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ON A NEW HALF-DISCRETE HILBERT-TYPE INEQUALITY WITH THE MULTIPLE UPPER LIMIT FUNCTION AND THE PARTIAL SUMS*

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Abstract By means of the weight coefficients, the Euler-Maclaurin summation formula and Abel's summation by parts formula, a new half-discrete Hilbert-type inequality with the power function as the interval variables as well as one multiple upper limit function and one partial sums is given. As applications, the equivalent conditions of the best possible constant factor in a particular inequality related to a few parameters and some particular cases are considered. We also obtain the equivalent forms and the operator expression in the case of $m = 0$.

Keywords Weight coefficient, Euler-Maclaurin summation formula, Half-dis-crete Hilbert-type inequality, parameter, partial sums, multiple upper limit function.

MSC(2010) 26D15.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the well known Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf [4], Theorem 315):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1.1}$$

In 2006, by introducing parameters $\lambda_i \in (0, 2]$ $(i = 1, 2), \lambda_1 + \lambda_2 = \lambda \in (0, 4]$, an extension of (1.1) was provided by [12] as follows:

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}\\ &< B(\lambda_1, \lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{1.2}$$

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where, the constant factor $B(\lambda_1, \lambda_2)$ is the best possible, and

$$B(u,v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt \ (u,v > 0) \tag{1.3}$$

is the beta function

For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (1.2) reduces to (1.1); for $p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (1.2) reduces to Yang's inequality in [24]. Recently, applying (1.2) and by the help of Abel's summation by parts formula, Adiyasuren et al. [1] gave a new inequality with the kernel $\frac{1}{(m+n)^\lambda}$ involving partial sums (cf. [3, 7, 8, 10, 12–14, 22, 24, 25, 27]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [4], Theorem 351): Assuming that $K(t)$ $(t > 0)$ is a decreasing function, $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^\infty K(t)t^{s-1}dt < \infty$, $a_n \geq 0$, such that $0 < \sum_{n=1}^\infty a_n^p < \infty$, we have

$$\int_0^\infty x^{p-1}(\sum_{n=1}^\infty K(nx)a_n)^p dx < \phi^p(\frac{1}{q})\sum_{n=1}^\infty a_n^p. \tag{1.4}$$

Some extensions of (1.4) were provided by [11, 16, 17, 26, 28].

In 2016, by using the techniques of real analysis, Hong et al. [7] considered some equivalent statements of the extensions of (1.1) with the best possible constant factor related to a few parameters. The other similar works about the extension of (1.2) and (1.4) were given by [5, 6, 9, 19–21, 29]. In a recent papers [29, 30], Yang et al. gave a reverse half-discrete Hardy-Hilbert's inequality as well as an extended Hardy-Hilbert's inequality, and dealt with their equivalent statements of the best possible constant factor related to several parameters as applications.

In this paper, based on the way of [4, 15, 31], by the use of the weight coefficients, the idea of introduced parameters, Euler-Maclaurin summation formula and Abel's summation by parts formula, a new half-discrete Hilbert-type inequality with the kernel as $\frac{1}{(x+n^\alpha)}$ and one multiple upper limit function as well as one partial sums is given. As applications, the equivalent conditions of the best possible constant factor in a particular inequality related to a few parameters are considered, and then some particular cases are obtained. We also provide the equivalent forms and the operator expressions in the case of $m = 0$..

2. Some lemmas

In what follows, we assume that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \mathbf{m} = 0, 1.2, 3, 4, \lambda \in (0.5 - m] \neq \Phi, \alpha \in (0, 1], \lambda_1 \in (0, \lambda + 1), \lambda_2 \in (0, \frac{2}{\alpha} - 1] \cap (0.\lambda + m)$, $\widehat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \widehat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$. We also assume that $f(x) := F_0(x)$ is a nonnegative Lebesgue integrable function in any interval $(0, b)\,(b > 0)$, define a multiple upper limit function as follows: $F_i(x) := \int_0^x F_{i-1}(t)dt$ $(x \geq 0)$, satisfying

$$F_i(0) = 0, F_i(x) = o(e^{tx}) \ (t > 0, x \to \infty; i = 1, \cdots, m),$$

and for $a_m \geq 0$, the partial sums is indicated as follows:

$$A_n := \sum_{i=1}^n a_i \ (n \in N := \{1, 2, \cdots\}),$$

satisfying $A_n = o(e^{tn^\alpha})$ $(t > 0, n \to \infty)$. For $\mathbf{m} = 0, 1.2, 3, 4$, we mark appointments that

$$0 < \int_0^\infty x^{-p(\widehat{\lambda}_1 - m - 1) - 1} F_m^p(x) dx < \infty, \text{ and}$$
$$0 < \sum_{n=1}^\infty n^{-q[\alpha(\widehat{\lambda}_2 + 1) - 1] - 1} A_n^q < \infty, \tag{2.1}$$

Lemma 2.1. *(i) (cf. [25], (2.2.3)) If* $(-1)^i \frac{d^i}{dt^i} g(t) > 0, t \in [n, \infty)$ $(n \in \mathbf{N})$ *with* $g^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3), P_i(t), B_i$ $(i \in \mathbf{N})$ *are Bernoulli functions and Bernoulli numbers of i-order, then we have*

$$\int_n^\infty P_{2q-1}(t) g(t) dt = -\varepsilon_q \frac{B_{2q}}{2q} g(n) \ (0 < \varepsilon_q < 1; q = 1, 2, \cdots). \tag{2.2}$$

In particular, for $q = 1$, *in view of* $B_2 = \frac{1}{6}$, *we have*

$$-\frac{1}{12} g(n) < \int_n^\infty P_1(t) g(t) dt < 0; \tag{2.3}$$

for $q = 2$, *in view of* $B_4 = -\frac{1}{30}$, *we have*

$$0 < \int_n^\infty P_3(t) g(t) dt < \frac{1}{120} g(n). \tag{2.4}$$

(ii) (cf. [25], (2.3.2)) If $h(t)(> 0) \in C^3[m, \infty), h^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3)$, *then we have the following Euler-Maclaurin summation formulas:*

$$\sum_{k=n}^\infty f(k) = \int_n^\infty f(t) dt + \frac{1}{2} f(n) + \int_n^\infty P_1(t) f'(t) dt, \tag{2.5}$$
$$\int_n^\infty P_1(t) f'(t) dt = -\frac{1}{12} f'(n) + \int_n^\infty P_3(t) f'''(t) dt. \tag{2.6}$$

Lemma 2.2. *For* $s \in (0, 6], s_2 \in (0, \frac{2}{\alpha}] \cap (0, s), k_s(s_2) = B(s - s_2, s_2)$, *define the following weight coefficient:*

$$\varpi(s_2, x) := \alpha s^{s - s_2} \sum_{n=1}^\infty \frac{n^{\alpha s_2 - 1}}{(x + n^\alpha)^s} \ (x \in \mathbf{R}_+ = (0, \infty)). \tag{2.7}$$

We have the following inequalities:

$$0 < k_s(s_2)(1 - O(\frac{1}{x^{s_2}})) < \varpi(s_2, x) < k_s(s_2), \tag{2.8}$$

where,

$$O(\frac{1}{(x^{s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{1}{x}} \frac{u^{s_2 - 1}}{(1 + u)^s} du > 0.$$

Proof. For fixed $x > 0$, we define the following function:

$$g_x(t) := \frac{\alpha t^{\alpha s_2 - 1}}{(x + t)^s} \ (t > 0).$$

By using (2.5), we have

$$\begin{aligned}\sum_{n=1}^{\infty} g_x(n) &= \int_1^{\infty} g_x(t)dt + \frac{1}{2}g_x(1) + \int_1^{\infty} P_1(t)g'_x(t)dt,\\ &= \int_0^{\infty} g_x(t)dt - h(x),\\ h(x) &:= \int_0^1 g_x(t)dt - \frac{1}{2}g_x(1) - \int_1^{\infty} P_1(t)g'_x(t)dt.\end{aligned}$$

We obtain $-\frac{1}{2}g_x(1) = \frac{-\alpha}{2(x+1)^s}$, and integration by parts, it follows that

$$\begin{aligned}\int_0^1 g_x(t)dt =&\alpha \int_0^1 \frac{t^{\alpha s_2 - 1}dt}{(x+t^\alpha)^s} = \int_0^1 \frac{u^{s_2-1}du}{(x+u)^s} = \frac{1}{s_2}\int_0^1 \frac{du^{s_2}}{(x+u)^s}\\ =&\frac{1}{s_2}\frac{u^{s_2}}{(x+u)^s}|_0^1 + \frac{s}{s_2}\int_0^1 \frac{u^{s_2}du}{(x+u)^{s+1}}\\ =&\frac{1}{s_2}\frac{1}{(x+1)^s} + \frac{s}{s_2(s_2+1)}\int_0^1 \frac{du^{s_2+1}}{(x+u)^{s+1}}\\ >&\frac{1}{s_2}\frac{1}{(x+1)^s} + \frac{s}{s_2(s_2+1)}\frac{u^{s_2+1}}{(x+u)^{s+1}}|_0^1\\ &+ \frac{s(s+1)}{s_2(s_2+1)(x+1)^{s+1}}\int_0^1 u^{s_2+1}du\\ =&\frac{1}{s_2}\frac{1}{(x+1)^s} + \frac{s}{s_2(s_2+1)}\frac{1}{(x+1)^{s+1}}\\ &+ \frac{s(s+1)}{s_2(s_2+1)(s_2+2)(x+1)^{s+2}},\end{aligned}$$

$$\begin{aligned}-g'_x(t) &= -\frac{\alpha(\alpha s_2 - 1)t^{\alpha s_2 - 2}}{(x+t^\alpha)^s} - \frac{\alpha^2 s t^{\alpha + \alpha s_2 - 2}}{(x+t^\alpha)^{s+1}}\\ &= \frac{\alpha(1-\alpha s_2)t^{\alpha s_2 - 2}}{(x+t^\alpha)^s} + \frac{\alpha^2 s(x+t^\alpha - x)t^{\alpha s_2 - 2}}{(x+t^\alpha)^{s+1}}\\ &= \frac{\alpha(\alpha s + 1 - \alpha s_2)t^{\alpha s_2 - 2}}{(x+t^\alpha)^s} - \frac{\alpha^2 s x t^{\alpha s_2 - 2}}{(x+t^\alpha)^{s+1}},\end{aligned}$$

and for $s_2 \in (0, \frac{2}{\alpha}] \cap (0, s)$ $(s \leq 6), \alpha \in (0, 1]$, we have

$$\begin{aligned}&(-1)^i \frac{d^i}{dt^i}[\frac{t^{\alpha s_2 - 2}}{(x+t^\alpha)^s}] > 0,\\ &(-1)^i \frac{d^i}{dt^i}[\frac{t^{\alpha s_2 - 2}}{(x+t^\alpha)^{s+1}}] > 0 \quad (t > \eta_2; \quad i = 0, 1, 2, 3).\end{aligned}$$

By (2.3)-(2.6), we obtain

$$\begin{aligned}&\alpha(\alpha s + 1 - \alpha s_2)\int_1^{\infty} P_1(t)\frac{t^{\alpha s_2 - 2}}{(x+t^\alpha)^s}dt > -\frac{\alpha(\alpha s + 1 - \alpha s_2)}{12(x+1)^s},\\ &-\alpha^2 x s\int_1^{\infty} P_1(t)\frac{t^{\alpha s_2 - 2}}{(x+t^\alpha)^{s+1}}dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^2 xs}{12(x+1)^{s+1}} - \frac{\alpha^2 xs}{6}\int_1^\infty P_1(t)[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}]''_t dt \\
&> \frac{\alpha^2 xs}{12(x+1)^{s+1}} - \frac{\alpha^2 xs}{720}[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}]''_{t=1} \\
&> \frac{\alpha^2(x+1-1)s}{12(x+1)^{s+1}} - \frac{\alpha^2(x+1)s}{720}\left[\frac{(s+1)(s+2)\alpha^2}{(x+1)^{s+3}}\right. \\
&\quad \left. + \frac{\alpha(s+1)(5-\alpha-2\alpha s_2)}{(x+1)^{s+2}} + \frac{(2-\alpha s_2)(3-\alpha s_2)}{(x+1)^{s+1}}\right] \\
&= \frac{\alpha^2 s}{12(x+1)^s} - \frac{\alpha^2 s}{12(x+1)^{s+1}} - \frac{\alpha^2 s}{720}\left[\frac{(s+1)(s+2)\alpha^2}{(x+1)^{s+2}}\right. \\
&\quad \left. + \frac{\alpha(s+1)(5-\alpha-2\alpha s_2)}{(x+1)^{s+1}} + \frac{(2-\alpha s_2)(3-\alpha s_2)}{(x+1)^s}\right],
\end{aligned}
$$

and then we have

$$
h(x) > \frac{h_1}{(x+1)^s} + \frac{sh_2}{(x+1)^{s+1}}\frac{s(s+1)h_3}{(x+1)^{s+2}},
$$

where, h_i $(i = 1, 2, 3)$ are indicated as

$$
\begin{aligned}
h_1 &:= \frac{1}{s_2} - \frac{\alpha}{2} - \frac{\alpha - \alpha^2 s_2}{12} - \frac{\alpha^2 s(2-\alpha s_2)(3-\alpha s_2)}{720}, \\
h_2 &:= \frac{1}{s_2(s_2+1)} - \frac{\alpha^2}{12} - \frac{\alpha^3(s+1)(5-\alpha-2\alpha s_2)}{720}, \quad \text{and} \\
h_3 &:= \frac{1}{s_2(s_2+1)(s_2+2)} - \frac{\alpha^4(s+2)}{720}.
\end{aligned}
$$

We obtain $h_1 \geq \frac{g(s_2)}{720 s_2}$, where, we indicate $g(\sigma)$ $(\sigma \in (0, \frac{2}{\alpha}])$ as follows:

$$
g(\sigma) := 720 - (420\alpha + 6s\alpha^2)\sigma + (60\alpha^2 + 5s\alpha^3)\sigma^2 - s\alpha^4\sigma^3.
$$

We obtain that for $\alpha \in (0, 1], s \in (0, 6]$,

$$
\begin{aligned}
g'(\sigma) &:= -(420\alpha + 6s\alpha^2) + 2(60\alpha^2 + 5s\alpha^3)\sigma - 3\alpha^4\sigma^2 \\
&\leq -420\alpha - 6s\alpha^2 + 2(60\alpha^2 + 5s\alpha^3)\frac{2}{\alpha} \\
&= (14s\alpha - 180)\alpha < 0,
\end{aligned}
$$

and then it follows that

$$
h_1 \geq \frac{g(s_2)}{720 s_2} \geq \frac{g(2/\alpha)}{720 s_2} = \frac{1}{6 s_2} > 0.
$$

we obtain that for $s_2 \in (0, \frac{2}{\alpha}], s \in (0, 6]$,

$$
\begin{aligned}
h_2 &> \frac{\alpha^2}{6} - \frac{\alpha^2}{12} - \frac{5\alpha^2(s+1)}{720} = (\frac{1}{12} - \frac{s+1}{140})\alpha^2 > 0, \\
h_3 &> (\frac{1}{24} - \frac{s+2}{720})\alpha^2 > 0.
\end{aligned}
$$

Hence, we have $h(x) > 0$, and then setting $t = x^{1/\alpha}u^{1/\alpha}$, it follows that

$$\varpi(s_2, x) = x^{s-s_2} \sum_{n=1}^{\infty} g_x(n) < x^{s-s_2} \int_0^{\infty} g_x(t)dt$$
$$= \alpha x^{s-s_2} \int_0^{\infty} \frac{t^{\alpha s_2 - 1}dt}{(x + t^\alpha)^s} = \int_0^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = k_s(s_2).$$

On the other hand, by (2.5), we also have

$$\sum_{n=1}^{\infty} g_x(n) = \int_1^{\infty} g_x(t)dt + \frac{1}{2}g_x(1) + \int_1^{\infty} P_1(t)g_x'(t)dt$$
$$= \int_1^{\infty} g_x(t)dt + H(x),$$
$$H(x) := \frac{1}{2}g_x(1) + \int_1^{\infty} P_1(t)g_x'(t)dt.$$

We have obtained that $\frac{1}{2}g_x(1) = \frac{a}{2(x+1)^s}$ and

$$g_x'(t) = -\frac{\alpha(\alpha s + 1 - \alpha s_2)t^{\alpha s_2 - 2}}{(x + t^\alpha)^s} + \frac{\alpha^2 s x t^{\alpha s_2 - 2}}{(x + t^\alpha)^{s+1}}.$$

For $s_2 \in (0, \frac{2}{\alpha}] \cap (0, s)$ ($s \in (0, 6]$), by (2.3), we find

$$-\alpha(\alpha s + 1 - \alpha s_2) \int_1^{\infty} P_1(t) \frac{t^{\alpha s_2 - 2}}{(x + t^\alpha)^s} dt > 0, \quad \text{and}$$
$$\alpha^2 s x \int_1^{\infty} P_1(t) \frac{t^{\alpha s_2 - 2}}{(x + t^\alpha)^{s+1}} dt > \frac{-\alpha^2 x s}{12(x+1)^{s+1}} > \frac{-\alpha^2 x s}{12(x+1)^s}.$$

Hence, we have

$$H(x) > \frac{\alpha}{2(x+1)^s} - \frac{\alpha^2 s}{12(x+1)^s} \geq \frac{\alpha}{2(x+1)^s} - \frac{2\alpha}{12(x+1)^s} > 0$$

and then we obtain

$$\varpi(s_2, x) = x^{s-s_2} \sum_{n=1}^{\infty} g_x(n) > x^{s-s_2} \int_1^{\infty} g_x(t)dt$$
$$= x^{s-s_2} \int_0^{\infty} g_x(t)dt - x^{s-s_2} \int_0^1 g_x(t)dt$$
$$= k_s(s_2)[1 - \frac{1}{k_s(s_2)} \int_0^{1/x} \frac{u^{s_2-1}du}{(1+u)^s}] > 0,$$

where, we set $O(\frac{1}{x^{s_2}}) = \frac{1}{k_s(s_2)} \int_0^{1/x} \frac{u^{s_2-1}du}{(1+u)^s}$, satisfying

$$0 < \int_0^{1/x} \frac{u^{s_2-1}}{(1+u)^s} du < \int_0^{1/x} u^{s_2-1} du = \frac{1}{s_2 x^{s_2}}.$$

Therefore, inequalities (2.8) follow.

The lemma is proved. □

Lemma 2.3. *For* $s \in (0,6], s_1 \in (0,s), s_2 \in (0,\frac{2}{\alpha}] \cap (0,s), k_s(s_i) = B(s-s_i, s_i)$ $(i=1,2)$, *we have the following half-discrete Hardy-Hilbert's inequality with the internal variable:*

$$I := \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{(x+n^\alpha)^s} dx \le (\frac{1}{\alpha} k_s(s_2))^{\frac{1}{p}} (k_s(s_1))^{\frac{1}{q}}$$
$$\times \left\{ \int_0^\infty t^{p[1-(\frac{s-s_2}{p}+\frac{s_1}{q})]-1} f^p(t)dt \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-(\frac{s-s_1}{q}+\frac{s_2}{p})]-1} a_n^q \right\}^{\frac{1}{q}}. \quad (2.9)$$

Proof. For $s_1 \in (0,s)$, setting $u = x/n^\alpha$, we obtain the following next weight coefficient:

$$\omega_\alpha(s_1,n) := n^{\alpha(s-s_1)} \int_0^\infty \frac{x^{s_1-1}dx}{(x+n^\alpha)^s} = \int_0^\infty \frac{u^{s_1-1}du}{(u+1)^s} = k_s(s_1) \ (n \in \mathbf{N}). \quad (2.10)$$

By Hölder's inequality (cf. [14]), we obtain

$$I = \int_0^\infty \sum_{n=1}^\infty \frac{1}{(x+n^\alpha)^s} \left[\frac{x^{(1-s_1)/q}(\alpha n^{\alpha-1})^{1/p}}{n^{\alpha(1-s_2)/p}} f(x) \right]$$
$$\times \left[\frac{n^{\alpha(1-s_2)/p} a_n}{x^{(1-s_1)/q}(\alpha n^{\alpha-1})^{1/p}} \right] dx$$
$$\le \left[\int_0^\infty \sum_{n=1}^\infty \frac{\alpha}{(x+n^\alpha)^s} \frac{x^{(1-s_1)(p-1)} n^{\alpha-1}}{n^{\alpha(1-s_2)}} f^p(x)dx \right]^{\frac{1}{p}}$$
$$\times \left[\sum_{n=1}^\infty \int_0^\infty \frac{1}{(x+n^\alpha)^s} \frac{n^{\alpha(1-s_2)(q-1)}}{x^{1-s_1}(\alpha n^{\alpha-1})^{(q-1)}} dx a_n^q \right]^{\frac{1}{q}}$$
$$= \frac{1}{\alpha^{1/p}} \left\{ \int_0^\infty \varpi(s_2,x) x^{p[1-(\frac{s-s_2}{p}+\frac{s_1}{q})-1]} f^p(x)dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ \sum_{n=1}^\infty \omega(s_1,n) n^{q[1-\alpha(\frac{s-s_1}{q}+\frac{s_2}{p})-1]} a_n^q \right\}^{\frac{1}{q}}.$$

Then by (2.8) and (2.10), we have (2.9).

The lemma is proved. □

Remark 2.1. For $s = \lambda+m+1 \in (m+1,6], \lambda \in (0,5-m] \neq \Phi\,(m=0,1,2,3)$, $s_1 = \lambda_1+m \in (k,s), \lambda_1 \in (0,\lambda+1), s_2 = \lambda_2+1 \in (1,\frac{2}{\alpha}), \lambda_2 \in (0,\frac{2}{\alpha}-1] \cap (0,\lambda+m)$ in (2.9), replacing $f(x)$ (resp. a_n) by $F_m(x)$ (resp. A_n), in view of (2.1,) we have

$$I_0 := \int_0^\infty \sum_{n=1}^\infty \frac{A_n F_m(x)dx}{(x+n^\alpha)^{\lambda+m+1}} \le (\frac{1}{\alpha} k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}$$
$$\times \left[\int_0^\infty x^{-p(\widehat{\lambda}_1+m-1)-1} F_m^p(x)dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{-q[\alpha(\widehat{\lambda}_2+1)-1]-1} A_n^q \right]^{\frac{1}{q}}. \quad (2.11)$$

Lemma 2.4. *For* $t > 0$, *we have the following expression and inequality:*

$$\int_0^\infty e^{-tx} f(x)dx = t^m \int_0^\infty e^{-tx} F_m(x)dx \ (m=0,1,2,3,4), \quad (2.12)$$

$$\sum_{n=1}^{\infty} e^{-tn^{\alpha}} a_n \leq t \sum_{n=1}^{\infty} e^{-tn^{\alpha}} A_n. \tag{2.13}$$

Proof. For $m = 0$, since $f(x) = F_0(x)$,(2.12) is valid; for $m = 1, 2, 3, 4$, integration by parts, in view of $F_i(0) = 0, F_i(x) = o(e^{tx})$ $(t > 0, x \to \infty; i = 1, \cdots, k)$, it follows that

$$\begin{aligned}\int_0^{\infty} e^{-tx} F_{i-1}(x)dx &= \int_0^{\infty} e^{-tx} dF_i(x) = e^{-tx}F_i(x)|_0^{\infty} - \int_0^{\infty} F_i(x)de^{-tx} \\ &= \lim_{x\to\infty} e^{-tx}F_i(x) + t\int_0^{\infty} e^{-tx}F_i(x)dx = t\int_0^{\infty} e^{-tx}F_i(x)dx,\end{aligned}$$

and then substitution of $i = 1, \cdots, m$, (2.12) follows.

In view of $A_n e^{-tn^{\alpha}} = o(1)$ $(n \to \infty)$, by Abel's summation by parts formula, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-tn^{\alpha}} a_n &= \lim_{n\to\infty} A_n e^{-tn^{\alpha}} + \sum_{n=1}^{\infty} [e^{-tn^{\alpha}} - e^{-t(n+1)^{\alpha}}]A_n \\ &= \sum_{n=1}^{\infty} [e^{-tn^{\alpha}} - e^{-t(n+1)^{\alpha}}]A_n.\end{aligned}$$

Since $1 - e^{-t} < t$ $(t > 0)$ and for $\alpha \in (0, 1]$,

$$\begin{aligned}& e^{-t(n+1)^{\alpha}} \geq e^{-t(n^{\alpha}+1)} \\ <=> \;& e^{-t[(n+1)^{\alpha}-n^{\alpha}-1]} \leq 1 \\ <=> \;& (n+1)^{\alpha} - n^{\alpha} - 1 = \alpha(n+\theta_{\alpha})^{\alpha-1} - 1 \ (\theta_{\alpha} \in (0,1)),\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-tn^{\alpha}} a_n &\leq \sum_{n=1}^{\infty} [e^{-tn^{\alpha}} - e^{-t(n^{\alpha}+1)}]A_n = (1 - e^{-t})\sum_{n=1}^{\infty} e^{-tn^{\alpha}} A_n \\ &\leq t\sum_{n=1}^{\infty} e^{-tn^{\alpha}} A_n,\end{aligned}$$

namely, (2.13) follows.

The lemma is proved. □

3. Main results and applications

In the following, for $m = 0$, we define $\prod_{i=0}^{m-1}(c+i) = 1$ $(c > 0)$.

Theorem 3.1. *We have the following Hilbert-type inequality:*

$$\begin{aligned}I &:= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{a_n f(x)}{(x+n^{\alpha})^{\lambda}} dx \\ &< \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} (\frac{1}{\alpha} k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}} \\ &\quad \times \left[\int_0^{\infty} x^{-p(\widehat{\lambda}_1+m-1)-1} F_m^p(x)dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{-q[\alpha(\widehat{\lambda}_2+1)-1]-1} A_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

In particular, for $\lambda_1+\lambda_2=\lambda$ ($\lambda_1\in(0,\lambda), \lambda_2\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda)$), we have

$$0<\int_0^\infty x^{-p(\lambda_1+m-1)-1}F_m^p(x)dx<\infty,\ 0<\sum_{n=1}^\infty n^{-q[\alpha(\lambda_2+1)-1]-1}A_n^q<\infty,$$

and the following inequality:

$$I=\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{(x+n^\alpha)^\lambda}dx<\frac{\lambda_2}{\alpha^{1/p}}\prod_{i=0}^{m-1}(\lambda_1+i)B(\lambda_1,\lambda_2)$$

$$\times\left[\int_0^\infty x^{-p(\lambda_1+m-1)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-q[\alpha(\lambda_2+1)-1]-1}A_n^q\right]^{\frac{1}{q}}. \quad (3.2)$$

Proof. Since we have

$$\frac{1}{(x+n^\alpha)^\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+n^\alpha)t}dt,$$

by (2.12) and (2.13), it follows that

$$\begin{aligned}
I&=\frac{1}{\Gamma(\lambda)}\int_0^\infty\sum_{n=1}^\infty a_nf(x)\int_0^\infty t^{\lambda-1}e^{-(x+n^\alpha)t}dt\\
&=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\int_0^\infty e^{-xt}f(x)dx\sum_{m=1}^\infty e^{-n^\alpha t}a_ndt\\
&\le\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+m}\int_0^\infty e^{-xt}F_m(x)dx\sum_{m=1}^\infty e^{-n^\alpha t}A_ndt\\
&=\frac{1}{\Gamma(\lambda)}\int_0^\infty\sum_{n=1}^\infty A_nF_m(x)\int_0^\infty t^{(\lambda+m+1)-1}e^{-(x+n^\alpha)t}dtdx\\
&=\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}\int_0^\infty\sum_{n=1}^\infty\frac{A_nF_m(x)dx}{(x+n^\alpha)^{\lambda+m+1}}=\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}I_0.
\end{aligned}$$

Then by (2.11), we have (3.1).

The theorem is proved. □

Remark 3.1. For $\alpha=1,\lambda_1\in(0,\lambda+1),\lambda_2\in(0,1]\cap(0,\lambda+m)$ in (3.1), we have the following half- discrete Hilbert-type inequality:

$$\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{(x+n)^\lambda}dx<\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}$$

$$\times\left[\int_0^\infty x^{-p(\widehat{\lambda}_1+m-1)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-q\widehat{\lambda}_2-1}A_n^q\right]^{\frac{1}{q}}. (3.3)$$

Theorem 3.2. *If $\lambda_1+\lambda_2=\lambda\,(\in(0,5-m]),(\lambda_1\in(0,\lambda),\lambda_2\in(0,1]\cap(0,\lambda))$, then the constant factor*

$$\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}$$

in (3.3) is the best possible. On the other hand, if the same constant factor in (3.3) is the best possible and $\lambda-\lambda_1\le 1$, then $\lambda_1+\lambda_2=\lambda\,(\in(0,5-m])$.

Proof. If $\lambda_1+\lambda_2=\lambda\ (\in(0,5-m]), (\lambda_1\in(0,\lambda), \lambda_2\in(0,1]\cap(0,\lambda))$, then we find

$$\begin{aligned}k_{\lambda+m+1}(\lambda_2+1)&=k_{\lambda+m+1}(\lambda_1+m)=B(\lambda_1+m,\lambda_2+1)\\&=\frac{\Gamma(\lambda_1+m)\Gamma(\lambda_2+1)}{\Gamma(\lambda+m+1)}=\frac{\lambda_2\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda+m+1)}\prod_{i=0}^{m-1}(\lambda_1+i),\end{aligned}$$

and then (3.3) reduces to

$$\begin{aligned}&\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{(x+n)^\lambda}dx\\&<\lambda_2B(\lambda_1,\lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i)\left[\int_0^\infty x^{-p(\lambda_1+m-1)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-q\lambda_2-1}A_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{3.4}$$

For any $0<\varepsilon<\min\{p\lambda_1,q\lambda_2\}$, we set

$$\widetilde{f}(x)=\begin{cases}0, & 0<x<1,\\ x^{\lambda_1-\frac{\varepsilon}{p}-1}, & x\geq1,\end{cases}\qquad \widetilde{a}_n=n^{\lambda_2-\frac{\varepsilon}{q}-1}\ (n\in\mathbf{N}).$$

Then it follows that

$$\begin{aligned}&\widetilde{F_j}(x):=\int_0^\infty\widetilde{F}_{j-1}(x)dx\leq\begin{cases}0, & 0<x<1,\\ \frac{x^{\lambda_1+j-\frac{\varepsilon}{p}-1}}{\prod_{i=0}^{j-1}(\lambda_1+i-\frac{\varepsilon}{p})}, & x\geq1,\end{cases}\\&(j=1,\cdots,m),\\&\widetilde{A}_n:=\sum_{i=1}^n\widetilde{a}_i=\sum_{i=1}^n i^{\lambda_2-\frac{\varepsilon}{q}-1}<\int_0^n t^{\lambda_2-\frac{\varepsilon}{q}-1}dt\\&\quad=\frac{n^{\lambda_2-\frac{\varepsilon}{q}}}{\lambda_2-\frac{\varepsilon}{q}}\ (n\in\mathbf{N}).\end{aligned}$$

If there exists a positive constant $M\leq\lambda_2B(\lambda_1,\lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i)$, such that (3.4) is valid when we replace $\lambda_2B(\lambda_1,\lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i)$ by M, then in particular, we have

$$\begin{aligned}\widetilde{I}&:=\int_0^\infty\sum_{n=1}^\infty\frac{\widetilde{a}_n\widetilde{f}(x)}{(x+n)^\lambda}dx\\&<M\left[\int_0^\infty x^{-p(\lambda_1+m-1)-1}\widetilde{F}_m^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-q\lambda_2-1}\widetilde{A}_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{3.5}$$

By (3.5) and the decreasingness property of series, we obtain

$$\begin{aligned}\widetilde{I}&<\frac{M}{\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}\frac{1}{\lambda_2-\frac{\varepsilon}{q}}\\&\times\left(\int_1^\infty x^{-p(\lambda_1+m-1)-1}x^{p\lambda_1+pm-p-\varepsilon}dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q\lambda_2-1}n^{q\lambda_2-\varepsilon}\right)^{\frac{1}{q}}.\end{aligned}$$

$$
\begin{aligned}
&= \frac{M}{\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}\frac{1}{\lambda_2-\frac{\varepsilon}{q}}\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left(1+\sum_{n=2}^\infty n^{-1-\varepsilon}\right)^{\frac{1}{q}} \\
&< \frac{M}{\varepsilon\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}\frac{1}{\lambda_2-\frac{\varepsilon}{q}}\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left(1+\int_1^\infty y^{-1-\varepsilon}dy\right)^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}\frac{1}{\lambda_2-\frac{\varepsilon}{q}}(\varepsilon+1)^{\frac{1}{q}}.
\end{aligned}
$$

By (2.10) (for $\alpha = 1$), setting $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \in (0, \lambda)$ $(0 < \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} < \lambda)$, we find

$$
\begin{aligned}
\widetilde{I} &= \sum_{n=1}^\infty \left[n^{\lambda_2+\frac{\varepsilon}{p}}\int_1^\infty \frac{x^{(\lambda_1-\frac{\varepsilon}{p})-1}}{(x+n)^\lambda}dx\right]n^{-\varepsilon-1} = \sum_{n=1}^\infty \omega(\widetilde{\lambda}_1, n)n^{-\varepsilon-1} \\
&= k_\lambda(\widetilde{\lambda}_1)\sum_{n=1}^\infty n^{-\varepsilon-1} > k_\lambda(\widetilde{\lambda}_1)\int_1^\infty y^{-\varepsilon-1}dy \\
&= \frac{1}{\varepsilon}B(\lambda_1-\frac{\varepsilon}{p}, \lambda_2+\frac{\varepsilon}{p}).
\end{aligned}
$$

Then in view of the above results, we have

$$
B(\lambda_1-\frac{\varepsilon}{p}, \lambda_2+\frac{\varepsilon}{p}) < \varepsilon\widetilde{I} < \frac{M}{\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})}\frac{1}{\lambda_2-\frac{\varepsilon}{q}}(\varepsilon+1)^{\frac{1}{q}}.
$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, we obtain

$$
\lambda_2 B(\lambda_1, \lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i) \le M.
$$

Hence, $M = \lambda_2 B(\lambda_1, \lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i)$ is the best possible constant factor in (3.4).

On the other hand, for $\widehat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$
\begin{aligned}
&\widehat{\lambda}_1 + \widehat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda, \\
&\widehat{\lambda}_2 \le \frac{1}{p} + \frac{1}{q} = 1, 0 < \widehat{\lambda}_1, \widehat{\lambda}_2 < \lambda,
\end{aligned}
$$

and $\widehat{\lambda}_2 B(\widehat{\lambda}_1, \widehat{\lambda}_2)\prod_{i=0}^{m-1}(\widehat{\lambda}_1+i) \in \mathbf{R}_+$.

If the constant factor

$$
\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}
$$

in (3.3) is the best possible, then by (3.4) (for $\lambda_i = \widehat{\lambda}_i, i = 1, 2$), we have the following inequality:

$$
\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}
$$

$$\leq \widehat{\lambda}_2 B(\widehat{\lambda}_1, \widehat{\lambda}_2) \prod_{i=0}^{m-1} (\widehat{\lambda}_1 + i) = \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} k_{\lambda+m+1}(\widehat{\lambda}_2 + 1),$$

which follows that

$$k_{\lambda+m+1}(\widehat{\lambda}_2 + 1) \geq (k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}. \tag{3.6}$$

By Hölder's inequality with weight, we obtain

$$\begin{aligned} 0 &< k_{\lambda+m+1}(\widehat{\lambda}_2+1) = k_{\lambda+m+1}(\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q} + 1) \\ &= \int_0^\infty \frac{u^{\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}}}{(1+u)^{\lambda+m+1}} du = \int_0^\infty \frac{(u^{\frac{\lambda_2}{p}})(u^{\frac{\lambda-\lambda_1}{q}})}{(1+u)^{\lambda+m+1}} du \\ &\leq \left[\int_0^\infty \frac{u^{\lambda_2}}{(1+u)^{\lambda+m+1}} du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda-\lambda_1}}{(1+u)^{\lambda+m+1}} du\right]^{\frac{1}{q}} \\ &= \left[\int_0^\infty \frac{u^{(\lambda_2+1)-1}}{(1+u)^{\lambda+m+1}} du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{v^{(\lambda_1+m)-1}}{(1+v)^{\lambda+m+1}} dv\right]^{\frac{1}{q}} \\ &= (k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}. \end{aligned} \tag{3.7}$$

By (3.6), we observe that (3.7) keeps the form of equality. Then there exist constants A and B, such that they are not both zero satisfying (cf. [14]) $Au^{\lambda_2} = Bu^{\lambda-\lambda_1}$ *a.e.* in. $\mathbf{R}_+$ Assuming that $A \neq 0$, we have $u^{\lambda_2-\lambda+\lambda_1} = B/A$ *a.e.* in. $\mathbf{R}_+$, and then $\lambda_2 - \lambda + \lambda_1 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda\, (\in (0, 5-k])$.

The theorem is proved. □

Remark 3.2. (i) For $m = 4$ in (3.4), we have $\lambda \in (0,1]$ and

$$\begin{aligned} &\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{(x+n)^\lambda} dx \\ &< \lambda_2 B(\lambda_1, \lambda_2) \prod_{i=0}^{3} (\lambda_1 + i) \left[\int_0^\infty x^{-p(\lambda_1+3)-1} F_4^p(x) dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{-q\lambda_2-1} A_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{3.8}$$

(ii) For $m = 1$ in (3.4), we have $\lambda \in (0,3]$ and

$$\begin{aligned} &\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{(x+n)^\lambda} dx \\ &< \lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \left[\int_0^\infty x^{-p\lambda_1-1} F_k^p(x) dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{-q\lambda_2-1} A_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{3.9}$$

In particular, for $\lambda = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}\, (\leq \frac{1}{2})$, we have the following Hilbert-type inequality with the best possible constant factor $\frac{\pi}{pq\sin(\pi/p)}$.

4. Equivalent form and operator expressions

For $m = 0$ in (3.3), we have $\lambda \in (0,5], \lambda_1 \in (0, \lambda+1), \lambda_2 \in (0,1] \cap (0,\lambda)$, and the following inequality:

$$I = \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x) dx}{(x+n)^\lambda}$$

$$<\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$$
$$\times\left[\int_0^\infty x^{-p(\widehat{\lambda}_1-1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q\widehat{\lambda}_2-1}A_n^q\right)^{\frac{1}{q}}. \tag{4.1}$$

Theorem 4.1. *For $\lambda\in(0,5],\lambda_1\in(0,\lambda+1),\lambda_2\in(0,1]\cap(0,\lambda),$ we have the following inequality equivalent to (4.1):*

$$J:=\left\{\int_0^\infty x^{q\widehat{\lambda}_1-1}\left[\sum_{n=1}^\infty\frac{a_n}{(x+n)^\lambda}\right]^q dx\right\}^{\frac{1}{q}}$$
$$<\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}\left(\sum_{n=1}^\infty n^{-q\widehat{\lambda}_2-1}A_n^q\right)^{\frac{1}{q}}. \tag{4.2}$$

In particular, for $\lambda_1+\lambda_2=\lambda(\in(0,5])\,(\lambda_1\in(0,\lambda),\lambda_2\in(0,1]\cap(0,\lambda)),$ we have

$$0<\int_0^\infty x^{-p(\lambda_1-1)-1}f^p(x)dx<0,0<\sum_{n=1}^\infty n^{-q\lambda_2-1}A_n^q<\infty,$$

and the following equivalent inequalities:

$$\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)dx}{(x+n)^\lambda}$$
$$<\lambda_2B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q\lambda_2-1}A_n^q\right)^{\frac{1}{q}}, \tag{4.3}$$
$$\left\{\int_0^\infty x^{q\lambda_1-1}\left[\sum_{n=1}^\infty\frac{a_n}{(x+n)^\lambda}\right]^q dx\right\}^{\frac{1}{q}}<\lambda_2B(\lambda_1,\lambda_2)\left(\sum_{n=1}^\infty n^{-q\lambda_2-1}A_n^q\right)^{\frac{1}{q}}. \tag{4.4}$$

Proof. Suppose that (4.2) is valid. By Hölder's inequality (cf. [14]), we have

$$I=\int_0^\infty x^{\frac{1}{q}-\widehat{\lambda}_1}f(x)\left[x^{\frac{-1}{q}+\widehat{\lambda}_1}\sum_{n=1}^\infty\frac{a_n}{(x+n)^\lambda}\right]dx$$
$$\le\left[\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}f^p(x)dx\right]^{\frac{1}{p}}J. \tag{4.5}$$

Then by (4.2), we have (4.1).

On the other hand, assuming that (4.1) is valid, we set

$$f(x):=x^{q\widehat{\lambda}_1-1}\left[\sum_{n=1}^\infty\frac{a_n}{(x+n)^\lambda}\right]^{q-1},\qquad x>0.$$

We have $J^q=\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}f^p(x)dx$. If $J=0$, then (4.2) is naturally valid; if $J=\infty$, then it is impossible that makes (4.2) valid, i.e. $J<\infty$. Suppose that $0<J<\infty$. By (4.1), we have

$$0<\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}f^p(x)dx=J^q=I$$

$$< \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} J^{q-1}\left(\sum_{n=1}^{\infty} n^{-q\widehat{\lambda}_2-1}A_n^q\right)^{\frac{1}{q}},$$

$$J < \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}\left(\sum_{n=1}^{\infty} n^{-q\widehat{\lambda}_2-1}A_n^q\right)^{\frac{1}{q}}.$$

Hence, (4.2) is valid, which is equivalent to (4.1).

The theorem is proved. □

Theorem 4.2. *If* $\lambda_1+\lambda_2=\lambda\in(0,5]$ $(\lambda_1\in(0,\lambda),\lambda_2\in(0,1]\cap(0,\lambda)$, *then the constant factor*

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$$

in (4.2) is the best possible. On the other hand, if the same constant factor in (4.2) is the best possible and $\lambda-\lambda_1\le 1$, *then we have* $\lambda_1+\lambda_2=\lambda$ $(\in(0,5])$.

Proof. If $\lambda_1+\lambda_2=\lambda\in(0,5]$ $(\lambda_1\in(0,\lambda),\lambda_2\in(0,1]\cap(0,\lambda))$, then the constant factor

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$$

in (4.1) is the best possible. By (4.5), we can show that the same constant factor in (4.2) is the best possible. On the other hand, if the same constant factor in (4.2) is the best possible, then by the equivalency of (4.2) and (4.1), in view of $J^q=I$, we can show that the same constant factor in (4.1) is the best possible. By Theorem 3.2, since $\lambda-\lambda_1\le 1$, we have $\lambda_1+\lambda_2=\lambda$ $(\in(0,5])$.

The theorem is proved. □

We set functions $\varphi(x):=x^{p(1-\widehat{\lambda}_1)-1}, \psi(n):=n^{-q\widehat{\lambda}_2-1}$, where from

$$\varphi^{1-q}(x):=x^{q\widehat{\lambda}_1-1}(x\in\mathbf{R}_+,n\in\mathbf{N}).$$

We also define the following normed spaces:

$$L_{p,\varphi}(\mathbf{R}_+):=\{f=f(x);||f||_{p,\varphi}=[\int_0^\infty \varphi(x)|f(x)|^p dx]^{\frac{1}{p}}<\infty\},$$

$$l_{q,\psi}:=\{a=\{a_n\}_{n=1}^\infty;||a||_{q,\psi}=[\sum_{n=1}^\infty \psi(n)|a_n|^q dx]^{\frac{1}{q}}<\infty\},$$

$$L_{q,\varphi^{1-q}}(\mathbf{R}_+):=\{g=g(x);||g||_{q,\varphi^{1-q}}=[\int_0^\infty \varphi^{1-q}(x)|g(x)|^q dx]^{\frac{1}{q}}<\infty\}.$$

Assuming that $a\in l_{q,\psi}, A\in l_{q,\psi}$, setting

$$g=g(x),g(x):=\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}, x\in\mathbf{R}_+,$$

we can rewrite (3.3) as follows:

$$||g||_{q,\varphi^{1-q}}<\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}||A||_{q,\psi}<\infty,$$

namely, $g\in L_{q,\varphi^{1-q}}(\mathbf{R}_+)$.

Definition 4.1. Define a half-discrete Hilbert-type operator $T : l_{q,\psi} \to L_{q,\varphi^{1-q}}(\mathbf{R}_+)$ as follows: For any $a \in l_{q,\psi}$, there exists a unique representation $g = Ta \in L_{q,\varphi^{1-q}}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+, Ta(x) = g(x)$. Define the formal inner product of Ta and $f \in L_{p,\varphi}(\mathbf{R}_+)$ and the norm of T as follows:

$$(Ta, f) := \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{(x+n)^\lambda} dx = I,$$

$$||T|| := \sup_{a(\neq 0)\in l_{q,\psi}} \frac{||Ta||_{q,\varphi^{1-q}}}{||a||_{q,\psi}}.$$

By Theorem 3.2, Theorem 4.1 and Theorem 4.2, we have

Theorem 4.3. *If $f(\geq 0) \in L_{p,\varphi}(\mathbf{R}_+), a(\geq 0) \in l_{q,\psi}, ||f||_{p,\varphi} > 0, ||a||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta, f) < \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} ||f||_{p,\varphi} ||A||_{q,\psi}, \tag{4.6}$$

$$||Ta||_{q,\varphi^{1-q}} < \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} ||A||_{q,\psi}. \tag{4.7}$$

Moreover, for $\lambda_1 + \lambda_2 = \lambda$, the constant factor $\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$ in (4.6) and (4.7) is the best possible, namely. $||T|| = \lambda_2 B(\lambda_1, \lambda_2)$. On the other hand, if the constant factor in (4.6) (or (4.7)) is the best possible and $\lambda - \lambda_1 \leq 1$, then we have $\lambda_1 + \lambda_2 = \lambda$.

5. Conclusions

In this paper, based on the way of [1, 7, 12], by means of the weight coefficients, the idea of introduced parameters, Euler-Maclaurin summation formula and Abel's summation by parts formula, a new half-discrete Hilbert-type inequality with the kernel as $\frac{1}{(x+n^\alpha)^\lambda}$ and one multiple upper limit function as well as one partial sums is given in Theorem 3.1. As applications, the equivalent conditions of the best possible constant factor in a particular inequality for the parameter $\alpha = 1$, related to a few parameters are considered in Theorem 3.2, and then some particular cases are obtained in Remark 3.3. We also provided the equivalent forms and the operator expressions in the case of $m = 0$ in Theorems 4.1-4.3. The lemmas and theorems provide an extensive account of this type of inequalities.

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