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WEYL ALMOST PERIODIC FUNCTIONS ON TIME SCALES AND WEYL ALMOST PERIODIC SOLUTIONS OF DYNAMIC EQUATIONS WITH DELAYS*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract Due to the incompleteness of the space composed of Weyl almost periodic functions, there are few results on the existence of Weyl almost periodic solutions of differential equations. In addition, as a discrete analogs of differential equations, there is almost no result of the existence of Weyl almost periodic solutions of difference equations. Because dynamic equations on time scales can unify the study of differential equations and difference equations. Therefore, in this paper, we first propose a concept of Weyl almost periodic functions on time scales. Then, taking a Clifford-valued neural network with time-varying delays on time scales as an example of dynamic equations on time scales, we study the existence and stability of Weyl almost periodic solutions of this neural network on time scales. Even when the system we consider degenerates into a real-valued system, our results are new. A numerical example is given to illustrate the feasibility of our results.

Keywords Weyl almost periodic functions on time scales, Clifford-valued neural networks, Weyl almost periodic solutions, global exponential stability, time scales.

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1. Introduction

The concept of almost periodicity was first introduced into mathematical research by H. Bohr [8,9]. Since H. Bohr put forward the concept of almost periodic functions, this concept has been extended by many mathematicians in various aspects, including its discrete analogs [5,23,36,39]. And studying the existence of almost periodic solutions in various senses of differential equations and difference equations has become an important research content of the qualitative theory of differential and difference equations [13,40,41]. Weyl almost periodic concept is a generalization of Bohr almost periodic concept and Stepanov almost periodic concept. Bohr almost periodic function space and Stepanov almost periodic function space are Banach spaces. However, the space composed of Weyl almost periodic functions

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is not a Banach space [23], so there are few results on the existence of Weyl almost periodic solutions of differential equations. There are almost no results on the existence of Weyl almost periodic solutions of difference equations.

On the one hand, we know that time scale calculus theory is a theory proposed by S. Hilger [17], which can unify continuous analysis and discrete analysis, and studying the dynamic equations on time scales can unify the problems of differential equations and difference equations [1, 31]. At present, various concepts of almost periodic functions on time scales have been proposed one after another, and the existence of almost periodic solutions of dynamic equations on time scales has been studied by many scholars [14–16, 19, 21, 24, 27–29, 33, 37, 38, 42, 43]. However, there is no Weyl almost periodic concept on time scales, so it is of great theoretical significance and potential application value to propose a Weyl almost periodic concept on time scales and study the Weyl almost periodic solutions of dynamic equations on time scales.

On the other hand, as a generalization of real-valued neural networks, complex-valued neural networks and quaternion-valued neural networks, Clifford-valued neural networks have been proved to have more advantages than real-valued neural networks in dealing with high-dimensional data and spatial transformation [10, 18, 32]. Nevertheless, because Clifford algebraic multiplication does not satisfy the commutative law, there are few results on the dynamics of Clifford-valued neural networks [2–4, 12, 22, 25, 30, 34, 35]. At the same time, there are few results using the direct method, that is, the method of not decomposing Clifford-valued systems into real-valued systems to study the dynamics of Clifford-valued neural networks. In addition, it is well known that almost periodic oscillation is one of the important dynamics of neural networks. Therefore, it is an important and challenging work to study the Weyl almost periodic oscillation of Clifford-valued neural networks by the direct method.

Inspired by the above discussion, the main purpose of this paper is to propose a concept of Weyl almost periodic functions on time scales, and then take the following Clifford-valued cellular neural network with time-varying delays on time scale $\mathbb{T}$:

$$\begin{aligned} x_i^{\Delta}(t) = &- a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\upsilon_{ij}(t))) + I_i(t) \end{aligned} \tag{1.1}$$

as an example of dynamic equations on time scales to study the existence and stability of Weyl almost periodic solutions of (1.1), where $i \in \{1, 2, \dots, n\} := \Lambda$, n corresponds to the number of units in the neural network; $x_i(t) \in \mathcal{A}$ corresponds to the state of the ith neuron at time t; $a_i(t) \in \mathcal{A}$ is the self-feedback connection weight; $b_{ij}(t), c_{ij}(t) \in \mathcal{A}$ are the delay connection weights from neuron j to neuron i at time t; $\tau_{ij}(t) \geq 0$ and $\upsilon_{ij}(t) \geq 0$ correspond to the transmission delays at time t and satisfy $t-\tau_{ij}(t)$ and $t-\upsilon_{ij}(t) \in \mathbb{T}$ for $t \in \mathbb{T}$; $I_i(t) \in \mathcal{A}$ denotes the external input at time t; $f_j, g_j : \mathcal{A} \to \mathcal{A}$ denote the activation functions of signal transmission.

The main contributions of this paper are (i) We put forward a concept of Weyl almost periodic functions on time scales. (ii) We use direct methods to study the existence and stability of Weyl almost periodic solutions of (1.1). (iii) Even when system (1.1) is a real-valued system, and our results are new. (iv) Our method of this paper can be used to study Weyl almost periodic solutions of other types of dynamic equations on time scales.

The rest of this paper is arranged as follows. In Section 2, we introduce some definitions and lemmas, and propose a concept of Weyl almost periodic functions on time scales. In Section 3, we study the existence and global exponential stability of Weyl almost periodic solutions of (1.1). In Section 4, a numerical example is given to verify the theoretical results. This paper ends with a brief conclusion in Section 5.

2. Preliminaries and the concept of Weyl almost periodicity on time scales

Let $\mathcal{A}$ be a real Clifford algebra over $\mathbb{R}^m$ with $e_\emptyset = e_0 = 1$ and e_p, $p = 1, 2, \ldots, m$ as its generators, where $e_p^2 = -1$, $p = 1, 2, \ldots, m$, and $e_pe_q + e_qe_p = 0$, $p \neq q$, $p, q = 1, 2, \ldots, m$. For convenience, we will denote the product of Clifford generators $e_{p_1}e_{p_2}\ldots e_{p_v}$ as $e_{p_1p_2\ldots p_v}$. Let $\Xi = \{\varnothing, 1, 2, \ldots, A, \ldots, 12\ldots m\}$, then $\mathcal{A} = \left\{\sum_{A\in\Xi} a^A e_A, a^A \in \mathbb{R}\right\}$. For $x = \sum_A x^A e_A \in \mathcal{A}$, we define the norm of x by $\|x\|_{\mathcal{A}} = \max_{A\in\Xi}\{|x^A|\}$ and for $y = (y_1, y_2, \ldots, y_n)^T \in \mathcal{A}^n$, we define $\|y\|_{\mathcal{A}^n} = \max_{p\in\Xi}\{\|y_p\|_{\mathcal{A}}\}$, then $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{A}^n, \|\cdot\|_{\mathcal{A}^n})$ are two Banach spaces.

Denote by $\mathbb{T}$ a time scale and by $\mathcal{R}^+$ the collection of positive regressive functions from $\mathbb{T}$ to $\mathbb{R}$. For $x = \sum_A x^A e_A \in C^1(\mathbb{T}, \mathbb{R})$, we define $x^\Delta(t) = \sum_A (x^A(t))^\Delta e_A$ for $t \in \mathbb{T}$. For more information about Clifford analysis and Time scale theory, we refer to [20] and [7], respectively.

Throughout this paper, we use $(\mathbb{X}, \|\cdot\|)$ to denote a Banach space.

Let $L^\infty(\mathbb{T}, \mathbb{X})$ be the set of all functions $f : \mathbb{T} \to \mathbb{X}$ that are strongly Δ-measurable and essentially bounded [11]. Then space $L^\infty(\mathbb{T}, \mathbb{X})$ is a Banach space with the norm

$$\|f\|_\infty := \inf\{D \geq 0 : \|f(t)\|_{\mathbb{X}} \leq D \ a.e. \ t \in \mathbb{T}\}.$$

Definition 2.1 ([11,29]). For $p \geq 1$, $f : \mathbb{T} \to \mathbb{X}$ is called locally L^p Δ-integrable if f is Δ-measurable and for any compact Δ-measurable set $\mathbb{E} \subset \mathbb{T}$, the Δ-integral

$$\int_{\mathbb{E}} \|f(s)\|_{\mathbb{X}}^p \Delta s < \infty.$$

The set of all locally L^p Δ-integrable functions is denoted by $L^p_{loc}(\mathbb{T}, \mathbb{X})$.

Lemma 2.1 ([6]). *Let f be Δ-integrable over $R = [a_1, b_1) \times [a_2, b_2)$ and assume that the single integral $I(t_1) = \int_{b_2}^{a_2} f(t_1, t_2)\Delta_2 t_2$ exists for each $t_1 \in [a_1, b_1)$. Then the iterated integral $\int_{b_1}^{a_1} I(t_1)\Delta_1 t_1$ exists and*

$$\int\int_R f(t_1, t_2)\Delta_1 t_1 \Delta_2 t_2 = \int_{b_1}^{a_1} \Delta_1 t_1 \int_{b_2}^{a_2} f(t_1, t_2)\Delta_2 t_2.$$

Definition 2.2 ([27]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi = \{\tau \in \mathbb{R} : \tau \pm t \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Throughout the rest of this paper, we always assume that $\mathbb{T}$ is an almost periodic time scale. Let $BC(\mathbb{T},\mathbb{X})$ be the set of all bounded continuous functions from $\mathbb{T}$ to $\mathbb{X}$.

Definition 2.3 ([27]). A function $f \in BC(\mathbb{T},\mathbb{X})$ is called Bohr almost periodic on $\mathbb{T}$ if for any $\epsilon > 0$, there exists a constant $l(\epsilon) > 0$ such that in every interval of length $l(\epsilon)$ contains at least one $\tau \in \Pi$ such that

$$\|f(t+\tau) - f(t)\|_{\mathbb{X}} < \epsilon, \quad t \in \mathbb{T}.$$

The ζ is called an ϵ-translation number of f. We will denote by $AP(\mathbb{T},\mathbb{X})$ the collection of all such functions.

For $f \in L^p_{loc}(\mathbb{T},\mathbb{X})$, we define the following seminorm:

$$\|f\|_{W^p} = \lim_{r\to+\infty} \sup_{\beta\in\mathbb{T}} \left(\frac{1}{r}\int_{\beta}^{\beta+r} \|f(t)\|^p_{\mathbb{X}}\Delta t\right)^{\frac{1}{p}}, \quad r \in \Pi.$$

Definition 2.4. A function $f \in L^p_{loc}(\mathbb{T},\mathbb{X})$ is said to be p-th Weyl almost periodic (W^p-almost periodic for short), if for every $\epsilon > 0$, there exists a constant $l = l(\epsilon) > 0$ such that in every interval of length $l(\epsilon)$ contains at least one $\zeta \in \Pi$ such that

$$\|f(t+\zeta) - f(t)\|_{W^p} < \epsilon.$$

This ζ is called an ϵ-translation number of f. The set of all such functions will be denoted by $APW^p(\mathbb{T},\mathbb{X})$.

Remark 2.1. Obviously, we have $AP(\mathbb{T},\mathbb{X}) \subset APW^p(\mathbb{T},\mathbb{X})$.

Similar to the proofs of the lemma on page 83 and the lemma on page 84 of [5], one can easily prove the following lemma.

Lemma 2.2. *If $f \in APW^p(\mathbb{T},\mathbb{X})$, then f is bounded and uniformly continuous on $\mathbb{T}$ with respect to the seminorn $\|\cdot\|_{W^p}$.*

One can easily prove the following lemma.

Lemma 2.3. *If $f_k \in APW^p(\mathbb{T},\mathbb{X})$, $k = 1,2,\dots,n$. Then, for every $\epsilon > 0$, there exist common ϵ-translation numbers for these functions.*

Lemma 2.4 ([26]). *If $-a \in \mathcal{R}^+$ and $t, s \in \mathbb{T}$, $\tau \in \Pi$, then*

$$\begin{aligned}&e_{-a}(t+\tau, \sigma(s+\tau)) - e_{-a}(t,\sigma(s))\\ =&\int_t^{\sigma(s)} e_{-a}(t,\sigma(\theta))(a(\theta+\tau) - a(\theta))e_{-a}(\theta+\tau, \sigma(s+\tau))\Delta\theta.\end{aligned}$$

Lemma 2.5. *If $\beta \in \mathbb{T}$, $r \in \Pi$, $a \in C(\mathbb{T},\mathbb{R}^+), -a \in \mathcal{R}^+$, $a^m > 0$ with $a^m\mu^+ < 1$ and $f \in APW^p(\mathbb{T},\mathbb{X})$, then*

$$\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\int_{-\infty}^t e_{-a}(t,\sigma(s))\|f(s)\|^p_{\mathbb{X}}\Delta s\Delta t \le \frac{1}{(1-a^m\mu^+)a^m}\|f\|^p_{W^p},$$

where $a^m := \inf_{t\in\mathbb{T}} |a(t)|$ and $\mu^+ := \sup_{t\in\mathbb{T}} \mu(t)$.

Proof. *From Lemma 2.1, we have*

$$\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\int_{-\infty}^t e_{-a}(t,\sigma(s))\|f(s)\|^p_{\mathbb{X}}\Delta s\Delta t$$

$$
\begin{aligned}
&\leq \frac{1}{1-a^m\mu^+}\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\int_{-\infty}^t e_{-a}(t,s)\|f(s)\|_{\mathbb{X}}^p\Delta s\Delta t\\
&\leq \frac{1}{1-a^m\mu^+}\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{-\infty}^{\beta+r} e_{-a}(0,s)\|f(s)\|_{\mathbb{X}}^p\Delta s\int_s^{\beta+r} e_{-a}(t,0)\Delta t\\
&\leq \frac{1}{(1-a^m\mu^+)a^m}\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{-\infty}^{\beta+r} e_{-a}(\beta+r,s)\|f(s)\|_{\mathbb{X}}^p\Delta s\\
&\leq \frac{1}{(1-a^m\mu^+)a^m}\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\sum_{k=0}^{\infty}\int_{\beta-kr}^{\beta-(k-1)r} e_{-a}(\beta+r,s)\|f(s)\|_{\mathbb{X}}^p\Delta s\\
&\leq \frac{1}{(1-a^m\mu^+)a^m}\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\sum_{k=0}^{\infty}e^{-a^mkr}\int_{\beta-kr}^{\beta-(k-1)r}\|f(s)\|_{\mathbb{X}}^p\Delta s\\
&\leq \frac{1}{(1-a^m\mu^+)a^m}\lim_{r\to+\infty}\frac{1}{1-e^{-a^mr}}\|f\|_{W^p}^p\\
&\leq \frac{1}{(1-a^m\mu^+)a^m}\|f\|_{W^p}^p.
\end{aligned}
$$

This completes the proof. □

3. Weyl almost periodic solutions to Clifford-valued cellular neural networks on time scales

In this section, on the basis of the previous section, we will discuss the existence and global exponential stability of Weyl almost periodic solutions of system (1.1).

About system (1.1), for $i\in\Lambda$, we denote $a_i(t)=\sum_A a_i^A(t)e_A\in\mathcal{A}$, $\check{a}_i(t)=\sum_{A\neq\varnothing}a_i^A(t)e_A$ and $a_i^\varnothing(t)=a_i(t)-\check{a}_i(t)$. For convenience, we introduce the following notations:

$$
a^-=\min_{i\in\Lambda}\{\inf_{t\in\mathbb{T}}a_i^\varnothing(t)\},\, a^+=\max_{i\in\Lambda}\{\sup_{t\in\mathbb{T}}a_i^\varnothing(t)\},\, \check{a}_i^+=\sup_{t\in\mathbb{T}}\|\check{a}_i(t)\|_{\mathcal{A}},\, b_{ij}^+=\sup_{t\in\mathbb{T}}\|b_{ij}(t)\|_{\mathcal{A}},
$$

$$
c_{ij}^+=\sup_{t\in\mathbb{T}}\|c_{ij}(t)\|_{\mathcal{A}},\ \tau_{ij}^+=\sup_{t\in\mathbb{T}}\{\tau_{ij}(t)\},\ \upsilon_{ij}^+=\sup_{t\in\mathbb{T}}\{\upsilon_{ij}(t)\},\ \tilde{\tau}=\max_{i,j\in\Lambda}\big\{\sup_{t\in\mathbb{T}}\{\tau_{ij}^\Delta(t)\},
$$

$$
\tilde{\upsilon}=\max_{i,j\in\Lambda}\big\{\sup_{t\in\mathbb{T}}\{\upsilon_{ij}^\Delta(t)\}\big\},\ \eta=\max_{i,j\in\Lambda}\left\{\sup_{t\in\mathbb{T}}\{\tau_{ij}(t)\},\ \sup_{t\in\mathbb{T}}\{\upsilon_{ij}(t)\}\right\}.
$$

The initial values of system (1.1) are given by

$$
x_i(s)=\varphi_i(s),\quad s\in[-\eta,0],
$$

where $\varphi_i\in C([-\eta,0],\mathcal{A}), i\in\Lambda$.

In what follows, we make the following assumptions:

(H_1) Functions $a_i^\varnothing\in AP(\mathbb{T},\mathbb{R}^+)$ with $-a_i^\varnothing\in\mathcal{R}^+, a^->0$ with $a^-\mu^+<1$, $\check{a}_i, b_{ij}, c_{ij}\in AP(\mathbb{T},\mathcal{A})$, $I_i\in APW^p(\mathbb{T},\mathcal{A})\cap L^\infty(\mathbb{T},\mathcal{A})$, τ_{ij} and $\upsilon_{ij}\in AP(\mathbb{T},\mathbb{R}^+)\cap C^1(\mathbb{T},\Pi)$ with $\tilde{\tau}<1$ and $\tilde{\upsilon}<1$, $i,j\in\Lambda$.

(H_2) There exist positive constants L_j^f and L_j^g such that for all $x,y\in\mathcal{A}$,

$$
\|f_j(x)-f_j(y)\|_{\mathcal{A}}\leq L_j^f\|x-y\|_{\mathcal{A}},\quad \|g_j(x)-g_j(y)\|_{\mathcal{A}}\leq L_j^g\|x-y\|_{\mathcal{A}},
$$

and $f_j(0)=0$, $g_j(0)=0$, $j\in\Lambda$.

(H_3) $\max\limits_{i\in\Lambda}\left\{\frac{2}{a^-}\left(\check{a}_i^+ + \sum\limits_{j=1}^n b_{ij}^+ L_j^f + \sum\limits_{j=1}^n c_{ij}^+ L_j^g\right)\right\} < 1.$

(H_4) For $p \geq 2$,

$$\max_{i\in\Lambda}\Bigg\{12\left(\frac{1}{a^-}\right)^p \frac{1}{1-a^-\mu^+}\Bigg[(\check{a}_i^+)^p + \frac{2e^{a^-\tau}}{1-\tilde{\tau}} \times \left(\sum_{j=1}^n b_{ij}^+ L_j^f\right)^p + \frac{2e^{a^-\upsilon}}{1-\tilde{\upsilon}}\left(\sum_{j=1}^n c_{ij}^+ L_j^g\right)^p\Bigg]\Bigg\} < 1.$$

Let $BUC(\mathbb{T},\mathcal{A}^n)$ be the collection of bounded and uniformly continuous functions from $\mathbb{T}$ to $\mathcal{A}^n$. Then, $BUC(\mathbb{T},\mathcal{A}^n)$ with the norm $\|x\|_0 = \sup\limits_{t\in\mathbb{T}} \|x(t)\|_{\mathcal{A}^n}$ is a Banach space, where $x \in BUC(\mathbb{T},\mathcal{A}^n)$.

Denote $\phi^0 = (\phi_1^0, \phi_2^0, \ldots, \phi_n^0)^T$, where

$$\phi_i^0(t) = \int_{-\infty}^t e_{-a_i^{\varnothing}}(t,\sigma(s)) I_i(s)\Delta s, \quad i \in \Lambda.$$

Take a positive constant $\kappa \geq \|\phi^0\|_0$. Define

$$\Omega = \big\{\phi \in BUC(\mathbb{T},\mathcal{A}^n) : \|\phi - \phi^0\|_0 \leq \kappa\big\}.$$

Then, for $\phi \in \Omega$, one has

$$\|\phi\|_0 \leq \|\phi - \phi^0\|_0 + \|\phi^0\|_0 \leq 2\kappa.$$

Theorem 3.1. *Assume (H_1)-(H_4) hold. Then system (1.1) has a unique W^p-almost periodic solution in the region Ω.*

Proof. It is easy to see that if $x = (x_1, x_2, \ldots, x_n)^T \in \Omega$ is a solution of the integral equation

$$x_i(t) = \int_{-\infty}^t e_{-a_i^{\varnothing}}(t,\sigma(s))\Bigg(-\check{a}_i(s)x_i(s) + \sum_{j=1}^n b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) + \sum_{j=1}^n c_{ij}(s) g_j(x_j(s-\upsilon_{ij}(s))) + I_i(s)\Bigg)\Delta s, \quad i \in \Lambda, \tag{3.1}$$

then x is a solution of system (1.1).

Define a nonlinear operator $T : \Omega \to \mathcal{A}^n$, $\phi \mapsto T\phi = (T_1\phi, T_2\phi, \ldots, T_n\phi)^T$, where

$$(T_i\phi)(t) = \int_{-\infty}^t e_{-a_i^{\varnothing}}(t,\sigma(s))\Bigg(-\check{a}_i(s)\phi_i(s) + \sum_{j=1}^n b_{ij}(s) f_j(\phi_j(s-\tau_{ij}(s))) + \sum_{j=1}^n c_{ij}(s) g_j(\phi_j(s-\upsilon_{ij}(s))) + I_i(s)\Bigg)\Delta s, \quad i \in \Lambda.$$

Next, we will prove that $T\phi$ is well defined. Indeed, we have

$$\|(T_i\phi)(t)\|_{\mathcal{A}}$$

$$
\begin{aligned}
&\leq \bigg\| \int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t,\sigma(s)) \bigg(-\check{a}_i(s)\phi_i(s) + \sum_{j=1}^{n} b_{ij}(s) f_j(\phi_j(s-\tau_{ij}(s))) \\
&\quad + \sum_{j=1}^{n} c_{ij}(s) g_j(\phi_j(s-\upsilon_{ij}(s))) \bigg) \Delta s \bigg\|_{\mathcal{A}} + \bigg\| \int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t,\sigma(s)) I_i(s) \Delta s \bigg\|_{\mathcal{A}} \\
&\leq \frac{1}{a^-} \bigg(\check{a}_i^+ + \sum_{j=1}^{n} b_{ij}^+ L_j^f + \sum_{j=1}^{n} c_{ij}^+ L_j^g \bigg) \|\phi\|_0 + \frac{1}{a^-} \|I_i\|_\infty \\
&< +\infty, \quad j \in \Lambda.
\end{aligned} \tag{3.2}
$$

That is, $T\phi$ is well defined.

We will divide the rest of the proof into four steps.

Step 1, we will prove that $T\phi \in BUC(\mathbb{T}, \mathcal{A}^n)$ for every $\phi \in \Omega$.

In fact, from (3.2), we see that $T\phi$ is bounded on $\mathbb{T}$. So, we only need to show that $T\phi$ is uniformly continuous on $\mathbb{T}$. Noticing that

$$
\|(T_i\phi)^\Delta(t)\|_{\mathcal{A}} \leq \bigg(1 + \frac{a^+}{a^-}\bigg) \bigg[\bigg(\check{a}_i^+ + \sum_{j=1}^{n} b_{ij}^+ L_j^f + \sum_{j=1}^{n} c_{ij}^+ L_j^g \bigg) \|\phi\|_0 + \|I_i\|_\infty \bigg], \quad i \in \Lambda.
$$

According to Corollary 1.68 in [7], we deduce that $(T_i\phi)$ is uniformly continuous on $\mathbb{T}$, $i \in \Lambda$. Therefore, $T\phi \in BUC(\mathbb{T}, \mathcal{A}^n)$.

Step 2, we prove that mapping T is a self-mapping from Ω to Ω.

In fact, for arbitrary $\phi \in \Omega$, by (H_1)-(H_3), we have

$$
\begin{aligned}
&\|T\phi - \phi^0\|_0 \\
&\leq \sup_{t \in \mathbb{R}} \bigg\{ \max_{i \in \Lambda} \bigg\| \int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t,\sigma(s)) \bigg(-\check{a}_i(s)\phi_i(s) + \sum_{j=1}^{n} b_{ij}(s) f_j(\phi_j(s-\tau_{ij}(s))) \\
&\quad + \sum_{j=1}^{n} c_{ij}(s) g_j(\phi_j(s-\upsilon_{ij}(s))) \Delta u \bigg) \Delta s \bigg\|_{\mathcal{A}} \bigg\} \\
&\leq \max_{i \in \Lambda} \bigg\{ \frac{1}{a^-} \bigg(\check{a}_i^+ + \sum_{j=1}^{n} b_{ij}^+ L_j^f + \sum_{j=1}^{n} c_{ij}^+ L_j^g \bigg) \bigg\} \|\phi\|_0 \leq \kappa,
\end{aligned}
$$

which implies that $T\phi \in \Omega$. Consequently, T is self-mapping from Ω to Ω.

Step 3, we will prove that T is contraction mapping.

As a matter of fact, in view of (H_1)-(H_3), for any $\phi, \nu \in \Omega$, we have

$$
\begin{aligned}
\|T\phi - T\nu\|_0 &\leq \sup_{t \in \mathbb{R}} \bigg\{ \max_{i \in \Lambda} \bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t,\sigma(s)) \bigg\| -\check{a}_i(s)(\phi_i(s) - \nu_i(s)) \\
&\quad + \sum_{j=1}^{n} b_{ij}(s) \Big(f_j(\phi_j(s-\tau_{ij}(s))) - f_j(\nu_j(s-\tau_{ij}(s))) \Big) \\
&\quad + \sum_{j=1}^{n} c_{ij}(s) \Big(g_j(\phi_j(s-\upsilon_{ij}(s))) - g_j(\nu_j(s-\upsilon_{ij}(s))) \Big) \bigg\|_{\mathcal{A}} \Delta s \bigg] \bigg\} \\
&\leq \max_{i \in \Lambda} \bigg\{ \frac{1}{a^-} \bigg(\check{a}_i^+ + \sum_{j=1}^{n} b_{ij}^+ L_j^f + \sum_{j=1}^{n} c_{ij}^+ L_j^g \bigg) \bigg\} \|\phi - \nu\|_0.
\end{aligned}
$$

Hence, it follows from this and (H_3) that

$$||T\phi - T\nu||_\infty \le \frac{1}{2}\|\phi - \nu\|_0.$$

That is, T is a contraction mapping. Consequently, system (1.1) has a solution in the region Ω.

Step 4, we show that the solution $x \in \Omega$ is W^p-almost periodic.

Indeed, since $x = (x_1, x_2 \dots, x_n)^T \in \Omega$, x is bounded and uniformly continuous. Hence, for every $\epsilon > 0$, there exists a $\delta \in (0, \epsilon)$ such that for any $t_1, t_2 \in \mathbb{T}$ with $|t_1 - t_2| < \delta$ and $i \in \Lambda$, we have

$$\|x_i(t_1) - x_i(t_2)\|_{\mathcal{A}} < \epsilon. \tag{3.3}$$

Also, for this δ, in view of (H_1) and Lemma 2.3, we see that there exists a common δ-translation number ζ such that

$$\lim_{r\to+\infty}\sup_{\beta\in\mathbb{R}}\left(\frac{1}{r}\int_\beta^{\beta+r}\|I_i(t+\zeta) - I_i(t)\|_{\mathcal{A}}^p\Delta t\right)^{\frac{1}{p}} < \epsilon, \tag{3.4}$$

$$\|b_{ij}(t+\zeta) - b_{ij}(t)\|_{\mathcal{A}} < \epsilon, \tag{3.5}$$

$$\|c_{ij}(t+\zeta) - c_{ij}(t)\|_{\mathcal{A}} < \epsilon, \tag{3.6}$$

$$|a_i^{\varnothing}(t+\zeta) - a_i^{\varnothing}(t)| < \epsilon, \quad \|\check{a}_i(t+\zeta) - \check{a}_i(t)\|_{\mathcal{A}} < \epsilon \tag{3.7}$$

$$|\tau_{ij}(t+\zeta) - \tau_{ij}(t)| < \delta, \quad |\upsilon_{ij}(t+\zeta) - \upsilon_{ij}(t)| < \delta, \tag{3.8}$$

where $i \in \Lambda$. Consequently, from (3.3) and (3.8), we get

$$\begin{cases} \|x_i(t - \tau_{ij}(t+\zeta)) - x_i(t - \tau_{ij}(t))\|_{\mathcal{A}} < \epsilon, \\ \|x_i(t - \upsilon_{ij}(t+\zeta)) - x_i(t - \upsilon_{ij}(t))\|_{\mathcal{A}} < \epsilon, \end{cases} \tag{3.9}$$

where $i \in \Lambda$. Since x is a solution of system (1.1), by (3.1), for $i \in \Lambda$, we have

$$\begin{aligned}
&\|x_i(t+\zeta) - x_i(t)\|_{\mathcal{A}} \\
\le&\left\|\int_{-\infty}^t e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta))(\check{a}_i(s+\zeta)x_i(s+\zeta) - \check{a}_i(s)x_i(s))\Delta s\right\|_{\mathcal{A}} \\
&+\left\|\int_{-\infty}^t e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta))\sum_{j=1}^n\Big(b_{ij}(s+\zeta)f_j(x_j(s+\zeta - \tau_{ij}(s+\zeta)))\right. \\
&\left.- b_{ij}(s)f_j(x_j(s - \tau_{ij}(s)))\Big)\Delta s\right\|_{\mathcal{A}} + \left\|\int_{-\infty}^t e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta))\sum_{j=1}^n\Big(c_{ij}(s+\zeta)\right. \\
&\left.\times g_j(x_j((x_j(s+\zeta - \upsilon_{ij}(s+\zeta))))) - c_{ij}(s)g_j(x_j((x_j(s - \upsilon_{ij}(s)))))\Big)\Delta s\right\|_{\mathcal{A}} \\
&+\left\|\int_{-\infty}^t e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta))(I_i(s+\zeta) - I_i(s))\Delta s\right\|_{\mathcal{A}} \\
&+\left\|\int_{-\infty}^t \Big(e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta)) - e_{-a_i^{\varnothing}}(t, \sigma(s))\Big)\right. \\
&\times\Big(\check{a}_i(s)x_i(s) + \sum_{j=1}^n b_{ij}(s)f_j(x_j(s - \tau_{ij}(s))) + \sum_{j=1}^n c_{ij}(s)
\end{aligned}$$

$$
\begin{aligned}
&\times g_j(x_j((x_j(s-\upsilon_{ij}(s)))))\Big)\Delta s\Big\|_{\mathcal{A}} + \Big\|\int_{-\infty}^{t}\Big(e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \\
&- e_{-a_i^{\varnothing}}(t,\sigma(s))\Big)I_i(s)\Delta s\Big\|_{\mathcal{A}} \\
:=& \sum_{l=1}^{6} F_{li}(t). \qquad (3.10)
\end{aligned}
$$

For $p > 2$, it follows from Hölder's inequality and (H_2) that

$$
\begin{aligned}
F_{2i}(t) \le & \int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \sum_{j=1}^{n} \big\|b_{ij}(s+\zeta)(f_j(x_j(s+\zeta-\tau_{ij}(s+\zeta))) \\
& - f_j(x_j(s-\tau_{ij}(s))))\big\|_{\mathcal{A}}\Delta s + \int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \sum_{j=1}^{n} \big\|(b_{ij}(s+\zeta) \\
& - b_{ij}(s))f_j(x_j(s-\tau_{ij}(s)))\big\|_{\mathcal{A}}\Delta s \\
\le & \left(\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\Delta s\right)^{\frac{p-2}{p}} \Bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \\
& \times \left(\sum_{j=1}^{n} b_{ij}^{+}\big\|f_j(x_j(s+\zeta-\tau_{ij}(s+\zeta))) - f_j(x_j(s-\tau_{ij}(s)))\big\|_{\mathcal{A}}\right)^{\frac{p}{2}}\Delta s\Bigg]^{\frac{2}{p}} \\
& + \left(\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\Delta s\right)^{\frac{p-2}{p}} \Bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \\
& \times \left(\sum_{j=1}^{n} 2L_j^f\kappa\big\|b_{ij}(s+\zeta) - b_{ij}(s)\big\|_{\mathcal{A}}\right)^{\frac{p}{2}}\Delta s\Bigg]^{\frac{2}{p}} \\
\le & \left(\frac{1}{a^-}\right)^{\frac{p-2}{p}} \Bigg\{\Bigg[\left(\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\Delta s\right)^{\frac{1}{2}} \Bigg(\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \\
& \times \left(\sum_{j=1}^{n} b_{ij}^{+}\big\|f_j(x_j(s+\zeta-\tau_{ij}(s+\zeta))) - f_j(x_j(s-\tau_{ij}(s)))\big\|_{\mathcal{A}}\right)^{p}\Delta s\Bigg)^{\frac{1}{2}}\Bigg]^{\frac{2}{p}} \\
& + \Bigg[\left(\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\Delta s\right)^{\frac{1}{2}} \Bigg(\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \\
& \times \left(\sum_{j=1}^{n} 2L_j^f\kappa\big\|b_{ij}(s+\zeta) - b_{ij}(s)\big\|_{\mathcal{A}}\right)^{p}\Delta s\Bigg)^{\frac{1}{2}}\Bigg]^{\frac{2}{p}}\Bigg\} \\
\le & \left(\frac{1}{a^-}\right)^{\frac{p-1}{p}} \Bigg\{\Bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\Bigg(\sum_{j=1}^{n} b_{ij}^{+}L_j^f\big\|x_j(s+\zeta-\tau_{ij}(s+\zeta)) \\
& - x_j(s-\tau_{ij}(s))\big\|_{\mathcal{A}}\Bigg)^{p}\Delta s\Bigg]^{\frac{1}{p}} + \Bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta)) \\
& \times \left(\sum_{j=1}^{n} 2L_j^f\kappa\big\|b_{ij}(s+\zeta) - b_{ij}(s)\big\|_{\mathcal{A}}\right)^{p}\Delta s\Bigg]^{\frac{1}{p}}\Bigg\}, \qquad (3.11)
\end{aligned}
$$

for $i \in \Lambda$. In a similar way, for $i \in \Lambda$, by Hölder's inequality and (H_2), one has

$$\begin{aligned}F_{1i}(t) \leq &\left(\frac{1}{a^-}\right)^{\frac{p-1}{p}} \Bigg\{ \check{a}_i^+ \bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta)) \|x_i(s+\sigma) - x_i(s)\|_{\mathcal{A}}^p \Delta s \bigg]^{\frac{1}{p}} \\ &+ 2\kappa \bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta)) \|\check{a}_i(s+\sigma) - \check{a}_i(s)\|_{\mathcal{A}}^p \Delta s \bigg]^{\frac{1}{p}} \Bigg\},\end{aligned} \tag{3.12}$$

$$\begin{aligned}F_{3i}(t) \leq &\left(\frac{1}{a^-}\right)^{\frac{p-1}{p}} \Bigg\{ \bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta)) \bigg(\sum_{j=1}^{n} c_{ij}^+ L_j^g \big\| x_j(s+\zeta - \upsilon_{ij}(s+\zeta)) \\ &- x_j(s - \upsilon_{ij}(s)) \big\|_{\mathcal{A}} \bigg)^p \Delta s \bigg]^{\frac{1}{p}} + \bigg[\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta)) \\ &\times \bigg(\sum_{j=1}^{n} 2L_j^g \kappa \big\| c_{ij}(s+\zeta) - c_{ij}(s) \big\|_{\mathcal{A}} \bigg)^p \Delta s \bigg]^{\frac{1}{p}} \Bigg\}\end{aligned} \tag{3.13}$$

and

$$F_{4i}(t) \leq \left(\frac{1}{a^-}\right)^{\frac{p-1}{p}} \left(\int_{-\infty}^{t} e_{-a_i^{\varnothing}}(t+\zeta, \sigma(s+\zeta)) \|I_i(s+\zeta) - I_i(s)\|_{\mathcal{A}}^p \Delta s \right)^{\frac{1}{p}}. \tag{3.14}$$

Moreover, by Lemma 2.4, (3.7), (H_1) and (H_2), for $i \in \Lambda$, we derive that

$$\begin{aligned}F_{5i}(t) \leq &\bigg(\check{a}_i^+ + \sum_{j=1}^{n} 2b_{ij}^+ L_j^f \kappa + \sum_{j=1}^{n} 2c_{ij}^+ L_j^g \kappa \bigg) \int_{-\infty}^{t} \bigg| \int_{t}^{\sigma(s)} e_{-a_i^{\varnothing}}(t, \sigma(\theta)) \\ &\times (a_i^{\varnothing}(\theta+\zeta) - a_i^{\varnothing}(\theta)) e_{-a_i^{\varnothing}}(\theta+\zeta, \sigma(s+\zeta)) \Delta\theta \bigg| \Delta s \\ \leq &\bigg(\check{a}_i^+ + \sum_{j=1}^{n} 2b_{ij}^+ L_j^f \kappa + \sum_{j=1}^{n} 2c_{ij}^+ L_j^g \kappa \bigg) \int_{-\infty}^{t} \bigg| \int_{t}^{\sigma(s)} e_{-a_i^{\varnothing}}(t, \sigma(\theta)) (a_i^{\varnothing}(\theta+\zeta) \\ &- a_i^{\varnothing}(\theta)) \Delta\theta \bigg| \Delta s \\ \leq &\frac{\epsilon}{(a^-)^2} \bigg(\check{a}_i^+ + \sum_{j=1}^{n} 2b_{ij}^+ L_j^f \kappa + \sum_{j=1}^{n} 2c_{ij}^+ L_j^g \kappa \bigg)\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}F_{6i}(t) \leq &\int_{-\infty}^{t} \bigg| \int_{t}^{\sigma(s)} e_{-a_i^{\varnothing}}(t, \sigma(\theta)) (a_i^{\varnothing}(\theta+\zeta) - a_i^{\varnothing}(\theta)) \\ &\times e_{-a_i^{\varnothing}}(\theta+\zeta, \sigma(s+\zeta)) \Delta\theta \bigg| \|I_i(s)\|_{\mathcal{A}} \Delta s \\ \leq &\int_{-\infty}^{t} \bigg| \int_{t}^{\sigma(s)} e_{-a_i^{\varnothing}}(t, \sigma(\theta)) (a_i^{\varnothing}(\theta+\zeta) - a_i^{\varnothing}(\theta)) \Delta\theta \bigg| \|I_i(s)\|_{\mathcal{A}} \Delta s\end{aligned}$$

$$\leq \frac{\epsilon}{(a^-)^2}\|I_i\|_\infty. \tag{3.16}$$

Therefore, together with Lemma 2.5, (3.7) and (3.12), we deduce that

$$\begin{aligned}
&\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}F_{1i}^p(t)\Delta t\\
\leq&\left(\frac{1}{a^-}\right)^{p-1}\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\left[\check{a}_i^+\left(\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\|x_i(s+\sigma)\right.\right.\\
&\left.-x_i(s)\|_{\mathcal{A}}^p\Delta s\right)^{\frac{1}{p}}+2\kappa\left(\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\right.\\
&\left.\left.\times\|\check{a}_i(s+\sigma)-\check{a}_i(s)\|_{\mathcal{A}}^p\Delta s\right)^{\frac{1}{p}}\right]^p\Delta t\\
\leq&2\left(\frac{1}{a^-}\right)^{p-1}\left[(\check{a}_i^+)^p\left(\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\|x_i(s+\sigma)\right.\right.\\
&\left.-x_i(s)\|_{\mathcal{A}}^p\Delta s\Delta t\right)+(2\kappa)^p\left(\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\right.\\
&\left.\left.\times\|\check{a}_i(s+\sigma)-\check{a}_i(s)\|_{\mathcal{A}}^p\Delta s\Delta t\right)\right]\\
\leq&2\left(\frac{1}{a^-}\right)^{p-1}\left[(\check{a}_i^+)^p\left(\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\|x(s+\sigma)\right.\right.\\
&\left.\left.-x(s)\|_{\mathcal{A}^n}^p\Delta s\right)+\frac{1}{a^-}(2\kappa\epsilon)^p\right]\\
\leq&2\left(\frac{1}{a^-}\right)^p\frac{(\check{a}_i^+)^p}{(1-a^-\mu^+)}\|x(t+\sigma)-x(t)\|_{W^p}^p+\rho_{1i},
\end{aligned}\tag{3.17}$$

where $\rho_{1i}:=2\left(\frac{1}{a^-}\right)^p(2\kappa\epsilon)^p$. By Lemma 2.5, (3.5), (3.9) and (3.11), we derive that

$$\begin{aligned}
&\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}F_{2i}^p(t)\Delta t\\
\leq&\left(\frac{1}{a^-}\right)^{p-1}\left\{\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\left[\left(\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\left(\sum_{j=1}^n b_{ij}^+L_j^f\right.\right.\right.\right.\\
&\left.\left.\times\left\|x_j(s+\zeta-\tau_{ij}(s+\zeta))-x_j(s-\tau_{ij}(s))\right\|_{\mathcal{A}}\right)^p\Delta s\right)^{\frac{1}{p}}\\
&\left.\left.+\left(\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\left(\sum_{j=1}^n 2L_j^f\kappa\left\|b_{ij}(s+\zeta)-b_{ij}(s)\right\|_{\mathcal{A}}\right)^p\Delta s\right)^{\frac{1}{p}}\right]^p\Delta t\right\}\\
\leq&2\left(\frac{1}{a^-}\right)^{p-1}\left[\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_\beta^{\beta+r}\int_{-\infty}^t e_{-a_i^\varnothing}(t+\zeta,\sigma(s+\zeta))\right.\\
&\times\left(\sum_{j=1}^n b_{ij}^+L_j^f\left\|x_j(s+\zeta-\tau_{ij}(s+\zeta))-x_j(s-\tau_{ij}(s))\right\|_{\mathcal{A}}\right)^p\Delta s\Delta t
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}\int_{-\infty}^{t}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\\
&\times\bigg(\sum_{j=1}^{n}2L_j^f\kappa\big\|b_{ij}(s+\zeta)-b_{ij}(s)\big\|_{\mathcal{A}}\bigg)^p\Delta s\Delta t\bigg]\\
\le&2\Big(\frac{1}{a^-}\Big)^{p-1}\bigg\{\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}\bigg[2\int_{-\infty}^{t}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\\
&\times\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\|x_j(s+\zeta-\tau_{ij}(s+\zeta))-x_j(s-\tau_{ij}(s+\zeta))\|_{\mathcal{A}}\bigg)^p\Delta s\\
&+2\int_{-\infty}^{t}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\\
&\times\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\|x_j(s-\tau_{ij}(s+\zeta))-x_j(s-\tau_{kl}(s))\|_{\mathcal{A}}\bigg)^p\Delta s\bigg]\Delta t\\
&+\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}\int_{-\infty}^{t}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\bigg(\sum_{j=1}^{n}2L_j^f\kappa\epsilon\bigg)^p\Delta s\Delta t\bigg\}\\
\le&2\Big(\frac{1}{a^-}\Big)^{p-1}\bigg\{\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}\bigg[\frac{2}{1-\tilde{\tau}}\int_{-\infty}^{t-\tau_{ij}(t+\zeta)}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(u+\tau+\zeta))\\
&\times\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\|x_j(u+\zeta)-x_j(u)\|_{\mathcal{A}}\bigg)^p\Delta u+2\int_{-\infty}^{t}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\\
&\times\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\epsilon\bigg)^p\Delta s\bigg]\Delta t+\frac{1}{a^-}\bigg(\sum_{j=1}^{n}2L_j^f\kappa\epsilon\bigg)^p\bigg\}\\
\le&2\Big(\frac{1}{a^-}\Big)^{p-1}\bigg\{\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}\bigg[\frac{2e^{a^-\tau}}{1-\tilde{\tau}}\int_{-\infty}^{t}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(u+\zeta))\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\\
&\times\|x_j(s+\zeta)-x_j(s)\|_{\mathcal{A}}\bigg)^p\Delta s\bigg]\Delta t+\frac{2}{a^-}\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\epsilon\bigg)^p+\frac{1}{a^-}\bigg(\sum_{j=1}^{n}2L_j^f\kappa\epsilon\bigg)^p\bigg\}\\
\le&2\Big(\frac{1}{a^-}\Big)^{p-1}\bigg[\frac{2e^{a^-\tau}}{1-\tilde{\tau}}\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\bigg)^p\bigg(\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}\int_{-\infty}^{t}e_{-a_i^{\varnothing}}(t+\zeta,\sigma(u+\zeta))\\
&\times\|x(s+\zeta)-x(s)\|_{\mathcal{A}^n}^p\Delta s\Delta t\bigg)+\frac{2}{a^-}\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\epsilon\bigg)^p+\frac{1}{a^-}\bigg(\sum_{j=1}^{n}L_j^f\epsilon\bigg)^p\bigg]\\
\le&\Big(\frac{1}{a^-}\Big)^{p}\frac{4e^{a^-\tau}}{(1-\tilde{\tau})(1-a^-\mu^+)}\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\bigg)^p\|x(t+\zeta)-x(t)\|_{W^p}^p+\rho_{2i},
\end{aligned}\tag{3.18}
$$

where

$$
\rho_{2i}:=2\Big(\frac{1}{a^-}\Big)^p\bigg[2\bigg(\sum_{j=1}^{n}b_{ij}^+L_j^f\bigg)^p+\bigg(\sum_{j=1}^{n}2L_j^f\kappa\bigg)^p\bigg]\epsilon^p.
$$

Similarly, by Lemma 2.5, (3.13) and (3.6), we can get

$$\begin{aligned}&\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}F_{3i}^{p}(t)\Delta t\\ \leq&\Big(\frac{1}{a^{-}}\Big)^{p}\frac{4e^{a^{-}\upsilon}}{(1-\tilde{\upsilon})(1-a^{-}\mu^{+})}\Big(\sum_{j=1}^{n}c_{ij}^{+}L_{j}^{g}\Big)^{p}\|x(t+\zeta)-x(t)\|_{W^{p}}^{p}+\rho_{3i},\end{aligned}\tag{3.19}$$

where

$$\rho_{3i}:=2\Big(\frac{1}{a^{-}}\Big)^{p}\Big[2\Big(\sum_{j=1}^{n}c_{ij}^{+}L_{j}^{g}\Big)^{p}+\Big(\sum_{j=1}^{n}2L_{j}^{g}\kappa\Big)^{p}\Big]\epsilon^{p}.$$

Based on Lemma 2.5, (3.4), (3.7), (3.16), (3.14) and (3.15), for $i\in\Lambda$, we can obtain

$$\begin{aligned}&\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}F_{4i}^{p}(t)dt\\ \leq&\Big(\frac{1}{a^{-}}\Big)^{p-1}\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}\int_{-\infty}^{t}e_{-a_{i}^{\varnothing}}(t+\zeta,\sigma(s+\zeta))\|I_{i}(s+\sigma)-I_{i}(s)\|_{\mathcal{A}}^{p}\Delta t\Delta s\\ \leq&\Big(\frac{1}{a^{-}}\Big)^{p}\frac{1}{1-a^{-}\mu^{+}}\epsilon_{i}^{p}:=\rho_{4i},\end{aligned}\tag{3.20}$$

$$\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}F_{5i}^{p}(t)\Delta t\leq\left(\frac{1}{a^{-}}\right)^{2p}\left(\breve{a}_{i}^{+}+\sum_{j=1}^{n}2b_{ij}^{+}L_{j}^{f}\kappa+\sum_{j=1}^{n}2c_{ij}^{+}L_{j}^{g}\kappa\right)^{p}\epsilon^{p}:=\rho_{5i}\tag{3.21}$$

and

$$\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}F_{6i}^{p}(t)\Delta t\leq\left(\frac{1}{a^{-}}\right)^{2p}\|I_{i}\|_{\infty}^{p}\epsilon^{p}:=\rho_{i6}.\tag{3.22}$$

Consequently, from (3.10) and (3.17)-(3.22) it follows that

$$\begin{aligned}\|x(t+\zeta)-x(t)\|_{W^{p}}^{p}\leq&\max_{i\in\Lambda}\left\{6\sum_{l=1}^{6}\left(\lim_{r\to+\infty}\sup_{\beta\in\mathbb{T}}\frac{1}{r}\int_{\beta}^{\beta+r}F_{6i}^{p}(t)\Delta t\right)\right\}\\ \leq&\rho+\gamma\|x(t+\zeta)-x(t)\|_{W^{p}}^{p},\end{aligned}$$

where

$$\begin{aligned}\rho=&6\sum_{l=1}^{6}\max_{i\in\Lambda}\{\rho_{li}\}\\ =&\max_{i\in\Lambda}\left\{6\Big(\frac{1}{a^{-}}\Big)^{p}\Big[2(2\kappa)^{p}+4\Big(\sum_{j=1}^{n}b_{ij}^{+}L_{j}^{f}\Big)^{p}+2\Big(\sum_{j=1}^{n}2L_{j}^{f}\kappa\Big)^{p}+4\Big(\sum_{j=1}^{n}c_{ij}^{+}L_{j}^{g}\Big)^{p}\right.\\ &+2\Big(\sum_{j=1}^{n}2L_{j}^{g}\kappa\Big)^{p}+\frac{1}{1-a^{-}\mu^{+}}+\Big(\frac{1}{a^{-}}\Big)^{p}\Big(\breve{a}_{i}^{+}+\sum_{j=1}^{n}2b_{ij}^{+}L_{j}^{f}\kappa+\sum_{j=1}^{n}2c_{ij}^{+}L_{j}^{g}\kappa\Big)^{p}\end{aligned}$$

$$+\left(\frac{1}{a^-}\right)^p\frac{1}{1-a^-\mu^+}\|I_i\|_\infty^p\Bigg]\Bigg\}\epsilon^p$$

and

$$\gamma=\max_{i\in\Lambda}\Bigg\{12\left(\frac{1}{a^-}\right)^p\frac{1}{1-a^-\mu^+}\Bigg[(\check{a}_i^+)^p+\frac{2e^{a^-\tau}}{1-\tilde{\tau}}\left(\sum_{j=1}^n b_{ij}^+L_j^f\right)^p$$
$$+\frac{2e^{a^-\upsilon}}{1-\tilde{\upsilon}}\left(\sum_{j=1}^n c_{ij}^+L_j^g\right)^p\Bigg]\Bigg\}.$$

By (H_4), we have $\gamma<1$. Thus, one has

$$\|x(t+\zeta)-x(t)\|_{W^p}^p\leq\frac{\rho}{1-\gamma}.$$

Consequently, $x\in W^p(\mathbb{T},\mathcal{A}^n)$.

For $p=2$, similar to the proof of the case of $p>2$, one can obtain

$$\|x(t+\zeta)-x(t)\|_{W^p}^2\leq\tilde{\rho}+\tilde{\gamma}\|x(t+\zeta)-x(t)\|_{W^p}^2,$$

where

$$\tilde{\rho}=6\sum_{l=1}^6\max_{i\in\Lambda}\{\tilde{\rho}_{li}\}$$
$$=\max_{i\in\Lambda}\Bigg\{6\left(\frac{1}{a^-}\right)^2\Bigg[8\kappa^2+4\left(\sum_{j=1}^n b_{ij}^+L_j^f\right)^2+2\left(\sum_{j=1}^n 2L_j^f\kappa\right)^2+4\left(\sum_{j=1}^n c_{ij}^+L_j^g\right)^2$$
$$+2\left(\sum_{j=1}^n 2L_j^g\kappa\right)^2+\frac{1}{1-a^-\mu^+}+\left(\frac{1}{a^-}\right)^2\left(\check{a}_i^++\sum_{j=1}^n 2b_{ij}^+L_j^f\kappa+\sum_{j=1}^n 2c_{ij}^+L_j^g\kappa\right)^2$$
$$+\left(\frac{1}{a^-}\right)^2\frac{1}{1-a^-\mu^+}\|I_i\|_\infty^2\Bigg]\Bigg\}\epsilon^2$$

and

$$\tilde{\gamma}=\max_{i\in\Lambda}\Bigg\{12\left(\frac{1}{a^-}\right)^2\frac{1}{1-a^-\mu^+}\Bigg[(\check{a}_i^+)^2+\frac{2e^{a^-\tau}}{1-\tilde{\tau}}\left(\sum_{j=1}^n b_{ij}^+L_j^f\right)^2$$
$$+\frac{2e^{a^-\upsilon}}{1-\tilde{\upsilon}}\left(\sum_{j=1}^n c_{ij}^+L_j^g\right)^2\Bigg]\Bigg\}.$$

By (H_4), we have $\tilde{\gamma}<1$. Thus, one has

$$\|x(t+\zeta)-x(t)\|_{W^2}^2\leq\frac{\tilde{\rho}}{1-\tilde{\gamma}}.$$

Hence, $x\in W^2(\mathbb{T},\mathcal{A}^n)$. The proof is complete. $\square$

Definition 3.1 ([26]). Let x be a solution of system (1.1) with initial value φ and y be an arbitrary solution of system (1.1) with initial value ψ, respectively. If there exist positive constants λ and M such that

$$\|x(t)-y(t)\|_{\mathcal{A}^n} \leq Me_{\ominus\lambda}(t,0)\|\varphi-\psi\|_\tau,\ t\in[0,+\infty)_{\mathbb{T}},$$

where $\|\varphi-\psi\|_\tau = \sup\limits_{t\in[-\tau,0]_{\mathbb{T}}} \|\varphi(t)-\psi(t)\|_{\mathcal{A}^n}$. Then the solution x of system (1.1) is said to be globally exponentially stable.

Using the same proof method as Theorem 2 in [26], one can prove that

Theorem 3.2. *Assume that* (H_1)-(H_3) *hold, then every solution of system* (1.1) *is globally exponentially stable.*

As a direct result of Theorems 3.1 and 3.2, we have

Corollary 3.1. *Assume that* (H_1)-(H_4) *hold, then system* (1.1) *has a unique* W^p*-almost periodic solution that is globally exponentially stable.*

4. A numerical example

Example 4.1. In system (1.1), let $m=3$, $n=2$ and take the coefficients as follows:

$$\begin{aligned}
f_j(x) =& \frac{1}{75}e_0\sin(x^0+2x^1)+\frac{1}{60}e_1\sin(x^1+x^2-x^{12})\\
&+\frac{2}{131}e_2\sin(x^0+2x^2)+\frac{1}{80}e_{12}\sin(3x^1),\\
g_j(x) =& \frac{3}{100}e_0\sin(x^0-x^2)+\frac{2}{121}e_1\sin(x^1-x^{12})\\
&+\frac{2}{117}e_2\sin(x^0-x^2)+\frac{6}{201}e_{12}\sin(x^1-x^3),\\
a_1(t) =& e_0\frac{1}{2}|\sin(2t)|+\frac{\sqrt{3}}{100}e_1\sin^2(4t)+\frac{3}{100}e_2|\cos(\sqrt{3}t)|+\frac{1}{135}e_{12}|\cos(\sqrt{5}t)|,\\
a_2(t) =& e_0\left(1-\frac{1}{3}\sin(2t)\right)+\frac{\sqrt{3}}{200}e_1\cos^2(2t)+\frac{1}{150}e_2\sin^4(\sqrt{2}t)+\frac{3}{200}e_{12}|\cos(\sqrt{7}t)|,\\
b_{11}(t) =& \frac{1}{100}e_0\sin(2t)+\frac{\sqrt{3}}{300}e_1\sin(4t)-\frac{\sqrt{3}}{305}e_2\cos(t)+\frac{\sqrt{2}}{201}e_{12}\sin(\sqrt{3}t),\\
b_{12}(t) =& \frac{1}{50}e_0\sin(\sqrt{2}t)+\frac{\sqrt{3}}{300}e_1\cos(4t)-\frac{\sqrt{3}}{315}e_2\sin(6t)+\frac{3}{100}e_{12}\sin(\sqrt{5}t),\\
b_{21}(t) =& \frac{13}{1000}e_0\cos(3t)+\frac{\sqrt{3}}{250}e_1\sin(4t)+\frac{3}{250}e_2\cos(\sqrt{3}t)+\frac{1}{100}e_{12}\cos(\sqrt{5}t),\\
b_{22}(t) =& \frac{9}{500}e_0\cos(3t)+\frac{17}{1000}e_1\sin(4t)+\frac{3}{250}e_2\cos(\sqrt{3}t)+\frac{2}{125}e_{12}\cos(\sqrt{5}t),\\
c_{11}(t) =& \frac{13}{500}e_0\cos(\sqrt{3}t)+\frac{3}{125}e_1\sin(4t)+\frac{2}{121}e_2\cos(\sqrt{3}t)+\frac{2}{125}e_{12}\cos(\sqrt{5}t),\\
c_{12}(t) =& \frac{3}{125}e_0\sin(\sqrt{3}t)+\frac{\sqrt{3}}{126}e_1\cos(2t)+\frac{1}{50}e_2\sin(\sqrt{2}t)+\frac{5}{251}e_{12}\cos(\sqrt{7}t),\\
c_{21}(t) =& \frac{\sqrt{3}}{143}e_0\cos(2t)+\frac{1}{66}e_1\cos(\sqrt{2}t)+\frac{1}{50}e_2\cos(\sqrt{3}t)+\frac{3}{100}e_{12}\sin(\sqrt{7}t),
\end{aligned}$$

$$c_{22}(t) = \frac{1}{50}e_0\cos(3t) + \frac{1}{55}e_1\sin(4t) + \frac{1}{63}e_2\cos(\sqrt{3}t) + \frac{1}{123}e_{12}\cos(\sqrt{5}t),$$
$$I_1(t) = \sqrt{2}e_0\sin 8t + \frac{1}{3}e_1e^{-|t|} + e_2\cos(\frac{1}{2}t) + \frac{2}{5}e_{12}\cos(9t),$$
$$I_2(t) = \frac{1}{2}e_0\sin 8t + \frac{5}{2}e_1\sin 7t - \frac{5}{6}e_{12}\cos 2t.$$

If $\mathbb{T} = \mathbb{R}$, then we take

$$\tau_{11}(t) = 2\sin^2(\frac{1}{2}t),\ \tau_{12}(t) = \frac{2}{17}\cos^2 t,\ \tau_{21}(t) = \frac{1}{3}\sin^2(\frac{1}{2}t),\ \tau_{22}(t) = \frac{2}{9}\sin^4(\frac{1}{2}t),$$
$$\upsilon_{11}(t) = \left|\cos(\frac{2}{3}t)\right|,\ \upsilon_{12}(t) = \frac{1}{19}\sin^2(\frac{1}{3}t),\ \upsilon_{21}(t) = \frac{1}{4}\sin^2(\frac{1}{5}t),\ \upsilon_{22}(t) = \frac{2}{5}\cos^4(\frac{1}{7}t),$$

and if $\mathbb{T} = \mathbb{Z}$, then we take

$$\tau_{11}(t) = \frac{1}{4}\sin\left(\pi t + \frac{\pi}{2}\right),\ \tau_{12}(t) = \frac{1}{5}\cos(\pi t),\ \tau_{21}(t) = \frac{1}{4}\sin(\frac{\pi}{2}t),$$
$$\tau_{22}(t) = \frac{2}{9}\cos(\frac{\pi}{2}t),\ \upsilon_{11}(t) = \frac{2}{13}\cos(\pi t),\ \upsilon_{12}(t) = \frac{1}{6}\sin\left(\pi t + \frac{\pi}{2}\right),$$
$$\upsilon_{21}(t) = \frac{1}{16}\cos(2\pi t),\ \upsilon_{22}(t) = \frac{2}{9}\sin(\frac{\pi}{2}t).$$

By computing,

$$L_1^f = L_2^f = \frac{1}{20}, L_1^g = L_2^g = \frac{3}{50}, a^- = \frac{1}{2}, \check{a}_1 = \frac{3}{100}, \check{a}_2 = \frac{3}{200}, b_{11}^+ = \frac{1}{100}, b_{12}^+ = \frac{3}{100},$$
$$b_{21}^+ = \frac{13}{1000}, b_{22}^+ = \frac{17}{1000}, c_{11}^+ = \frac{13}{500}, c_{12}^+ = \frac{3}{125}, c_{21}^+ = \frac{3}{100}, c_{22}^+ = \frac{1}{50}.$$

Thus, we obtain that

$$\max_{1\le i\le 2}\left\{\frac{2}{a^-}\left(\check{a}_i^+ + \sum_{j=1}^{2} b_{ij}^+L_j^f + \sum_{j=1}^{2} c_{ij}^+L_j^g\right)\right\} = 0.14 < 1.$$

When $\mathbb{T} = \mathbb{R}$, take $p = 2$, it is easy to obtain that $\tau = 2$,$\tilde{\tau} = \frac{1}{2}$, $\upsilon = 1$,$\tilde{\upsilon} = \frac{2}{3}$ and

$$\max_{1\le i\le 2}\Bigg\{12\left(\frac{1}{a^-}\right)^p\frac{1}{1-a^-\mu^+}\Bigg[(\check{a}_i^+)^p + \frac{2e^{a^-\tau}}{1-\tilde{\tau}}$$
$$\times\left(\sum_{j=1}^{n} b_{ij}^+L_j^f\right)^p + \frac{2e^{a^-\upsilon}}{1-\tilde{\upsilon}}\left(\sum_{j=1}^{n} c_{ij}^+L_j^g\right)^p\Bigg]\Bigg\} \approx 0.0496 < 1.$$

When $\mathbb{T} = \mathbb{Z}$, take $p = 3$, it is easy to obtain that $\tau = \frac{1}{4}$,$\tilde{\tau} = \frac{1}{2}$, $\upsilon = \frac{1}{6}$,$\tilde{\upsilon} = \frac{1}{3}$ and

$$\max_{1\le i\le 2}\Bigg\{12\left(\frac{1}{a^-}\right)^p\frac{1}{1-a^-\mu^+}\Bigg[(\check{a}_i^+)^p + \frac{2e^{a^-\tau}}{1-\tilde{\tau}}$$
$$\times\left(\sum_{j=1}^{n} b_{ij}^+L_j^f\right)^p + \frac{2e^{a^-\upsilon}}{1-\tilde{\upsilon}}\left(\sum_{j=1}^{n} c_{ij}^+L_j^g\right)^p\Bigg]\Bigg\} \approx 0.0052 < 1.$$

Thus, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, all of the conditions of Corollary 3.1 are satisfied. Hence, system (1.1) has a unique W^p-almost periodic solution, which is globally exponentially stable (see Figures 1-6).

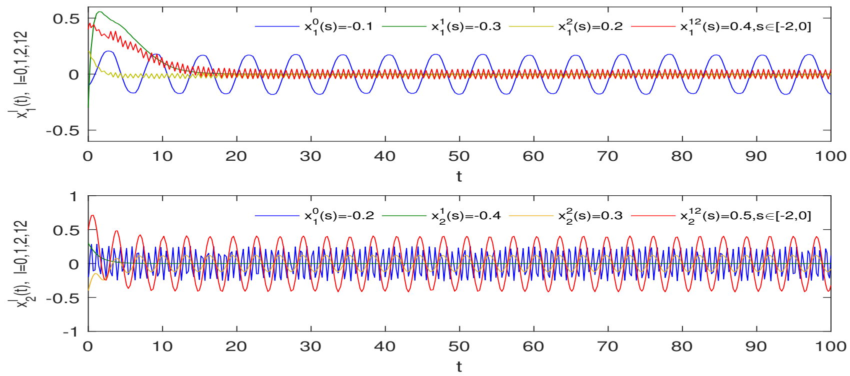

Figure 1. $\mathbb{T} = \mathbb{R}$, states $x_1^l(t)$ and $x_2^l(t)$ of system (1.1) with different initial values.

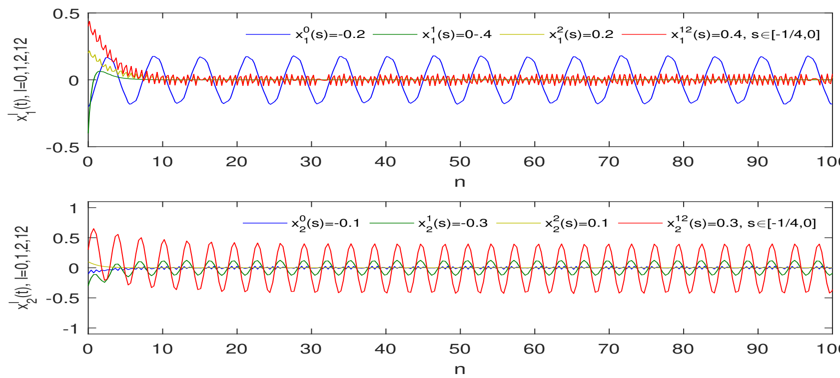

Figure 2. $\mathbb{T} = \mathbb{Z}$, states $x_1^l(t)$ and $x_2^l(t)$ of system (1.1) with different initial values.

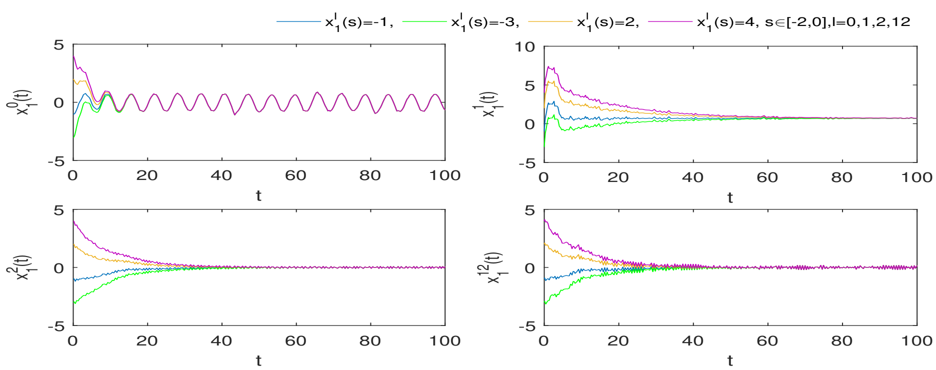

Figure 3. $\mathbb{T} = \mathbb{R}$, the global exponential stability of states $x_1^0(t), x_1^1(t), x_1^2(t)$ and $x_1^{12}(t)$ of system (1.1) with different initial values.

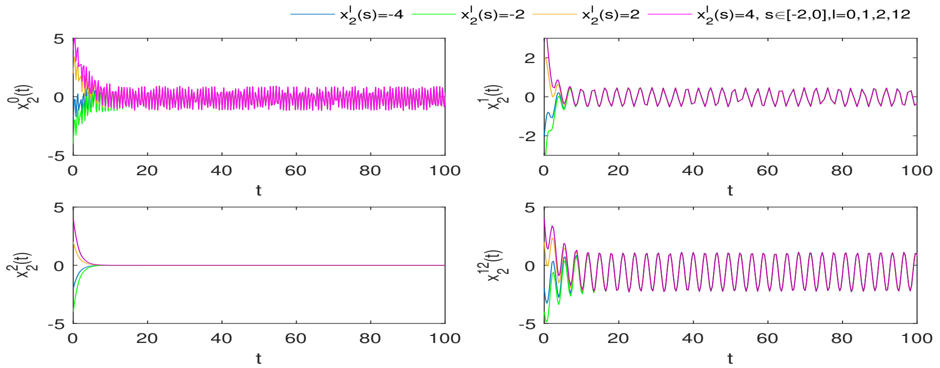

Figure 4. $\mathbb{T} = \mathbb{R}$, the global exponential stability of states $x_2^0(t), x_2^1(t), x_2^2(t)$ and $x_2^{12}(t)$ of system (1.1) with different initial values.

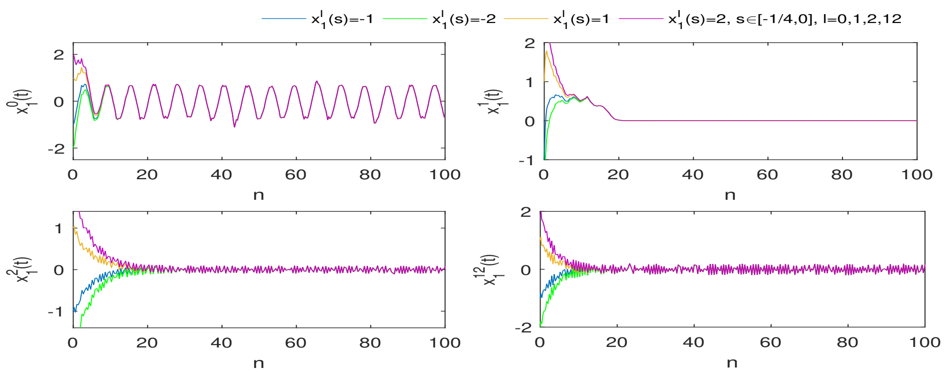

Figure 5. $\mathbb{T} = \mathbb{Z}$, the global exponential stability of states $x_1^0(t), x_1^1(t), x_1^2(t)$ and $x_1^{12}(t)$ of system (1.1) with different initial values.

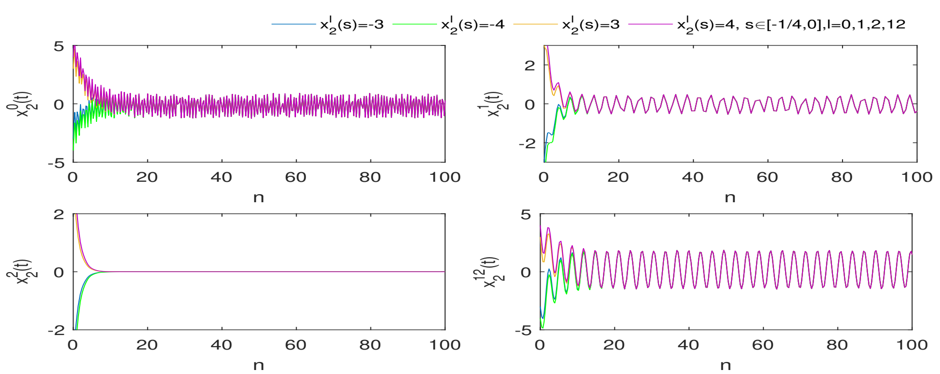

Figure 6. $\mathbb{T} = \mathbb{Z}$, the global exponential stability of states $x_2^0(t), x_2^1(t), x_2^2(t)$ and $x_2^{12}(t)$ of system (1.1) with different initial values.

5. Conclusion

In this paper, the concept of Weyl almost periodic on time scales is proposed. Taking a Clifford-valued neural network with time-varying delays on time scales as an example of dynamic equations on time scales, the existence and global exponential stability of Weyl almost periodic solutions of the network on time scales are established. Even when the system we consider is a real-value system, our results are brand-new. In addition, the method of this paper can also be used to study the existence of Weyl almost periodic solutions of other types of dynamic equations on time scales.

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