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EFFICIENT NUMERICAL SOLUTION OF TWO-DIMENSIONAL TIME-SPACE FRACTIONAL NONLINEAR DIFFUSION-WAVE EQUATIONS WITH INITIAL SINGULARITY*

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Abstract In this paper, we present an efficient linearized alternating direction implicit (ADI) scheme for two-dimensional time-space fractional nonlinear diffusion-wave equations with initial singularity. First, the original problem is equivalently transformed into its partial integro-differential form. Then, for the time discretization, the Crank-Nicolson technique combined with the midpoint formula and the second order convolution quadrature formula are used. Meanwhile, the classical central difference formula and fractional central difference formula are adopted to approximate the second order derivative and the Riesz derivative in space, respectively. The unconditional stability and convergence of the proposed scheme are proved by the energy method. Numerical experiments support the theoretical results.

Keywords Fractional nonlinear diffusion-wave equations, finite difference scheme, linearized ADI scheme, stability, convergence.

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1. Introduction

In this paper, the following two-dimensional time-space fractional nonlinear diffusion-wave equation with homogeneous initial boundary conditions will be considered

$$
\begin{aligned}
{}_0^C D_t^{\alpha} u(x,y,t) =& \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \frac{\partial^{\beta} u(x,y,t)}{\partial |x|^{\beta}} + \frac{\partial^{\beta} u(x,y,t)}{\partial |y|^{\beta}} \\
& + g(u(x,y,t)) + f(x,y,t), \qquad (1.1)
\end{aligned}
$$

where $(x,y) \in (0,L_x) \times (0,L_y)$, $t \in (0,T]$, $1 < \alpha, \beta < 2$, $f(x,y,t)$ is a known function, $g(u)$ is a nonlinear function of u with $g(0) = 0$ and satisfies the Lipschitz condition, and ${}_0^C D_t^{\alpha} u(x,y,t)$ denotes the temporal Caputo derivative with order α

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defined as

$$ {}_0^C D_t^\alpha u(x,y,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u(x,y,s)}{\partial s^2} ds. $$

And $\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta}$ is the Riesz fractional derivative of order β in x defined as

$$ \frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta} = -\frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)} \left({}_0^{RL} D_x^\beta u(x,y,t) + {}_x^{RL} D_{L_x}^\beta u(x,y,t) \right), $$

where ${}_0^{RL} D_x^\beta u(x,y,t)$ and ${}_x^{RL} D_{L_x}^\beta u(x,y,t)$ are the left and the right Riemann-Liouville derivatives defined by

$$ {}_0^{RL} D_x^\beta u(x,y,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_0^x (x-z)^{1-\beta} u(z,y,t) dz $$

and

$$ {}_x^{RL} D_{L_x}^\beta u(x,y,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_x^{L_x} (z-x)^{1-\beta} u(z,y,t) dz, $$

respectively. $\frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta}$ is the Riesz fractional derivative of order β in y, which can be similarly defined.

It is possible to interpolate diffusion and wave phenomena as well as processes with spatial non-local dependence using Eq. (1.1), which is derived from the classical diffusion or wave equation by substituting fractional derivatives of order α, β for the second order time and space derivatives. Therefore, such models are widely used for description of viscoelastic damping materials, diffusion images of human brain tissues, etc. [15, 20, 27]. However, it is often difficult or impossible to solve fractional diffusion-wave equations analytically (see [24, 26] for examples), thus numerical methods are necessary. As a result, interest in developing numerical methods for solving fractional diffusion-wave equations has grown, see [2,9,21,25,28] and the references therein.

In recent years, numerous numerical methods for one-dimensional time-space fractional diffusion-wave equations (TSFDWEs) have been developed, see [1, 3, 6, 7, 10–12, 17, 33] and the references therein. Bhrawy et al. [1] solved second and fourth order time fractional diffusion-wave equations by using a spectral tau algorithm based on Jacobi operational matrix. Using the fractional trapezoidal rule and the generalized Newton-Gregory formula, Zeng [33] proposed second order in time and space and conditionally stable finite difference schemes for the time fractional diffusion-wave equation. Ebadian et al. [6] proposed triangular function (TFs) approaches for solving a class of multi-term time fractional nonlinear diffusion-wave equations, where they deduced a fractional operational matrix of integration for the TFs. Huang et al. [11] presented two convolution quadrature methods for fractional nonlinear diffusion-wave equations. The stability and convergence of the methods were rigorously proved. Huang et al. [12] proposed efficient scheme and alternating direction implicit (ADI) schemes for solving one-dimensional and two-dimensional nonlinear time fractional diffusion-wave equations, and then proved their unconditional stability and convergence with first order accuracy in time and second order accuracy in space.

On the other hand, there are still few works on the numerical methods for solving the two-dimensional TSFDWEs, see [8, 14, 31, 34]. Wang et al. [31] constructed an ADI scheme for solving TSFDWEs with second order accuracy in both space and time. Fan et al. [8] developed a fully discrete numerical technique for two-dimensional multi-term TSFDWEs on an irregular convex domain using a mixed difference scheme in time and an unstructured mesh finite element method in space. The proposed numerical scheme's stability and convergence were proved. Zhang et al. [34] developed two numerical techniques for the one-dimensional and two-dimensional time-space fractional vibration equations. The proposed scheme's convergence and unconditional stability were also extensively proved. Huang et al. [14] extended the ADI scheme in [12] for the two-dimensional single-term time fractional diffusion-wave equations to the two-dimensional multi term time-space fractional ones. Then, the solvability, unconditionally stability and convergence with first-order accuracy in time and second-order accuracy in space were proved.

Due to the fact that the initial singularity of the solution of the TSFDWEs often generates a singular source, solving the equation numerically becomes more complicated. Therefore, the majority of numerical analysis results for TSFDWEs in the literature are valid under the smooth solution assumption. Recently, an increasing number of scholars have focused on the singularity in time fractional models, see [5, 13, 19, 29]. Thus, let us consider the analytical solution of Problem (1.1) with the following regularity assumption in time, namely

$$\left|\frac{\partial^i u(x,y,t)}{\partial t^i}\right| \le C t^{\sigma-i},\ \ i=0,1,2, \tag{1.2}$$

where $1<\sigma<\alpha$ is a regularity parameter, which depends on the order of the Caputo fractional derivative α.

Remark 1.1. Applying the bound (1.2) on $u(x,y,t)$ in the Eq. (1.1) and assuming $\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}+\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta}+\frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta}$ is bounded. Hence Eq. (1.1) gives $f(x,y,t)=\mathcal{O}(t^{\sigma-\alpha})$ which blows up at $t=0$.

Herein, we propose and analyze a linearized ADI scheme for two-dimensional time-space fractional nonlinear diffusion-wave equations with initial singularity. In order to be more explicit, we first utilize the Riemann-Liouville integral operator to transform Eq (1.1) into their equivalent partial integro-differential equations. Second, we construct the linearized ADI scheme by using the Crank-Nicolson technique combined with the second order convolution quadrature formula and the midpoint formula in time, the classical central difference formula and the fractional central difference formula approximations in space. Finally, the linearized ADI scheme is proved to be unconditional stable and convergence.

The rest of this paper is organized as follows. In Section 2, the linearized ADI finite difference scheme is constructed. In Section 3, the stability and convergence of the linearized ADI finite difference scheme are proved. Numerical experiments are provided to support the theoretical results in Section 4. The article ends with a brief conclusion in Section 5.

2. Derivation of the Linearized ADI Scheme

2.1. Preliminaries

In this subsection, we introduce some fundamental definitions, notations, and lemmas that will be utilized to construct a linearized ADI scheme for Problem (1.1).

In order to implement discretizations, we introduce the temporal step size $\tau = T/N$, $t_n = n\tau$, $t_{n+1/2} = (n+1/2)\tau$. For spatial discretizations, let $h_x = L_x/M_x$ and $h_y = L_y/M_y$ for positive integers M_x and M_y, $x_i = ih_x, i = 0, 1, \cdots, M_x$, and $y_j = jh_y, j = 0, 1, \cdots, M_y$.

Lemma 2.1. *If $u(t)$ satisfies* (1.2) *, then the following results*

$$\begin{aligned} u_t(t_{n+1/2}) &= \frac{u(t_{n+1}) - u(t_n)}{\tau} + \mathcal{O}(t_{n+1}^{\sigma-3}\tau^2) \\ &= \delta_t u^{n+\frac{1}{2}} + \mathcal{O}(t_{n+1}^{\sigma-3}\tau^2) \end{aligned} \tag{2.1}$$

and

$${}_0J_t^{\alpha-1}u(t_{n+1/2}) = \frac{1}{2}\left[{}_0J_t^{\alpha-1}u(t_{n+1}) + {}_0J_t^{\alpha-1}u(t_n)\right] + \mathcal{O}(t_{n+1}^{\sigma+\alpha-3}\tau^2) \tag{2.2}$$

hold, where ${}_0J_t^{\alpha-1}$ is the Riemann-Liouville fractional integral operator defined by

$${}_0J_t^{\alpha-1}u(x,y,t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}u(x,y,s)ds.$$

Proof. For $-1 \le \gamma \le 1$ and $n = 0, 1, \cdots, N-1$, one can find that by the Taylor expansion

$$\left(t_{n+\frac{1}{2}}\right)^{\sigma-\gamma} = \frac{1}{2}\left[\left(t_{n+1}^{\sigma-\gamma}\right) + \left(t_n^{\sigma-\gamma}\right)\right] + O\left(t_{n+1}^{\sigma-\gamma-2}\tau^2\right). \tag{2.3}$$

Since $u(t) = \mathcal{O}(t^\sigma)$, we easily deduce that ${}_0J_t^{\alpha-1}u(t_{n+\frac{1}{2}}) = \mathcal{O}\left(t_{n+\frac{1}{2}}^{\sigma+\alpha-1}\right)$. Therefore, (2.2) is obtained by letting $\gamma = 1-\alpha$ in (2.3). Similarly, (2.1) can be obtained by letting $\gamma = 1$ in (2.3). $\square$

The convolution quadrature [22, 23] approximation for the Riemann-Liouville integral is given below.

Lemma 2.2. *Let $1 < \sigma < 2$ and $\omega_k^{(\alpha-1)}$ be the weights from generating function $\left(3/2 - 2z + z^2/2\right)^{1-\alpha}$, under the Assumption* (1.2)*, then*

$$\left|{}_0J_{t_{n+1}}^{\alpha-1}u(t) - \tau^{\alpha-1}\sum_{k=0}^{n+1}\omega_{n+1-k}^{(\alpha-1)}u(t_k)\right| \le Ct_{n+1}^{\sigma+\alpha-3}\tau^2.$$

Lemma 2.3 (see Page 5 of [16])**.** *Suppose $u(t)$ satisfies the Assumption* (1.2)*, then the following approximation holds*

$$u(t_{n+1}) = 2u(t_n) - u(t_{n-1}) + \mathcal{O}(t_n^{\sigma-2}\tau^2).$$

Lemma 2.4 (see Lemma 1.2 in [30])**.** *Suppose $u(x) \in C^4([x_{i-1}, x_{i+1}])$, let $\zeta(s) = u^{(4)}(x_i + sh_x) + u^{(4)}(x_i - sh_x)$, then*

$$\delta_x^2 u(x_i) = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h_x^2} = u_{xx}(x_i) + \frac{h_x^2}{24}\int_0^1 \zeta(s)(1-s)^3 ds.$$

Lemma 2.5 (see Celik and Duman [4]). *Let $1 < \beta < 2$, $u(x) \in C^5(\mathbb{R})$ and all its derivative up to the order five belong to $L(\mathbb{R})$. If $u(x) = 0$ when $x \notin (0, L)$, then*

$$\frac{\partial^\beta u(x)}{\partial |x|^\beta} = -\delta_x^\beta u(x) + \mathcal{O}(h_x^2),$$

where

$$\delta_x^\beta u(x) := \frac{1}{h_x^\beta} \sum_{j=-\lceil \frac{L-x}{h_x} \rceil}^{\lceil \frac{x}{L} \rceil} \frac{(-1)^j \, \Gamma(\beta+1)}{\Gamma(\beta/2 - j + 1)\Gamma(\beta/2 + j + 1)} u(x - jh_x),$$

where $\frac{\partial^\beta u(x)}{\partial |x|^\beta}$ is the Riesz derivative with order β.

2.2. Construction of the Linearized ADI Scheme

In this subsection, an ADI finite difference scheme for Problem (1.1) will be derived under the Assumption (1.2).

Firstly, we multiply ${}_0J_t^{\alpha-1}$ on both sides of Eq. (1.1), then Eq. (1.1) is equivalent to the following partial integro-differential equation

$$\begin{aligned}
\frac{\partial u(x,y,t)}{\partial t} =& {}_0J_t^{\alpha-1} \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right) \\
&+ {}_0J_t^{\alpha-1} \left(\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta} \right) \\
&+ {}_0J_t^{\alpha-1} g(u(x,y,t)) + F(x,y,t),
\end{aligned} \tag{2.4}$$

where $F(x,y,t) = {}_0J_t^{\alpha-1} f(x,y,t)$.

Assume $u(x,y,\cdot) \in C_{x,y}^{5,5}([0, L_x] \times [0, L_y])$ with $u(0,\cdot,\cdot) = u(L_x,\cdot,\cdot) = u(\cdot,0,\cdot) = u(\cdot,L_y,\cdot) = 0$ and consider Eq. (2.4) at the point $(x_i, y_j, t_{n+1/2})$, that is

$$\begin{aligned}
\left.\frac{\partial u(x_i,y_j,t)}{\partial t}\right|_{t=t_{n+\frac{1}{2}}} =& {}_0J_{t_{n+\frac{1}{2}}}^{\alpha-1} \left(\frac{\partial^2 u(x_i,y_j,t)}{\partial x^2} + \frac{\partial^2 u(x_i,y_j,t)}{\partial y^2} \right) \\
&+ {}_0J_{t_{n+\frac{1}{2}}}^{\alpha-1} \left(\frac{\partial^\beta u(x_i,y_j,t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x_i,y_j,t)}{\partial |y|^\beta} \right) \\
&+ {}_0J_{t_{n+\frac{1}{2}}}^{\alpha-1} g(u(x_i,y_j,t)) + F(x_i,y_j,t_{n+\frac{1}{2}}).
\end{aligned}$$

The Crank-Nicolson technique and Lemma 2.1 for the above equation yield

$$\begin{aligned}
\frac{u(x_i,y_j,t_{n+1}) - u(x_i,y_j,t_n)}{\tau} =& \frac{1}{2} {}_0J_{t_{n+1}}^{\alpha-1} \left(\frac{\partial^2 u(x_i,y_j,t)}{\partial x^2} + \frac{\partial^2 u(x_i,y_j,t)}{\partial y^2} \right) \\
&+ \frac{1}{2} {}_0J_{t_n}^{\alpha-1} \left(\frac{\partial^2 u(x_i,y_j,t)}{\partial x^2} + \frac{\partial^2 u(x_i,y_j,t)}{\partial y^2} \right) \\
&+ \frac{1}{2} {}_0J_{t_{n+1}}^{\alpha-1} \left(\frac{\partial^\beta u(x_i,y_j,t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x_i,y_j,t)}{\partial |y|^\beta} \right) \\
&+ \frac{1}{2} {}_0J_{t_n}^{\alpha-1} \left(\frac{\partial^\beta u(x_i,y_j,t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x_i,y_j,t)}{\partial |y|^\beta} \right)
\end{aligned}$$

$$+\frac{1}{2}\left[{}_0J^{\alpha-1}_{t_{n+1}}g(u(x_i,y_j,t)) + {}_0J^{\alpha-1}_{t_n}g(u(x_i,y_j,t))\right] + F(x_i,y_j,t_{n+\frac{1}{2}}) + \mathcal{O}(t^{\sigma-3}_{n+1}\tau^2).$$

We apply Lubich's convolution quadrature approximation of Lemma 2.2 to discretize the Riemann-Liouville integrals, apply Lemma 2.4 to discretize the second order derivatives, and apply the fractional centered difference of Lemma 2.5 to discretize the Riesz derivatives, it achieves that

$$\begin{aligned}\frac{u^{n+1}_{ij}-u^n_{ij}}{\tau} =& \frac{\tau^{\alpha-1}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^2+\delta_y^2\right)u_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^2+\delta_y^2\right)u_{ij}^{n-k}\right]\\ &-\frac{\tau^{\alpha-1}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^\beta+\delta_y^\beta\right)u_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^\beta+\delta_y^\beta\right)u_{ij}^{n-k}\right]\\ &+\frac{\tau^{\alpha-1}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}g(u_{ij}^{n+1-k})+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}g(u_{ij}^{n-k})\right]\\ &+F_{ij}^{n+\frac{1}{2}}+\mathcal{O}(t_{n+1}^{\sigma-3}\tau^2+h_x^2+h_y^2),\end{aligned}\tag{2.5}$$

where $u^n_{ij}=u(x_i,y_j,t_n)$ and $F_{ij}^{n+\frac{1}{2}}=F(x_i,y_j,t_{n+\frac{1}{2}})$.

To construct the ADI scheme, a small term $\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^\beta\right)\left(\delta_y^2-\delta_y^\beta\right)\delta_t u_{ij}^{n+\frac{1}{2}}$ is added to the both sides of Eq (2.5), then after multiplying τ in the both sides, we have

$$\begin{aligned}&u^{n+1}_{ij}-u^n_{ij}+\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^\beta\right)\left(\delta_y^2-\delta_y^\beta\right)\left(u^{n+1}_{ij}-u^n_{ij}\right)\\ =&\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^\beta\right)u_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^\beta\right)u_{ij}^{n-k}\right]\\ &+\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^\beta\right)u_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^\beta\right)u_{ij}^{n-k}\right]\\ &+\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}g(u_{ij}^{n+1-k})+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}g(u_{ij}^{n-k})\right]\\ &+\tau F_{ij}^{n+\frac{1}{2}}+\mathcal{O}(t_{n+1}^{\sigma-3}\tau^3+\tau h_x^2+\tau h_y^2).\end{aligned}\tag{2.6}$$

It is clear that Eq. (2.6) is a nonlinear system with respect to the unknown u_{ij}^{n+1}. To linearly solve Eq. (2.6), we use $u^1_{ij}=u^0_{ij}+\tau(u_t)^0_{ij}+\mathcal{O}\left(\tau^2\ t^{\sigma-1}\big|_{t_0}^{t_1}\right)$ and Lemma 2.3 to linearize Eq. (2.6) for $n=0$ and $1\le n\le N-1$, respectively, i.e.,

$$\begin{aligned}&u^{1}_{ij}-u^0_{ij}+\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^\beta\right)\left(\delta_y^2-\delta_y^\beta\right)\left(u^{1}_{ij}-u^0_{ij}\right)\\ =&\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{1}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^\beta\right)u_{i}^{1-k}+\omega_0^{(\alpha-1)}\left(\delta_x^2-\delta_x^\beta\right)u_{ij}^{0}\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{1}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)u_{ij}^{1-k}+\omega_0^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)u_{ij}^{0}\right]\\
&+\frac{\tau^{\alpha}}{2}\left[\omega_0^{(\alpha-1)}g(u_{ij}^0+\tau(u_t)_{ij}^0)+\omega_1^{(\alpha-1)}g(u_{ij}^0)+\omega_0^{(\alpha-1)}g(u_{ij}^0)\right]\\
&+\tau F_{ij}^{n+\frac{1}{2}}+R_{ij}^{*}
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&u_{ij}^{n+1}-u_{ij}^{n}+\left(\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^{\beta}\right)\left(\delta_y^2-\delta_y^{\beta}\right)\left(u_{ij}^{n+1}-u_{ij}^{n}\right)\\
=&\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)u_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)u_{ij}^{n-k}\right]\\
&+\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)u_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)u_{ij}^{n-k}\right]\\
&+\frac{\tau^{\alpha}}{2}\left[\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)}g(u_{ij}^{n+1-k})+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}g(u_{ij}^{n-k})\right]\\
&+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}g(2u_{ij}^{n}-u_{ij}^{n-1})+\tau F_{ij}^{n+\frac{1}{2}}+R_{ij}^{*},
\end{aligned}
\tag{2.8}
$$

where $R_{ij}^{*}=\mathcal{O}(t_{n+1}^{\sigma-3}\tau^3+\tau h_x^2+\tau h_y^2)$.

Noting $(u_t)_{ij}^0=0$, neglecting the truncation error term R_{ij}^{*} in both above equations, and replacing the u_{ij}^{n} with its numerical solution U_{ij}^{n} , we deduce the following linearized finite difference schemes for Problem (2.4)

$$
\begin{aligned}
&U_{ij}^{1}-U_{ij}^{0}+\left(\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^{\beta}\right)\left(\delta_y^2-\delta_y^{\beta}\right)\left(U_{ij}^{1}-U_{ij}^{0}\right)\\
=&\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{1}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)U_{ij}^{1-k}+\omega_0^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)U_{ij}^{0}\right]\\
&+\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{1}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)U_{ij}^{1-k}+\omega_0^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)U_{ij}^{0}\right]\\
&+\tau^{\alpha}\omega_0^{(\alpha-1)}g(U_{ij}^0)+\frac{\tau^{\alpha}}{2}\omega_1^{(\alpha-1)}g(U_{ij}^0)+\tau F_{ij}^{n+\frac{1}{2}}
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
&U_{ij}^{n+1}-U_{ij}^{n}+\left(\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^{\beta}\right)\left(\delta_y^2-\delta_y^{\beta}\right)\left(U_{ij}^{n+1}-U_{ij}^{n}\right)\\
=&\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)U_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)U_{ij}^{n-k}\right]\\
&+\frac{\tau^{\alpha}}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)U_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)U_{ij}^{n-k}\right]
\end{aligned}
$$

$$+\frac{\tau^{\alpha}}{2}\left[\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)}g(U_{ij}^{n+1-k})+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}g(U_{ij}^{n-k})\right]$$
$$+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}g(2U_{ij}^{n}-U_{ij}^{n-1})+\tau F_{ij}^{n+\frac{1}{2}}. \tag{2.10}$$

Schemes (2.9) and (2.10) can be written into more compact forms, namely

$$\left[1-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\left(\delta_x^2-\delta_x^{\beta}\right)\right]\left[1-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\left(\delta_y^2-\delta_y^{\beta}\right)\right]U_{ij}^{1}=G_{ij}^{0}$$

and

$$\left[1-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\left(\delta_x^2-\delta_x^{\beta}\right)\right]\left[1-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\left(\delta_y^2-\delta_y^{\beta}\right)\right]U_{ij}^{n+1}=G_{ij}^{n},$$

respectively, where

$$G_{ij}^{0}=\left[1+\left(\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^{\beta}\right)\left(\delta_y^2-\delta_y^{\beta}\right)\right]U_{ij}^{0}$$
$$+\frac{\tau^{\alpha}}{2}\left(\omega_0^{(\alpha-1)}+\omega_1^{(\alpha-1)}\right)\left(\delta_x^2-\delta_x^{\beta}\right)U_{ij}^{0}+\frac{\tau^{\alpha}}{2}\left(\omega_0^{(\alpha-1)}+\omega_1^{(\alpha-1)}\right)\left(\delta_y^2-\delta_y^{\beta}\right)U_{ij}^{0}$$
$$+\tau^{\alpha}\omega_0^{(\alpha-1)}g(u_{ij}^{0})+\frac{\tau^{\alpha}\omega_1^{(\alpha-1)}}{2}g(u_{ij}^{0})+\tau F_{ij}^{\frac{1}{2}}$$

and

$$G_{ij}^{n}=\left[1+\left(\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^{\beta}\right)\left(\delta_y^2-\delta_y^{\beta}\right)\right]U_{ij}^{n}$$
$$+\frac{\tau^{\alpha}}{2}\left[\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)U_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^2-\delta_x^{\beta}\right)U_{ij}^{n-k}\right]$$
$$+\frac{\tau^{\alpha}}{2}\left[\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)U_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_y^2-\delta_y^{\beta}\right)U_{ij}^{n-k}\right]$$
$$+\frac{\tau^{\alpha}}{2}\left[\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)}g(U_{ij}^{n+1-k})+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}g(U_{ij}^{n-k})\right]$$
$$+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}g(2U_{ij}^{n}-U_{ij}^{n-1})+\tau F_{ij}^{n+\frac{1}{2}}.$$

According to the Peaceman-Rachford strategy, the intermediate variables $U_{ij}^{*}=\left(1-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\left(\delta_y^2-\delta_y^{\beta}\right)\right)U_{ij}^{n+1}$ can be introduced. Then, the numerical solutions of Eq. (2.9) and (2.10) are obtained by solving two independent sets of one-dimensional linear systems. Thus, the ADI scheme is described as follows. For fixed $j\in\{1,2,\cdots,M_y-1\}$, we can solve the following systems to obtain blue$\{U_{ij}^{*}\}$ for $1\le i\le M_x-1$,

$$\begin{cases}\left(1-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\left(\delta_x^2-\delta_x^{\beta}\right)\right)U_{ij}^{*}=G_{ij}^{n},\ 0\le n\le N-1,\\ U_{0j}^{*}=0,\ U_{M_xj}^{*}=0.\end{cases} \tag{2.11}$$

Once $\{U^*_{ij}\}$ is available, we alternate the spatial direction to solve the following system for fixed $i \in \{1, 2, \cdots, M_x - 1\}$,

$$\begin{cases} \left(1 - \frac{\tau^\alpha \omega_0^{(\alpha-1)}}{2}\left(\delta_y^2 - \delta_y^\beta\right)\right) U_{ij}^{n+1} = U^*_{ij}, \, 0 \le n \le N-1, \\ U_{i0}^{n+1} = 0, \; U_{iM_y}^{n+1} = 0. \end{cases} \tag{2.12}$$

3. Analysis of the Linearized ADI Scheme

In this section, the convergence and stability of the proposed ADI Schemes (2.9) and (2.10) will be proved.

3.1. Convergence

In this subsection, we shall consider the convergence of the linearized ADI Schemes (2.9) and (2.10). We first define a grid function space

$$\Theta_h = \{v_{ij}^n | 0 \le n \le N, 0 \le i \le M_x, 0 \le j \le M_y, bluev_{0j}^n = v_{M_xj}^n = v_{i0}^n = v_{iM_y}^n = 0\}.$$

For two vectors $u_{ij}^n, v_{ij}^n \in \Theta_h$, we define the following inner product and norm, that is,

$$\langle u^n, v^n\rangle = h_x h_y \sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1} u_{ij}^n v_{ij}^n, \; ||u^n||^2 = \langle u^n, u^n\rangle.$$

Lemma 3.1 (see Lemma 3.4 in [32]). *For $1 < \beta < 2$ and the operators δ_x^β and δ_y^β definied in Lemma 2.5, there exist linear difference operators, denoted by $\delta_x^{\beta/2}$ and $\delta_y^{\beta/2}$, such that*

$$\langle \delta_x^\beta u^n, v^n\rangle = \langle \delta_x^{\beta/2} u^n, \delta_x^{\beta/2} v^n\rangle$$

and

$$\langle \delta_y^\beta u^n, v^n\rangle = \langle \delta_y^{\beta/2} u^n, \delta_y^{\beta/2} v^n\rangle,$$

where $u_{ij}^n, \; v_{ij}^n \in \Theta_h red.$

Lemma 3.2 (see Lemma 4.2.2 in [18]). *For any grid function $u_{ij}^n, \; v_{ij}^n \in \Theta_h$, it holds*

$$\langle \delta_x^2 u^n, v^n\rangle = -\langle \delta_x u^n, \delta_x v^n\rangle \quad \text{and} \quad \langle \delta_y^2 u^n, v^n\rangle = -\langle \delta_y u^n, \delta_y v^n\rangle.$$

Lemma 3.3 (see Lemma 2.5 in [11]). *Let $\{\omega_j^{(\alpha-1)}\}_{j=0}^\infty$ be the weights defined in Lemma 2.2. Then, for any positive integer K and any real vector $(V_1, blueV_2, \cdots, V_K)$, the inequlality*

$$\sum_{n=0}^{K-1}\left(\sum_{j=0}^{n} blue\omega_j^{(\alpha-1)} V_{n+1-j}\right) V_{n+1} \ge 0$$

holds.

We now proceed to prove the convergence of Schemes (2.9) and (2.10) under the regularity Assumption (1.2).

Theorem 3.1. *Assume* $u(\cdot,\cdot,t)=\mathcal{O}(t^\sigma)$ *and* $u(0,\cdot,\cdot)=u(L_x,\cdot,\cdot)$, $u(\cdot,0,\cdot)$, $u(\cdot,L_y,\cdot)=0$. *And let* $u(x,y,t)$ *and* $\{U_{ij}^n \mid 0\le i\le M_x, 0\le j\le M_y, 1\le n\le N\}$ *be the exact solution of Eq.* (1.1) *and the numerical solution of Schemes* (2.9) *and* (2.10)*, respectively. Then, for* $1\le n\le N$*, it holds that*

$$\|U^n-u^n\|\le C\left(\tau^\sigma+h_x^2+h_y^2\right).$$

Proof. Let us start by analyzing the error of (2.10). Subtracting Eq. (2.10) from Eq. (2.8), we have

$$\begin{aligned}
&e_{ij}^{n+1}-e_{ij}^n+\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\left(\delta_x^2-\delta_x^\beta\right)\left(\delta_y^2-\delta_y^\beta\right)\left(e_{ij}^{n+1}-e_{ij}^n\right)\\
=&\frac{\tau^\alpha}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^2+\delta_y^2\right)e_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^2+\delta_y^2\right)e_{ij}^{n-k}\right]\\
&-\frac{\tau^\alpha}{2}\left[\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\delta_x^\beta+\delta_y^\beta\right)e_{ij}^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\delta_x^\beta+\delta_y^\beta\right)e_{ij}^{n-k}\right]\\
&+\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\left[g(u_{ij}^{n-k})-g(U_{ij}^{n-k})\right]\\
&+\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\left[g(2u_{ij}^n-u_{ij}^{n-1})-g(2U_{ij}^n-U_{ij}^{n-1})\right]+R^*,
\end{aligned}$$

where $e_{ij}^n=u_{ij}^n-U_{ij}^n$. Multiplying the both sides of the above equation by $h_xh_y(e_{ij}^{n+1}+e_{ij}^n)$ and summing over $1\le i\le M_x-1, 1\le j\le M_y-1$, we have

$$\begin{aligned}
&\|e^{n+1}\|^2-\|e^n\|^2+\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\langle\left(\delta_x^2-\delta_x^\beta\right)\left(\delta_y^2-\delta_y^\beta\right)\left(e^{n+1}-e^n\right),e^{n+1}+e^n\rangle\\
=&\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\left(\delta_x^2+\delta_y^2\right)\left(e^{n+1-k}+e^{n-k}\right),e^{n+1}+e^n\rangle\\
&-\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\left(\delta_x^\beta+\delta_y^\beta\right)\left(e^{n+1-k}+e^{n-k}\right),e^{n+1}+e^n\rangle\\
&+\frac{\tau^\alpha}{2}\left[\omega_{n+1}^{(\alpha-1)}\langle\left(\delta_x^2+\delta_y^2\right)e^0,e^{n+1}+e^n\rangle-\omega_{n+1}^{(\alpha-1)}\langle\left(\delta_x^\beta+\delta_y^\beta\right)e^0,e^{n+1}+e^n\rangle\right]\\
&+\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\langle g(u^{n-k})-g(U^{n-k}),e^{n+1}+e^n\rangle\\
&+\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\langle g(2u^n-u^{n-1})-g(2U^n-U^{n-1}),e^{n+1}+e^n\rangle+\langle R^*,e^{n+1}+e^n\rangle.
\end{aligned}$$

Since $e_{ij}^0=0$ for $0\le i\le M_x$ and $0\le j\le M_y$, sum the above equation over n from 1 to $J-1$ and use Lemmas 3.1 and 3.2, we get

$$\|e^J\|^2-\|e^1\|^2+\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\|\eta_1\|^2$$

$$
\begin{aligned}
&\leq -\frac{\tau^{\alpha}}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle(\delta_x+\delta_y)\left(e^{n+1-k}+e^{n-k}\right),(\delta_x+\delta_y)\left(e^{n+1}+e^{n}\right)\rangle\\
&\quad-\frac{\tau^{\alpha}}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\left(\delta_x^{\beta/2}+\delta_y^{\beta/2}\right)\left(e^{n+1-k}+e^{n-k}\right),\left(\delta_x^{\beta/2}+\delta_y^{\beta/2}\right)\left(e^{n+1}+e^{n}\right)\rangle\\
&\quad+\frac{\tau^{\alpha}}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\langle g(u^{n-k})-g(U^{n-k}),e^{n+1}+e^{n}\rangle\\
&\quad+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\sum_{n=1}^{J-1}\langle g(2u^{n}-u^{n-1})-g(2U^{n}-U^{n-1}),e^{n+1}+e^{n}\rangle\\
&\quad+\sum_{n=1}^{J-1}\langle R^{*},e^{n+1}+e^{n}\rangle, \qquad (3.1)
\end{aligned}
$$

where

$$
\begin{aligned}
\|\eta_1\|^2 =&\|\delta_x\delta_y e^J\|^2+\|\delta_x\delta_y^{\beta/2}e^J\|^2+\|\delta_x^{\beta/2}\delta_y e^J\|^2+\|\delta_x^{\beta/2}\delta_y^{\beta/2}e^J\|^2\\
&-\|\delta_x\delta_y e^1\|^2-\|\delta_x\delta_y^{\beta/2}e^1\|^2-\|\delta_x^{\beta/2}\delta_y e^1\|^2-\|\delta_x^{\beta/2}\delta_y^{\beta/2}e^1\|^2.
\end{aligned}
$$

Now, we turn to analyze $\|e^1\|$. Subtracting Eq. (2.9) from Eq. (2.7) and using the same deductions as above, we can derive that

$$
\begin{aligned}
\|e^1\|^2 =&\left(\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\right)^2\|\eta_2\|^2-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\|\left(\delta_x+\delta_y\right)e^1\|^2\\
&-\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\|\left(\delta_x^{\beta/2}+\delta_y^{\beta/2}\right)e^1\|^2+\tau^{\alpha}\omega_0^{(\alpha-1)}\langle g(u^0)-g(U^0),e^1\rangle\\
&+\frac{\tau^{\alpha}\omega_1^{(\alpha-1)}}{2}\langle g(u^0)-g(U^0),e^1\rangle+\langle R^{*},e^1\rangle, \qquad (3.2)
\end{aligned}
$$

where

$$
\|\eta_2\|^2=\|\delta_x\delta_y e^1\|^2+\|\delta_x\delta_y^{\beta/2}e^1\|^2+\|\delta_x^{\beta/2}\delta_y e^1\|^2+\|\delta_x^{\beta/2}\delta_y^{\beta/2}e^1\|^2.
$$

Sum (3.1) and (3.2) and use Lemma 3.3, it deduces that

$$
\begin{aligned}
\|e^J\|^2\leq&-\left(\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\right)^2\|\eta_3\|^2\\
&+\frac{\tau^{\alpha}}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\langle g(u^{n-k})-g(U^{n-k}),e^{n+1}+e^{n}\rangle\\
&+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\sum_{n=1}^{J-1}\langle g(2u^{n}-u^{n-1})-g(2U^{n}-U^{n-1}),e^{n+1}+e^{n}\rangle\\
&+\tau^{\alpha}\omega_0^{(\alpha-1)}\langle g(u^0)-g(U^0),e^1\rangle+\frac{\tau^{\alpha}\omega_1^{(\alpha-1)}}{2}\langle g(u^0)-g(U^0),e^1\rangle\\
&+\sum_{n=1}^{J-1}\langle R^{*},e^{n+1}+e^{n}\rangle, \qquad (3.3)
\end{aligned}
$$

where

$$\|\eta_3\|^2 = \|\delta_x\delta_y e^J\|^2 + \|\delta_x\delta_y^{\beta/2}e^J\|^2 + \|\delta_x^{\beta/2}\delta_y e^J\|^2 + \|\delta_x^{\beta/2}\delta_y^{\beta/2}e^J\|^2.$$

Using the blueLipschitz condition of g and exchanging the order of the two blue-summations in the second term in the right-hand side of the above inequlaity, we obtain

$$\|e^J\|^2 \le -\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2 \|\eta_3\|^2 + C\tau^\alpha \sum_{k=0}^{J-1}\sum_{n=k}^{J-1}\left(w_{n+1-k}^{(\alpha-1)} + w_{n-k}^{(\alpha-1)}\right)\|e^k\|\|e^{n+1}+e^n\| + C\tau^\alpha\sum_{n=1}^{J-1}\|e^n\|\|e^{n+1}+e^n\| + C\sum_{n=1}^{J-1}R^*\|e^{n+1}+e^n\|.$$

Due to $-\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\|\eta_3\|^2$ is negative, $\tau^\alpha\sum_{n=k}^{J-1}\left(\omega_{n+1-k}^{(\alpha-1)} + \omega_{n-k}^{(\alpha-1)}\right)$ is bounded and assume that $\|e^P\| = \max_{0\le J\le N}\|e^J\|$, then it holds

$$\begin{aligned}\|e^P\| \le & C\sum_{n=0}^{P-1}\left(t_{n+1}^{\sigma-3}\tau^3 + \tau h_x^2 + \tau h_y^2\right)\\ \le & C\left(\sum_{n=0}^{P-1}(n+1)^{\sigma-3}\tau^\sigma + h_x^2 + h_y^2\right).\end{aligned}$$

Since $\sum_{n=0}^{P-1}(n+1)^{\sigma-3}$ is bounded, we obtain that

$$\|e^P\| \le C\left(\tau^\sigma + h_x^2 + h_y^2\right).$$

The proof is completed. □

Remark 3.1. Although Theorem 3.1 shows the linearized ADI Schemes (2.9) and (2.10) have temporal accuracy $\mathcal{O}(\tau^\sigma)$. However, the global truncation error in the temporal direction of Eq. (2.5) is $\mathcal{O}\left(t_{n+1}^{\sigma-3}\tau^2\right)$, but if t is far away from t_0, the global truncation error in the temporal direction can be $\mathcal{O}\left(\tau^2\right)$. The results of numerical experiments in Section 4 are consistent with this remark. Thus, $t = t_0$, linearized ADI Schemes (2.9) and (2.10) have a temporal accuracy $\mathcal{O}(\tau^\sigma)$ and become $\mathcal{O}\left(\tau^2\right)$ when t_{n+1} is far away from t_0.

3.2. Stability

We can derive the stability of the linearized ADI Schemes (2.9) and (2.10) as similarly as proving Theorem 3.1.

Theorem 3.2. *Let* $\{U_{ij}^n | 0 \le i \le M_x, 0 \le j \le M_y, 0 \le n \le N\}$ *be the numerical solution of Schemes* (2.9) *and* (2.10) *for Problem* (2.4). *Then for* $1 \le K \le N$, *it holds*

$$\|U^K\| \le C\left(\max_{0\le n\le N}\|g(U^n)\| + \max_{0\le n\le N-1}\|F^{n+\frac{1}{2}}\|\right).$$

Proof. Multiplying Eq. (2.10) by $h_x h_y(U_{ij}^{n+1}+U_{ij}^{n})$ and summing over $1\le i\le M_x-1, 1\le j\le M_y-1$, we have

$$
\begin{aligned}
&\|U^{n+1}\|^2-\|U^n\|^2+\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\\
&\times\langle\left(\delta_x^2-\delta_x^\beta\right)\left(\delta_y^2-\delta_y^\beta\right)\left(U^{n+1}-U^n\right),U^{n+1}+U^n\rangle\\
=&\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\left(\delta_x^2+\delta_y^2\right)\left(U^{n+1-k}+U^{n-k}\right),U^{n+1}+U^n\rangle\\
&-\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\left(\delta_x^\beta+\delta_y^\beta\right)\left(U^{n+1-k}+U^{n-k}\right),U^{n+1}+U^n\rangle\\
&+\frac{\tau^\alpha}{2}\left[\omega_{n+1}^{(\alpha-1)}\langle\left(\delta_x^2+\delta_y^2\right)U^0,U^{n+1}+U^n\rangle-\omega_{n+1}^{(\alpha-1)}\langle\left(\delta_x^\beta+\delta_y^\beta\right)U^0,U^{n+1}+U^n\rangle\right]\\
&+\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\langle g(U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\langle g(2U^n-U^{n-1}),U^{n+1}+U^n\rangle+\tau\langle F^{n+\frac12},U^{n+1}+U^n\rangle.
\end{aligned}
$$

Note that Eq. (1.1) is equipped with the homogeneous initial conditions, thus it deduces

$$
\begin{aligned}
&\|U^{n+1}\|^2-\|U^n\|^2+\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\\
&\times\langle\left(\delta_x^2-\delta_x^\beta\right)\left(\delta_y^2-\delta_y^\beta\right)\left(U^{n+1}-U^n\right),U^{n+1}+U^n\rangle\\
=&\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\left(\delta_x^2+\delta_y^2\right)\left(U^{n+1-k}+U^{n-k}\right),U^{n+1}+U^n\rangle\\
&-\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\left(\delta_x^\beta+\delta_y^\beta\right)\left(U^{n+1-k}+U^{n-k}\right),U^{n+1}+U^n\rangle\\
&+\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\langle g(U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\langle g(2U^n-U^{n-1}),U^{n+1}+U^n\rangle+\tau\langle F^{n+\frac12},U^{n+1}+U^n\rangle.
\end{aligned}
$$

Applying the similar deductions to get Eq. (3.3), it achieves that

$$
\begin{aligned}
\|U^J\|^2\le&-\left(\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\right)^2\|\eta_3\|^2+C\tau\sum_{k=0}^{J-1}\|g(U^k)\|\left(\|U^{n+1}\|+\|U^n\|\right)\\
&+\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\sum_{n=1}^{J-1}\|g(2U^n-U^{n-1})\|\left(\|U^{n+1}\|+\|U^n\|\right)\\
&+C\tau\sum_{n=1}^{J-1}\|F^{n+\frac12}\|\left(\|U^{n+1}\|+\|U^n\|\right).
\end{aligned}\tag{3.4}
$$

One can estimate $\|g(2U^n - U^{n-1})\|$ as following

$$\begin{aligned}\|g(2U^n - U^{n-1})\| =&\|g(2U^n - U^{n-1}) - g(U^n) + g(U^n)\| \\ \leq&\|g(2U^n - U^{n-1}) - g(U^n)\| + \|g(U^n)\| \\ \leq&C(\|U^n\| + \|U^{n-1}\|) + \|g(U^n)\|. \end{aligned} \tag{3.5}$$

Substituting Eq. (3.5) into (3.4), omitting the non-positive terms and using Young's inequality, then we have

$$\|U^J\|^2 \leq C\tau \sum_{n=0}^{J-1} \|U^n\|^2 + C \max_{0\leq n\leq N} \|g(U^n)\|^2 + C \max_{0\leq n\leq N-1} \|F^{n+\frac{1}{2}}\|^2. \tag{3.6}$$

By applying the Gronwall inequality to (3.6), it becomes

$$\|U^J\|^2 \leq C \left(\max_{0\leq n\leq N} \|g(U^n)\|^2 + \max_{0\leq n\leq N-1} \|F^{n+\frac{1}{2}}\|^2 \right),$$

and thus completes the proof. □

4. Numerical Experiments

In this section, we carry out numerical experiments for our linearized ADI finite difference scheme.

Example 4.1. Consider the following problem with exact solution $u(x, y, t) = t^\sigma x^2(1 - x)^2 y^2(1 - y)^2$.

$$\begin{aligned}{}_0^C D_t^\alpha u(x, y, t) =&\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + \frac{\partial^\beta u(x, y, t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x, y, t)}{\partial |y|^\beta} + g(u) \\ &+ f(x, y, t),\end{aligned}$$

where $T = 1$, $(x, y) \in (0, 1) \times (0, 1)$, $0 < t \leq T$, and $1 < \sigma < \alpha$. The nonlinear function $g(u) = u^2$ and $f(x, y, t)$ is

$$\begin{aligned}f(x, y, t) =&\frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + 1)} t^{\sigma-\alpha} x^2(1 - x)^2 y^2(1 - y)^2 \\ &- t^\sigma y^2(1 - y)^2 \left(12x^2 - 12x + 2\right) - t^\sigma x^2(1 - x)^2(12y^2 - 12y + 2) \\ &+ \frac{t^\sigma}{2\cos\left(\frac{\beta\pi}{2}\right)} \Gamma(5 - \beta) \left[h(y, x, \beta) + h(x, y, \beta)\right] - t^{2\sigma} x^4(1-x)^4 y^4(1-y)^4,\end{aligned}$$

where

$$\begin{aligned}h(v, w, \beta) =&v^2 \left(1 - v^2\right) \Big[12 \left(w^{4-\beta} + (1 - w)^{4-\beta}\right) - 6 (4 - \beta) \left(w^{3-\beta} + (1 - w)^{3-\beta}\right) \\ &+ (4 - \beta) (3 - \beta) \left(w^{2-\beta} + (1 - w)^{2-\beta}\right)\Big].\end{aligned}$$

It is worth noting that the exact solution satisfies the smoothness condition in Theorem 3.1. At $t = 0$, the right-hand side source function $f(x, y, t)$ is singular. In Figure 1, we compare the exact solution with numerical solution of linearized

ADI finite difference Schemes (2.9) and (2.10). We can see from Figure 2 that the errors are small and hence our numerical solutions can approximate the exact solutions well. Theorem 3.1 shows that the temporal numerical convergence order is σ. To verify this, we set $h = 0.01$, a value is small enough such that the spatial discretization errors are negligible as compared with the temporal errors, and choose different time step size. In Table 1, we fix $\beta = 1.5$, $\alpha = 1.8$, and present the errors at t_1 and time convergence order. In Table 2, we fix $\beta = 1.5$, $\sigma = 1.3$, and present the errors at t_1 and time convergence order. We conclude that the temporal numerical convergence order approach to σ. To verify Remark 3.1, we compute the L_2-errors and the temporal numerical convergence orders at the final t_N for $h = 0.01$ and $\beta = 1.5$ in Tables 3 and 4. It is clearly visible that at t_N, the numerical convergence is close to 2.

On other hand, Table 5 shows the L_2-errors, space convergence order with $\sigma = 1.4$ and $\alpha = 1.6$. We set $\tau = 0.01$, a value is small enough such that the temporal discretization errors are negligible as compared with the spatial errors, and choose different space step size. From all scenarios, we conclude that the spatial convergence order is 2. This is consistent with the theoretical analysis.

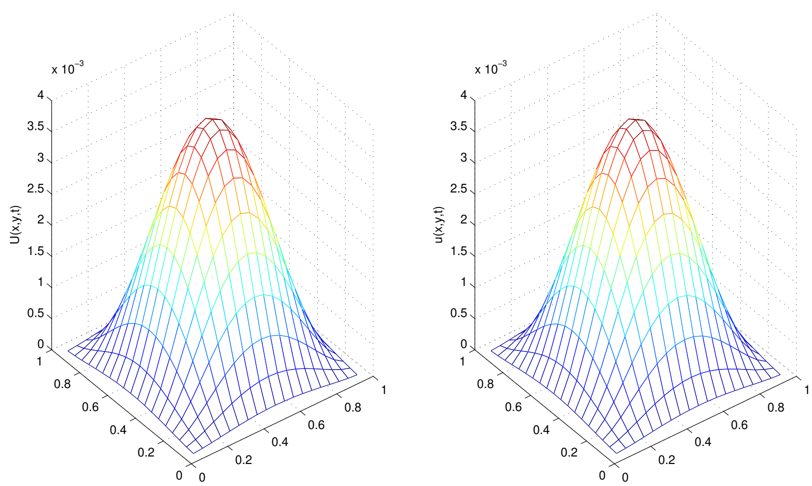

Figure 1. Numrical solution (left column) and exact solution (right column) when $\tau = 1/16$, $h = 1/20$, $\sigma = 1.4$, $\alpha = 1.6$, and $\beta = 1.5$.

Table 1. The errors for different σ and temporal numerical convergence orders for Schemes (2.9) and (2.10) at t_1, $h = 0.01$, $\beta = 1.5$, and $\alpha = 1.8$.

τ	$\sigma = 1.3$		$\sigma = 1.5$		$\sigma = 1.7$	
	error	order	error	order	error	order
1/64	3.6383×10^{-7}		1.7160×10^{-7}		5.5330×10^{-8}	
1/128	1.5762×10^{-7}	1.2069	6.4760×10^{-8}	1.4058	1.8619×10^{-8}	1.5713
1/256	6.5212×10^{-8}	1.2732	2.3330×10^{-8}	1.4729	5.8797×10^{-9}	1.6630
1/512	2.6626×10^{-8}	1.2923	8.2930×10^{-9}	1.4922	1.8231×10^{-9}	1.6894

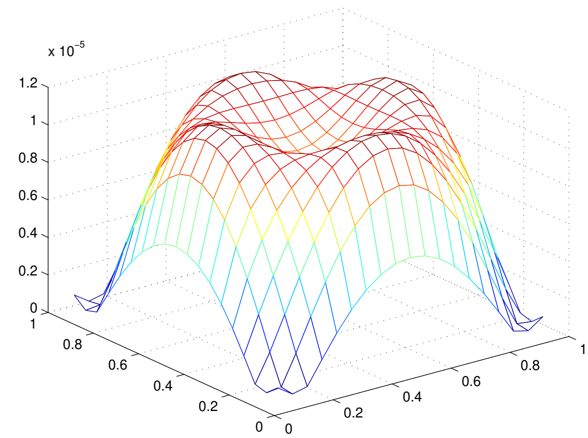

Figure 2. The error surface between numerical solutions and exact solutions for $\tau = 1/16$, $h = 1/20$, $\sigma = 1.4$, $\alpha = 1.6$, and $\beta = 1.5$.

Table 2. The errors for different α and temporal numerical convergence orders for Schemes (2.9) and (2.10) at t_1, $h = 0.01$, $\beta = 1.5$, and $\sigma = 1.3$.

τ	$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.9$	
	error	order	error	order	error	order
1/64	2.6765×10^{-7}		3.2750×10^{-7}		3.7487×10^{-7}	
1/128	1.3896×10^{-7}	0.9470	1.5165×10^{-8}	1.1108	1.5916×10^{-7}	1.2359
1/256	6.2023×10^{-8}	1.1638	6.4313×10^{-8}	1.2375	6.5416×10^{-8}	1.2828
1/512	2.6106×10^{-8}	1.2481	2.6496×10^{-9}	1.2794	2.6652×10^{-8}	1.2954

Table 3. The errors for different σ and temporal numerical convergence orders for Schemes (2.9) and (2.10) at t_N, $h = 0.01$, $\beta = 1.5$, and $\alpha = 1.8$.

τ	$\sigma = 1.3$		$\sigma = 1.5$		$\sigma = 1.7$	
	error	order	error	order	error	order
1/5	9.2221×10^{-5}		2.9769×10^{-5}		9.3783×10^{-5}	
1/10	2.5085×10^{-5}	1.8783	2.4008×10^{-5}	1.9027	2.3030×10^{-5}	2.0258
1/20	6.6822×10^{-6}	1.9084	6.2278×10^{-6}	1.9467	5.7155×10^{-6}	2.0106
1/40	1.7526×10^{-6}	1.9308	1.5729×10^{-6}	1.9853	1.3939×10^{-6}	2.0358

5. Conclusion

In this paper, we blueconstruct a linearized ADI scheme for two-dimensional time-space fractional nonlinear diffusion-wave equations with initial singularity. To reduce the smoothness requirement in time, the proposed scheme is constructed based on the equivalent partial integro-differential equations. Then, the Crank-Nicolson technique, the midpoint formula, the second order convolution formula, the clas-

Table 4. The errors for different α and temporal numerical convergence orders for Schemes (2.9) and (2.10) at t_N, $h = 0.01$, $\beta = 1.5$, and $\sigma = 1.3$.

τ	$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.9$	
	error	order	error	order	error	order
1/5	1.5173×10^{-4}		8.9262×10^{-5}		9.3689×10^{-5}	
1/10	4.4183×10^{-5}	1.7799	2.2582×10^{-5}	1.9829	2.7237×10^{-5}	1.7823
1/20	1.2255×10^{-5}	1.8501	5.6456×10^{-6}	2.0000	7.4858×10^{-6}	1.8634
1/40	3.1356×10^{-6}	1.9666	1.4212×10^{-6}	1.9900	2.0115×10^{-6}	1.8959

Table 5. The errors for different β and spatial numerical convergence orders of Schemes (2.9) and (2.10) for $\tau = 0.01$, $\sigma = 1.4$, and $\alpha = 1.6$.

h	$\beta = 1.3$		$\beta = 1.5$		$\beta = 1.7$	
	error	order	error	order	error	order
1/8	8.9601×10^{-5}		9.0426×10^{-5}		9.4047×10^{-5}	
1/16	2.2120×10^{-5}	2.0182	2.2181×10^{-5}	2.0274	2.2984×10^{-5}	2.0328
1/32	5.4113×10^{-6}	2.0313	5.3863×10^{-6}	2.0419	5.5482×10^{-6}	2.0505

sical central difference formula, and the fractional central difference formula were applied to construct the proposed scheme. Theoretically, the convergence and the unconditional stability of the proposed scheme are proved and discussed. All of the numerical experiments support our theoretical results.

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