## EXISTENCE OF KINK WAVES TO PERTURBED DISPERSIVE K(3,1) EQUATION\*

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**Abstract** This paper concerns on the existence problem of traveling wave solutions to perturbed dispersive K(3, 1) equation by using geometric singular perturbation technique. Based on the analogy between solitary wave solutions and heteroclinic orbits of the associated ordinary differential equations, kink and antikink waves persistent is concluded when the perturbed parameter is small sufficiently in perturbed nonlinear wave equation.

**Keywords** Heteroclinic orbits, geometric singular perturbation theory, Melnikov function.

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### 1. Introduction

The well-known nonlinear dispersive K(m, n) equation

$$U_t + \sigma(U^m)_x + (U^n)_{xxx} = 0, \quad m > 0, n \ge 1, \tag{1.1}$$

was proposed by Rosenau and Hyman [18]. Eq. (1.1) was studied as a role of nonlinear dispersion in the formation of patterns in liquid drops. It was formally derived in [4, 13] that the delicate interaction between nonlinear convection and genuine nonlinear dispersion generates solitary waves with compact support. Unlike soliton that narrows as the amplitude increases, the compacton<sub>i</sub><sup>-</sup>s width is independent on the amplitude.

In 1997, Rosenau [19] found that (1.1) hold a number of dispersive effects. In reference [21], Wu et al. investigated the traveling wave solutions by using the bifurcation method of dynamic systems. When m = 3, n = 1, the expressions of kink and antikink wave solutions were given

$$U(x - ct) = u(\xi) = \pm \sqrt{\frac{c}{\sigma}} \tanh(\sqrt{-\frac{c}{2}}\xi), \qquad (1.2)$$

where c < 0 is the wave speed.

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On discussing kink and anti-kink solitons, Cheng and K $\ddot{u}$ pper [2] used perturbation methods basing on inverse scattering transform to study the dynamical behavior of the kink and anti-kink solitons exhibited by a driven-damped sine-Gordon equation with infinite boundary condition. It is shown that such solitons may be forced by damping and external forces to remain in a finite region instead of approaching infinity. On using the geometric singular perturbation theory and the regular perturbation analysis for a Hamiltonian system, Yan et al. [22] discussed a perturbed generalized KdV equation, they proved that solitary wave solutions and periodic wave solutions persist for sufficiently small perturbed parameter.

In reference [1], the existence of solitary waves and periodic waves for a perturbed generalized BBM equation was established by using geometric singular perturbation theory. More recently, Wen [20] studied the existence of kink and antikink wave solutions of singularly perturbed Gardner equation from the geometric perspective. In 2020, Cosgun and Sari [3] obtained different traveling wave solutions of the kink type for singularly perturbed generalized Burgers Huxley and Burgers Fisher equations. Ghazaryan et al. [8] proved the existence of traveling fronts in this regime and investigated their stability by using geometric singular perturbation theory. Du and Qiao [5] constructed a locally invariant manifold for a associated traveling wave equation and obtained the traveling wave fronts for the equation by employing geometric singular perturbation theory.

Motivated by the mentioned references, in this paper, the following singularly perturbed dispersive K(3, 1) equation is considered

$$U_t + \sigma(U^3)_x + U_{xxx} + \varepsilon(U_{xx} + U_{xxxx}) = 0, \qquad (1.3)$$

where  $\sigma$  is a real parameter,  $\varepsilon$  is a positive parameter. We shall use Fenichel persistence theorem [6,7,9,12,15,16] to study the existence of kink wave and antikink solutions for equation (1.3).

The remaining part of presented paper is organized as follows: In next section, we perform a useful tool Fenichel theory which is important to obtain our main results. Singularly perturbation analysis to dispersive K(3,1) equation (1.3), the existence of kink wave solutions for (1.3) are obtained in Section 3. Finally, numerical simulation is presented in Section 4 to verify the theoretical prediction.

#### 2. Fenichel theory

In this section, we review the necessary theory which will be used for our discussion. Taking the exposition in Jones [12], for details, one can resort to Fenichel [7], or Jones [12]. Geometric singular perturbation theory was first given by Fenichel [7], and was often referred to as Fenichel theory. A comprehensive overview of the theory, as well as new proofs of many theorems and detailed examples of applications are given by [10].

Consider the standard fast-slow system:

$$\begin{cases} \dot{f}(t) = F(f(t), g(t), \varepsilon), \\ \dot{g}(t) = \varepsilon G(f(t), g(t), \varepsilon), \end{cases}$$
(2.1)

where  $\cdot$  is the derivative with respect to  $t, 0 < \varepsilon \ll 1$  is a real parameter,  $f = (f_1, f_2, \dots, f_{n_f})^T \in \mathbb{R}^{n_f}, g = (g_1, g_2, \dots, g_{n_g})^T \in \mathbb{R}^{n_g}, n_f + n_g = n, \max ||f|| =$ 

max  $\|\dot{g}\|$ , and f corresponds to fast directions and g corresponds to slow directions. f, g are  $C^{\infty}$  on set  $U \times V$ , where  $U \subset \mathbb{R}^n$  and V is an open interval containing 0. Assume that for  $\varepsilon = 0$ , the system has a compact normally hyperbolic manifold  $M_0$  which is contained in a set  $\{f(x, y, 0) = 0\}$ . The manifold  $M_0$  is said to be normally hyperbolic if the linearization of (2.1) at each point in  $M_0$  has exactly  $n_g$ eigenvalues with zero real part, where  $n_g$  is the dimension of the center dimensions.

**Proposition 2.1** (Feichel's Persistence Theorem 1). Under assumption above, if  $\varepsilon > 0$  is sufficiently small, there exists a function  $M_{\varepsilon}$  defined on D such that the manifold  $M_{\varepsilon}$  is locally invariant under the flow of (2.1). Moreover,  $M_{\varepsilon}$  is  $C^{r}$  smooth for any  $r < +\infty$ ,  $M_{\varepsilon} = \{(x, y) | x = h^{\varepsilon}(y)\}$  for some  $C^{r}$  function  $h^{\varepsilon}$  and y in some compact set K.

With a change of time-scale  $\tau = \varepsilon t$ ,  $\ell = \frac{d}{d\tau}$ , system (2.1) can be changed to

$$\begin{cases} \varepsilon f' = F(f, g, \varepsilon), \\ g' = \varepsilon G(f, g, \varepsilon), \end{cases}$$
(2.2)

when  $\varepsilon \neq 0$ , system (2.1) and (2.2) are equivalent, system (2.1) is called the fast system and (2.2) is called the slow system. Geometric singular perturbation theory exploits a differential equation's geometric structures, such as its slow (center) manifolds and their fast stable and unstable fibers.

# 3. Singularly perturbation analysis to dispersive $\mathbf{K}(3,1)$ equation

For given constant c < 0, substituting  $U(x,t) = u(x - ct) = u(\xi)$  into Equation (1.1), integrating and setting the integral constant to be zero, we have

$$-cu + \sigma u^{3} + u'' + \varepsilon (u' + u''') = 0, \qquad (3.1)$$

Then ODE (3.1) can be rewritten to a three-dimensional system as follows

$$\begin{cases} \frac{du}{d\xi} = y, \\ \frac{dy}{d\xi} = w, \\ \varepsilon \frac{dw}{d\xi} = cu - \sigma u^3 - w - \varepsilon y. \end{cases}$$
(3.2)

System (3.2) is obviously formulated on a slow time scale because of the location of the small parameter  $\varepsilon$ . The corresponding fast system is

$$\begin{aligned}
\frac{du}{d\tau} &= \varepsilon y, \\
\frac{dy}{d\tau} &= \varepsilon w, \\
\frac{dw}{d\tau} &= cu - \sigma u^3 - w - \varepsilon y.
\end{aligned}$$
(3.3)

Generally, system (3.2) is referred to as the slow system since the time scale  $\xi$  is slow, and system (3.3) is referred to as the fast system since the time scale  $\tau$  is fast. The two systems are equivalent when  $\varepsilon > 0$ . On the following, we shall find the equilibrium points of system (3.3) and discuss the local dynamical behavior in a neighborhood of the given equilibria. Let  $Z = (u, y, w)^T$  and

$$G(Z,c,\varepsilon) = \begin{pmatrix} \varepsilon y \\ \varepsilon w \\ cu - \sigma u^3 - w - \varepsilon y \end{pmatrix},$$

system (3.3) can be formulated as  $\frac{dZ}{d\tau} = G(Z, c, \varepsilon)$ . For  $c\sigma > 0$ , there are three equilibria satisfying  $G(Z, c, \varepsilon) = 0$ , which are  $Z_0 = (0, 0, 0)$ ,  $Z_1 = \left(\sqrt{\frac{c}{\sigma}}, 0, 0\right)$ ,  $Z_2 = \left(-\sqrt{\frac{c}{\sigma}}, 0, 0\right)$ .

Note that  $A_i$  is the coefficient matrix of the linearized system for  $(3.3)_{\varepsilon=0}$  at equilibrium points  $Z_i(u_i, 0, 0)$  for (i = 1, 2), with  $u_0 = 0, u_1 = \sqrt{\frac{c}{\sigma}}, u_2 = -\sqrt{\frac{c}{\sigma}}$ . We obtain

$$A_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c - 3\sigma u_i^2 & 0 & -1 \end{pmatrix},$$

since  $M_0$  can be characterized as the graph of a function, we have that

$$M_{\varepsilon} = \{(u, y, w) | w = g(u, y, \varepsilon)\},\$$

where g is a analytic function depending smoothly on  $\varepsilon$  and satisfies  $g(u, y, 0) = cu - \sigma u^3$ . Therefore, system (3.2) can be reduced to the following differential equation on  $M_{\varepsilon}$ 

$$\begin{cases} \frac{du}{d\xi} = y, \\ \frac{dy}{d\xi} = g(u, v, \varepsilon), \end{cases}$$
(3.4)

which has the limiting form on  $M_0$  of

$$\begin{cases} \frac{du}{d\xi} = y, \\ \frac{dy}{d\xi} = cu - \sigma u^3. \end{cases}$$
(3.5)

It is necessary to state that  $M_0$  is normally hyperbolic by examining the linearization of fast system (3.5). According to the geometric singular perturbation theory, it is easy to know that  $M_{\varepsilon}$  can be characterized as the graph of a function, and g(u, y, w) be expanded with respect to  $\varepsilon$  for  $\varepsilon > 0$  sufficiently small. Assume that

$$w = g(u, y, \varepsilon) = cu - \sigma u^3 + \varepsilon g_1(u, y) + O(\varepsilon^2).$$
(3.6)



Figure 1. Heteroclinic orbits of system (3.5)) under the condition c < 0,  $\sigma < 0$ .

Substituting (3.6) into the last equation of slow system (3.2), comparing coefficients of  $\varepsilon$  yields

$$g_1(u,y) = -(c+1-3\sigma u^2)y.$$
(3.7)

Thus, dynamics on the slow manifold  $M_{\varepsilon}$  for system (3.2) becomes

$$\begin{cases} \frac{du}{d\xi} = y, \\ \frac{dy}{d\xi} = cu - \sigma u^3 - \varepsilon (c + 1 - 3\sigma u^2)y + O(\varepsilon^2). \end{cases}$$
(3.8)

Note that when  $\varepsilon = 0$ , system (3.8) reduces to system (3.5). By the bifurcation theory of dynamical systems [14], it is easy to find that the equilibrium point  $u_1(\sqrt{\frac{c}{\sigma}}, 0)$  and  $u_2(-\sqrt{\frac{c}{\sigma}}, 0)$  are saddle points, then there exists a heteroclinic orbit  $L_1$  and another heteroclinic orbit exists, which denoted by  $L_2$  (see Figure 1).

However, with a small perturbation, the heteroclinic orbits will persist or blow up? In the present paper, we wonder the sufficient conditions to guarantee the persistence of heteroclinic orbits of system (3.8). As  $u_1 = (\sqrt{\frac{c}{\sigma}}, 0)$  and  $u_2 = (-\sqrt{\frac{c}{\sigma}}, 0)$  are two saddles of (3.8), let  $L_3$  be an unstable manifold of  $u_1$  and  $L_4$  be a stable manifold of  $u_2$  for system (3.8) for  $0 < \varepsilon \ll 1$ . In order to investigate the existence of a heteroclinic orbit connecting the saddle points  $u_1$  and  $u_2$  near  $L_1$  for  $0 < \varepsilon \ll 1$ , suppose that l is a segment normal line, which intersects with  $L_1$  at a point  $P_1 \in L_1$ , intersects  $L_3$  at a point  $P_2$ , and intersects  $L_4$  at  $P_3$  (see Figure 2).

Note that  $d(L_1, \varepsilon) = -\vec{n} \cdot \overrightarrow{P_2P_3}$ , where  $\vec{n} = (H_u(P_1), H_y(P_1))/|(H_y(P_1), -H_u(P_1))|$ , thus  $d(L_1, \varepsilon)$  is quoted to measure the distance between  $L_3$  and  $L_4$ . If  $d(L_1, \varepsilon) = 0$ , we conclude that system (3.8) has a heteroclinic orbit near  $L_1$ , which connecting  $u_2$  to  $u_1$  for  $0 < \varepsilon \ll 1$ . Similar procedure can be applied to study whether system (3.8) has another heteroclinic orbit near  $L_2$  for  $0 < \varepsilon \ll 1$ .

From [11, 17], the Melnikov function for system (3.8) can be expressed by

$$M(L) = \int_{L} F(u, y(u)) du, \qquad (3.9)$$



Figure 2. Diagram of the connection of saddles of system (3.8) with  $0 < \varepsilon \ll 1$ .

where  $F(u, y) = -(c + 1 - 3\sigma u^2)y + O(\varepsilon)$  and y(u) is the expression of  $L_1$ .

Note that  $L_1$  and  $L_2$  possess the following expressions

$$y(u) = \pm \alpha_0 (u^2 - A^2), \quad -\sqrt{\frac{c}{\sigma}} < u < \sqrt{\frac{c}{\sigma}}, \tag{3.10}$$

where  $\alpha_0$  is a coefficient, A is the root of the equation  $u^4 - \frac{2c}{\sigma}u^2 - \frac{4}{\sigma}h_1 = 0$ . Therefore, it is obtained

$$M(L_{1}) = \int_{-\sqrt{\frac{c}{\sigma}}}^{\sqrt{\frac{c}{\sigma}}} F(u, y(u)) du = \pm 2\alpha_{0} \int_{0}^{\sqrt{\frac{c}{\sigma}}} [(c+1-3\sigma u^{2})(u^{2}-A^{2}) + O(\varepsilon)] du$$
$$= \pm \frac{2\alpha_{0}}{15} \sqrt{\frac{c}{\sigma}} (4c^{2}-5c+15A^{2}\sigma) + O(\varepsilon).$$
(3.11)

Consequently, we have following Lemma:

**Lemma 3.1.** For  $\varepsilon > 0$  sufficiently small, it has

$$d(\varepsilon, L_1) = \varepsilon \cdot M(L_1) + O(\varepsilon^2),$$

where  $M(L_1) = \pm \frac{2\alpha_0}{15} \sqrt{\frac{c}{\sigma}} (4c^2 - 5c + 15A^2\sigma) + O(\varepsilon).$ 

On the following, we will prove that there exists two heteroclinic orbits connects the two equilibria, and then the perturbed dispersive K(3,1) equation has kink and antikink wave solution connecting  $u_2$  to  $u_1$ . It is obvious that when  $c = \frac{5-\sqrt{25-240A^2\sigma}}{8}$  and  $\varepsilon = 0$ ,  $M(L_1) \equiv 0$ . Furthermore,

$$\frac{\partial M(L_1)}{\partial c}\Big|_{c=\frac{5-\sqrt{25-240A^2\sigma}}{8},\varepsilon=0} = \pm \frac{\alpha_0}{12\sqrt{c\sigma}} (5-48A^2\sigma - \sqrt{25-240A^2\sigma}) \neq 0.$$
(3.12)

Hence, from the implicit function theorem and the definition of the function  $d(\varepsilon, L_1)$ , we conclude that perturbed dispersive K(3, 1) equation has a heteroclinic orbit for  $0 < \varepsilon \ll 1$ . In other words, system (1.3) has a kink wave solution. Therefore, we have the following theorem:



**Figure 3.** Simulations: (a) shows the existence of heteroclinic orbits of system (3.2) for  $\varepsilon = 0.01$ , (b) shows the heteroclinic orbits blow up for  $\varepsilon = 0.1$ .

**Theorem 3.1.** For the singularly perturbed dispersive K(3,1) equation with  $0 < \varepsilon \ll 1$  and c < 0,  $\sigma < 0$ , we conclude that system (1.3) has kink wave solution and antikink wave solution, when the wave speed satisfies  $c = \frac{5-\sqrt{25-240A^2\sigma}}{8} + O(\varepsilon)$ .

### 4. Numerical simulations

We now numerically simulate the persistence of heteroclinic orbits of the singularly perturbed dispersive K(3, 1) equation (1.3) through system (3.2). Setting  $\sigma = -1$ , c = -1, and taking the initial condition  $u_0 = -1, y_0 = \pm 0.001, w_0 = 0.001$ , the persistence of heteroclinic orbits for sufficient small  $\varepsilon = 0.01$  is shown in Figure 3(a), and the heteroclinic orbits blow up for  $\varepsilon = 0.1$  is shown in Figure 3(b).

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