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A REVERSE MORE ACCURATE HARDY-HILBERT'S INEQUALITY*

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Abstract By means of the weight coefficients, the idea of introduced parameters, Hermite-Hadamard's inequality and Euler-Maclaurin summation formula, a reverse more accurate Hardy-Hilbert's inequality and the equivalent forms are given. The equivalent statements of the best possible constant factor related to a few parameters are also considered, and some particular reverse inequalities are obtained.

Keywords Weight coefficient, Hermite-Hadamard's inequality, Hardy-Hilbert's inequality, reverse, equivalent statement, parameter.

MSC(2010) 26D15.

1. Introduction

Assuming that $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m, b_n \geq 0, 0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, we have the following Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf [6], Theorem 315):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1.1}$$

In 2006, by introducing parameters $\lambda_i \in (0,2]$ $(i=1,2), \lambda_1+\lambda_2=\lambda\in(0,4]$, an extension of (1.1) was provided by Krnić et al. [13] as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{1.2}$$

where, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible, and

$$B(u,v):=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt \ (u,v>0)$$

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is the beta function. For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (1.2) reduces to (1.1); for $p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (2) reduces to Yang's inequality in [25]. Recently, applying (1.2), Adiyasuren et al. [1] gave a new Hardy-Hilbert's inequality with the kernel as $\frac{1}{(m+n)^\lambda}$ involving partial sums.

If $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(y)dy < \infty$, then we still have the following Hardy -Hilbert's integral inequality (cf. [6], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \tag{1.3}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1), (1.2) and (1.3) with their extensions and reverses are important in analysis and its applications (cf. [2, 3, 5, 7, 8, 14, 16, 17, 21–23, 26, 29–31]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [6], Theorem 351): If $K(t)$ $(t > 0)$ is decreasing, $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^\infty K(t)t^{s-1}dt < \infty$, then for $a_n \geq 0, 0 < \sum_{n=1}^\infty a_n^p < \infty$, we have

$$\int_0^\infty x^{p-1} (\sum_{n=1}^\infty K(nx)a_n)^p dx < \phi^p(\frac{1}{q}) \sum_{n=1}^\infty a_n^p. \tag{1.4}$$

In recent years, some new extensions of (1.4) with the reverses were provided by [18–20, 27, 28].

In 2016, by means of the techniques of real analysis, Hong et al. [9] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. The other similar works were given by [4, 10–12, 23].

In this paper, following the way of [9, 13], by the use of the weight coefficients, the idea of introduced parameters, Hermite-Hadamard's inequality, Euler-Maclaurin summation formula and the techniques of real analysis, a reverse more accurate Hardy-Hilbert's inequality as well as the equivalent forms are given. The equivalent statements of the best possible constant factor related to several parameters are considered, and and some particular reverse inequalities are obtained.

2. Some lemmas

In what follows, we assume that $0 < p < 1 (q < 0), \frac{1}{p} + \frac{1}{q} = 1, \mathbf{N} := \{1.2. \cdots\}, \eta_i \in [0, \frac{1}{4}]$ $(i = 1, 2), \eta_1 + \eta_2 = \eta \in [0, \frac{1}{2}], \lambda \in (0.3], \lambda_i \in [0, \frac{3}{2}] \cap (0, \lambda)$,

$$k_\lambda(\lambda_i) = B(\lambda_i, \lambda - \lambda_i) (i = 1, 2).$$

We also assume that $a_m, b_n \geq 0$, such that

$$0 < \sum_{m=1}^\infty (m - \eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} a_m^p < \infty \text{ and}$$
$$0 < \sum_{n=1}^\infty (n - \eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} b_n^q < \infty, \tag{2.1}$$

Lemma 2.1. *For $\lambda_2 \in [0, \frac{3}{2}] \cap (0, \lambda)$, define the following weight coefficient:*

$$\varpi(\lambda_2, m) := (m - \eta_1)^{\lambda - \lambda_2} \sum_{n=1}^{\infty} \frac{(n - \eta_2)^{\lambda_2 - 1}}{(m + n - \eta)^{\lambda}} \ (m \in \mathbf{N}). \tag{2.2}$$

We have the following inequalities:

$$k_\lambda(\lambda_2) \left[1 - O(\frac{1}{(m - \eta_1)^{\lambda_2}})\right] < \varpi(\lambda_2, m) < k_\lambda(\lambda_2) \ (m \in \mathbf{N}), \tag{2.3}$$

where, $O(\frac{1}{(m-\eta_1)^{\lambda_2}})$ is indicated as

$$O(\frac{1}{(m - \eta_1)^{\lambda_2}}) := \frac{1}{k_\lambda(\lambda_2)} \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{\lambda_2 - 1}}{(1 + u)^{\lambda}} du \ (> 0).$$

Proof. For fixed $m \in \mathbf{N}$, we set the following real function:

$$g(m, t) := \frac{(t - \eta_2)^{\lambda_2 - 1}}{(m - \eta + t)^{\lambda}} \ (t > \eta_2).$$

In the following we divide two cases to prove (2.3).

(i) For $\lambda_2 \in [0, 1] \cap (0, \lambda)$, since

$$(-1)^i g^{(i)}(m, t) > 0 \ (t > \eta_2; i = 0, 1, 2),$$

by Hermite-Hadamard's inequality (cf. [15]), setting $u = \frac{t-\eta_2}{m-\eta_1}$, we have

$$\begin{aligned}\varpi(\lambda_2, m) &= (m - \eta_1)^{\lambda - \lambda_2} \sum_{n=1}^{\infty} g(m, n) < (m - \eta_1)^{\lambda - \lambda_2} \int_{\frac{1}{2}}^{\infty} g(m, t) dt \\ &= (m - \eta_1)^{\lambda - \lambda_2} \int_{\frac{1}{2}}^{\infty} \frac{(t - \eta_2)^{\lambda_2 - 1} dt}{(m - \eta_1 + t - \eta_2)^{\lambda}} = \int_{\frac{\frac{1}{2} - \eta_2}{m - \eta_1}}^{\infty} \frac{u^{\lambda_2 - 1}}{(1 + u)^{\lambda}} du \\ &\leq \int_0^{\infty} \frac{u^{\lambda_2 - 1}}{(1 + u)^{\lambda}} du = B(\lambda_2, \lambda - \lambda_2) = k_\lambda(\lambda_2).\end{aligned}$$

In view of the decreasingness property of series, we obtain

$$\begin{aligned}\varpi(\lambda_2, m) &= (m - \eta_1)^{\lambda - \lambda_2} \sum_{n=1}^{\infty} g(m, n) > (m - \eta_1)^{\lambda - \lambda_2} \int_1^{\infty} g(m, t) dt \\ &= \int_{\frac{1-\eta_2}{m-\eta_1}}^{\infty} \frac{u^{\lambda_2 - 1} du}{(1 + u)^{\lambda}} = k_\lambda(\lambda_2)[1 - O(\frac{1}{(m - \eta_1)^{\lambda_2}})] > 0,\end{aligned}$$

where, $O(\frac{1}{(m-\eta_1)^{\lambda_2}}) = \frac{1}{k_\lambda(\lambda_2)} \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du > 0$, satisfying

$$0 < \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{\lambda_2 - 1}}{(1 + u)^{\lambda}} du < \int_0^{\frac{1-\eta_2}{m-\eta_1}} u^{\lambda_2 - 1} du = \frac{1}{\lambda_2} (\frac{1 - \eta_2}{m - \eta_1})^{\lambda_2} \ (m \in \mathbf{N}).$$

Hence, we obtain (2.3).

(ii) For $\lambda_2 \in (1, \frac{3}{2}] \cap (0, \lambda)$, by using Euler-Maclaurin summation formula and Bernoulli function of 1-order as $\rho(t) := t - [t] - \frac{1}{2}$ (cf. [13]), we have

$$\begin{aligned}\sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \frac{1}{2}g(m,1) + \int_1^{\infty} \rho(t)g'(m,t)dt \\ &= \int_{\eta_2}^{\infty} g(m,t)dt - h(m),\end{aligned}$$

where,

$$h(m) := \int_{\eta_2}^{1} g(m,t)dt - \frac{1}{2}g(m,1) - \int_1^{\infty} \rho(t)g'(m,t)dt.$$

We obtain $-\frac{1}{2}g(m,1) = \frac{-(1-\eta_2)^{\lambda_2-1}}{2(m-\eta+1)^{\lambda}}$, and integrating by parts, it follows that

$$\begin{aligned}\int_{\eta_2}^{1} g(m,t)dt &= \int_{\eta_2}^{1} \frac{(t-\eta_2)^{\lambda_2-1}dt}{(m-\eta+t)^{\lambda}} = \frac{1}{\lambda_2}\int_{\eta_2}^{1} \frac{d(t-\eta_2)^{\lambda_2}}{(m-\eta+t)^{\lambda}} \\ &= \frac{1}{\lambda_2}\frac{(t-\eta_2)^{\lambda_2}}{(m-\eta+t)^{\lambda}}\Big|_{\eta_2}^{1} + \frac{\lambda}{\lambda_2}\int_{\eta_2}^{1} \frac{(t-\eta_2)^{\lambda_2}dt}{(m-\eta+t)^{\lambda+1}} \\ &= \frac{1}{\lambda_2}\frac{(1-\eta_2)^{\lambda_2}}{(m-\eta+1)^{\lambda}} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\int_{\eta_2}^{1} \frac{d(t-\eta_2)^{\lambda_2+1}}{(m-\eta+t)^{\lambda+1}} \\ &> \frac{1}{\lambda_2}\frac{(1-\eta_2)^{\lambda_2}}{(m-\eta+1)^{\lambda}} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\frac{(t-\eta_2)^{\lambda_2+1}}{(m-\eta+t)^{\lambda+1}}\Big|_{\eta_2}^{1} \\ &> \frac{1}{\lambda_2}\frac{(1-\eta_2)^{\lambda_2}}{(m-\eta+1)^{\lambda}} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\frac{(t-\eta_2)^{\lambda_2+1}}{(m-\eta+t)^{\lambda+1}}\Big|_{\eta_2}^{1} \\ &\quad + \frac{\lambda(\lambda+1)}{\lambda_2(\lambda_2+1)(m-\eta+1)^{\lambda+2}}\int_{\eta_2}^{1}(t-\eta_2)^{\lambda_2+1}dt \\ &= \frac{1}{\lambda_2}\frac{(1-\eta_2)^{\lambda_2}}{(m-\eta+1)^{\lambda}} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\frac{(1-\eta_2)^{\lambda_2+1}}{(m-\eta+1)^{\lambda+1}} \\ &\quad + \frac{\lambda(\lambda+1)}{\lambda_2(\lambda_2+1)\lambda_2+2)}\frac{(1-\eta_2)^{\lambda_2+2}}{(m-\eta+1)^{\lambda+2}}.\end{aligned}$$

We find

$$\begin{aligned}-g'(m,t) &= \frac{(\lambda_2-1)(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda}} + \frac{\lambda(t-\eta_2)^{\lambda_2-1}}{(m-\eta+t)^{\lambda+1}} \\ &= \frac{(\lambda+1-\lambda_2)(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda}} - \frac{\lambda(m-\eta_1)(t-\eta_2)^{\lambda_2-1}}{(m-\eta+t)^{\lambda+1}},\end{aligned}$$

and for $1 \leq \lambda_2 \leq \frac{3}{2}$, $\lambda_2 < \lambda \leq 3$, $i = 0, 1, 2, 3$, it follows that

$$(-1)^i \frac{d^i}{dt^i}[\frac{(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda}}] > 0, (-1)^i \frac{d^i}{dt^i}[\frac{(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda+1}}] > 0 \ (t > \eta_2).$$

Still by Euler-Maclaurin summation formula (cf. [13]), setting $a := 1-\eta_2 \in [\frac{3}{4}, 1]$, we obtain

$$(\lambda+1-\lambda_2)\int_1^{\infty} \rho(t)\frac{(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda}}dt > -\frac{(\lambda+1-\lambda_2)(1-\eta_2)^{\lambda_2-2}}{12(m-\eta+1)^{\lambda}},$$

$$
\begin{aligned}
&-(m-\eta_1)\lambda\int_1^\infty \rho(t)\frac{(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda+1}}dt \\
&> \frac{(m-\eta_1)\lambda a^{\lambda_2-2}}{12(m-\eta+1)^{\lambda+1}} - \frac{(m-\eta_1)\lambda}{720}[\frac{(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda+1}}]''_{t=1} \\
&= \frac{\lambda a^{\lambda_2-2}}{12(m-\eta+1)^{\lambda}} - \frac{\lambda a^{\lambda_2-1}}{12(m-\eta+1)^{\lambda+1}} - \frac{\lambda a^{\lambda_2-4}}{720} \\
&\quad\times[\frac{(\lambda+1)(\lambda+2)a^2}{(m-\eta+1)^{\lambda+2}} + \frac{2(\lambda+1)(2-\lambda_2)a}{(m-\eta+1)^{\lambda+1}} + \frac{(2-\lambda_2)(3-\lambda_2)}{(m-\eta+1)^{\lambda}}],
\end{aligned}
$$

and then we have

$$
h(m) > \frac{a^{\lambda_2-4}}{(m-\eta+1)^{\lambda}}h_1 + \frac{a^{\lambda_2-3}}{(m-\eta+1)^{\lambda+1}}h_2 + \frac{a^{\lambda_2-2}}{(m-\eta+1)^{\lambda+2}}h_3,
$$

where, h_i $(i=1,2,3)$ are indicated as

$$
\begin{aligned}
h_1 &:= \frac{a^4}{\lambda_2} - \frac{a^3}{2} - \frac{(1-\lambda_2)a^2}{12} - \frac{\lambda(2-\lambda_2)(3-\lambda_2)}{720}, \\
h_2 &:= \frac{a^4}{\lambda_2(\lambda_2+1)} - \frac{a^2}{12} - \frac{(\lambda+1)(2-\lambda_2)}{360} \text{ and} \\
h_3 &:= \frac{a^4}{\lambda_2(\lambda_2+1)(\lambda_2+2)} - \frac{\lambda+2}{720}.
\end{aligned}
$$

For $\lambda\in(0,3], \lambda_2\in[1,\frac{3}{2}], a\in[\frac{3}{4},1]$, we find

$$
h_1 > \frac{a^2}{12\lambda_2}[\lambda_2^2-(6a+1)\lambda_2+12a^2] - \frac{1}{120}.
$$

In view of

$$
\frac{\partial}{\partial a}[\lambda_2^2-(6a+1)\lambda_2+12a^2] = 6(-\lambda_2+4a) > 0 \ \left(a \geq \frac{3}{4} > \frac{\lambda_2}{4}\right),
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial\lambda_2}[\lambda_2^2-(6a+1)\lambda_2+12a^2] \\
&= 2\lambda_2-(6a+1) \leq 2\times\frac{3}{2} - (6\times\frac{3}{4}+1) = 3-\frac{9}{2} < 0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
h_1 &> \frac{(3/4)^2}{12(3/2)}[(\frac{3}{2})^2 - (6\times\frac{3}{4}+1)\frac{3}{2} + 12(\frac{3}{4})^2] - \frac{1}{120} \\
&= \frac{3}{128} - \frac{1}{120} > 0, \\
h_2 &\geq a^2(\frac{4a^2}{15} - \frac{1}{12}) - \frac{1}{90} \geq (\frac{3}{4})^2[\frac{4(3/4)^2}{15} - \frac{1}{12}] - \frac{1}{90} \\
&= \frac{3}{80} - \frac{1}{90} > 0,
\end{aligned}
$$

$$h_3 \geq \frac{8a^4}{105} - \frac{5}{720} \geq \frac{8(3/4)^4}{105} - \frac{5}{720} = \frac{27}{1120} - \frac{1}{114} > 0,$$

and then $h(m) > 0$ $(m \in \mathbf{N})$.

On the other hand, we also have

$$\sum_{n=1}^{\infty} g(m,n) = \int_1^{\infty} g(m,t)dt + \frac{1}{2}g(m,1) + \int_1^{\infty} \rho(t)g'(m,t)dt$$
$$= \int_1^{\infty} g(m,t)dt + H(m),$$

where, $H(m)$ is indicated as

$$H(m) := \frac{1}{2}g(m,1) + \int_1^{\infty} \rho(t)g'(m,t)dt.$$

We have obtained $\frac{1}{2}g(m,1) = \frac{a^{\lambda_2-1}}{2(m-\eta+1)^{\lambda}}$ and

$$g'(m,t) = \frac{-(\lambda+1-\lambda_2)(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda}} + \frac{\lambda(m-\eta_1)(t-\eta_2)^{\lambda_2-1}}{(m-\eta+t)^{\lambda+1}}.$$

For $\lambda \in (0,3], \lambda_2 \in (1,\frac{3}{2}] \cap (0,\lambda)$, by Euler-Maclaurin summation formula (cf. [13]), we obtain

$$-(\lambda+1-\lambda_2)\int_1^{\infty} \rho(t)\frac{(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda}}dt > 0,$$
$$(m-\eta_1)\lambda\int_1^{\infty} \rho(t)\frac{(t-\eta_2)^{\lambda_2-2}}{(m-\eta+t)^{\lambda+1}}dt > \frac{-(m-\eta_1)\lambda a^{\lambda_2-2}}{12(m-\eta+1)^{\lambda+1}}$$
$$= -\frac{\lambda a^{\lambda_2-2}}{12(m-\eta+1)^{\lambda}} + \frac{\lambda a^{\lambda_2-1}}{12(m-\eta+1)^{\lambda+1}} > -\frac{\lambda a^{\lambda_2-2}}{12(m-\eta+1)^{\lambda}}.$$

Hence, we have

$$H(m) > \frac{a^{\lambda_2-1}}{2(m-\eta+1)^{\lambda}} - \frac{\lambda a^{\lambda_2-2}}{12(m-\eta+1)^{\lambda}}$$
$$= \frac{(\frac{a}{2}-\frac{\lambda}{12})a^{\lambda_2-2}}{(m-\eta+1)^{\lambda}} \geq \frac{(\frac{3/4}{2}-\frac{3}{12})a^{\lambda_2-2}}{(m-\eta+1)^{\lambda}} > 0,$$

and the following inequalities:

$$\int_1^{\infty} g(m,t)dt < \sum_{n=1}^{\infty} g(m,n) < \int_{\eta_2}^{\infty} g(m,t)dt.$$

In view of the the results in the case (i), we still can obtain (2.3).

The lemma is proved. □

Lemma 2.2. *We have the following reverse more accurate Hardy-Hilbert's inequality:*

$$I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}}$$

$$
\begin{aligned}
&> B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \\
&\quad \times \left\{ \sum_{m=1}^{\infty} \left[1 - O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right] (m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{n=1}^{\infty} (n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q \right\}^{\frac{1}{q}}. \qquad (2.4)
\end{aligned}
$$

Proof. In the same way of obtaining (2.3), for $\lambda_1 \in [0, \frac{3}{2}] \cap (0, \lambda)$, we still find the following inequality of another weight coefficient:

$$
\omega(\lambda_1, n) := (n-\eta_2)^{\lambda-\lambda_1} \sum_{m=1}^{\infty} \frac{(m-\eta_1)^{\lambda_1-1}}{(m+n-\eta)^{\lambda}} < B(\lambda_1, \lambda-\lambda_1) \ (n \in \mathbf{N}). \qquad (2.5)
$$

By the reverse Hölder's inequality (cf. [15]), we obtain

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n-\eta)^{\lambda}} \left[\frac{(n-\eta_2)^{(\lambda_2-1)/p} a_m}{(m-\eta_1)^{(\lambda_1-1)/q}} \right] \left[\frac{(m-\eta_1)^{(\lambda_1-1)/q} b_n}{(n-\eta_2)^{(\lambda_2-1)/p}} \right] \\
&\geq \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n-\eta)^{\lambda}} \frac{(n-\eta_2)^{\lambda_2-1} a_m^p}{(m-\eta_1)^{(\lambda_1-1)(p-1)}} \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n-\eta)^{\lambda}} \frac{(m-\eta_1)^{\lambda_1-1} b_n^q}{(n-\eta_2)^{(\lambda_2-1)(q-1)}} \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{m=1}^{\infty} \varpi(\lambda_2, m) (m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{n=1}^{\infty} \omega(\lambda_1, n) \sum_{n=1}^{\infty} (n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q \right\}^{\frac{1}{q}}.
\end{aligned}
$$

Then by (2.3) and (2.5), in view of $0 < p < 1, q < 0$, we have (2.4).

The lemma is proved. □

Remark 2.1. By (2.4), for $\lambda_1 + \lambda_2 = \lambda \in (0, 3], 0 < \lambda_i \leq \frac{3}{2} \, (i = 1, 2)$, we find $\omega(\lambda_1, n) < B(\lambda_1, \lambda_2)$,

$$
B(\lambda_1, \lambda_2) \left[1 - O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right] < \varpi(\lambda_2, m) < B(\lambda_1, \lambda_2) \ (m, n \in \mathbf{N}),
$$

$$
0 < \sum_{m=1}^{\infty} (m-\eta_1)^{p(1-\lambda_1)-1} a_m^p < \infty, 0 < \sum_{n=1}^{\infty} (n-\eta_2)^{q(1-\lambda_2)-1} b_n^q < \infty,
$$

and the following reverse inequality:

$$
\begin{aligned}
I &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}} \\
&> B(\lambda_1, \lambda_2) \left\{ \sum_{m=1}^{\infty} \left[1 - O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right] (m-\eta_1)^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{p}}
\end{aligned}
$$

$$\times \left[\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{2.6}$$

Lemma 2.3. *The constant fctor $B(\lambda_1,\lambda_2)$ in (2.6) is the best possible.*

Proof. For any $0<\varepsilon<p\lambda_1$, we set

$$\widetilde{a}_m := (m-\eta_1)^{\lambda_1-\frac{\varepsilon}{p}-1}, \widetilde{b}_n := (n-\eta_2)^{\lambda_2-\frac{\varepsilon}{q}-1}\ (m,n\in\mathbf{N}).$$

If there exists a constant $M\geq B(\lambda_1,\lambda_2)$, such that (2.6) is valid when we replace $B(\lambda_1,\lambda_2)$ by M, then in particular, substitution of $a_m=\widetilde{a}_m$ and $b_n=\widetilde{b}_n$ in (2.6), we have

$$\begin{aligned}\widetilde{I} &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{(m+n-\eta)^{\lambda}}\\ &> M\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p(1-\lambda_1)-1}\widetilde{a}_m^p\right\}^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}.\end{aligned}$$

By the decreasingness property of series, we obtain

$$\begin{aligned}\widetilde{I} &> M\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p(1-\lambda_1)-1}(m-\eta_1)^{p\lambda_1-\varepsilon-p}\right\}^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\lambda_2)-1}(n-\eta_2)^{q\lambda_2-\varepsilon-q}\right]^{\frac{1}{q}}\\ &= M\left[\sum_{m=1}^{\infty}(m-\eta_1)^{-\varepsilon-1}-\sum_{m=1}^{\infty}O(\frac{1}{(m-\eta_1)^{\lambda_2+\varepsilon+1}})\right]^{\frac{1}{p}}\\ &\quad\times\left[a^{-\varepsilon-1}+\sum_{n=2}^{\infty}(n-\eta_2)^{-\varepsilon-1}\right]^{\frac{1}{q}}\\ &> M\left[\int_1^{\infty}(x-\eta_1)^{-\varepsilon-1}dx-O(1)\right]^{\frac{1}{p}}\left[a^{-\varepsilon-1}+\int_1^{\infty}(y-\eta_2)^{-\varepsilon-1}dy\right]^{\frac{1}{q}}\\ &= \frac{M}{\varepsilon}\left[(1-\eta_1)^{-\varepsilon}-\varepsilon O(1)\right]^{\frac{1}{p}}(\varepsilon a^{-\varepsilon-1}+a^{-\varepsilon})^{\frac{1}{q}}.\end{aligned}$$

By (2.5), setting $\widehat{\lambda}_1:=\lambda_1-\frac{\varepsilon}{p}\in(0,\frac{3}{2})\cap(0,\lambda)$ $(0<\widehat{\lambda}_2:=\lambda_2+\frac{\varepsilon}{p}<\lambda)$, we find

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty}\left[(n-\eta_2)^{\lambda_2+\frac{\varepsilon}{p}}\sum_{m=1}^{\infty}\frac{(m-\eta_1)^{(\lambda_1-\frac{\varepsilon}{p})-1}}{(m+n-\eta)^{\lambda}}\right](n-\eta_2)^{-\varepsilon-1}\\ &= \sum_{n=1}^{\infty}\omega(\widehat{\lambda}_1,n)(n-\eta_2)^{-\varepsilon-1} < B(\widehat{\lambda}_1,\widehat{\lambda}_2)\left[a^{-\varepsilon-1}+\sum_{n=2}^{\infty}(n-\eta_2)^{-\varepsilon-1}\right]\end{aligned}$$

$$< B(\widehat{\lambda}_1, \widehat{\lambda}_2) \left[a^{-\varepsilon-1} + \int_1^\infty (y-\eta_2)^{-\varepsilon-1} dy \right] = \frac{1}{\varepsilon} B(\widehat{\lambda}_1, \widehat{\lambda}_2)(\varepsilon a^{-\varepsilon-1} + a^{-\varepsilon}).$$

Then we have

$$B(\widehat{\lambda}_1, \widehat{\lambda}_2)(\varepsilon a^{-\varepsilon-1} + a^{-\varepsilon}) > M \left[(1-\eta_1)^{-\varepsilon} - \varepsilon O(1) \right]^{\frac{1}{p}} (\varepsilon a^{-\varepsilon-1} + a^{-\varepsilon})^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, we find $B(\lambda_1, \lambda_2) \geq M$. Hence, $M = B(\lambda_1, \lambda_2)$ is the best possible constant factor of (2.6).

The lemma is proved. □

Remark 2.2. (i) Setting $\widetilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \widetilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$ in (2.4), we find

$$\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda.$$

(ii) For $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we obtain

$$\widetilde{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} > \frac{(1-p)\lambda_1}{p} + \frac{\lambda_1}{q} = 0,$$
$$\widetilde{\lambda}_1 < \frac{\lambda_1 + p(\lambda-\lambda_1)}{p} + \frac{\lambda_1}{q} = \lambda, 0 < \widetilde{\lambda}_2 = \lambda - \widetilde{\lambda}_1 < \lambda,$$

and then we have $B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \in R_+ = (0, \infty)$.

(iii) For $\lambda - \lambda_1 - \lambda_2 \in [p(\lambda - \lambda_1 - \frac{3}{2}), p(\frac{3}{2} - \lambda_1)]$, we obtain $\widetilde{\lambda}_i \leq \frac{3}{2}$.

Hence, in view of (i), (ii) and (iii), we still can reduce (2.6) to the following:

$$\begin{aligned} I :=& \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n-\eta)^\lambda} \\ >& B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \left\{ \sum_{m=1}^\infty \left[1 - O(\frac{1}{(m-\eta_1)^{\widetilde{\lambda}_2}}) \right] (m-\eta_1)^{p(1-\widetilde{\lambda}_1)-1} a_m^p \right\}^{\frac{1}{p}} \\ & \times \left[\sum_{n=1}^\infty (n-\eta_2)^{q(1-\widetilde{\lambda}_2)-1} b_n^q \right]^{\frac{1}{q}}. \end{aligned} \quad (2.7)$$

Lemma 2.4. *If the constant factor* $B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ *in (2.4) is the best possible, then for*

$$\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1)) \cap \left[p(\lambda - \lambda_1 - \frac{3}{2}), p(\frac{3}{2} - \lambda_1) \right] (\supset \{0\}), \quad (2.8)$$

we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. If the constant factor $B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ in (2.4) is the best possible, then in view of (2.8) and (2.7), we have the following inequality:

$$B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \geq B(\widetilde{\lambda}_1, \widetilde{\lambda}_2).$$

By the reverse Hölder's inequality (cf. [15]), we find

$$B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) = B(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})$$

$$= \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}}{(1+u)^\lambda} du = \int_0^\infty \frac{(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})}{(1+u)^\lambda} du$$
$$\geq \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda} du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda} du\right]^{\frac{1}{q}}$$
$$= B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1). \tag{2.9}$$

Hence, we have $B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) = B(\widetilde{\lambda}_1, \widetilde{\lambda}_2)$, namely, (2.9) keeps the form of equality.

We observe that (2.9) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [15])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } R_+.$$

Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_2-\lambda_1} = B/A$ a.e. in R_+., and then $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

The lemma is proved. □

3. Main results and some particular inequalities

Theorem 3.1. *Inequality (2.4) is equivalent to the following inequalities:*

$$J := \left\{\sum_{n=1}^\infty (n-\eta_2)^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1} \left[\sum_{m=1}^\infty \frac{a_m}{(m+n-\eta)^\lambda}\right]^p\right\}^{\frac{1}{p}}$$
$$> B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \tag{3.1}$$
$$\times \left\{\sum_{m=1}^\infty \left[1 - O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right] (m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}},$$

$$J_1 := \left\{\sum_{m=1}^\infty \frac{(m-\eta_1)^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})]^{q-1}} \left[\sum_{n=1}^\infty \frac{b_n}{(m+n-\eta)^\lambda}\right]^q\right\}^{\frac{1}{q}}$$
$$> B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \left\{\sum_{n=1}^\infty (n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}. \tag{3.2}$$

If the constant factor in (2.4) is the best possible, then, so is the constant factor in (3.1) and (3.2).

Proof. Suppose that (3.1) is valid. By the reverse Hölder's inequality, we have

$$I = \sum_{n=1}^\infty \left[(n-\eta_2)^{-\frac{1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})} \sum_{m=1}^\infty \frac{a_m}{(m+n-\eta)^\lambda}\right] \left[(n-\eta_2)^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})} b_n\right]$$
$$\geq J \left\{\sum_{n=1}^\infty (n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}. \tag{3.3}$$

Then by (3.1), we obtain (2.4). On the other hand, assuming that (2.4) is valid, we set

$$b_n := (n-\eta_2)^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1} \left[\sum_{m=1}^\infty \frac{a_m}{(m+n-\eta)^\lambda}\right]^{p-1}, \quad n \in \mathbf{N}.$$

If $J=\infty$, then (3.1) is naturally valid; if $J=0$, then it is impossible to make (3.1) valid, namely, $J>0$. Suppose that $0<J<\infty$. By (2.4), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q = J^p = I\\
&>B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\\
&\quad\times\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}J^{p-1},\\
&J=\left\{\sum_{n=1}^{\infty}(n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{p}}\\
&\quad>B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\\
&\quad\quad\times\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}},
\end{aligned}$$

namely, (3.1) follows, which is equivalent to (2.4).

Suppose that (3.2) is valid. By the reverse Hölder's inequality, we have

$$\begin{aligned}
I &= \sum_{m=1}^{\infty}\left[(1-O(\frac{1}{(m-\eta_1)^{\lambda_2}}))^{\frac{1}{p}}(m-\eta_1)^{\frac{1}{q}-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}a_m\right]\\
&\quad\times\left[\frac{(m-\eta_1)^{\frac{-1}{q}+(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}}{(1-O(\frac{1}{(m-\eta_1)^{\lambda_2}}))^{\frac{1}{p}}}\sum_{n=1}^{\infty}\frac{b_n}{(m+n-\eta)^{\lambda}}\right]\\
&\geq\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}J_1. \qquad (3.4)
\end{aligned}$$

Then by (3.2), we obtain (2.4). On the other hand, assuming that (2.4) is valid, we set

$$a_m := \frac{(m-\eta_1)^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m+n-\eta)^{\lambda}}\right]^{q-1}, \quad m\in\mathbf{N}.$$

If $J_1=\infty$, then (3.2) is naturally valid; if $J_1=0$, then it is impossible to make (3.2) valid, namely,.$J_1>0$. Suppose that $0<J_1<\infty$ By (2.4), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p = J_1^q = I\\
&>B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)J_1^{q-1}\left\{\sum_{n=1}^{\infty}(n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}},\\
&J_1=\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{q}}\\
&\quad> B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left\{\sum_{n=1}^{\infty}(n-\eta_2)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}},
\end{aligned}$$

namely, (3.2) follows, which is equivalent to (2.4), and then inequalities (2.4), (3.1) and (3.2) are equivalent.

If the constant factor in (2.4) is the best possible, then so is the constant factor in (3.1) and (3.2). Otherwise, by (3.3) (or (3.4)), we would reach a contradiction that the constant factor in (2.4) is not the best possible.

The theorem is proved. □

Theorem 3.2. *The following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) Both $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ and $B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})$ are independent of p,q;

(ii) $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)=B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})$;

(iii) $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ in (2.4) is the best possible constant factor;

(iv) if $\lambda-\lambda_1-\lambda_2\in(-p\lambda_1,p(\lambda-\lambda_1))\cap[p(\lambda-\lambda_1-\frac{3}{2}),p(\frac{3}{2}-\lambda_1)]$, then $\lambda_1+\lambda_2=\lambda$.

If the statement (iv) follows, namely, $\lambda_1+\lambda_2=\lambda$, then we have (2.6) and the following equivalent inequalities with the best possible constant factor $B(\lambda_1,\lambda_2)$:

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}(n-\eta_2)^{p\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n-\eta)^{\lambda}}\right]^p\right\}^{\frac{1}{p}}\\ >&B(\lambda_1,\lambda_2)\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})\right](m-\eta_1)^{p(1-\lambda_1)-1}a_m^p\right\}^{\frac{1}{p}},\end{aligned}\tag{3.5}$$

$$\begin{aligned}&\left\{\sum_{m=1}^{\infty}\frac{(m-\eta_1)^{q\lambda_1-1}}{[1-O(\frac{1}{(m-\eta_1)^{\lambda_2}})]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m+n-\eta)^{\lambda}}\right]^q\right\}^{\frac{1}{q}}\\ >&B(\lambda_1,\lambda_2)\left[\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{3.6}$$

Proof. $(i)=>(ii)$. By (i), in view of the continuity of the beta function, we have

$$\begin{aligned}&B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\\ =&\lim_{p\to1^-}\lim_{q\to-\infty}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)=B(\lambda_2,\lambda-\lambda_1),\\ &B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})\\ =&\lim_{p\to1^-}\lim_{q\to-\infty}B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})=B(\lambda_2,\lambda-\lambda_1),\end{aligned}$$

namely, $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)=B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})$.

$(ii)=>(iv)$. By (ii), (2.9) keeps the form of equality. In view of the proof of Lemma 2.4, it follows that $\lambda_1+\lambda_2=\lambda$.

$(iv)=>(i)$. If $\lambda_1+\lambda_2=\lambda$, then

$$\begin{aligned}&B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\\ =&B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})=B(\lambda_1,\lambda_2),\end{aligned}$$

which is independent of p,q. Hence, $(i)<=>(ii)<=>(iv)$.

$(iii) => (iv)$. By Lemma 2.4, we have $\lambda_1 + \lambda_2 = \lambda$.

$(iv) => (iii)$. By Lemma 2.3, for $\lambda_1 + \lambda_2 = \lambda$,

$$B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)(= B(\lambda_1, \lambda_2))$$

is the best possible constant factor of (2.4). Therefore, we have $(iii) <=> (iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent.

The theorem is proved. □

Remark 3.1. (i) For $\eta_i = 0, \lambda_i = \frac{\lambda}{2} \in (0, \frac{3}{2}]$ $(i = 1, 2; 0 < \lambda \leq 3)$ in (2.6), (3.5) and (3.6), we have the following equivalent inequalities with the best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$:

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}\\ >&B(\frac{\lambda}{2},\frac{\lambda}{2})\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{m^{\lambda/2}})\right]m^{p(1-\frac{\lambda}{2})-1}a_m^p\right\}^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}n^{p\frac{\lambda}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^{\lambda}}\right]^p\right\}^{\frac{1}{p}}\\ >&B(\frac{\lambda}{2},\frac{\lambda}{2})\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{m^{\lambda/2}})\right]m^{p(1-\frac{\lambda}{2})-1}a_m^p\right\}^{\frac{1}{p}},\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\left\{\sum_{m=1}^{\infty}\frac{m^{q\frac{\lambda}{2}-1}}{[1-O(\frac{1}{m^{\lambda/2}})]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m+n)^{\lambda}}\right]^q\right\}^{\frac{1}{q}}\\ >&B(\frac{\lambda}{2},\frac{\lambda}{2})\left[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{3.9}$$

(ii) For $\eta_i = \frac{1}{4}, \lambda_i = \frac{\lambda}{2} \in (0, \frac{3}{2}]$ $(i = 1, 2; 0 < \lambda \leq 3)$ in (2.6), (3.5) and (3.6), we have the following revere more accurate equivalent inequalities with the best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$:

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-\frac{1}{2})^{\lambda}}\\ >&B(\frac{\lambda}{2},\frac{\lambda}{2})\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\frac{1}{4})^{\lambda/2}})\right](m-\frac{1}{4})^{p(1-\frac{\lambda}{2})-1}a_m^p\right\}^{\frac{1}{p}}\\ &\times\left[\sum_{n=1}^{\infty}(n-\frac{1}{4})^{q(1-\frac{\lambda}{2})-1}b_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}(n-\frac{1}{4})^{p\frac{\lambda}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n-\frac{1}{2})^{\lambda}}\right]^p\right\}^{\frac{1}{p}}\\ >&B(\frac{\lambda}{2},\frac{\lambda}{2})\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\frac{1}{4})^{\lambda/2}})\right](m-\frac{1}{4})^{p(1-\frac{\lambda}{2})-1}a_m^p\right\}^{\frac{1}{p}},\end{aligned} \tag{3.11}$$

$$\left\{\sum_{m=1}^{\infty}\frac{(m-\frac{1}{4})^{q\frac{\lambda}{2}-1}}{[1-O(\frac{1}{(m-\frac{1}{4})^{\lambda/2}})]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m+n-\frac{1}{2})^{\lambda}}\right]^q\right\}^{\frac{1}{q}}$$
$$>B(\frac{\lambda}{2},\frac{\lambda}{2})\left[\sum_{n=1}^{\infty}(n-\frac{1}{4})^{q(1-\frac{\lambda}{2})-1}b_n^q\right]^{\frac{1}{q}}. \tag{3.12}$$

In particular, for $\lambda = 3$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{8}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-\frac{1}{2})^3}$$
$$>\frac{\pi}{8}\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\frac{1}{4})^{3/2}})\right](m-\frac{1}{4})^{-\frac{p}{2}-1}a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left[\sum_{n=1}^{\infty}(n-\frac{1}{4})^{-\frac{q}{2}-1}b_n^q\right]^{\frac{1}{q}}, \tag{3.13}$$

$$\left\{\sum_{n=1}^{\infty}(n-\frac{1}{4})^{\frac{3p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n-\frac{1}{2})^3}\right]^p\right\}^{\frac{1}{p}}$$
$$>\frac{\pi}{8}\left\{\sum_{m=1}^{\infty}\left[1-O(\frac{1}{(m-\frac{1}{4})^{3/2}})\right](m-\frac{1}{4})^{-\frac{p}{2}-1}a_m^p\right\}^{\frac{1}{p}}, \tag{3.14}$$

$$\left\{\sum_{m=1}^{\infty}\frac{(m-\frac{1}{4})^{\frac{3q}{2}-1}}{[1-O(\frac{1}{(m-\frac{1}{4})^{3/2}})]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m+n-\frac{1}{2})^3}\right]^q\right\}^{\frac{1}{q}}$$
$$>\frac{\pi}{8}\left[\sum_{n=1}^{\infty}(n-\frac{1}{4})^{-\frac{q}{2}-1}b_n^q\right]^{\frac{1}{q}}. \tag{3.15}$$

4. Conclusions

In this paper, by means of the weight coefficients, the idea of introduced parameters, Hermite-Hadamard's inequality and Euler-Maclaurin summation formula, a reverse more accurate Hardy-Hilbert's inequality as well as the equivalent forms are given in Lemma 2.2 and Theorem 3.1. The equivalent statements of the best possible constant factor related to several parameters are considered in Theorem 3.2, and some particular reverse inequalities are obtained in Remark 3.1. The lemmas and theorems provide an extensive account of this type of inequalities.

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