrho+1)}\right\} + \frac{T^{\varrho-\frac{1}{2}}}{\Gamma(\varrho)(2\varrho-1)^{\frac{1}{2}}\sqrt{|\cos\pi\varrho|}},
\end{aligned}
$$

and $A_{1\varrho}$ is defined in (2.5).

Proposition 2.4 ([24]). *If $\frac{1}{2} < \varrho \le 1$, the sequence $\{v_k\}$ converges weakly to v in E_ϱ, namely $v_k \rightharpoonup v$. Then $v_k \to v$ in $C([0,T], \mathbb{R}^N)$, namely $\|v_k - v\|_\infty \to 0$ as $k \to \infty$.*

Hereinafter, we will probe into FADE (1.1) in the E_ϱ with the corresponding norm $\|v\|$ which is defined in (2.3).

To facilitate this discussion, we denote

$$\mathfrak{D}^\varrho(v(x)) = \frac{1}{2}\ {}_0D_x^{\varrho-1}({}_0^cD_x^\varrho v(x)) - \frac{1}{2}\ {}_xD_T^{\varrho-1}({}_x^cD_T^\varrho v(x)). \tag{2.7}$$

As discussed in [16], FADE (1.1) transforms to the following form

$$\begin{cases} -\dfrac{d}{dt}\mathfrak{D}^\varrho(v(x)) + K(x)v(x) = f(x,v(x)) + \displaystyle\sum_{i=1}^m l_i(x,v(x)),\ a.e.\ x\in[0,T], \\ \alpha v(0) - \beta\mathfrak{D}^\varrho(v(0)) = 0,\ \gamma v(T) + \sigma\mathfrak{D}^\varrho(v(T)) = 0, \end{cases} \tag{2.8}$$

where $\varrho = 1 - \frac{\zeta}{2} \in (\frac{1}{2}, 1]$.

Definition 2.4. A function $v \in E_\varrho$ is termed as a weak solution of FADE (1.1) if

$$\begin{aligned} &-\frac{1}{2}\int_0^T [({}_0^cD_x^\alpha v(x),\ {}_x^cD_T^\alpha\omega(x)) + (\ {}_0^cD_x^\alpha\omega(x),\ {}_x^cD_T^\alpha v(x))] + (K(x)v(x),\omega(x))dx \\ &+\frac{\gamma}{\sigma}v(T)\omega(T) + \frac{\alpha}{\beta}v(0)\omega(0) \\ =&\int_0^T\left[f(x,v(x))\omega(x) + \sum_{i=1}^m l_i(x,v(x))\omega(x)\right]dx, \end{aligned}$$

holds for every $\omega \in E_\varrho$.

The energy functional $\Theta : E_\varrho \to \mathbb{R}$ associated with FADE (1.1) is defined by

$$\begin{aligned} \Theta(v) =& \int_0^T \frac{1}{2}\left[-({}_0^cD_x^\varrho v(x),\ {}_x^cD_T^\varrho v(x)) + (K(x)v(x),v(x))\right]dx + \frac{\gamma}{2\sigma}(v(T))^2 \\ &+ \frac{\alpha}{2\beta}(v(0))^2 - \int_0^T\left[F(x,v(x)) + \sum_{i=1}^m L_i(x,v(x))\right]dx \\ =&\frac{1}{2}\|v\|^2 - \int_0^T\left[F(x,v(x)) + \sum_{i=1}^m L_i(x,v(x))\right]dx, \end{aligned} \tag{2.9}$$

where $F(x,v) = \int_0^v f(x,\xi)d\xi$ and $L_i(x,v) = \int_0^v l_i(x,\xi)d\xi$ $(i = 1,2,\ldots,m)$.

Due to the properties of F, L_i $(i = 1,2,\ldots,m)$, which manifest that $\Theta \in C^1(E_\varrho,\mathbb{R})$ and for every $\omega \in E_\varrho$,

$$\begin{aligned} \langle\Theta'(v),\omega\rangle =& -\frac{1}{2}\int_0^T ({}_0^cD_x^\varrho v(x),\ {}_x^cD_T^\varrho\omega(x)) + ({}_0^cD_x^\varrho\omega(x),\ {}_x^cD_T^\varrho v(x))dx \\ &+\int_0^T (K(x)v(x),\omega(x))dx + \frac{\gamma}{\sigma}v(T)\omega(T) + \frac{\alpha}{\beta}v(0)\omega(0) \\ &-\int_0^T\left[f(x,v(x))\omega(x) + \sum_{i=1}^m l_i(x,v(x))\omega(x)\right]dx. \end{aligned} \tag{2.10}$$

Let W be a Banach space endued with the norm $\|\cdot\|$ and $W = \overline{\oplus_{j\in\mathbf{N}}w_j}$ with $\dim w_j < \infty$ for $j \in \mathbf{N}$. Put $M_k^* = \oplus_{j=1}^k w_j, N_k^* = \overline{\oplus_{j=k}^\infty w_j}$ and $Z_k^* = \{v \in M_k^* : \|v\| \le R_k^*\}, S_k^* = \{v \in N_k^* : \|v\| = r_k^*\}$ for $R_k^* > r_k^* > 0$.

Lemma 2.4 ([31]). *The C^1 functional $\Theta_{\widetilde{\lambda}} : W \to \mathbb{R}$ defined by $\Theta_{\widetilde{\lambda}}(v) := \Lambda(v) - \widetilde{\lambda}\Psi(v), \widetilde{\lambda} \in [1,2]$, satisfies:*

(i) $\Theta_{\widetilde{\lambda}}$ *maps bounded sets to bounded sets uniformly for* $\widetilde{\lambda} \in [1,2]$. *Furthermore,* $\Theta_{\widetilde{\lambda}}(-v) = \Theta_{\widetilde{\lambda}}(v)$ *for all* $(\widetilde{\lambda}, v) \in [1,2] \times W$;

(ii) $\Psi(v) \geq 0$ *for all* $v \in \mathbb{R}$, $\Psi(v) \to \infty$ *as* $\|v\| \to \infty$ *on any finite dimensional subspace of* W;

(iii) there exist $R_k^* > r_k^* > 0$ *such that*

$$\widetilde{a_k^*}(\widetilde{\lambda}) := \inf_{v \in N_k^*, \|v\| = R_k^*} \Theta_{\widetilde{\lambda}}(v) \geq 0 > \widetilde{b_k^*}(\widetilde{\lambda}) := \max_{v \in M_k^*, \|v\| = r_k^*} \Theta_{\widetilde{\lambda}}(v), \forall\, \widetilde{\lambda} \in [1,2],$$

and

$$\widetilde{d_k^*}(\widetilde{\lambda}) := \inf_{v \in N_k^*, \|v\| \leq R_k^*} \Theta_{\widetilde{\lambda}}(v) \to 0, \ \text{as } k \to \infty \text{ uniformly for } \widetilde{\lambda} \in [1,2].$$

Then, there exist $\widetilde{\lambda}_{n_*} \to 1, v(\widetilde{\lambda}_{n_*}) \in M_{n_*}$ *such that*

$$\Theta'_{\widetilde{\lambda}_{n_*}}|_{M_{n_*}}(v(\widetilde{\lambda}_{n_*})) = 0, \ \Theta_{\widetilde{\lambda}_{n_*}}(v(\widetilde{\lambda}_{n_*})) \to \widetilde{c_k^*} \in [\widetilde{d_k^*}(2), \widetilde{b_k^*}(1)], \ \text{as } n_* \to \infty.$$

Specially, if $\{v(\widetilde{\lambda}_{n_*})\}$ *has a convergent subsequence for every* k, *then* Θ_1 *has infinitely many nontrivial critical points* $\{v_k\} \in W\backslash\{0\}$ *satisfying* $\Theta_1(v_k) \to 0^-$ *as* $k \to \infty$.

Lemma 2.5 ([31]). *The C^1 functional $\Theta_{\widetilde{\lambda}} : W \to \mathbb{R}$ defined by $\Theta_{\widetilde{\lambda}}(v) := \Lambda(v) - \widetilde{\lambda}\Psi(v), \widetilde{\lambda} \in [1,2]$, satisfies:*

(i) $\Theta_{\widetilde{\lambda}}$ *maps bounded sets to bounded sets uniformly for* $\widetilde{\lambda} \in [1,2]$. *Additionally,* $\Theta_{\widetilde{\lambda}}(-v) = \Theta_{\widetilde{\lambda}}(v)$ *for all* $(\widetilde{\lambda}, v) \in [1,2] \times W$;

(ii) $\Psi(v) \geq 0$ *for all* $v \in W$, $\Lambda(v) \to \infty$ *or* $\Psi(v) \to \infty$ *as* $\|v\| \to \infty$; *or*

(ii′) $\Psi(v) \leq 0$ *for all* $v \in W$, $\Psi(v) \to -\infty$ *as* $\|v\| \to \infty$;

(iii) There exist $R_k^* > r_k^* > 0$ *such that*

$$\widetilde{b_k^*}(\widetilde{\lambda}) = \inf_{v \in N_k^*, \|v\| = r_k^*} \Theta_{\widetilde{\lambda}}(v) > \widetilde{a_k^*}(\widetilde{\lambda}) = \max_{v \in M_k^*, \|v\| = R_k^*} \Theta_{\widetilde{\lambda}}(v), \ \forall\, \widetilde{\lambda} \in [1,2].$$

Then,

$$\widetilde{b_k^*}(\widetilde{\lambda}) \leq \widetilde{c_k^*}(\widetilde{\lambda}) = \inf_{\Upsilon \in \Gamma_k} \max_{v \in Z_k^*} \Theta_{\widetilde{\lambda}}(\Upsilon(v)), \ \forall\, \widetilde{\lambda} \in [1,2],$$

where $\Gamma_k = \{\Upsilon \in C(Z_k^*, W) : \Upsilon \text{ is odd}, \Upsilon|_{\partial Z_k^*} = id\}(k \geq 2)$. *Furthermore, for almost every* $\widetilde{\lambda} \in [1,2]$, *there exists a sequence* $\{v_{n_*}^k(\widetilde{\lambda})\}$ *such that*

$$\sup_{n_*} \|v_{n_*}^k(\widetilde{\lambda})\| < \infty, \ \Theta'_{\widetilde{\lambda}}(v_{n_*}^k(\widetilde{\lambda})) \to 0, \ \Theta_{\widetilde{\lambda}}(v_{n_*}^k(\widetilde{\lambda})) \to \widetilde{c_k^*}(\widetilde{\lambda}), \ \text{as } n_* \to \infty.$$

Since E_ϱ is a separable Banach space in the light of Proposition 2.3. We choose an orthonormal basis $\{\ell_j\}$ of E_ϱ and write $w_j = span\{\ell_j\}$, $M_k^* = \oplus_{j=1}^k w_j, N_k^* = \overline{\oplus_{j=k}^\infty w_j}$. Consider $\Theta_{\widetilde{\lambda}} : E_\varrho \to \mathbb{R}$ defined by

$$\Theta_{\widetilde{\lambda}}(v) := \Lambda(v) - \widetilde{\lambda}\Psi(v) \tag{2.11}$$

$$=\frac{1}{2}\|v\|^2-\int_0^T\sum_{i=1}^m L_i(x,v(x))dx-\widetilde{\lambda}\int_0^T F(x,v(x))dx,\ \forall v\in E_\varrho,\ \widetilde{\lambda}\in[1,2].$$

Namely, $\Lambda(v)=\frac{1}{2}\|v\|^2-\int_0^T\sum_{i=1}^m L_i(x,v(x))dx$, $\Psi(v)=\int_0^T F(x,v(x))dx$.

To facilitate this discussion, henceforth, we will repeatedly use the letter $\widetilde{\mathfrak{c}^*}$ to bespeak varieties of positive constants whose exact value is irrelevant.

3. Main result

We give some assumptions of f, l_i as follows:
(A_1) $f, l_i\in C([0,T]\times\mathbb{R},\mathbb{R})$ are odd in v for $i=1,2,\dots,m$.
(A_2) There exist $\varsigma^*,\tau^*\in(1,2), b_1^*>0, b_2^*>0, b_3^*>0$ such that

$$b_1^*|v|^{\tau^*}\le f(s,v)v\le b_2^*|v|^{\tau^*}+b_3^*|v|^{\varsigma^*},\ a.e.\ s\in[0,T], v\in\mathbb{R}.$$

(A_3) There exists $2\le\widetilde{\sigma}_i<\infty$ such that $|l_i(s,v)|\le\widetilde{b}_i^*(1+|v|^{\widetilde{\sigma}_i-1})$ for $a.e.$ $s\in[0,T], v\in\mathbb{R}$. Furthermore, $\lim_{v\to 0}\frac{l_i(s,v)}{v}=0$ uniformly for $s\in[0,T], i=1,2,\dots,m$.
(A_4) Suppose one of the following conditions hold

- (i) $\lim_{|v|\to\infty}\frac{l_i(s,v)}{v}=0$ uniformly for $s\in[0,T], i=1,2,\dots,m$;
- (i') $\lim_{|v|\to\infty}\frac{l_i(s,v)}{v}=-\infty$ uniformly for $s\in[0,T], i=1,2,\dots,m$. Moreover, $\frac{l_i(s,v)}{v}$ and $\frac{f(s,v)}{v}$ are decreasing in v for v large enough;
- (ii') $\lim_{|v|\to\infty}\frac{l_i(s,v)}{v}=\infty$ uniformly for $s\in[0,T], i=1,2,\dots,m$, $\frac{l_i(s,v)}{v}$ is increasing in v for v large enough. Additionally, there exists $\wp_i>\wp>\varsigma^*+\tau^*>\max\{\varsigma^*,\tau^*\}$ such that $\liminf_{|v|\to\infty}\frac{l_i(s,v)v-2L_i(s,v)}{|v|^{\wp_i}}\ge\mathfrak{c}_i>0$ uniformly for $s\in[0,T], i=1,2,\dots,m$, where $\wp=\min_{i=1,2,\dots,m}\{\wp_i\}$.

Theorem 3.1. *Suppose that* (A_1), (A_2), (A_3) *and* (A_4) *hold, then FADE* (1.1) *has infinitely many solutions* $\{v_k\}$ *satisfying*

$$\Theta(v_k)=\frac{1}{2}\|v_k\|^2-\int_0^T\left[F(x,v_k(x))+\sum_{i=1}^m L_i(x,v_k(x))\right]dx\to 0^-,\ as\ k\to\infty.$$

Proof. Evidently, $\Psi(v)\ge 0$ and $\Psi(v)\to\infty$ as $\|v\|\to\infty$ on any finite dimensional subspace. Combining (A_2) and (A_3), it is easily seen that $\Theta_{\widetilde{\lambda}}$ maps bounded sets into bounded sets uniformly for $\widetilde{\lambda}\in[1,2]$. What's more, by virtue of (A_1), $\Theta_{\widetilde{\lambda}}(-v)=\Theta_{\widetilde{\lambda}}(v)$ for all $(\widetilde{\lambda},v)\in[1,2]\times E_\varrho$. On account of (A_3), for any $\varepsilon_i>0$, there exists C_{ε_i} such that $|L_i(x,v(x))|\le\varepsilon_i|v|^2+C_{\varepsilon_i}|v|^{\widetilde{\sigma}_i}$, $(i=1,2,\dots,m)$. Hence, for $\|v\|$ small enough,

$$\begin{aligned}\Theta_{\widetilde{\lambda}}(v)\ge&\frac{1}{2}\|v\|^2-T\sum_{i=1}^m\varepsilon_i A_{3\varrho}^2\|v\|^2-\widetilde{\mathfrak{c}^*}\|v\|_{\tau^*}^{\tau^*}-\widetilde{\mathfrak{c}^*}\|v\|_{\varsigma^*}^{\varsigma^*} &(3.1)\\ \ge&\left[\frac{1}{2}-Tm\varepsilon A_{3\varrho}^2\right]\|v\|^2-\widetilde{\mathfrak{c}^*}\|v\|_{\tau^*}^{\tau^*}-\widetilde{\mathfrak{c}^*}\|v\|_{\varsigma^*}^{\varsigma^*}\\ \ge&\frac{1}{4}\|v\|^2-\widetilde{\mathfrak{c}^*}\|v\|_{\tau^*}^{\tau^*}-\widetilde{\mathfrak{c}^*}\|v\|_{\varsigma^*}^{\varsigma^*},\end{aligned}$$

combining with (A_2), where $\varepsilon := \max_{i=1,2,\ldots,m}\{\varepsilon_i\} \le \frac{1}{4TmA_{3\varrho}^2}$. Suppose $\tau^* \le \varsigma^*$ and set $\Omega_k(\tau^*) := \sup_{v\in N_k^*,\|v\|=1}\|v\|_{\tau^*}$, $\Omega_k(\varsigma^*) := \sup_{v\in N_k^*,\|v\|=1}\|v\|_{\varsigma^*}$, then $\Omega_k(\tau^*) \to 0, \Omega_k(\varsigma^*) \to 0$ as $k \to \infty$. It is similar to the proof of Lemma 3.5 in [2] and Lemma 3.2 in [26], we omit the proving processes. Accordingly, for $\|v\| = R_k^* := (8\widetilde{\mathfrak{c}^*}\Omega_k^{\tau^*}(\tau^*) + 8\widetilde{\mathfrak{c}^*}\Omega_k^{\varsigma^*}(\varsigma^*))^{1/(2-\tau^*)}$, we get $\Theta_{\widetilde{\lambda}}(v) \ge \frac{(R_k^*)^2}{8} > 0$. For another thing, if $v \in M_k^*$ with $\|v\|$ small enough, we acquire that

$$
\begin{aligned}
\Theta_{\widetilde{\lambda}}(v) \le& \frac{1}{2}\|v\|^2 - \widetilde{\mathfrak{c}^*}\int_0^T |v|^{\tau^*}dx + \sum_{i=1}^m\int_0^T \varepsilon_i|v|^2 + C_{\varepsilon_i}|v|^{\widetilde{\sigma}_i}dx \\
\le& \widetilde{\mathfrak{c}^*}\|v\|^2 + \sum_{i=1}^m \widetilde{\mathfrak{c}^*}\|v\|^{\widetilde{\sigma}_i} - \widetilde{\mathfrak{c}^*}\|v\|^{\tau^*} \\
<& 0,
\end{aligned} \tag{3.2}
$$

for $\tau^* \in (1,2)$, $2 \le \widetilde{\sigma}_i < \infty$, $i = 1,2,\ldots,m$, due to $(A_2), (A_3)$, and the equivalence of norm in finite dimensional space. The above discussions indicate that $b_k^*(\widetilde{\lambda}) < 0 \le a_k^*(\widetilde{\lambda})$ for $\widetilde{\lambda} \in [1,2]$. Additionally, if $v \in N_k^*$ with $\|v\| \le R_k^*$, we can acquire that

$$
\Theta_{\widetilde{\lambda}}(v) \ge -\widetilde{\mathfrak{c}^*}\|v\|_{\tau^*}^{\tau^*} - \widetilde{\mathfrak{c}^*}\|v\|_{\varsigma^*}^{\varsigma^*} \ge -\widetilde{\mathfrak{c}^*}\Omega_k^{\tau^*}(\tau^*)R_k^{\tau^*} - \widetilde{\mathfrak{c}^*}\Omega_k^{\varsigma^*}(\varsigma^*)R_k^{\varsigma^*} \to 0,
$$

as $k \to \infty$, in terms of (3.1). Consequently, $d_k^*(\widetilde{\lambda}) \to 0$ as $k \to \infty$. By Lemma 2.4, we get there exist $\widetilde{\lambda}_{n_*} \to 1, v(\widetilde{\lambda}_{n_*}) \in M_{n_*}$ such that

$$
\Theta'_{\widetilde{\lambda}_{n_*}}|_{M_{n_*}}(v(\widetilde{\lambda}_{n_*})) = 0,\ \Theta_{\widetilde{\lambda}_{n_*}}(v(\widetilde{\lambda}_{n_*})) \to \widetilde{c_k^*} \in [\widetilde{d_k^*}(2), \widetilde{b_k^*}(1)],\ as\ n_* \to \infty. \tag{3.3}
$$

Subsequently, we certify that $\{v(\widetilde{\lambda}_{n_*})\}$ is bounded in E_ϱ. On account of

$$
\Theta'_{\widetilde{\lambda}_{n_*}}|_{M_{n_*}}(v(\widetilde{\lambda}_{n_*})) = 0,
$$

then

$$
1 = \int_0^T \frac{\widetilde{\lambda}_{n_*}f(x,v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) + \sum_{i=1}^m l_i(x,v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*})}{\|v(\widetilde{\lambda}_{n_*})\|^2}dx.
$$

If, up to a subsequence, $\|v(\widetilde{\lambda}_{n_*})\| \to \infty$ as $n_* \to \infty$, thus, as a result of (A_2), we acquire

$$
1 + o(1) = \int_0^T \frac{\sum_{i=1}^m l_i(x,v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*})}{\|v(\widetilde{\lambda}_{n_*})\|^2}dx,
$$

where $o(1) \to 0$ as $n_* \to \infty$. Distinctly, it is a contradiction if (A_4)(i) holds.

In addition, choose $\varpi_{n_*} = \frac{v(\widetilde{\lambda}_{n_*})}{\|v(\widetilde{\lambda}_{n_*})\|}$, then, $\varpi_{n_*} \rightharpoonup \varpi$ in E_ϱ, $\varpi_{n_*} \to \varpi$ in $L^2([0,T])$ and $\varpi_{n_*}(x) \to \varpi(x)$ for *a.e.* $x \in [0,T]$.

Case 1. If, $\varpi \ne 0$ in E_ϱ, and $\lim_{|v|\to\infty}\frac{l_i(x,v)}{v} = -\infty$ in (A_4) (i'), for n_* large enough, via Fatou's Lemma, we get that

$$
-1 + o(1) = \int_0^T \frac{-\sum_{i=1}^m l_i(x,v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*})}{|v(\widetilde{\lambda}_{n_*})|^2}|\varpi_{n_*}|^2dx
$$

$$
\begin{aligned}
&\geq \widetilde{\mathfrak{c}^*} + \int_{\{\varpi\neq 0\}\cap\{|v(\widetilde{\lambda}_{n_*})|\geq\widetilde{\mathfrak{c}^*}\}} \frac{-\sum_{i=1}^m l_i(x,v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*})}{|v(\widetilde{\lambda}_{n_*})|^2}|\varpi_{n_*}|^2dx\\
&\to\infty,
\end{aligned}
$$

a contradiction. It is similar if $\lim_{|v|\to\infty}\frac{l_i(s,v)}{v}=\infty$ in $(A_4)(i'')$.

Case 2. If, $\varpi=0$ in E_ϱ, we denote

$$
\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})) := \max_{\widetilde{s}\in[0,1]}\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s}v(\widetilde{\lambda}_{n_*})), \tag{3.4}
$$

then

$$
\lim_{n_*\to\infty}\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}))=\infty, \qquad \langle\Theta'_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})),\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})\rangle=0.
$$

As a matter of fact, for any $\widetilde{\mathfrak{c}^*}>0$ and $\vartheta^*_{n_*}:=(4\widetilde{\mathfrak{c}^*})^{\frac{1}{2}}\varpi_{n_*}$, due to $(4\widetilde{\mathfrak{c}^*})^{\frac{1}{2}}\|v(\widetilde{\lambda}_{n_*})\|^{-1}\in(0,1)$ and $\int_0^T L_i(x,\vartheta^*_{n_*}(x))dx\to 0$ $(i=1,2,\ldots,m)$, $\int_0^T F(x,\vartheta^*_{n_*}(x))dx\to 0$, we obtain, for n_* large enough, that

$$
\begin{aligned}
\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})) \geq& \Theta_{\widetilde{\lambda}_{n_*}}(\vartheta^*_{n_*})\\
=& 2\widetilde{\mathfrak{c}^*} - \int_0^T\sum_{i=1}^m L_i(x,\vartheta^*_{n_*}(x))dx - \widetilde{\lambda}_{n_*}\int_0^T F(x,\vartheta^*_{n_*}(x))dx\\
\geq& \widetilde{\mathfrak{c}^*},
\end{aligned}
$$

which means that

$$
\lim_{n_*\to\infty}\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}))=\infty. \tag{3.5}
$$

We note that $\Theta_{\widetilde{\lambda}_{n_*}}(0)=0$ and (3.3) holds, combining with (3.5), we observe that there exists $\widetilde{s_{n_*}}\in(0,1)$, and so by (3.4), we deduce

$$
\frac{d}{d\widetilde{s}}|_{\widetilde{s}=\widetilde{s_{n_*}}}\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s}v(\widetilde{\lambda}_{n_*}))=0
$$

for any $n_*\in\mathbf{N}$. Thereby, we have

$$
\langle\Theta'_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})),\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})\rangle = \widetilde{s_{n_*}}\frac{d}{d\widetilde{s}}|_{\widetilde{s}=\widetilde{s_{n_*}}}\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s}v(\widetilde{\lambda}_{n_*}))=0.
$$

It follows that

$$
\begin{aligned}
\infty =& \lim_{n_*\to\infty}\left(\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})) - \frac{1}{2}\langle\Theta'_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})),\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*})\rangle\right)\\
\leq& \lim_{n_*\to\infty}\widetilde{\lambda}_{n_*}\int_0^T\frac{1}{2}f(x,\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}))\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}) - F(x,\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}))dx\\
&+\sum_{i=1}^m\int_0^T\frac{1}{2}l_i(x,\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}))\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}) - L_i(x,\widetilde{s_{n_*}}v(\widetilde{\lambda}_{n_*}))dx.
\end{aligned}
$$

If (A_4) (i') holds, we have that

$$
\frac{1}{2}f(s,\widetilde{\chi}v)\widetilde{\chi}v - F(s,\widetilde{\chi}v) + \sum_{i=1}^m\left[\frac{1}{2}l_i(s,\widetilde{\chi}v)\widetilde{\chi}v - L_i(s,\widetilde{\chi}v)\right]\leq\widetilde{\mathfrak{c}^*}
$$

for all $\widetilde{\chi} > 0$ and $v \in \mathbb{R}$, which is a contradiction.

If $(A_4)(i'')$ holds, we have that

$$\infty \leq \widetilde{\mathfrak{c}^*} \int_0^T |v(\widetilde{\lambda}_{n_*})|^{\tau^*} dx + \int_0^T \frac{1}{2} l_i(x, v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) - L_i(x, v(\widetilde{\lambda}_{n_*}))dx,$$

which means that

$$\int_0^T \frac{1}{2} l_i(x, v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) - L_i(x, v(\widetilde{\lambda}_{n_*}))dx \to \infty.$$

Whereas, owing to the property of $v(\widetilde{\lambda}_{n_*})$, we know that

$$\begin{aligned}
\widetilde{b_k^*}(1) \geq & \widetilde{\lambda}_{n_*} \int_0^T \frac{1}{2} f(x, v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) - F(x, v(\widetilde{\lambda}_{n_*}))dx \\
& + \int_0^T \sum_{i=1}^m \frac{1}{2} l_i(x, v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) - L_i(x, v(\widetilde{\lambda}_{n_*}))dx \\
\geq & \frac{1}{2} \int_0^T \sum_{i=1}^m \frac{1}{2} l_i(x, v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) - L_i(x, v(\widetilde{\lambda}_{n_*}))dx \\
& + \frac{1}{2} \sum_{i=1}^m \frac{\mathfrak{c}_i}{2} \int_0^T |v(\widetilde{\lambda}_{n_*})|^{\wp_i} dx - \frac{1}{2} \widetilde{\mathfrak{c}^*} \int_0^T |v(\widetilde{\lambda}_{n_*})|^{\tau^*} dx - \frac{1}{2} \widetilde{\mathfrak{c}^*} \int_0^T |v(\widetilde{\lambda}_{n_*})|^{\varsigma^*} dx \\
\geq & \frac{1}{2} \int_0^T \sum_{i=1}^m \frac{1}{2} l_i(x, v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) - L_i(x, v(\widetilde{\lambda}_{n_*}))dx \\
& + \frac{1}{2} \widetilde{\mathfrak{c}^*} \int_0^T |v(\widetilde{\lambda}_{n_*})|^{\wp} dx - \frac{1}{2} \widetilde{\mathfrak{c}^*} \int_0^T |v(\widetilde{\lambda}_{n_*})|^{\tau^*} dx - \frac{1}{2} \widetilde{\mathfrak{c}^*} \int_0^T |v(\widetilde{\lambda}_{n_*})|^{\varsigma^*} dx \\
\geq & \widetilde{\mathfrak{c}^*} \int_0^T \sum_{i=1}^m \frac{1}{2} l_i(x, v(\widetilde{\lambda}_{n_*}))v(\widetilde{\lambda}_{n_*}) - L_i(x, v(\widetilde{\lambda}_{n_*}))dx - \widetilde{\mathfrak{c}^*},
\end{aligned}$$

where $\wp = \min_{i=1,2,\ldots,m}\{\wp_i\}$, $\widetilde{\mathfrak{c}^*}$ represents different positive exact values that are unrelated, which contradicts the prior estimate. The above discussions suggest that $\{v(\widetilde{\lambda}_{n_*})\}$ is bounded. Applying similar arguments of the proof of Theorem 12 in [26] or Theorem 3.1 in [2], we have $\{v(\widetilde{\lambda}_{n_*})\}$ has a convergent subsequence for every k. According to Lemma 2.4, we are aware of that $\Theta = \Theta_1$ has infinitely many nontrivial critical points $\{v_k\} \in W\backslash\{0\}$ satisfying $\Theta_1(v_k) \to 0^-$ as $k \to \infty$. The proof is complete. □

Remark 3.1. Conditions (A_3) and (A_4)(i) involve the case of $l_i \equiv 0$ $(i = 1, 2, \ldots, m)$.

In what follows, we consider the case that $l_i \equiv 0$ $(i = 1, 2, \ldots, m)$, and list some hypotheses which are different from Theorem 3.1. The assumptions are made as follows:

(B_1) $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $|f(s,v)| \leq \ell^*(1 + |v|^{\eta^*-1})$, *a.e.* $s \in [0,T], v \in \mathbb{R}$ with $\eta^* \in (2, \infty), \ell^* > 0$, and $f(s,v)v \geq 0$ for all $v > 0$.

(B_2) $\liminf_{|v|\to\infty} \frac{f(s,v)v}{|v|^{\iota}} \geq \widehat{d^*} > 0$ uniformly for $s \in [0,T]$, where $\iota > 2$.

(B_3) $\lim_{|v|\to 0} \frac{f(s,v)}{v} = 0$ uniformly for $s \in [0,T]$; $\frac{f(s,v)}{v}$ is an increasing function of v for every $s \in [0,T]$.

(B_4) $f(s,-v) = -f(s,v)$ for any $v \in E_{\varrho}, s \in [0,T]$.

Theorem 3.2. *Suppose that* (B_1), (B_2), (B_3) *and* (B_4) *hold, then FADE* (1.1) *with* $l_i \equiv 0$ $(i = 1, 2, \ldots, m)$ *has infinitely many solutions* $\{v_k\}$ *satisfying*

$$\Theta(v_k) = \frac{1}{2}\|v_k\|^2 - \int_0^T F(x, v_k(x))dx \to \infty, \ \ as \ k \to \infty.$$

Proof. Consider $\Theta_{\widetilde{\lambda}} : E_\varrho \to \mathbb{R}$ defined by

$$\Theta_{\widetilde{\lambda}}(v) := \Lambda(v) - \widetilde{\lambda}\Psi(v) = \frac{1}{2}\|v\|^2 - \widetilde{\lambda}\int_0^T F(x, v(x))dx, \ \forall v \in E_\varrho, \ \widetilde{\lambda} \in [1, 2]. \quad (3.6)$$

Namely, $\Lambda(v) = \frac{1}{2}\|v\|^2$, $\Psi(v) = \int_0^T F(x, v(x))dx$. Obviously, $\Psi(v) \geq 0, \Lambda(v) \to \infty$ as $\|v\| \to \infty$, $\Theta_{\widetilde{\lambda}}(-v) = \Theta_{\widetilde{\lambda}}(v)$ for all $(\widetilde{\lambda}, v) \in [1, 2] \times E_\varrho$.

Clearly, by conditions (B_1), (B_2), (B_3), for any $\widetilde{\varepsilon} > 0$, there exists $C^*_{\widetilde{\varepsilon}}$ such that $f(s, v)v \geq C^*_{\widetilde{\varepsilon}}|v|^\iota - \widetilde{\varepsilon}|v|^2, \forall v \in \mathbb{R}$. Consequently, it can be easily proved, for certain $R^*_k > 0$ large enough, that $\widetilde{a^*_k}(\widetilde{\lambda}) := \max_{v \in M^*_k, \|v\| = R^*_k} \Theta_{\widetilde{\lambda}}(v) \leq 0$ uniformly for $\widetilde{\lambda} \in [1, 2]$. For another, due to (B_3), for any $\widetilde{\varepsilon} > 0$, there exists $C^*_{\widetilde{\varepsilon}} > 0$ such that $|f(s, v)| \leq C^*_{\widetilde{\varepsilon}}|v|^{\eta^*-1} + \widetilde{\varepsilon}|v|, \forall s \in [0, T], v \in \mathbb{R}$. Consider $\Omega_k(\eta^*) := \sup_{v \in N^*_k, \|v\|=1} \|v\|_{\eta^*}$, then $\Omega_k(\eta^*) \to 0$ as $k \to \infty$ (see [2, 26]). Hence, for $v \in N^*_k$ and $\widetilde{\varepsilon}$ small enough, we have

$$\begin{aligned}
\Theta_{\widetilde{\lambda}}(v) =& \frac{1}{2}\|v\|^2 - \widetilde{\lambda}\int_0^T F(x, v(x))dx \\
\geq& \frac{1}{2}\|v\|^2 - \frac{\widetilde{\lambda}\widetilde{\varepsilon}}{2}\|v\|_2^2 - \frac{\widetilde{\lambda}C^*_{\widetilde{\varepsilon}}}{\eta^*}\|v\|_{\eta^*}^{\eta^*} \\
\geq& \frac{1}{4}\|v\|^2 - \widetilde{\mathfrak{c}^*}\|v\|_{\eta^*}^{\eta^*} \\
\geq& \frac{1}{4}\|v\|^2 - \widetilde{\mathfrak{c}^*}\Omega_k^{\eta^*}(\eta^*)\|v\|^{\eta^*}.
\end{aligned}$$

If we take $r^*_k = (4\widetilde{\mathfrak{c}^*}\eta^*\Omega_k^{\eta^*}(\eta^*))^{\frac{1}{2-\eta^*}}$, then for $v \in N^*_k$ with $\|v\| = r^*_k$, we have

$$\Theta_{\widetilde{\lambda}}(v) \geq (4\widetilde{\mathfrak{c}^*}\eta^*\Omega_k^{\eta^*})^{\frac{2}{2-\eta^*}}\left(\frac{1}{4} - \frac{1}{4\eta^*}\right) := \mathbf{b}_k,$$

which indicates that $\widetilde{b^*_k}(\widetilde{\lambda}) := \inf_{v \in N^*_k, \|v\| = r^*_k} \geq \mathbf{b}_k \to \infty$ uniformly for $\widetilde{\lambda}$ as $k \to \infty$. Thus, in virtue of Lemma 2.5, for *a.e.* $\widetilde{\lambda} \in [1, 2]$, there exists a sequence $\{v^k_{n_*}(\widetilde{\lambda})\}_{n_*=1}^\infty$ such that

$$\sup_{n_*}\|v^k_{n_*}(\widetilde{\lambda})\| < \infty, \ \Theta'_{\widetilde{\lambda}}(v^k_{n_*}(\widetilde{\lambda})) \to 0, \ \Theta_{\widetilde{\lambda}}(v^k_{n_*}(\widetilde{\lambda})) \to \widetilde{c^*_k}(\widetilde{\lambda}) \geq \widetilde{b^*_k}(\widetilde{\lambda}) \geq \mathbf{b}_k, \ n_* \to \infty.$$

Additionally, since $\widetilde{c^*_k}(\widetilde{\lambda}) \leq \sup_{v \in Z^*_k} \Theta(v) := \mathbf{c}_k$ and E_ϱ is embedded compactly to $L^2([0, T])$), which is similar to the proof of Lemma 12 in [11] or Lemma 2.7 in [20], we omit the proving processes, and by standard argument (see [2, 26]), $\{v^k_{n_*}(\widetilde{\lambda})\}_{n_*=1}^\infty$ has a convergent subsequence. Therefore, there exists $\hbar^k(\widetilde{\lambda})$ such that $\Theta'_{\widetilde{\lambda}}(\hbar^k(\widetilde{\lambda})) = 0$ and $\Theta_{\widetilde{\lambda}}(\hbar^k(\widetilde{\lambda})) \in [\mathbf{b}_k, \mathbf{c}_k]$. Obviously, we may find $\widetilde{\lambda}_{n_*} \to 1$ as $n_* \to \infty$, and $\{\hbar_{n_*}\}_{n_*=1}^\infty \subset E_\varrho$ such that

$$\Theta'_{\widetilde{\lambda}_{n_*}}(\hbar_{n_*}) = 0, \ \Theta_{\widetilde{\lambda}_{n_*}}(\hbar_{n_*}) \in [\mathbf{b}_k, \mathbf{c}_k], \quad (3.7)$$

with $\mathbf{c}_k > \mathbf{b}_k > 0$.

From now on, we attest that $\{\hbar_{n_*}\}_{n_*=1}^{\infty}$ is bounded. Suppose that $\|\hbar_{n_*}\| \to \infty$ as $n_* \to \infty$. Consider $\Xi_{n_*} := \frac{\hbar_{n_*}}{\|\hbar_{n_*}\|}$. Accordingly, up to a subsequence, we gain $\Xi_{n_*} \rightharpoonup \Xi$ in E_ϱ, $\Xi_{n_*} \to \Xi$ in $L^2([0,T])$, $\Xi_{n_*}(s) \to \Xi(s)$ *a.e.* $s \in [0,T]$.

Case 1. $\Xi \neq 0$ in E_ϱ. On account of $\Theta'_{\widetilde{\lambda}_{n_*}}(\hbar_{n_*}) = 0$, we get that

$$\int_0^T \frac{f(s,\hbar_{n_*})\hbar_{n_*}}{\|\hbar_{n_*}\|^2} ds \leq \widetilde{\mathfrak{c}^*}.$$

In addition, via Fatou's Lemma and conditions (B_1) and (B_2), we get that

$$\int_0^T \frac{f(s,\hbar_{n_*})\hbar_{n_*}}{\|\hbar_{n_*}\|^2} ds = \int_{\{\Xi_{n_*}(s)\neq 0\}} |\Xi_{n_*}(s)|^2 \frac{f(s,\hbar_{n_*})\hbar_{n_*}}{|\hbar_{n_*}(s)|^2} ds \to \infty,$$

a contradiction.

Case 2. $\Xi = 0$ in E_ϱ. Similar to the arguments in Theorem 3.1, we set

$$\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}\hbar_{n_*}) := \max_{\widetilde{s}\in[0,1]} \Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s}\hbar_{n_*}). \tag{3.8}$$

For any $\widetilde{\mathfrak{c}^*} > 0$ and $\widetilde{\Xi}_{n_*} := (4c)^{\frac{1}{2}}\Xi_{n_*}$, n_* large enough, we gain that

$$\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}\hbar_{n_*}) \geq \Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{\Xi}_{n_*}) = 2\widetilde{\mathfrak{c}^*} - \widetilde{\lambda}_{n_*}\int_0^T F(s,\widetilde{\Xi}_{n_*})ds \geq \widetilde{\mathfrak{c}^*},$$

which means that

$$\lim_{n_*\to\infty} \Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}\hbar_{n_*}) = \infty. \tag{3.9}$$

We notice that $\Theta_{\widetilde{\lambda}_{n_*}}(0) = 0$ and (3.7) holds, combining with (3.9), we see that there exists $\widetilde{s_{n_*}} \in (0,1)$, and so by (3.8), we deduce $\frac{d}{d\widetilde{s}}|_{\widetilde{s}=\widetilde{s_{n_*}}}\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s}\hbar_{n_*}) = 0$ for any $n_* \in \mathbf{N}$. Thereby, we have

$$\langle \Theta'_{\widetilde{\lambda}_{n_*}}(\widetilde{s_{n_*}}\hbar_{n_*}, \widetilde{s_{n_*}}\hbar_{n_*}\rangle = \widetilde{s_{n_*}}\frac{d}{d\widetilde{s}}|_{\widetilde{s}=\widetilde{s_{n_*}}}\Theta_{\widetilde{\lambda}_{n_*}}(\widetilde{s}\hbar_{n_*}) = 0.$$

It follows that

$$\int_0^T \frac{1}{2}f(t,\widetilde{s_{n_*}}\hbar_{n_*})\widetilde{s_{n_*}}\hbar_{n_*} - F(t,\widetilde{s_{n_*}}\hbar_{n_*})dt \to \infty.$$

Via condition (B_3), $\Im(x) = \frac{1}{2}x^2 f(t,y)y - F(t,xy)$ is increasing in $x \in [0,1]$, thereby, $\frac{1}{2}f(t,y)y - F(t,y)$ is increasing in $y > 0$. Together with the oddness of f, the following inequality is obtained

$$\int_0^T \frac{1}{2}f(t,\hbar_{n_*})\hbar_{n_*} - F(t,\hbar_{n_*})dt \geq \int_0^T \frac{1}{2}f(t,\widetilde{s_{n_*}}\hbar_{n_*})\widetilde{s_{n_*}}\hbar_{n_*} - F(t,\widetilde{s_{n_*}}\hbar_{n_*})dt \to \infty,$$

it is a contraction since

$$\widetilde{\lambda}_{n_*}\int_0^T \frac{1}{2}f(t,\hbar_{n_*})\hbar_{n_*} - F(t,\hbar_{n_*})dt = \Theta_{\widetilde{\lambda}_{n_*}}(\hbar_{n_*}) - \frac{1}{2}\langle\Theta'_{\widetilde{\lambda}_{n_*}}(\hbar_{n_*}),\hbar_{n_*}\rangle$$

$$=\Theta_{\widetilde{\lambda}_{n_*}}(\hbar_{n_*}) \in [\mathbf{b}_k, \mathbf{c}_k].$$

According to $\mathbf{b}_k \to \infty$ as $k \to \infty$, we acquire that FADE (1.1) with $l_i \equiv 0$ $(i = 1, 2, \dots, m)$ has infinitely many nontrivial high energy solutions. The proof is complete. □

Combining the above two theorems and their proof, we have the following corollary.

Corollary 3.1. *Suppose that* (A_1), (A_2), (A_3) *hold, and* $\frac{l_i(s,v)}{v}$ *is increasing in* v *for* v *large enough. Moreover, suppose that*

$$\liminf_{|v|\to\infty} \frac{l_i(s,v)v}{|v|^{\iota_i}} \geq \mathfrak{d}_\mathfrak{i} > 0, \ \liminf_{|v|\to\infty} \frac{l_i(s,v)v - 2L_i(s,v)}{|v|^{\wp_i}} \geq \mathfrak{c}_i > 0$$

uniformly for $s \in [0,T]$, *where* $\iota_i > 2$, $i = 1, 2, \dots, m$, $\wp_i > \wp > \varsigma^* + \tau^* > \max\{\varsigma^*, \tau^*\}$, $\wp = \min_{i=1,2,\dots,m}\{\wp_i\}$. *Then FADE* (1.1) *has two sequences* $\{v_k\}$ *and* $\{\widetilde{v}_k\}$ *of nontrivial solutions such that*

$$\Theta(v_k) \to 0^-, \quad \Theta(\widetilde{v}_k) \to \infty, \ \text{as } k \to \infty.$$

Remark 3.2. Boundary value conditions include the following four cases:

(i) $\beta = 0, \alpha \neq 0$, the boundary value conditions of FADE (1.1) convert into $v(0) = 0, \gamma v(T) + \sigma \left[\frac{1}{2}\ {}_0D_x^{-\zeta}(v'(T)) + \frac{1}{2}\ {}_xD_T^{-\zeta}(v'(T))\right] = 0$.

(ii) $\sigma = 0, \gamma \neq 0$, the boundary value conditions of FADE (1.1) convert into $v(T) = 0, \alpha v(0) - \beta \left[\frac{1}{2}\ {}_0D_x^{-\zeta}(v'(0)) + \frac{1}{2}\ {}_xD_T^{-\zeta}(v'(0))\right] = 0$.

(iii) $\beta = \sigma = 0$, the boundary value conditions of FADE (1.1) convert into $v(0) = 0, v(T) = 0$.

(iv) If $\zeta = \beta = \sigma = 0$, FADE (1.1) convert into second-order differential equations with Dirichlet boundary value problem.

To sum up, our study is more general than the previous literature.

Example 3.1. Choose $f(s,v) = v|v|^{\tau^*-2}\ln(2+|v|)$, $l_i(s,v) = \widetilde{\kappa}_i v \ln(1+|v|), (i = 1, 2, \dots, m)$ where $\tau^* \in (1,2)$. It is easily to see that (A_1), (A_2), (A_3) and $(A_4)(i')$ hold while $\widetilde{\kappa}_i < 0$ $(i = 1, 2, \dots, m)$; (A_1), (A_2), (A_3) and $(A_4)(i'')$ hold with $\wp_i = 2$ while $\widetilde{\kappa}_i > 0$ $(i = 1, 2, \dots, m)$. Take

$$l_i(s,v) = \begin{cases} v^3, & |v| \leq 1, \\ \widetilde{\mathfrak{c}}_\mathfrak{i}^*|v|^{-\frac{1}{2}}\ln(1+|v|), & |v| \geq 1, \end{cases}$$

where $\mathfrak{c}_\mathfrak{i}^*$ $(i = 1, 2, \dots, m)$ is well chosen, then (A_1), (A_2), (A_3) and (A_4)(i) hold. By Theorem 3.1, FADE (1.1) has infinitely many solutions.

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