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AFFINE-PERIODIC SOLUTIONS FOR PERTURBED SYSTEMS

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Abstract In this paper, we try to study the existence and uniqueness of affine-periodic solutions for the perturbed affine-periodic system. We prove that, under certain conditions, if the coefficient of the forced term is sufficiently small, then the system admits affine-periodic solutions which have the form of $z(t+T,\mu) = Qz(t,\mu)$ with some nonsingular matrix Q. Depending on the structure of Q, they may be periodic, anti-periodic, quasi-periodic or even unbounded spiral motions. The main tools we used are the theory of exponential dichotomy and Banach contraction mapping principle.

Keywords Affine-periodic solutions, perturbed systems, exponential dichotomy, Banach contraction mapping principle.

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1. Introduction

Periodicity is a very common phenomenon in nature. In the real world, many phenomena show some periodicity to varying degrees. So the periodicity problem is one of center topics in the study of dynamic system theory. But not all natural phenomena can be described alone by periodicity, some natural phenomena often exhibit symmetry besides time periodicity. For example, rigid body motion, spiral wave, typhoon motion and other physical phenomena. In 2013, Li and his coauthors [18] proposed the concept of "affine periodic solution" based on this phenomenon, and gave a strict mathematical definition. Consider the system

$$\dot{x} = f(t,x), \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and for some $Q \in GL_n(\mathbb{R}^n)$ satisfies the following affine symmetry:

$$f(t+T,x) = Qf(t,Q^{-1}x).$$

We call it a (Q,T)-affine-periodic system. For this (Q,T)-affine-periodic system, we are concerned with the existence of (Q,T)-affine-periodic solutions $x(t)$ with

$$x(t+T) = Qx(t) \quad \forall t \in \mathbb{R}.$$

It should be pointed out that when $Q = I$ (identity matrix) or $Q = -I$, the solutions are just the pure periodic solutions or antiperiodic ones; When Q is a power identity

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matrix, i.e., $Q^k = I$ for some integer $k \neq 0$, this kind of solutions is subharmonic; If $Q \in O(n)$, i.e., Q is an orthogonal matrix, the solutions are a special quasi-periodic; When $Q \in GL(n) \backslash O(n)$, an affine-periodic solution might be spiral like $(e^{at} \cos \omega t, e^{at} \sin \omega t)$.

In recent years, the study of affine periodic solutions has attracted great attention of scholars. The existence of affine periodic solutions for different systems was discussed and investigated in the literature [2, 7, 8, 14–16, 18]. In 2015, Li and Huang studied Levinson's problem on affine-periodic solutions where the system is dissipative-repulsive [7]; In 2016, Cheng, Huang and Li proved that every (Q, T)-affine-periodic differential equation has a (Q, T)-affine-periodic solution if the corresponding homogeneous linear equation admits exponential dichotomy or exponential trichotomy [2]; Wang, Yang and Li are discussed the existence of affine-periodic solutions of systems without those conditions such as dissipativeness [14]. In 2017, Wang, Yang and Li gave the LaSalle type stationary oscillation theorems for affine-periodic [15]; In 2018, Li, Wang and Yang studied Fink type conjecture on affine-periodic solutions and levinson's conjecture to newtonian systems [8]; Xing, Yang and Li gave the existence of affine-periodic solutions for perturbed affine-periodic systems by using the averaging method [16].

In this paper, we try to study the existence of affine-periocic solutions for the perturbed affine-periodic system with the following type

$$\dot{z} = g(z) + \mu h(t, z, \mu),$$

where $g(z)$ and $h(t, z, \mu)$ are continuous functions and, μ is a small parameter. And for some $Q \in GL_n(\mathbb{R}^n)$, the following affine symmetry holds:

$$g(z) = Qg(Q^{-1}z), \ h(t + T, z, \mu) = Qh(t, Q^{-1}z, \mu).$$

We will prove that, under certain conditions, the perturbed system admits affine-periocic solutions. Our approach is based on the theory of exponential dichotomy and Banach contraction mapping principle.

This paper is organized as follows. In Section 2, we introduce some basic concepts about exponential dichotomy and affine-periodic solution. In Section 3, we state our main result and give its proof. In Section 4, some examples are given to illustrate the main findings.

Throughout this paper, we need the following notations.

$\mathbb{R}^n$:	n-dimensional Euclidean space,
$\mathbb{R}^+$:	Positive real number set,
$\mathbb{R}^-$:	Negative real number set,
$GL_n(\mathbb{R}^n)$:	General linear group on $\mathbb{R}^n$,
I :	Identity matrix,
Q^{-1} :	Inverse of matrix Q,

$g_z(z)$:	Jacobian matrix of function g with respect to z,
μ, δ, ε :	Small parameters.

2. Preliminaries

Exponential dichotomy is one of the most basic concepts arising in the theory of dynamical systems. It plays a central role in some complicated behaviors, such as homoclinic, heteroclinic orbits. Now, we recall the definition of exponential dichotomies:

Definition 2.1. Let $A(t)$ be a real $n \times n$ matrix function, piecewise continuous on an interval J. We consider linear system

$$\dot{x} = A(t)x,$$

where $x \in \mathbb{R}^n$. We say the system admits an exponential dichotomy on interval $J \subset \mathbb{R}$ if there exist constants $K, \alpha > 0$, projection P and fundamental matrix $X(t)$ satisfying

$$\begin{aligned} &|X(t)PX^{-1}(s)| \leq Ke^{-\alpha(t-s)} \qquad && s \leq t, \\ &|X(t)(I-P)X^{-1}(s)| \leq Ke^{-\alpha(s-t)} \qquad && t \leq s, \end{aligned} \tag{2.1}$$

for $t, s \in J$.

Besides, we also need definitions below.

Definition 2.2. We denote by $GL_n(\mathbb{R}^n)$ the n-dimensional general linear group over $\mathbb{R}^n$ and consider the system

$$\dot{x} = f(t, x), \tag{2.2}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. If there exists $Q \in GL_n(\mathbb{R}^n)$ and $T > 0$ such that

$$f(t+T, x) = Qf(t, Q^{-1}x)$$

holds for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, we call the system (2.2) a (Q, T)-affine-periodic system (APS for short).

For APS (2.2), we define its affine-periodic solutions as follows:

Definition 2.3. If $x(t)$ is a solution of APS (2.2) on $\mathbb{R}$ and

$$x(t+T) = Qx(t) \quad \forall t \in \mathbb{R},$$

then $x(t)$ is said to be a (Q, T)-affine-periodic solution.

3. Main Result

In this section, we are ready to state and prove our main results.

We consider the system of the following form:

$$\dot{z} = g(z) + \mu h(t, z, \mu), \tag{3.1}$$

where μ is a small parameter. Furthermore, we make the following assumptions.

(H1) Let $g(z)$ be a twice continuously differentiable vector function defined in $\mathbb{R}^n$ and $g_z(z)$ be bounded. Suppose that the system

$$\dot{z} = g(z) \tag{3.2}$$

has a (Q, T)-affine-periodic solution $\xi(t)$, and the variational equation

$$\dot{z} = g_z(\xi(t))z \tag{3.3}$$

has an exponential dichotomy on $\mathbb{R}$ with constants K, α, where $Q \in GL_n(\mathbb{R}^n)$, $T > 0$ is a constant.

(H2) Let $h(t, z, \mu)$ be a continuous vector function defined for $t \in \mathbb{R}, z \in \mathbb{R}^n, |\mu| < \delta(\mu \in \mathbb{R})$, such that the partial derivative $h_t, h_z, h_\mu, h_{zz}, h_{z\mu}, h_{\mu z}$, are continuous in t for each fixed z, μ and continuous in z, μ uniformly with respect to t, z, μ.

(H3) The following affine symmetry holds:

$$g(z) = Qg(Q^{-1}z), \;\; h(t + T, z, \mu) = Qh(t, Q^{-1}z, \mu) \tag{3.4}$$

for all $(t, z) \in \mathbb{R} \times \mathbb{R}^n$.

(H4) For sufficiently small μ, there exist positive constants $M_1, M_2 > 0$ such that for

$$\frac{K(2M_1 + \mu M_2)}{\alpha}(e^{\alpha T} - 1)(\sum_{k=1}^{+\infty}(|Q^{-k}| + |Q^k|)e^{-\alpha kT} + 2e^{-\alpha T}) < 1,$$

where $M_1 = \sup_z \|g_z(z)\|, M_2 = \max\{\sup_{t\in[0,T]} |h(t, z, \mu)|, \sup_{t\in[0,T]} |h_z(t, z, \mu)|\}$.

Theorem 3.1. *Suppose $(H1) - (H4)$ are satisfied. Then for sufficiently small μ, the system (3.1) has a unique affine-periodic solution $z(t, \mu)$.*

Remark 3.1. For (H3), we notice that $h(t, z, \mu)$ is affine periodic, thus $|h(t, z, \mu)|$ and $|h_z(t, z, \mu)|$ may be unbounded on $t \in \mathbb{R}$, but they can be bounded on $t \in [0, T]$, i.e., $\|h(t, z, \mu)\| := \sup_{t\in[0,T]} |h(t, z, \mu)|$, $\|h_z(t, z, \mu)\| := \sup_{t\in[0,T]} |h_z(t, z, \mu)|$ can be bounded. In our theorem, we only need $|h(t, z, \mu)|$ and $|h_z(t, z, \mu)|$ to be bounded on $t \in [0, T]$. For details, please refer to the proof of theorem. Therefore, the assumption about M_2 in (H4) is reasonable.

Remark 3.2. For (H1), we notice that the variational equation has an exponential dichotomy on $\mathbb{R}$. The reason for this assumption is that $\xi(t)$ is an affine-periodic solution. If $\xi(t)$ is homoclinic or heteroclinic orbits, we only need to assume that the variational equation has an exponential dichotomy on the both $\mathbb{R}^+$ and $\mathbb{R}^-$.

We give the following lemma, which is useful to our proof.

Lemma 3.1 (See [2])**.** *Let $Q \in GL(n)$. For a fixed μ,*

$$C_T = \{y(\cdot, \mu) \in C(\mathbb{R}^1, \mathbb{R}^n) : y(t + T, \mu) = Qy(t, \mu), \;\; for\ all\ t \in \mathbb{R}\}.$$

Then $\{C_T, \|\cdot\|\}$ is a Banach space with the norm $\|y\| = \sup_{t\in[0,T]} |y(t, \mu)|$.

Proof. Firstly, we need to prove that the norm is well defined.

(1)$\|y\| \geq 0, \forall y \in C_T$. If $y \in C_T$ such that $\|y\| = \sup\limits_{t\in[0,T]} |y(t,\mu)| = 0$, then we get that $y(t,\mu)$ is zero vector for all $t \in [0,T]$. For any $k \in \mathbb{Z}$, if $t \in [kT,(k+1)T]$, then we have $y(t,\mu) = Q^k y(t-kT,\mu)$, which means that $y(t,\mu)$ is zero vector for all $t \in \mathbb{R}$.

(2)$\|cy\| = \sup\limits_{t\in[0,T]} |cy(t,\mu)| = |c| \sup\limits_{t\in[0,T]} |y(t,\mu)| = |c|\|y\|, \forall c \in \mathbb{R}, y \in C_T$.

(3)$\|y_1 + y_2\| = \sup\limits_{t\in[0,T]} |y_1(t,\mu) + y_2(t,\mu)| \leq \sup\limits_{t\in[0,T]} |y_1(t,\mu)| + \sup\limits_{t\in[0,T]} |y_2(t,\mu)| = \|y_1\| + \|y_2\|, \forall y_1, y_2 \in C_T$.

Thus the norm is well defined.

Secondly, we prove that C_T is complete.

Let $\{y_n\}$ be a Cauchy sequence in C_T. For all n, denote by $\bar{y}_n$ the restriction of y_n on the interval $[0,T]$. Then $\bar{y}_n$ is a Cauchy sequence in $C([0,T])$, which is a Banach space, and there exists a $\bar{y}^* \in C([0,T])$ such that $\lim\limits_{n\to+\infty} \|\bar{y}_n - \bar{y}^*\| = 0$. Define a continuous (Q,T)-affine-periodic function

$$y^*(t,\mu) = \begin{cases} \bar{y}^*(t,\mu), & t \in [0,T], \\ Q^k \bar{y}^*(t-kT,\mu), & t \in [kT,(k+1)T], k \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{3.5}$$

Then we have $\lim\limits_{n\to+\infty} \|y_n - y^*\| = \lim\limits_{n\to+\infty} \|\bar{y}_n - \bar{y}^*\| = 0$, which means that C_T is complete. Thus $\{C_T, \|\cdot\|\}$ is a Banach space. □

Now we prove Theorem 3.1.

Proof. Let $z(t,\mu) = \xi(t) + x(t,\mu)$, then (3.1) becomes

$$\dot{\xi}(t) + \dot{x}(t,\mu) = g(\xi(t) + x(t,\mu)) + \mu h(t, \xi(t) + x(t,\mu), \mu),$$

and

$$\begin{aligned} \dot{x}(t,\mu) &= g(\xi(t) + x(t,\mu)) - g(\xi(t)) + \mu h(t, \xi(t) + x(t,\mu), \mu) \\ &= g_z(\xi(t))x + g(\xi(t) + x(t,\mu)) - g(\xi(t)) - g_z(\xi(t))x \\ &\quad + \mu h(t, \xi(t) + x(t,\mu), \mu) \\ &= g_z(\xi(t))x + W(t,x,\mu), \end{aligned} \tag{3.6}$$

where

$$W(t,x,\mu) = g(\xi(t) + x(t,\mu)) - g(\xi(t)) - g_z(\xi(t))x + \mu h(t, \xi(t) + x(t,\mu), \mu).$$

From $g(z) = Qg(Q^{-1}z)$, we have $g_z(z) = Qg_z(Q^{-1}z)Q^{-1}$.

Denote $\hat{g}(t,z) = g_z(\xi(t))z$. Then

$$\begin{aligned} \hat{g}(t+T,z) &= g_z(\xi(t+T))z \\ &= g_z(Q\xi(t))z \\ &= Qg_z(Q^{-1}Q\xi(t))Q^{-1}z \\ &= Qg_z(\xi(t))Q^{-1}z \\ &= Q\hat{g}(t,Q^{-1}z). \end{aligned} \tag{3.7}$$

Let $\Phi(t) = Q^{-1}X(t+T)X^{-1}(T)Q$, then

$$\begin{aligned}\frac{d(\Phi(t))}{dt} &= Q^{-1}\frac{d(X(t+T))}{d(t+T)}X^{-1}(T)Q \\ &= Q^{-1}g_z(\xi(t+T))X(t+T)X^{-1}(T)Q \\ &\overset{(3.7)}{=} Q^{-1}Qg_z(\xi(t))Q^{-1}X(t+T)X^{-1}(T)Q \\ &= g_z(\xi(t))\Phi(t).\end{aligned}\tag{3.8}$$

Since $\Phi(0) = I$, by the uniqueness of solutions, we get $\Phi(t) = X(t)$.

Thus

$$X(t+T) = QX(t)Q^{-1}X(T).$$

Consider the following nonhomogeneous linear equation

$$\dot{x}(t,\mu) = g_z(\xi(t))x + W(t, y(t,\mu), \mu),\tag{3.9}$$

where $y(t,\mu)$ is a continuous function.

From (H1), we know that $\dot{x}(t,\mu) = g_z(\xi(t))x$ has an exponential dichotomy on $\mathbb{R}$, then there is a projection P and constants $K, \alpha > 0$ such that

$$\begin{aligned}&|X(t)PX^{-1}(s)| \le Ke^{-\alpha(t-s)}, && s \le t,\\ &|X(t)(I-P)X^{-1}(s)| \le Ke^{-\alpha(s-t)}, && t \le s.\end{aligned}$$

Then (3.9) has the following solution:

$$\begin{aligned}x(t,\mu) = &\int_{-\infty}^{t} X(t)PX^{-1}(s)W(s,y,\mu)ds \\ &- \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)W(s,y,\mu)ds.\end{aligned}\tag{3.10}$$

We show that $x(t,\mu)$ is (Q,T)-affine-periodic if $y(t,\mu)$ is (Q,T)-affine-periodic.

$$\begin{aligned}x(t+T,\mu) = &\int_{-\infty}^{t+T} X(t+T)PX^{-1}(s)W(s,y,\mu)ds \\ &- \int_{t+T}^{\infty} X(t+T)(I-P)X^{-1}(s)W(s,y,\mu)ds,\end{aligned}$$

where

$$\begin{aligned}&W(t+T, y(t+T,\mu), \mu) \\ &= g(\xi(t+T) + y(t+T,\mu)) - g(\xi(t+T)) - g_z(\xi(t+T))y(t+T,\mu) \\ &\quad + \mu h(t+T, \xi(t+T) + y(t+T,\mu), \mu) \\ &= Qg(Q^{-1}(\xi(t+T) + y(t+T,\mu))) - Qg(Q^{-1}\xi(t+T)) \\ &\quad - Qg_z(Q^{-1}\xi(t+T))Q^{-1}y(t+T,\mu) \\ &\quad + \mu Qh(t, Q^{-1}(\xi(t+T) + y(t+T,\mu)), \mu) \\ &= Qg(\xi(t) + y(t,\mu)) - Qg(\xi(t)) - Qg_z(\xi(t))y(t,\mu) \\ &\quad + \mu Qh(t, \xi(t) + y(t,\mu), \mu) \\ &= QW(t, y(t,\mu), \mu),\end{aligned}\tag{3.11}$$

and
$$Q^{-1}X(T)P = PQ^{-1}X(T), Q^{-1}X(T)(I-P) = (I-P)Q^{-1}X(T).$$
Let $s = \tau + T$, we have

$$\begin{aligned}
x(t+T,\mu) &= \int_{-\infty}^{t} X(t+T)PX^{-1}(\tau+T)W(\tau+T, y(\tau+T,\mu),\mu)d\tau \\
&\quad - \int_{t}^{\infty} X(t+T)(I-P)X^{-1}(\tau+T)W(\tau+T, y(\tau+T,\mu),\mu)d\tau \\
&= \int_{-\infty}^{t} QX(t)Q^{-1}X(T)P(QX(\tau)Q^{-1}X(T))^{-1}QW(\tau,y,\mu)d\tau \\
&\quad - \int_{t}^{\infty} QX(t)Q^{-1}X(T)(I-P)(QX(\tau)Q^{-1}X(T))^{-1}QW(\tau,y,\mu)d\tau \\
&= \int_{-\infty}^{t} QX(t)PX^{-1}(\tau)W(\tau,y,\mu)d\tau \\
&\quad - \int_{t}^{\infty} QX(t)(I-P)X^{-1}(\tau)W(\tau,y,\mu)d\tau \\
&= Qx(t,\mu),
\end{aligned}$$

which means that $x(t,\mu)$ is (Q,T)-affine-periodic.

Next, we prove that the integral (3.10) exists.

$$\begin{aligned}
\|W(t,y,\mu)\| &= \|g(\xi+y) - g(\xi) - g_z(\xi)y + \mu h(t,\xi+y,\mu)\| \\
&\le \|(g_z(\xi+y') - g_z(\xi))y\| + \mu\|h(t,\xi+y,\mu)\| \\
&\le \|g_z(\xi+y') - g_z(\xi)\|\|y\| + \mu M_2 \\
&\le 2M_1\|y\| + \mu M_2,
\end{aligned}$$

where $M_1 = \sup_z \|g_z(z)\|, M_2 = \max\{\|h\|, \|h_z\|\}$, which means

$$\sup_{t\in[0,T]} |W(t,y(t,\mu),\mu)| \le 2M_1 \sup_{t\in[0,T]} |y(t,\mu)| + \mu M_2 \le 2CM_1 + \mu M_2.$$

Define a map $H : C_T \to C_T$ by

$$\begin{aligned}
H(y)(t,\mu) &= \int_{-\infty}^{t} X(t)PX^{-1}(s)W(s,y,\mu)ds \\
&\quad - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)W(s,y,\mu)ds.
\end{aligned}$$

Note that H is well defined. For $y(t,\mu) \in C_T$, we have

$$\begin{aligned}
|H(y)(t,\mu)| &= |\int_{-\infty}^{t} X(t)PX^{-1}(s)W(s,y(s,\mu),\mu)ds \\
&\quad - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)W(s,y(s,\mu),\mu)ds| \\
&\le \int_{-\infty}^{t} |X(t)PX^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)|ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^{\infty}|X(t)(I-P)X^{-1}(s)|\cdot|W(s,y(s,\mu),\mu)|ds\\
&\leq K\{\int_{-\infty}^{t}e^{-\alpha(t-s)}|W(s,y(s,\mu),\mu)|ds\\
&+\int_t^{\infty}e^{-\alpha(s-t)}|W(s,y(s,\mu),\mu)|ds\}.
\end{aligned}
$$

For a fixed t, if $t\in[0,T]$, the first term as follows

$$
\begin{aligned}
&\int_{-\infty}^{t}e^{-\alpha(t-s)}|W(s,y(s,\mu),\mu)|ds\\
&=\sum_{k=-1}^{-\infty}\int_{kT}^{(k+1)T}e^{-\alpha(t-s)}|W(s,y(s,\mu),\mu)|ds+\int_0^t e^{-\alpha(t-s)}|W(s,y(s,\mu),\mu)|ds\\
&\leq\left(\sum_{k=-1}^{-\infty}|Q^k|\int_0^T e^{-\alpha(t-s-kT)}ds+\int_0^t e^{-\alpha(t-s)}ds\right)\|W(s,y(s,\mu),\mu)\|\\
&\leq(2CM_1+\mu M_2)\left(\sum_{k=-1}^{-\infty}|Q^k|\int_0^T e^{-\alpha(t-s-kT)}ds+\int_0^t e^{-\alpha(t-s)}ds\right)\\
&=\frac{2CM_1+\mu M_2}{\alpha}\left(\sum_{k=-1}^{-\infty}|Q^k|e^{\alpha kT-\alpha t}(e^{\alpha T}-1)+1-e^{-\alpha t}\right).
\end{aligned}
\tag{3.12}
$$

Furthemore,

$$
\begin{aligned}
&\int_t^{+\infty}e^{-\alpha(s-t)}|W(s,y(s,\mu),\mu)|ds\\
&=\sum_{k=1}^{+\infty}\int_{kT}^{(k+1)T}e^{-\alpha(s-t)}\cdot|W(s,y(s,\mu),\mu)|ds+\int_t^T e^{-\alpha(s-t)}|W(s,y(s,\mu),\mu)|ds\\
&\leq\left(\sum_{k=1}^{+\infty}|Q^k|\int_0^T e^{-\alpha(s-t+kT)}ds+\int_t^T e^{-\alpha(s-t)}ds\right)\|W(s,y(s,\mu),\mu)\|\\
&\leq(2CM_1+\mu M_2)\left(\sum_{k=1}^{+\infty}|Q^k|\int_0^T e^{-\alpha(s-t+kT)}ds+\int_t^T e^{-\alpha(s-t)}ds\right)\\
&=\frac{2CM_1+\mu M_2}{\alpha}(\sum_{k=1}^{+\infty}|Q^k|e^{\alpha(t-kT)}(1-e^{-\alpha T})+1-e^{\alpha(t-T)}).
\end{aligned}
\tag{3.13}
$$

Thus

$$
\begin{aligned}
&|H(y)(t,\mu)|\\
&\leq\frac{K(2CM_1+\mu M_2)}{\alpha}(\sum_{k=-1}^{-\infty}|Q^k|e^{\alpha kT-\alpha t}(e^{\alpha T}-1)\\
&+\sum_{k=1}^{+\infty}|Q^k|e^{\alpha(t-kT)}(1-e^{-\alpha T})+2-e^{-\alpha t}-e^{\alpha(t-T)})
\end{aligned}
$$

$$\leq \frac{K(2CM_1+\mu M_2)}{\alpha} \sup_{t\in[0,T]} (\sum_{k=-1}^{-\infty} |Q^k| e^{\alpha kT - \alpha t}(e^{\alpha T}-1)$$

$$+ \sum_{k=1}^{+\infty} |Q^k| e^{\alpha(t-kT)}(1-e^{-\alpha T}) + 2 - e^{-\alpha t} - e^{\alpha(t-T)})$$

$$= \frac{K(2CM_1+\mu M_2)}{\alpha} (\sum_{k=-1}^{-\infty} |Q^k| e^{\alpha kT}(e^{\alpha T}-1)$$

$$+ \sum_{k=1}^{+\infty} |Q^k| e^{\alpha(T-kT)}(1-e^{-\alpha T}) + 2 - 2e^{-\alpha T})$$

$$= \frac{K(2CM_1+\mu M_2)}{\alpha} (e^{\alpha T}-1)(\sum_{k=1}^{+\infty}(|Q^{-k}|+|Q^k|)e^{-\alpha kT} + 2e^{-\alpha T}).$$

$$\overset{H4}{<} \infty.$$

If $t < 0$,

$$\begin{aligned}
|H(y)(t,\mu)| &= |\int_{-\infty}^{t} X(t)PX^{-1}(s)W(s,y(s,\mu),\mu)ds \\
&\quad - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)W(s,y(s,\mu),\mu)ds| \\
&\leq \int_{-\infty}^{t} |X(t)PX^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)|ds \\
&\quad + \int_{t}^{\infty} |X(t)(I-P)X^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)|ds \\
&= \int_{-\infty}^{t} |X(t)PX^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)|ds \\
&\quad + \int_{t}^{0} |X(t)(I-P)X^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)| \\
&\quad + \int_{0}^{\infty} |X(t)(I-P)X^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)|ds \\
&\leq \int_{-\infty}^{0} |X(t)PX^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)|ds \\
&\quad + \int_{t}^{0} |X(t)(I-P)X^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)| \\
&\quad + \int_{0}^{\infty} |X(t)(I-P)X^{-1}(s)| \cdot |W(s,y(s,\mu),\mu)|ds \\
&< \infty.
\end{aligned}$$

If $t > T$,

$$\begin{aligned}
|H(y)(t,\mu)| &= |\int_{-\infty}^{t} X(t)PX^{-1}(s)W(s,y(s,\mu),\mu)ds \\
&\quad - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)W(s,y(s,\mu),\mu)ds|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{-\infty}^{t} |X(t)PX^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| ds \\
&\quad + \int_{t}^{\infty} |X(t)(I-P)X^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| ds \\
&= \int_{-\infty}^{T} |X(t)PX^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| ds \\
&\quad + \int_{T}^{t} |X(t)PX^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| \\
&\quad + \int_{t}^{\infty} |X(t)(I-P)X^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| ds \\
&\le \int_{-\infty}^{T} |X(t)PX^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| ds \\
&\quad + \int_{T}^{t} |X(t)(I-P)X^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| \\
&\quad + \int_{T}^{\infty} |X(t)(I-P)X^{-1}(s)| \cdot |W(s, y(s,\mu), \mu)| ds \\
&< \infty.
\end{aligned}
$$

From the above discussion, we get the integral (3.10) exsists.

In order to prove the existence of (Q,T)-affine-periodic solutions of equation (3.6), we only need to prove that there exists a fixed point of H in C_T. For any $y_1, y_2 \in C_T$,

$$
\begin{aligned}
&\|H(y_1)(\cdot) - H(y_2)(\cdot)\| \\
= &\sup_{t\in[0,T]} \Big| \int_{-\infty}^{t} X(t)PX^{-1}(s)(W(s,y_1,\mu) - W(s,y_2,\mu)) ds \\
&- \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)(W(s,y_1,\mu) - W(s,y_2,\mu)) ds \Big|, \\
\le &\sup_{t\in[0,T]} \Big\{ \int_{-\infty}^{t} |X(t)PX^{-1}(s)| \cdot |W(s,y_1,\mu) - W(s,y_2,\mu)| ds \\
&+ \int_{t}^{\infty} |X(t)(I-P)X^{-1}(s)| \cdot |W(s,y_1,\mu) - W(s,y_2,\mu)| ds \Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|W(s,y_1,\mu) - W(s,y_2,\mu)\| \\
= &\|g(\xi+y_1) - g(\xi) - g_z(\xi)y_1 + \mu h(t,\xi+y_1,\mu) \\
&- (g(\xi+y_2) - g(\xi) - g_z(\xi)y_2 + \mu h(t,\xi+y_2,\mu))\| \\
= &\|g(\xi+y_1) - g(\xi+y_2) - g_z(\xi)y_1 + g_z(\xi)y_2 \\
&+ \mu h(t,\xi+y_1,\mu) - \mu h(t,\xi+y_2,\mu))\| \\
\le &\|(g_z(\xi+y_0) - g_z(\xi))(y_1-y_2)\| + \mu\|h_z(t,\xi+\hat{y},\mu)(y_1-y_2)\| \\
\le &(\|g_z(\xi+y_0) - g_z(\xi)\| + \mu\|h_z(t,\xi+\hat{y})\|) \cdot \|y_1-y_2\| \\
\le &(2M_1 + \mu M_2)\|y_1-y_2\|.
\end{aligned}
$$

Firstly,

$$
\begin{aligned}
&\int_{-\infty}^{t} |X(t)PX^{-1}(s)| \cdot |W(s,y_1,\mu) - W(s,y_2,\mu)|ds \\
\leq\ & K\int_{-\infty}^{t} e^{-\alpha(t-s)}|W(s,y_1,\mu) - W(s,y_2,\mu)|ds \\
=\ & K\{\sum_{k=-1}^{-\infty}\int_{kT}^{(k+1)T} e^{-\alpha(t-s)}|W(s,y_1,\mu) - W(s,y_2,\mu)|ds \\
&+ \int_{0}^{t} e^{-\alpha(t-s)}|W(s,y_1,\mu) - W(s,y_2,\mu)|ds\} \qquad (3.14)\\
\leq\ & K(\sum_{k=-1}^{-\infty}|Q^k|\int_0^T e^{-\alpha(t-s-kT)}ds + \int_0^t e^{-\alpha(t-s)}ds)\|W(s,y_1,\mu) - W(s,y_2,\mu)\| \\
\leq\ & K(2M_1+\mu M_2)(\sum_{k=-1}^{-\infty}|Q^k|\int_0^T e^{-\alpha(t-s-kT)}ds + \int_0^t e^{-\alpha(t-s)}ds)\|y_1-y_2\| \\
=\ & \frac{K(2M_1+\mu M_2)}{\alpha}\left(\sum_{k=-1}^{-\infty}|Q^k|e^{\alpha kT-\alpha t}(e^{\alpha T}-1)+1-e^{-\alpha t}\right)\|y_1-y_2\|.
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
&\int_{-\infty}^{t} |X(t)(I-P)X^{-1}(s)| \cdot |W(s,y_1,\mu) - W(s,y_2,\mu)|ds \\
\leq\ & K\int_{t}^{+\infty} e^{-\alpha(s-t)}|W(s,y_1,\mu) - W(s,y_2,\mu)|ds \\
=\ & K(\sum_{k=1}^{+\infty}\int_{kT}^{(k+1)T} e^{-\alpha(s-t)} \cdot |W(s,y_1,\mu) - W(s,y_2,\mu)|ds \\
&+ \int_{t}^{T} e^{-\alpha(s-t)}|W(s,y_1,\mu) - W(s,y_2,\mu)|ds) \qquad (3.15)\\
\leq\ & K(\sum_{k=1}^{+\infty}|Q^k|\int_0^T e^{-\alpha(s-t+kT)}ds + \int_t^T e^{-\alpha(s-t)}ds)\|W(s,y_1,\mu) - W(s,y_2,\mu)\| \\
\leq\ & K(2M_1+\mu M_2)(\sum_{k=1}^{+\infty}|Q^k|\int_0^T e^{-\alpha(s-t+kT)}ds + \int_t^T e^{-\alpha(s-t)}ds)\|y_1-y_2\| \\
=\ & \frac{K(2M_1+\mu M_2)}{\alpha}(\sum_{k=1}^{+\infty}|Q^k|e^{\alpha(t-kT)}(1-e^{-\alpha T})+1-e^{\alpha(t-T)})\|y_1-y_2\|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|H(y_1)(\cdot) - H(y_2)(\cdot)\| \\
\overset{\substack{(3.14)\\(3.15)}}{\leq}\ & \frac{K(2M_1+\mu M_2)}{\alpha}\sup_{t\in[0,T]}\{(\sum_{k=-1}^{-\infty}|Q^k|e^{\alpha kT-\alpha t}(e^{\alpha T}-1)+1-e^{-\alpha t})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{+\infty}|Q^k|e^{\alpha(t-kT)}(1-e^{-\alpha T})+1-e^{\alpha(t-T)})\}\|y_1-y_2\| \\
&\leq \frac{K(2M_1+\mu M_2)}{\alpha}\{\sum_{k=-1}^{-\infty}|Q^k|e^{\alpha kT}(e^{\alpha T}-1) \\
&+\sum_{k=1}^{+\infty}|Q^k|e^{\alpha(T-kT)}(1-e^{-\alpha T})+2-2e^{-\alpha T}\}\|y_1-y_2\| \\
&\leq \frac{K(2M_1+\mu M_2)}{\alpha}\{\sum_{k=1}^{+\infty}|Q^{-k}|e^{-\alpha kT}(e^{\alpha T}-1) \\
&+\sum_{k=1}^{+\infty}|Q^k|e^{-\alpha kT}(e^{\alpha T}-1)+2(e^{\alpha T}-1)e^{-\alpha T}\}\|y_1-y_2\| \\
&\leq \frac{K(2M_1+\mu M_2)}{\alpha}(e^{\alpha T}-1)(\sum_{k=1}^{+\infty}(|Q^{-k}|+|Q^k|)e^{-\alpha kT}+2e^{-\alpha T})\|y_1-y_2\|.
\end{aligned}
$$

From $H4$, for sufficiently small μ, $H(y)$ is a contraction mapping on C_T. From Banach Fixed Point Theorem, it follows that H admits a unique fixed point $x(t,\mu) \in C_T$ which is the unique (Q,T)-affine-periodic solution of equation (3.6) and $z(t,\mu)$ is the unique (Q,T)-affine-periodic solution of equation (3.1). □

4. Examples

In this section, we will show two examples and prove the existence of affine-periodic solutions.

Example 4.1. Consider the system

$$
\begin{aligned}
x' &= -\varepsilon x + \mu e^{-\frac{\varepsilon}{2}t}, \\
y' &= \varepsilon y,
\end{aligned}
\tag{4.1}
$$

where $\varepsilon \ll 1$.

Let $z=(x,y)^T$, we have $g(z)=(-\varepsilon x, \varepsilon y)^T$, $h(t,z,\mu)=(e^{-\frac{\varepsilon}{2}t},0)^T$.

We will verify the assumptions of Theorem 3.1 hold.

(H1) $g(z)$ be a twice continuously differentiable vector function on $\mathbb{R}^n$. And

$$
g_z(z)=\begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}
$$

is bounded. Furthemore, $\dot{z}=g(z)$ has a solution $\xi(t)=(0,0)^T$. The eigenvalues of $g_z(\xi(t))$ are $\pm\varepsilon$, so the variational equation $\dot{z}=g_z(\xi(t))z$ has an exponential dichotomy on $\mathbb{R}$ with $K=1,\ \alpha=\varepsilon$.

(H2) is obviously true.

(H3)Put

$$
Q=\begin{pmatrix} e^{-\varepsilon\pi} & 0 \\ 0 & 1 \end{pmatrix},
$$

$$T = 2\pi.$$

Thus we have

$$\begin{aligned}
&\xi(t+2\pi) = Q\xi(t),\\
&Q^{-1} = \begin{pmatrix} e^{\varepsilon\pi} & 0 \\ 0 & 1 \end{pmatrix},\\
&Q^{-1}z = (e^{\varepsilon\pi}x, y)^T.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
Qg(Q^{-1}z) &= \begin{pmatrix} e^{-\varepsilon\pi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\varepsilon e^{\varepsilon\pi}x \\ \varepsilon y \end{pmatrix}\\
&= \begin{pmatrix} -\varepsilon x \\ \varepsilon y \end{pmatrix}\\
&= g(z).\\
Qh(t, Q^{-1}z, \mu) &= \begin{pmatrix} e^{-\varepsilon\pi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{\varepsilon}{2}t} \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} e^{-\frac{\varepsilon}{2}(t+2\pi)} \\ 0 \end{pmatrix}\\
&= h(t+2\pi, z, \mu).
\end{aligned}$$

(H4) We can get

$$\begin{aligned}
&M_1 = \sup \|g_z(z)\| = \sqrt{2}\varepsilon,\\
&M_2 = \max\{\sup_{t\in[0,2\pi]} |h(t,z,\mu)|, \sup_{t\in[0,2\pi]} |h_z(t,z,\mu)|\} = 1.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{K(2M_1+\mu M_2)}{\alpha}(e^{\alpha T}-1)(\sum_{k=1}^{+\infty}(|Q^{-k}|+|Q^k|)e^{-\alpha kT} + 2e^{-\alpha T})\\
&= \frac{2\sqrt{2}\varepsilon+\mu}{\varepsilon}(e^{2\varepsilon\pi}-1)(\sum_{k=1}^{+\infty}((e^{2k\varepsilon\pi}+1)^{\frac{1}{2}} + (e^{-2k\varepsilon\pi}+1)^{\frac{1}{2}})e^{-2k\varepsilon\pi} + 2e^{-2\varepsilon\pi})\\
&= (2\sqrt{2}+\frac{\mu}{\varepsilon})(e^{2\varepsilon\pi}-1)(\sum_{k=1}^{+\infty}((e^{2k\varepsilon\pi}+1)^{\frac{1}{2}} + (e^{-2k\varepsilon\pi}+1)^{\frac{1}{2}})e^{-2k\varepsilon\pi} + 2e^{-2\varepsilon\pi})\\
&< 1, \qquad \text{if } \mu \text{ is sufficiently small.}
\end{aligned}$$

because of

$$e^{2\varepsilon\pi} - 1 \to 0, \quad \varepsilon \to 0,$$

$$\frac{\mu}{\varepsilon}(e^{2\varepsilon\pi}-1)\to 2\pi\mu, \varepsilon\to 0,$$
$$\sum_{k=1}^{+\infty}((e^{2k\varepsilon\pi}+1)^{\frac{1}{2}}+(e^{-2k\varepsilon\pi}+1)^{\frac{1}{2}})e^{-2k\varepsilon\pi}+2e^{-2\varepsilon\pi}<\infty.$$

By Theorem 3.1, for sufficiently small μ, the system (4.1) admits a unique affine-periodic solution.

Example 4.2. Consider the system

$$\begin{aligned} x^{'} &= \varepsilon x+\varepsilon y+\mu\sin t,\\ y^{'} &= -\varepsilon x+\varepsilon y+\mu\cos t, \end{aligned} \tag{4.2}$$

where $\varepsilon \ll 1$. Let $z=(x,y)^T$, we have $g(z)=(\varepsilon x+\varepsilon y, -\varepsilon x+\varepsilon y)^T$, $h(t,z,\mu)=(\sin t,\cos t)^T$.

(H1) $g(z)$ be a twice continuously differentiable vector function on $\mathbb{R}^n$. $\dot z=g(z)$ has a solution $\xi(t)=(0,0)^T$. The real part of eigenvalues of $g_z(\xi(t))$ is ε, so the variational equation $\dot z=g_z(\xi(t))z$ has an exponential dichotomy on $\mathbb{R}$ with $K=1,\ \alpha=\varepsilon$. And

$$g_z(z)=\begin{pmatrix}\varepsilon & \varepsilon\\ -\varepsilon & \varepsilon\end{pmatrix}$$

is bounded.

(H2) is obviously true.

(H3) Put

$$Q=\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix},$$
$$T=\beta.$$

Thus we have

$$\begin{aligned} &\xi(t+\beta)=Q\xi(t),\\ &Q^{-1}=\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix},\\ &Q^{-1}z=(x\cos\beta-y\sin\beta, x\sin\beta+y\cos\beta)^T. \end{aligned}$$

And there exist constant C such that $|Q^{-k}|+|Q^k|<C$.

Furthermore,

$$\begin{aligned} Qg(Q^{-1}z)=&\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix}\\ &\times\begin{pmatrix}\varepsilon(x\cos\beta-y\sin\beta)+\varepsilon(x\sin\beta+y\cos\beta)\\ -\varepsilon(x\cos\beta-y\sin\beta)+\varepsilon(x\sin\beta+y\cos\beta)\end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \varepsilon x + \varepsilon y \\ -\varepsilon x + \varepsilon y \end{pmatrix} \\
&= g(z).
\end{aligned}
$$

$$
\begin{aligned}
Qh(t, Q^{-1}z, \mu) &= \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \\
&= \begin{pmatrix} \sin(t+\beta) \\ \cos(t+\beta) \end{pmatrix} \\
&= h(t+\beta, z, \mu).
\end{aligned}
$$

(H4)We can get

$$
\begin{aligned}
&M_1 = \sup \|g_z(z)\| = 2\varepsilon, \\
&M_2 = \max\{\sup_{t\in[0,2\pi]} |h(t,z,\mu)|, \sup_{t\in[0,2\pi]} |h_z(t,z,\mu)|\} = \sqrt{2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{K(2M_1 + \mu M_2)}{\alpha}(e^{\alpha T} - 1)(\sum_{k=1}^{+\infty}(|Q^{-k}| + |Q^k|)e^{-\alpha k T} + 2e^{-\alpha T}) \\
= {}& (4 + \frac{\sqrt{2}\mu}{\varepsilon})(e^{\varepsilon\beta} - 1)(\sum_{k=1}^{+\infty} Ce^{-k\varepsilon\beta} + 2e^{-\varepsilon\beta}) \\
< {}& 1, \qquad \text{if } \mu \text{ is sufficiently small.}
\end{aligned}
$$

because of

$$
\begin{aligned}
&e^{\varepsilon\beta} - 1 \to 0, \varepsilon \to 0, \\
&\frac{\sqrt{2}\mu}{\varepsilon}(e^{\varepsilon\beta} - 1) \to \sqrt{2}\mu\beta, \varepsilon \to 0, \\
&\sum_{k=1}^{+\infty} Ce^{-k\varepsilon\beta} + 2e^{-\varepsilon\beta} < \infty.
\end{aligned}
$$

By Theorem 3.1, for sufficiently small μ, the system (4.2) admits a unique affine-periodic solution.

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