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PLANAR INTEGRABLE NONLINEAR OSCILLATORS HAVING A STABLE LIMIT CYCLE*

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Abstract In this short paper, by improving some conclusions given by [2], we show that planar integrable nonlinear oscillators can have a stable limit cycle. We also obtain these xact parametric representations of these limit cycles.

Keywords Limit cycle, nonlinear oscillatory type equation, exact solution, integrable planar polynomial system.

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1. Introduction

It is well known that if a planar polynomial Hamiltonian system has an equilibrium point of center type at $E_c(x_c, 0)$, then there exists a period annulus of enclosing this center. Here, a period annulus means an open set consisting of uncountably infinitely many periodic orbits of this system. Since the divergence of a planar Hamiltonian vector field is equivalently equal zero, it can not have any limit cycle.

For the planar integrable systems, as we well known that some artificially constructing models such as

$$\frac{dx}{dt} = -y - x(x^2 + y^2 - a^2), \quad \frac{dy}{dt} = x - y(x^2 + y^2 - a^2)$$

has a stable limit cycle $x^2 + y^2 = a^2$.

Now the question arises as to whether there exists a planar integrable polynomial oscillator, which has a limit cycle? To our knowledge, we have not found such an example in the published literature. The aim of this short paper is to answer the mentioned question positively.

2. A fifth-order nonlinear oscillator

The authors of [2] proved that the following nonlinear oscillatory type equation

$$x'' + (k_1 x^2 + k_2)x' + \frac{k_1^2}{16}x^5 + \frac{k_1 k_2}{4}x^3 + \lambda_1 x = 0 \tag{2.1}$$

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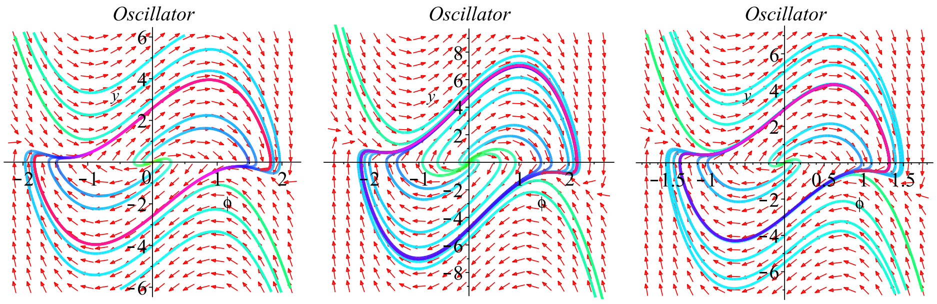


(a) $k_1 = 3, k_2 = -2.5, \lambda_1 = 2$. (b) $k_1 = 3, k_2 = -3.5, \lambda_1 = 5$. (c) $k_1 = 8, k_2 = -3.5, \lambda_1 = 5$.

Figure 1. Three phase portraits of system (2) for given parameter group (k_1, k_2, λ_1)

is integrable, where k_1, k_2 and λ_1 are arbitrary real parameters. Clearly, equation (2.1) is equivalent to a planar dynamical system of the form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -(k_1x^2 + k_2)y - x\left(\frac{k_1^2}{16}x^4 + \frac{k_1k_2}{4}x^2 + \lambda_1\right). \tag{2.2}$$

From [1] one can see that system (2.2) has the first integrals

$$I_{\pm}(x, y, t) = e^{\mp\omega t}\left(\frac{4y + 2(k_2 \pm \omega)x + k_1x^3}{4y + 2(k_2 \mp \omega)x + k_1x^3}\right), \tag{2.3}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$.

Write that $f(X) = \frac{k_1^2}{16}X^2 + \frac{k_1k_2}{4}X + \lambda_1$. Clearly, when $k_1 \neq 0$, then $f(X)$ has two zeros at $X_{1,2} = \frac{1}{k_1^2}\left(-k_1k_2 \mp |k_1|\sqrt{\Delta}\right)$ if $\Delta = k_2^2 - 4\lambda_1 > 0$, has only one zero at $X_0 = -\frac{k_2}{k_1}$ if $\Delta = 0$ and has no real zero if $\Delta < 0$.

Now we assume that $k_1 > 0$, $k_2 < 0$, and $k_2^2 < 4\lambda_1$, i.e., $\Delta < 0$. Then, system (2.2) has a unique singular point at the origin $O(0, 0)$ which is an unstable focus. By using Mapple, we can draw the phase portraits of system (2.2) as shown in Fig.1. Obviously, we see that system (2.2) has an unique stable limit cycle. The existence and uniqueness of the limit cycle can be seen from the following discussion of the exact solutions of system (2.2). Notice that the vector field defined by system (2.2) is centrally symmetric and that the divergence of the vector field is

$$\mathrm{Div}(2) = -(k_1x^2 + k_2).$$

Hence, the orbit of the limit cycle must intersect with the two straight lines $x = \mp\sqrt{\frac{-k_2}{k_1}}$.

When $\Delta > 0$, the authors of [2] used the first integral (2.3) to find a general solution of equation (1) as follows

$$x(t) = \left(\frac{8k_2\lambda_1(e^{\omega t} - C_1)^2}{C_1^2k_1k_2(-k_2+\omega) - k_1k_2(k_2+\omega)e^{2\omega t} + 8C_2k_2\lambda_1e^{(k_2+\omega)t} + 8C_1k_1\lambda_1e^{\omega t}}\right)^{\frac{1}{2}}, \tag{2.4}$$

where C_1, C_2 are two arbitrary constants. We can check that (2.4) satisfies equation (2.1). Unfortunately, when $\Delta < 0$ the solution (100) in [2] is not correct since it does not satisfy equation (2.1). In fact, if instead of (2.4) we take

$$x(t) = \left(\frac{8k_2\lambda_1(e^{i\omega_0 t} - C_1)^2}{C_1^2 k_1 k_2(-k_2 + i\omega_0) - k_1 k_2(k_2 + i\omega_0)e^{2i\omega_0 t} + 8C_2 k_2 \lambda_1 e^{(k_2 + i\omega_0)t} + 8C_1 k_1 \lambda_1 e^{i\omega_0 t}} \right)^{\frac{1}{2}}, \tag{2.5}$$

where $\omega_0 = \sqrt{4\lambda_1 - k_2^2} = \sqrt{-\Delta}$, then (2.5) satisfies equation (2.1).

We next separate the real part and imaginary part of $x(t)$ given by (2.5). Write that

$$f_1(t) = -C_1^2 k_1 k_2^2 - k_1 k_2^2 \cos(2\omega_0 t) + k_1 k_2 \omega_0 \sin(2\omega_0 t) + 8\lambda_1(C_2 k_2 e^{k_2 t} + C_1 k_1)\cos(\omega_0 t),$$
$$f_2(t) = C_1^2 k_1 k_2 \omega_0 - k_1 k_2^2 \sin(2\omega_0 t) - k_1 k_2 \omega_0 \cos(2\omega_0 t) + 8\lambda_1(C_2 k_2 e^{k_2 t} + C_1 k_1)\sin(\omega_0 t).$$

Then, (2.5) can be rewritten as

$$x(t) = \frac{-2\sqrt{2|k_2|\lambda_1}(\sin(\omega_0 t) + i(C_1 - \cos(\omega_0 t))}{\sqrt{f_1(t) + i f_2(t)}}. \tag{2.6}$$

Notice that

$$\frac{1}{\sqrt{f_1(t) + i f_2(t)}} = \left(\frac{2}{\sqrt{f_1^2(t) + f_2^2(t)}} + \frac{2f_1(t)}{f_1^2(t) + f_2^2(t)} \right)^{\frac{1}{2}} \mp i \left(\frac{2}{\sqrt{f_1^2(t) + f_2^2(t)}} - \frac{2f_1(t)}{f_1^2(t) + f_2^2(t)} \right)^{\frac{1}{2}}. \tag{2.7}$$

Therefore, we can derive the general solutions of system (2.2) when $\Delta < 0$. Because the formula is too long, we omit it.

Gonzalez and Piro [3] cited the result given by Bellman's book [1] that when $\Delta < 0$, equation (2.1) has the exact general solution depending on two arbitrary constants as follows:

$$x(t) = \frac{\cos(\omega_0 t + C_2)}{\sqrt{Q(t)}}, \tag{2.8}$$

where

$$Q(t) = C_1 e^{k_2 t} - \frac{k_1 \omega_0^2}{k_2(4\omega_0^2 + k_2^2)} \left[1 + \frac{k_2^2}{2\omega_0^2}\cos^2(\omega_0 t + C_2) - \frac{k_2}{2\omega_0}\sin 2(\omega_0 t + C_2) \right], \tag{2.9}$$

where $\omega_0 = \frac{1}{2}\sqrt{4\lambda_1 - k_2^2}$, C_1 and C_2 are two arbitrary constants. It is easy to check that the function (2.8) satisfies equation (2.1).

Because of $k_2 < 0$, in the term $e^{k_2 t}$ of (2.8), taking $t \to \infty$, we obtain the exact solution of the limit cycle about $x-$component as follows:

$$L(t) = \frac{\sqrt{-k_2(4\omega_0^2 + k_2^2)}\cos(\omega_0 t + C_2)}{\sqrt{k_1}\omega_0 \left[1 - \frac{k_2}{2\omega_0}\cos(\omega_0 t + C_2)\left(-\frac{k_2}{\omega_0}\cos(\omega_0 t + C_2) + 2\sin(\omega_0 t + C_2) \right) \right]^{\frac{1}{2}}}, \tag{2.10}$$

which satisfies equation (2.1). Thus, system (2.2) has the exact parametric representation of the limit cycle given by $(L(t), L'(t))$. From the above global exact

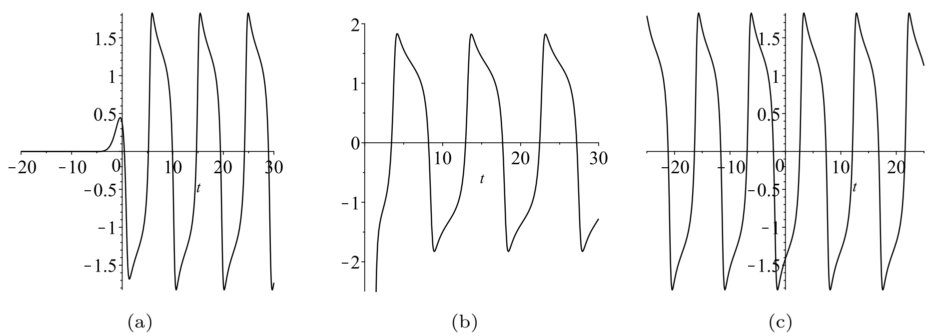


Figure 2. Three graphs of $x(t)$ and $L(t)$ of system (2.2) for given parameter group (k_1, k_2, λ_1)

solutions, we now the existence and uniqueness of the stable limit cycle of system (2.2).

By taking $k_1 = 3, k_2 = -2.5, \lambda_1 = 2$ in (2.8), (2.9) and (2.10) to draw graphs of $L(t)$, we obtain Fig.2 (a), (b) and (c).

Parameter group $(k_1, k_2, \lambda_1) = (3, -2.5, 2)$.

(a) The graph of $x(t)$ given by (2.8), $C_1 = 3, C_2 = -5$.

(b) The graph of $x(t)$ given by (2.8), $C_1 = -13, C_2 = 15$.

(c) The graph of limit cycle given by $L(t)$ in (3.2), $C_2 = -5$.

Obviously, the initial value of Fig.2 (a) is taken inside of limit cycle, while the initial value of Fig.2 (b) is taken outside of limit cycle. The $\alpha-$limt set of two orbits are zero and $-\infty$, respectively.

To sum up, we have the following conclusion.

Theorem 2.1. *When* $\Delta = k_2^2 - 4\lambda_1 < 0, k_2 < 0, k_1 > 0, \lambda_1 > 0$, *depending on the change of the parameter group* (k_1, k_2, λ_1), *planar nonlinear oscillator* (2.1) *has a family of stable limit cycles for which the* $x-$*component has the exact parametric representation given by* (2.10).

3. $2q + 1$-order oscillators $(q > 2)$

Chandrasekar, et, al. [2] also stated that the following high-order nonlinear oscillatory type equation

$$x'' + ((q+2)k_1 x^q + k_2)x' + k_1^2 x^{2q+1} + k_1 k_2 x^{q+1} + \lambda_1 x = 0 \tag{3.1}$$

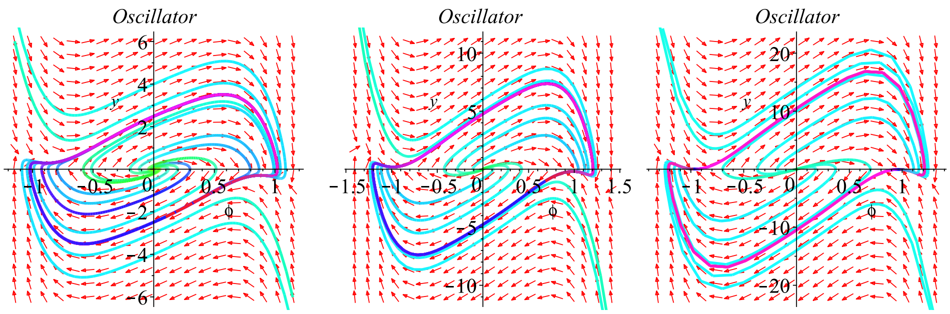
is integrable, where k_1, k_2 and λ_1 are arbitrary real parameters. Equation (3.1) is equivalent to planar dynamical system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -((q+2)k_1 x^q + k_2)y - x\left(k_1^2 x^{2q} + k_1 k_2 x^q + \lambda_1\right). \tag{3.2}$$

System (3.2) has the first integral

$$I(x, y, t) = e^{\mp\omega t}\left(\frac{y - \frac{1}{2}(-k_2 \mp \omega)x + \frac{k_1}{q+2}x^{q+1}}{y + \frac{1}{2}(-k_2 \pm \omega)x + \frac{k_1}{q+2}x^{q+1}}\right), \tag{3.3}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$.



(a) $k_1 = 3, k_2 = -2.5, \lambda_1 = 4$. (b) $k_1 = 3, k_2 = -5, \lambda_1 = 8.5$. (c) $k_1 = 7, k_2 = -12, \lambda_1 = 36.5$.

Figure 3. Three phase portraits of system (3.2) for given parameter group (k_1, k_2, λ_1)

Clearly, when $\Delta = k_2^2 - 4\lambda_1 < 0$, system (3.4) has only one singular point $O(0,0)$. It is unstable focus.

In 1961, Smith [5] obtained the following exact general solution of equation (3.1):

$$x(t) = \frac{\cos(\omega_0 t + C_2)}{(Q_q(t))^{\frac{1}{q}}}, \tag{3.4}$$

where

$$Q_q(t) = e^{\frac{1}{2}k_2 q t}\left(C_1 + qk_1 \int e^{-\frac{1}{2}k_2 q s}\cos^q(\omega_0 s + C_2)ds\right) \tag{3.5}$$

and $\omega_0 = \frac{1}{2}\sqrt{4\lambda_1 - k_2^2}, C_1$ and C_2 are two arbitrary constants. We remark that the formula (126) given by Chandrasekar, et, al., [2] is incorrect.

By using the recursion formulas

$$\int e^{at}\cos^q(bt)dt = \frac{e^{at}\cos^{q-1}(bt)}{a^2+q^2b^2}(a\cos(bt)+qb\sin(bt)) + \frac{q(q-1)b^2}{a^2+b^2q^2}\int e^{at}\cos^{q-2}(bt)dt, \tag{3.6}$$

we can calculate (3.5). Thus, when $\Delta = k_2^2 - 4\lambda_1 < 0, k_2 < 0, k_1 > 0, \lambda_1 > 0$ and $q = 2m$ is an even number, we can obtain the parametric representations of the limit cycles of system (3.2).

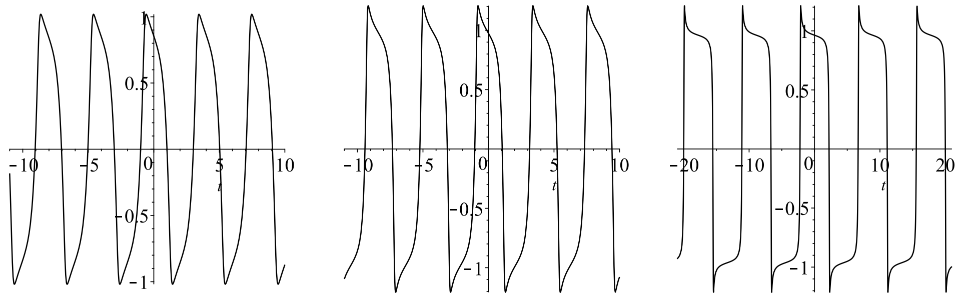
For an example, taking $q = 4$, we have the phase portraits of system (3.2) as following Fig. 3.

In addition, notice that $\lambda_1 = \frac{1}{4}(k_2^2 + 4\omega_0^2)$, by calculating (3.7), we obitain following result:

$$\begin{aligned} Q_4(t) =& C_1 e^{2k_2 t} + \frac{k_1}{4\lambda_1}\Bigg[-2k_2\cos^2(\omega_0 t + C_2)\left(\cos^2(\omega_0 t + C_2) + \frac{3\omega_0^2}{k_2^2+\omega_0^2}\right) \\ & + \omega_0 \sin 2(\omega_0 t + C_2)\left(2\cos^2(\omega_0 t + C_2) + \frac{3\omega_0^2}{k_2^2+\omega_0^2}\right) - \frac{3\omega_0^4}{k_2(k_2^2+\omega_0^2)}\Bigg]. \end{aligned} \tag{3.7}$$

Thus, the limit cycle of system (3.4) has the parametric representation $(L_4(t), L_4'(t))$, where

$$L_4(t) = \frac{\cos(\omega_0 t + C_2)}{(Q_{04}(t))^{\frac{1}{4}}} \tag{3.8}$$



(a) $k_1 = 3, k_2 = -2.5, \lambda_1 = 4$. (b) $k_1 = 3, k_2 = -5, \lambda_1 = 8.5$. (c) $k_1 = 7, k_2 = -12, \lambda_1 = 36.5$.

Figure 4. Three graphs of the $x-$component $L_4(t)$ of the limit cycles for system (12)

and

$$Q_{04}(t) = \frac{k_1}{4\lambda_1}\Bigg[-2k_2\cos^2(\omega_0 t + C_2)\left(\cos^2(\omega_0 t + C_2) + \frac{3\omega_0^2}{k_2^2+\omega_0^2}\right) + \omega_0 \sin 2(\omega_0 t + C_2)\left(2\cos^2(\omega_0 t + C_2) + \frac{3\omega_0^2}{k_2^2+\omega_0^2}\right) - \frac{3\omega_0^4}{k_2(k_2^2+\omega_0^2)}\Bigg]. \tag{3.9}$$

By using (3.8) to draw the graphs for the $x-$component $L_4(t)$, corresponding to the phase portraits in Fig. 3, we obtain Fig. 4.

Thus, the following conclusion holds.

Theorem 3.1. *When* $\Delta = k_2^2 - 4\lambda_1 < 0, k_2 < 0, k_1 > 0, \lambda_1 > 0, q = 2m$, *depending on the change of the parameter group* (k_1, k_2, λ_1), *planar integrable system* (3.2) *has a family of stable limit cycles which has the exact parametric representation given by the limit orbit of* (3.2) *(for example, when* $q = 4$, *the* $x-$*component is given by* (3.8).

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