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EXISTENCE OF OSCILLATORY SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAYS*

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Abstract In this paper, under weaker hypothesis, we use the Schauder-Tychonoff theorem to obtain new sufficient condition for the global existence of oscillatory solutions of fractional differential equations with distributed delays.

Keywords Fractional differential equation, Liouville derivative, distributed delays, oscillatory solutions, global existence.

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1. Introduction

In this paper we study the existence of oscillatory solutions for the nonlinear fractional differential equations with distributed delays

$$D_t^\alpha[r(t)\Phi(x'(t))] + \int_a^b p(t,\tau)f(x(t-\tau))d\tau = q(t), \quad t \geq t_0, \tag{1.1}$$

where D_t^α is Liouville fractional derivative of order $\alpha \geq 0$ on the half-axis, $r \in C\big([t_0,\infty), R^+\big), p \in C\big([t_0,\infty) \times [a,b], R\big), q \in C\big([t_0,\infty), R\big), f \in C\big([t_0,\infty), R\big), 0 < a < b$, $\Phi(u)$ are continuously increasing real function with respect to u defined on R, and $\Phi^{-1}(u)$ satisfys the local *Lipschitz* condition.

Fractional differential equations have been widely applied in fluid science, chemical physics, electronic networks, fluid mechanics, and economics. In recent years, the research on fractional order ordinary differential equations and partial differential equations have been widely carried out and some favorable results have been obtained in [4, 10, 13].

There has been an increasing interest in oscillatory theory of differential equations, since its theoretical and practical value. So far, many authors have contributed to the subject and got many results about functional differential equations of integer order [1, 5–7, 11].

Recently, Grace, Agarwal, Wong,et al. [8], Bolat [2], Duan, Wang and Fu [3], Harikrishnan, Prakash and Nieto [9] investigated oscillation and forced oscillation of fractional order delay differential equations.zhou et al. [16], zhou et al. [17], Sun,

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Zhao [14], investigated the nonoscillatory theory for fractional differential equations. However, the existence of oscillatory solutions for fractional functional differential equations with distributed delays has been scarcely studied. Finally we will consider this problem.

2. Preliminaries

In this section, we introduce preliminary details which are used throughout this paper.

Definition 2.1. As usual, a solution of Equation (1.1) is a function $x(t)$ defined on $[t_0 - b, \infty)$ such that $x(t)$ and $r(t)\Phi(x'(t))$ are continuously differentiable on $[t_0 - b, \infty)$. Our attention will be restricted to those solution $x(t)$ of (1.1) which satisfys $\sup |x(t)| > 0$, for $t \geq T \geq t_0 - b$. Such a solution is said to be oscillatory if it has a sequence of zeros tending to infinity. Otherwise, it is said to be nonoscillatory.

Definition 2.2 ([10]). The Liouville fractional derivative on the half-axis is defined by

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f(s) ds,$$

where $t \in R$ and $\alpha \in [0, \infty)$.

Definition 2.3 ([10]). The Liouville fractional derivative on the half-axis is defined by

$$D_t^{\alpha} f(t) = \frac{d^n}{dt^n}(D_t^{-(n-\alpha)} f(t)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^{\infty} (s-t)^{n-\alpha-1} f(s) ds,$$

where $n = [\alpha] + 1, \alpha \in [0, \infty), [\alpha]$ denotes the integer part of α and $t \in R$. In particular, if $\alpha = n \in N$, then$D_t^n f(t) = f^{(n)}(t)$, where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n.

Property 2.1. ([10]) *For* $\alpha > 0$, $D_t^{\alpha}(D_t^{-\alpha} f)(t) = f(t)$.

We will prove a qeneral result about equation (1.1) on the existence of oscillatory solutions.

Here are some notations. For a constant $\gamma > 0$, $\theta_\gamma = \max_{|x| \leq \gamma} |f(x)|$, $t \geq t_0$. L_γ denote the local Lipischitz constants of functions $\Phi^{-1}(u)$.

3. The main results

Lemma 3.1 ([15]). *Let $S \subset K$, $K \subset X$, S be compact, K be nonvoid and convex, and X be a locally convex space, For given a continuous map $F : K \to S$, then there exists $\tilde{x} \in S$ such that $F(\tilde{x}) = \tilde{x}$.*

Theorem 3.1. *Assume there exists $\gamma > 0$,*

$$\frac{1}{r(t)} \int_t^{\infty} s^{\alpha-1} q(s) ds \text{ is integrable on } [t_0, \infty), \tag{3.1}$$

$$\frac{1}{r(t)} \int_t^{\infty} s^{\alpha-1} \int_a^b p(s,\tau) d\tau ds \text{ is integrable on } [t_0, \infty), \tag{3.2}$$

moreover, there exist two increasing divergent sequences $\{t_n\}$ *and* $\{s_n\}, t_n, s_n$ *such that*

$$\int_{t_n}^{\infty} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^{\infty}(u-t)^{\alpha-1}(q(u)+\theta_\gamma\int_a^b p(u,\tau)d\tau)du\right)ds < 0, \tag{3.3}$$

$$\int_{s_n}^{\infty} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^{\infty}(u-t)^{\alpha-1}(q(u)-\theta_\gamma\int_a^b p(u,\tau)d\tau)du\right)ds > 0, \tag{3.4}$$

then equation (1.1) has an oscillatory solution $x(t)$ *defined on* $[t_0,\infty)$ *with* $|x|\le\gamma$, *and* $\lim\limits_{t\to\infty} x(t)=0$.

Proof. The proof is based on an application of the well known Schauder-Tychonoff fixed point theorem.

From (3.1)and (3.2), for any $\gamma>0$ we choose a large $T_\gamma \ge T$ such that for all $t\ge T_\gamma$,

$$\int_{t}^{\infty} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^{\infty}(u-t)^{\alpha-1}(q(u)+\theta_\gamma\int_a^b p(u,\tau)d\tau)du\right)ds \le \gamma, \tag{3.5}$$

$$\int_{t}^{\infty} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^{\infty}(u-t)^{\alpha-1}(q(u)-\theta_\gamma\int_a^b p(u,\tau)d\tau)du\right)ds \ge -\gamma. \tag{3.6}$$

Let $C[t_0-b,\infty)$ denote the locally convex space of all continuous functions with topology of uniform convergence on compact subsets of $[t_0-b,\infty)$. Let $S=\{x\in C[t_0-b,\infty), |x(t)|\le\gamma\}$. Clearly, S is a close convex subset of $C[t_0-b,\infty)$.

Introduce an operator F by,

$$(Fx)(t)=\begin{cases}\displaystyle\int_t^{\infty}\Phi^{-1}\Bigg(\frac{1}{\Gamma(\alpha)r(s)}\int_s^{\infty}(u-t)^{\alpha-1}(q(u)\\ \displaystyle+\int_a^b p(u,\tau)f(x(u-\tau))d\tau)du\Bigg)ds,\ t>T_\gamma,\\ (Fx)(T_\gamma), \qquad t_0-b\le t\le T_\gamma.\end{cases}$$

It is easy to see that for any $x\in S$, $(Fx)(t)$ is continuous and well defined on $[t_0-b,\infty)$. From (3.5) and (3.6) we obtain

$$\begin{aligned}(Fx)(t) &\le \int_t^{\infty}\Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^{\infty}(u-t)^{\alpha-1}(q(u)+\theta_\gamma\int_a^b p(u,\tau)d\tau)du\right)ds\\ &\le \gamma, \quad t\ge t_0-b,\end{aligned}$$

and

$$\begin{aligned}(Fx)(t) &\ge \int_t^{\infty}\Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^{\infty}(u-t)^{\alpha-1}(q(u)-\theta_\gamma\int_a^b p(u,\tau)d\tau)du\right)ds\\ &\ge -\gamma, \quad t\ge t_0-b.\end{aligned}$$

Since $|(Fx)(t)|\le\gamma$, we have $FS\subset S$ and Fx is uniformly bounded on S.

Let T_1 be large constant with $T_1 > T$, for any $\epsilon > 0$ such that

$$\int_{T_1}^{\infty} \frac{1}{\Gamma(\alpha)r(s)} \left(\int_s^{\infty} (s-t)^{\alpha-1} \int_a^b p(s,\tau) d\tau du \right) ds < \frac{\epsilon}{3\theta_\gamma L_\gamma}. \tag{3.7}$$

Let$\{x_n\}_{n=1}^{\infty} \in S$ be any sequence and $x_0 \in S$ with $\lim_{n\to\infty} x_n = x_0$. From the compactness of the domain of f , there exist a large number $N(\epsilon) > 0$ and a constant $\delta(\epsilon) > 0$. Let $t \in [t_0 - b, T_1]$ and $n \geq N$, when $|x_n - x_0| < \delta(\epsilon)$,

$$\max_{t_0 - b \leq t \leq T_1} \{|f(x_n(t-\tau)) - f(x_0(t-\tau))|\} \leq \frac{\epsilon}{3ML_\gamma}, \tag{3.8}$$

where $M = \int_{t_0-b}^{T_1} \frac{(s-t_0+b)^{\alpha-1}}{\Gamma(\alpha)r(s)} \int_a^b p(s,\tau) d\tau ds$. By virtue of (1.1), (3.1)-(3.8), we have that for any $t \geq t_0 - b$ and $|x_n - x_0| < \delta$,

$$\begin{aligned}
&|(Fx_n)(t) - (Fx_0)(t)| \\
=&|\int_t^{\infty} \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^{\infty} (u-t)^{\alpha-1}(q(u) + \int_a^b p(u,\tau) f(x_n(u-\tau)) d\tau) du] ds \\
&- \int_t^{\infty} \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^{\infty} (u-t)^{\alpha-1}(q(u) + \int_a^b p(u,\tau) f(x_0(u-\tau)) d\tau) du] ds| \\
\leq& \int_t^{\infty} |\Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^{\infty} (u-t)^{\alpha-1}(q(u) + \int_a^b p(u,\tau) f(x_n(u-\tau)) d\tau) du] \\
&- \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^{\infty} (u-t)^{\alpha-1}(q(u) + \int_a^b p(u,\tau) f(x_0(u-\tau)) d\tau) du]| ds \\
\leq& \int_t^{\infty} |\frac{L_\gamma}{\Gamma(\alpha)r(s)}| \int_s^{\infty} (u-t)^{\alpha-1} (\int_a^b |p(u,\tau)||f(x_n(u-\tau)) - f(x_0(u-\tau))| d\tau) du ds \\
\leq& \int_{t_0-b}^{T_1} |\frac{L_\gamma}{\Gamma(\alpha)r(s)}| \int_s^{\infty} (u-t)^{\alpha-1} (\int_a^b |p(u,\tau)||f(x_n(u-\tau)) - f(x_0(u-\tau))| d\tau) du ds \\
&+ \int_{T_1}^{\infty} |\frac{L_\gamma}{\Gamma(\alpha)r(s)}| \int_s^{\infty} (u-vt)^{\alpha-1} (\int_a^b |p(u,\tau)||f(x_n(u-\tau)) - f(x_0(u-\tau))| d\tau) du ds \\
\leq& \int_{t_0-b}^{T_1} \frac{L_\gamma (s-t_0+b)^{\alpha-1}}{\Gamma(\alpha)r(s)} (\int_a^b |p(u,\tau)||f(x_n(u-\tau)) - f(x_0(u-\tau))| d\tau) ds \\
&+ 2\theta_\gamma \int_{T_1}^{\infty} \frac{L_\gamma}{\Gamma(\alpha)r(s)} \int_s^{\infty} (u-t)^{\alpha-1} (\int_a^b |p(u,\tau)| d\tau) du ds \\
<& \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon,
\end{aligned}$$

which means F is contious on S. Moreover, for all $t_2, t_1 > t_0 - b$,

$$\begin{aligned}
&(Fx)(t_2) - (Fx)(t_1) \\
=& \int_{t_1}^{t_2} \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^{\infty} (u-t)^{\alpha-1}(q(u) + \int_a^b p(u,\tau) f(x(u-\tau)) d\tau) du] ds \\
\leq& \int_{t_1}^{t_2} \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^{\infty} (u-t)^{\alpha-1}(q(u) + \theta_\gamma \int_a^b p(u,\tau) d\tau) du] ds \\
\leq& \alpha(t_2 - t_1),
\end{aligned}$$

where $\alpha = \sup_{t\geq t_0} \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)+\theta_\gamma \int_a^b p(u,\tau)d\tau)du]$, and

$$\begin{aligned}&(Fx)(t_2)-(Fx)(t_1)\\ =&\int_t^\infty \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)+\int_a^b p(u,\tau)f(x(u-\tau))d\tau)du]ds\\ \geq& \int_{t_1}^{t_2} \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)-\theta_\gamma \int_a^b p(u,\tau)d\tau)du]ds\\ \geq&\beta(t_2-t_1),\end{aligned}$$

where $\beta = \inf_{t\geq t_0} \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)+\theta_\gamma \int_a^b p(u,\tau)d\tau)du]$.

We get

$$|(Fx)(t_2)-(Fx)(t_1)| \leq M|t_2-t_1|,$$

where $M = \max\{|\alpha|, |\beta|\}$. This implies Fx is equicontinuous. Hence by the Ascoli-Arzela Theorem the operator is a completely continuous on S.

By Lemma, there exists $\widetilde{x} \in S$ satisfying

$$\begin{aligned}&\widetilde{x}(t) = (F\widetilde{x})(t)\\ =&\begin{cases}\displaystyle\int_t^\infty \Phi^{-1}[\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)+\int_a^b p(u,\tau)f(\widetilde{x}(u-\tau))d\tau)du]ds, & t>T_\gamma,\\ (F\widetilde{x})(T_\gamma), \quad t_0-b\leq t<T_\gamma.\end{cases}\end{aligned}$$

On the other hand, from (3.3) and (3.4), we find

$$\widetilde{x}(t_n) \leq \int_{t_n}^\infty \Phi^{-1}(\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)+\theta_\gamma \int_a^b p(u,\tau)d\tau)du)ds < 0,$$

and

$$\widetilde{x}(s_n) \geq \int_{s_n}^\infty \Phi^{-1}(\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)-\theta_\gamma \int_a^b p(u,\tau)d\tau)du)ds > 0,$$

which implies that $\widetilde{x}(t)$ is a bounded oscillatory solution of equation (1.1) and $\lim_{t\to\infty} \widetilde{x}(t) = 0$. The proof is completed. □

Corollary 3.1. *Assume that (3.1) and (3.2) of Theorem hold. Specially,* $\Phi(u) = u^\alpha$*, for* $\alpha \geq 1$ *is the ratio of two positive odd integers, and there exist two increasing divergent sequences* $\{t_n\}$ *and* $\{s_n\}$*, such that*

$$\int_{t_n}^\infty \left(\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)+\theta_\gamma \int_a^b p(u,\tau)d\tau)du\right)^{\frac{1}{\alpha}} ds < 0,$$

and

$$\int_{s_n}^\infty \left(\frac{1}{\Gamma(\alpha)r(s)} \int_s^\infty (u-t)^{\alpha-1}(q(u)-\theta_\gamma \int_a^b p(u,\tau)d\tau)du\right)^{\frac{1}{\alpha}} ds > 0.$$

Then equation (1.1) has an oscillatory solution $x(t)$ *defined on* $[t_0,\infty)$ *for* $|x| \leq \gamma$*, and* $\lim_{t\to\infty} x(t) = 0$*.*

4. Remark

We consider the existence of oscillatory solutions of equation (1.1) for any order $\alpha > 0$. Especially, for $\alpha = 1$ the equation (1.1) reduces to equation (1) of reference [12].

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