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STABILITY SWITCHING CURVES AND HOPF BIFURCATION ON A THREE SPECIES FOOD CHAIN WITH TWO DELAYS*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract A three species food chain with two time delays and double Holling type-II functional responses is investigated. The conditions for the existence of positive equilibrium and Hopf bifurcation are presented. The stability area of positive equilibrium is surrounded by coordinate axis and stability switching curves. By using the theory of Hassard, Hopf bifurcation directions are determined analytically. Numerical simulations are presented on the frontier of stability to explain and support the analytic results.

Keywords Three species food chain, two time delays, stability switching curves, Hopf bifurcation.

MSC(2010) 34A34, 34C23, 34C25, 34D20, 37G15.

1. Introduction

The models of three species food chain have long attracted attention because of their complex dynamical behaviors [3–6, 10, 12, 13, 15, 18, 20, 22–26]. Hastings and Powell [10] studied a three species food chain with double Holling type-II functional responses,

$$\begin{cases} \dot{X} = RX(1 - \frac{X}{K}) - C_1F_1(X)Y, \\ \dot{Y} = F_1(X)Y - F_2(Y)Z - D_1Y, \\ \dot{Z} = C_2F_2(Y)Z - D_2Z, \end{cases} \tag{1.1}$$

where X, Y, Z express prey, mid-level predator and top-level predator respectively, $F_i(U) = \frac{A_iU}{B_i+U}, i = 1, 2$ are Holling type II functional responses, K and R denote the carrying capacity and intrinsic growth rate of species X, D_1 and D_2 denote the death rates for Y and Z, C_1 and C_2 are the conversion rates of prey-to-predator for Y and Z, and the constants coefficients are all positive.

Since then, literature has studied the dynamical behaviors of this system. For example, Lv and Zhao [13], based on biologically feasible parameters, obtain bifur-

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cation diagrams and showed that food chain system had rich and complex features. Lonngren et al. [12] studied the dynamical behaviors of a food chain system and showed that the system could be synchronized with another food chain system. Varriale and Gomes [23] got the asymptotic states from numerical integration and by using the embedding procedure, local Lyapunov exponents were obtained. Gakkhar and Singh [5] investigated the model involving another predator of top prey. Fang and Lin [4] studied (1.1) with one time delay and chaotic behaviors were observed. Yang et al. [26] studied a periodically kicked three species food chain with one time delay and rank-one chaos was observed. Cui and Yan [3] investigated the stability and bifurcation behaviors on a two delayed three-species food chain with Lotka-Volterra functional response, they dealt with the delays as $\tau = \tau_1 + \tau_2$.

In reality, multiple delays arise in many subject such as biology, ecology, epidemiology, physics, chemistry, and engineering disciplines. However, systems with multiple delays are of great interest [1, 16, 17, 19, 21, 27]. The method of stability switching is an effective method to understand the stable region for system with two delays, and realize the bifurcation behaviors [2, 7, 8, 11]. Recently, Matsumto and Szidarovszky [14] studied a two species two delayed Lotka-Volterra competition model with some symmetries, and the stability switching curve was investigated.

In the present paper, considering that the mid-level predator and the top-level predator take time τ_1 and τ_2 to convert the food into their growth, we devote our attention to the bifurcating phenomenons of above system with two time delays described by

$$\begin{cases} \dot{X} = RX(1-\frac{X}{K}) - C_1F_1(X)Y, \\ \dot{Y} = F_1(X(T-\tau_1))Y - F_2(Y)Z - D_1Y, \\ \dot{Z} = C_2F_2(Y(T-\tau_2))Z - D_2Z, \end{cases} \tag{1.2}$$

where $\tau_1 > 0$ and $\tau_2 > 0$.

To simplify the study, using the following nondimensional variables:

$$t = RT, x = \frac{X}{K}, y = \frac{C_1Y}{K}, z = \frac{C_1Z}{C_2K},$$

then system (1.2) can be described by:

$$\begin{cases} \dot{x}(t) = x(1-x) - \dfrac{a_1xy}{1+b_1x}, \\ \dot{y}(t) = \dfrac{a_1x(t-\tau_1)y}{1+b_1x(t-\tau_1)} - \dfrac{a_2yz}{1+b_2y} - d_1y, \\ \dot{z}(t) = \dfrac{a_2y(t-\tau_2)z}{1+b_2y(t-\tau_2)} - d_2z, \end{cases} \tag{1.3}$$

where $a_1 = \frac{KA_1}{RB_1}, b_1 = \frac{K}{B_1}, d_1 = \frac{D_1}{R}, a_2 = \frac{C_2KA_2}{C_1RB_1}, b_2 = \frac{K}{C_1B_2}, d_2 = \frac{D_2}{R}$.

We first study the existence and local stability of the positive equilibrium, then obtain the conditions of switching curve in the plane of $\tau_1 - \tau_2$. Bifurcation direction and stability of bifurcating periodic solution are determined analytically. The numerical simulation shows the stability switching curve. Hassard's theory is combined to obtain bifurcation direction and stability of periodic solution on the switching curve.

2. Stability analysis for two delays

We first consider the existence of the positive equilibrium, and then determine the stability switching curves and the conditions of Hopf bifurcations.

For the equilibrium analysis, since the equilibria of two delays are the same as one delay, so we just introduce it from [26]. System (1.3) has equilibria:

$E_0(0,0,0)$, which is unstable.

$E_1=(1,0,0)$, which is stable if $\frac{a_1}{1+b_1}<d_1$, and unstable if $d_1<\frac{a_1}{1+b_1}$.

$E_2=(\frac{d_1}{a_1-b_1d_1},\frac{a_1-(b_1+1)d_1}{(a_1-b_1d_1)^2},0)$, which exists if $d_1<\frac{a_1}{1+b_1}$.

Suppose that

$$(C1)\quad d_1<\frac{a_1[(b_1-1)+\sqrt{\Delta}]}{b_1[(b_1+1)+\sqrt{\Delta}]},\quad d_2<\frac{a_2}{a_1+b_2}.$$

$$(C2)\quad d_1<\frac{a_1[(b_1-1)-\sqrt{\Delta}]}{b_1[(b_1+1)-\sqrt{\Delta}]},\quad \frac{a_2}{a_1+b_2}<d_2<\frac{a_2(b_1+1)^2}{b_2(b_1+1)^2+4a_1b_1},\quad b_1>1.$$

If (C1) is satisfied, then system (1.3) has an unique positive equilibrium $E_{*1}=(x_{*1},y_{*1},z_{*1})$. If (C2) is satisfied, then system (1.3) has two positive equilibria $E_{*i}=(x_{*i},y_{*i},z_{*i}), i=1,2$. Where $x_{*i}=\frac{(b_1-1)-(-1)^i\sqrt{\Delta}}{2b_1}$, $y_{*i}=\frac{d_2}{a_2-b_2d_2}$, $z_{*i}=\frac{(a_1-b_1d_1)x_{*i}-d_1}{(a_2-b_2d_2)(1+b_1x_{*i})}$, and $\Delta=(b_1-1)^2+\frac{4b_1[a_2-(a_1+b_2)d_2]}{a_2-b_2d_2}$.

We rewrite the positive equilibrium E_{*i} as E_*. The corresponding characteristic equation of system (1.3) at E_* can be written by

$$\lambda^3+\alpha_1\lambda^2+\alpha_2\lambda+\alpha_3+(\alpha_4\lambda+\alpha_5)e^{-\lambda\tau_1}+(\alpha_6\lambda+\alpha_7)e^{-\lambda\tau_2}=0, \tag{2.1}$$

where

$$\alpha_1=-(p_1+p_2+p_3),\quad \alpha_2=p_1p_2+p_1p_3+p_2p_3,\quad \alpha_3=-p_1p_2p_3,$$
$$\alpha_4=\frac{a_1^2x_*y_*}{(1+b_1x_*)^3},\ \alpha_5=-p_3\frac{a_1^2x_*y_*}{(1+b_1x_*)^3},\ \alpha_6=\frac{a_2^2y_*z_*}{(1+b_2y_*)^3},\ \alpha_7=-p_1\frac{a_2^2y_*z_*}{(1+b_2y_*)^3}.$$
$$p_1=1-2x_*-\frac{a_1y_*}{(1+b_1x_*)^2},\ p_2=\frac{a_1x_*}{1+b_1x_*}-\frac{a_2z_*}{(1+b_2y_*)^2}-d_1,\ p_3=\frac{a_2y_*}{1+b_2y_*}.$$

When $\tau_1=\tau_2=0$, we have the theorem by the Routh-Hurwitz criteria:

Theorem 2.1. *Suppose that $\tau_1=\tau_2=0$ and the condition (C1) or (C2) holds, then the equilibrium E_* is locally asymptotically stable if the following conditions are satisfied:*

(H1) $\quad \alpha_1>0,\quad \alpha_3+\alpha_5+\alpha_7>0,\quad \alpha_1(\alpha_2+\alpha_4+\alpha_6)>\alpha_3+\alpha_5+\alpha_7.$

We rewrite characteristic equation (2.1) as

$$Q(\lambda,\tau_1,\tau_2)\equiv Q_0(\lambda)+Q_1(\lambda)e^{-\lambda\tau_1}+Q_2(\lambda)e^{-\lambda\tau_2}=0, \tag{2.2}$$

where

$$Q_0(\lambda)=\lambda^3+\alpha_1\lambda^2+\alpha_2\lambda,$$
$$Q_1(\lambda)=\alpha_3\lambda^2+\alpha_4\lambda+\alpha_5,$$
$$Q_2(\lambda)=\alpha_6\lambda+\alpha_7.$$

We restrict that equation (2.2) satisfies:

(I) $\deg(Q_0(\lambda)) \geq \max\{\deg(Q_1(\lambda)), \deg(Q_2(\lambda))\}$,
(II) $Q_0(0) + Q_1(0) + Q_2(0) \neq 0$,
(III) The polynomials $Q_0(\lambda), Q_1(\lambda)$ and $Q_2(\lambda)$ have not any common zeros,
(IV) $\lim_{\lambda\to\infty}(|Q_1(\lambda)/Q_0(\lambda)| + |Q_2(\lambda)/Q_0(\lambda)|) < 1$.
From above conditions, we have following lemma

Lemma 2.1. *For each $\omega > 0$, $Q_0(i\omega) \neq 0$, $(\tau_1, \tau_2) \in R_+^2$, $\lambda = i\omega$ can be a solution of $Q(\lambda, \tau_1, \tau_2) = 0$ if and only if*

$$(H2) \qquad |Q_1(i\omega)|+|Q_2(i\omega)| \geq |Q_0(i\omega)|, \tag{2.3}$$

and

$$-|Q_0(i\omega)| \leq |Q_1(i\omega)| - |Q_2(i\omega)| \leq |Q_0(i\omega)|. \tag{2.4}$$

The proof of Lemma 1 can be found in [7].
Let

$$\begin{aligned} F1 &= |Q_1(i\omega)| - |Q_2(i\omega)| - |Q_0(i\omega)|, \\ F2 &= |Q_2(i\omega)| - |Q_1(i\omega)| - |Q_0(i\omega)|, \\ F3 &= |Q_0(i\omega)| - |Q_1(i\omega)| - |Q_2(i\omega)|, \end{aligned}$$

then we know that $\lambda = i\omega$ is a solution of $Q(\lambda, \tau_1, \tau_2) = 0$ if and only if $F1 \leq 0, F2 \leq 0, F3 \leq 0$ simultaneously.

Denote Ω as the switching set of all $\omega > 0$ which satisfy (2.3) and (2.4). Then for given $\omega_0 \in \Omega, Q_k(i\omega_0) \neq 0, k = 0, 1, 2$, we know from (2.3),(2.4) that (τ_1, τ_2) satisfying

$$\tau_{10} = \tau_1^{u\pm}(\omega_0) = (\arg \frac{Q_1(i\omega_0)}{Q_0(i\omega_0)} + (2u-1)\pi \pm \psi_1)/\omega_0 \geq 0, u = u_0^\pm, u_0^\pm + 1, ..., \tag{2.5}$$

$$\tau_{20} = \tau_2^{v\pm}(\omega_0) = (\arg \frac{Q_2(i\omega_0)}{Q_0(i\omega_0)} + (2v-1)\pi \mp \psi_2)/\omega_0 \geq 0, v = v_0^\pm, v_0^\pm + 1, ..., \tag{2.6}$$

where $\psi_1, \psi_2 \in [0, \pi]$ can be calculated as

$$\psi_1 = \cos^{-1}\left(\frac{|Q_0(i\omega_0)|^2 + |Q_1(i\omega_0)|^2 - |Q_2(i\omega_0)|^2}{2|Q_0(i\omega_0)||Q_1(i\omega_0)|}\right), \tag{2.7}$$

$$\psi_2 = \cos^{-1}\left(\frac{|Q_0(i\omega_0)|^2 + |Q_2(i\omega_0)|^2 - |Q_1(i\omega_0)|^2}{2|Q_0(i\omega_0)||Q_2(i\omega_0)|}\right), \tag{2.8}$$

and $u_0^+, u_0^-, v_0^+, v_0^-$ are the smallest possible integers (may be negative and may depend on ω_0) such that the corresponding $\tau_1^{u_0^+ +}, \tau_1^{u_0^- -}, \tau_2^{v_0^+ +}, \tau_2^{v_0^- -}$ calculated are nonnegative. Let $\omega_0 \in \Omega$, we can obtain τ_{10}, τ_{20} from (2.5), (2.6) to obtain stability switching curves on $\tau_1 - \tau_2$ plane.

For obtaining transversal conditions, we get derivative of (2.1) with respect to τ_1 as

$$Re[\frac{d\lambda}{d\tau_1}]^{-1}_{\tau_1=\tau_{10}} = \frac{-\alpha_4\omega_0^2 N_{11} + \alpha_5\omega_0 N_{12}}{\alpha_4^2\omega_0^4 + \alpha_5^2\omega_0^2}, \tag{2.9$_1$}$$

where

$$\begin{aligned} N_{11} =& (-3\omega_0^2 + \alpha_2)\cos\omega_0\tau_{10} - 2\alpha_1\omega_0 \sin\omega_0\tau_{10} + \alpha_4 \\ &+ (\alpha_6 - \tau_{20}\alpha_7)\cos\omega_0(\tau_{10} - \tau_{20}) + \tau_{20}\alpha_6\omega_0 \sin\omega_0(\tau_{10} - \tau_{20}), \end{aligned}$$

$$N_{12} = (-3\omega_0^2 + \alpha_2)\sin\omega_0\tau_{10} + 2\alpha_1\omega_0\cos\omega_0\tau_{10} + (\alpha_6 - \tau_{20}\alpha_7)\sin\omega_0(\tau_{10} - \tau_{20}) - \tau_{20}\alpha_6\omega_0\cos\omega_0(\tau_{10} - \tau_{20}).$$

With respect to τ_2 we have

$$Re[\frac{d\lambda}{d\tau_2}]^{-1}_{\tau_2=\tau_{20}} = \frac{-\alpha_6\omega_0^2 N_{21} + \alpha_7\omega_0 N_{22}}{\alpha_6^2\omega_0^4 + \alpha_7^2\omega_0^2}, \tag{2.9$_2$}$$

where

$$N_{21} = (-3\omega_0^2 + \alpha_2)\cos\omega_0\tau_{20} - 2\alpha_1\omega_0\sin\omega_0\tau_{20} + \alpha_6 + (\alpha_4 - \tau_{10}\alpha_5)\cos\omega_0(\tau_{20} - \tau_{10}) + \tau_{10}\alpha_4\omega_0\sin\omega_0(\tau_{20} - \tau_{10}),$$
$$N_{22} = (-3\omega_0^2 + \alpha_2)\sin\omega_0\tau_{20} + 2\alpha_1\omega_0\cos\omega_0\tau_{20} + (\alpha_4 - \tau_{10}\alpha_5)\sin\omega_0(\tau_{20} - \tau_{10}) - \tau_{10}\alpha_4\omega_0\cos\omega_0(\tau_{20} - \tau_{10}).$$

Let

$$(H3) \qquad -\alpha_4\omega_0^2 N_{11} + \alpha_5\omega_0 N_{12} \neq 0, \quad -\alpha_6\omega_0^2 N_{21} + \alpha_7\omega_0 N_{22} \neq 0.$$

Denote $T^j = \{(\tau_{10}^j(\omega), \tau_{20}^j(\omega)), \omega \in \Omega\}, j = 1, 2, ..., m$ are m sections of continuous curves defined on Ω, T° is the internal region surrounded by $T = \bigcup_{j=1}^m T^j$ with coordinate axis $\tau_1 = 0$ and $\tau_2 = 0$. Then we can obtain following

Theorem 2.2. *Assume the conditions* $(H1)$, $(H2)$ *hold,*

(I) When $(\tau_1, \tau_2) \in T^\circ$, *then system* (1.3) *has a locally asymptotically stable positive equilibrium* E_*.

(II) When (τ_1, τ_2) *crossing* T *and* $(H3)$ *holds, then system* (1.3) *undergoes Hopf bifurcation at* E_* *when* $(\tau_1, \tau_2) = (\tau_{10}, \tau_{20}) \in T$.

We name $T^j, j = 1, 2, ..., m$ as stability switching curves.

3. Bifurcating directions and stability of periodic solution

Next, using the theory of Hassard et al. [9] we derive the explicit formulae for determining the properties of the Hopf bifurcation at the critical value (τ_{10}, τ_{20}). We consider two cases, (i) $0 < \tau_{20} \leqslant \tau_{10}$, (ii) $0 < \tau_{10} < \tau_{20}$.

(i) $0 < \tau_{20} \leqslant \tau_{10}$

Let $\mu = \tau_1 - \tau_{10}$, then $\mu = 0$ is the Hopf bifurcation value of system (1.3). Let $t = \tau_1\bar{t}$, and omit " $-$ " above t, then system (1.3) can be rewritten as

$$\dot{u}(t) = L_{1\mu}u_t + f_1(\mu, u_t), \tag{3.1}$$

where $u(t) = (x(t), y(t), z(t))^T \in \mathbf{R}^3$, $u_t(\theta) = u(t + \theta)$, $\theta \in [-1, 0]$. $L_{1\mu} : C \to \mathbf{R}^3$, $f_1 : \mathbf{R} \times C \to \mathbf{R}^3$

$$L_{1\mu}\phi = (\tau_{10} + \mu)[A\phi(0) + B\phi(-\tau_{20}/\tau_1) + C\phi(-1)]. \tag{3.2}$$
$$f_1(\mu, \phi) = (\tau_{10} + \mu)(f_{11}, f_{12}, f_{13})^\top,$$

where

$$A=\begin{pmatrix} p_1 & -\frac{a_1x_*}{1+b_1x_*} & 0\\ 0 & p_2 & -\frac{a_2y_*}{1+b_2y_*}\\ 0 & 0 & p_3\end{pmatrix},\quad B=\begin{pmatrix}0&0&0\\0&0&0\\0&\frac{a_2z_*}{(1+b_2y_*)^2}&0\end{pmatrix},$$

$$C=\begin{pmatrix}0&0\,0\\ \frac{a_1y_*}{(1+b_1x_*)^2}&0\,0\\ 0&0\,0\end{pmatrix},\quad \phi(\theta)=(\phi_1(\theta),\phi_2(\theta),\phi_3(\theta))^\top.$$

$$\begin{aligned} f_{11}=&[\frac{a_1b_1y_*}{(1+b_1x_*)^3}-1]\phi_1^2(0)-\frac{a_1}{(1+b_1x_*)^2}\phi_1(0)\phi_2(0)-\frac{a_1b_1^2y_*}{(1+b_1x_*)^4}\phi_1^3(0)\\ &+\frac{a_1b_1}{(1+b_1x_*)^3}\phi_1^2(0)\phi_2(0)+\cdots,\\ f_{12}=&-\frac{a_1b_1y_*}{(1+b_1x_*)^3}\phi_1^2(-1)+\frac{a_1}{(1+b_1x_*)^2}\phi_1(-1)\phi_2(0)+\frac{a_1b_1^2y_*}{(1+b_1x_*)^4}\phi_1^3(-1)\\ &-\frac{a_1b_1}{(1+b_1x_*)^3}\phi_1^2(-1)\phi_2(0)+\frac{a_2b_2z_*}{(1+b_2y_*)^3}\phi_2^2(0)-\frac{a_2}{(1+b_2y_*)^2}\phi_2(0)\phi_3(0)\\ &-\frac{a_2b_2^2z_*}{(1+b_2y_*)^4}\phi_2^3(0)+\frac{a_2b_2}{(1+b_2y_*)^3}\phi_2^2(0)\phi_3(0)+\cdots,\\ f_{13}=&-\frac{a_2b_2z_*}{(1+b_2y_*)^3}\phi_2^2(-\tau_{20}/\tau_1)+\frac{a_2}{(1+b_2y_*)^2}\phi_2(-\tau_{20}/\tau_1)\phi_3(0)\\ &+\frac{a_2b_2^2z_*}{(1+b_2y_*)^4}\phi_2^3(-\tau_{20}/\tau_1)-\frac{a_2b_2}{(1+b_2y_*)^3}\phi_2^2(-\tau_{20}/\tau_1)\phi_3(0)+\cdots.\end{aligned}$$

By the Riesz representation theorem, there exits a function $\eta(\theta,\mu)$ of bounded variation for $\theta\in[-1,0]$, such that

$$L_{1\mu}\phi=\int_{-1}^0 d\eta(\theta,\mu)\phi(\theta),\qquad \phi\in C, \tag{3.3}$$

where

$$\eta(\theta,\mu)=(\tau_{10}+\mu)[A\delta(\theta)+B\delta(\theta+\tau_{20}/\tau_1)+C\delta(\theta+1), \tag{3.4}$$

$\delta(\theta)$ is the Dirac delta function.

For $\phi\in C^1([-1,0],\mathbf{R}^3)$, define

$$\mathcal{A}(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^0 d\eta(s,\mu)\phi(s), & \theta=0,\end{cases}$$

and

$$R(\mu)\phi=\begin{cases}0, & \theta\in[-1,0),\\ f(\mu,\phi), & \theta=0.\end{cases}$$

Then system (3.1) is equivalent to

$$\dot{u}_t=\mathcal{A}(\mu)u_t+R(\mu)u_t. \tag{3.5}$$

Denote $\mathcal{A} = \mathcal{A}(0)$,

$$\mathcal{A}^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} d\eta^\top(t,0)\psi(-t), & s = 0, \end{cases}$$

and

$$\langle\psi(s), \phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta,0)\phi(\xi)d\xi, \tag{3.6}$$

where $\psi \in C^*([0,1],(R^3)^*)$. Then $\mathcal{A}$ and $\mathcal{A}^*$ are adjoint operators. We know that if $\pm i\omega_k\tau_k$ are eigenvalues of $\mathcal{A}$, then they are eigenvalues of $\mathcal{A}^*$.

Suppose that $q(\theta) = (1,\beta,\gamma)^\top e^{i\omega_0\tau_{10}\theta}$ is an eigenvector of $\mathcal{A}$ corresponding to $i\omega_0\tau_{10}$, that is $\mathcal{A}q(\theta) = i\omega_0\tau_{10}q(\theta)$. Then we obtain that

$$\beta = \frac{(p_1 - i\omega_0)(1+b_1x_*)}{a_1x_*}, \qquad \gamma = \frac{a_2z_*(p_1-i\omega_0)(1+b_1x_*)e^{-i\omega_0\tau_{20}}}{a_1x_*(i\omega_0 - p_3)(1+b_2y_*)^2}.$$

Let $q^*(s) = D(1,\beta^*,\gamma^*)e^{i\omega_0\tau_{10}s}$ is an eigenvector of $\mathcal{A}^*$ corresponding to $-i\omega_0\tau_{10}$, then we have

$$\beta^* = -\frac{(p_1+i\omega_0)(1+b_1x_*)^2e^{i\omega_0\tau_{10}}}{a_1y_*}, \qquad \gamma^* = -\frac{a_2y_*(p_1+i\omega_0)(1+b_1x_*)^2e^{i\omega_0\tau_{10}}}{a_1y_*(i\omega_0+p_3)(1+b_2y_*)}.$$

By(3.6), we have

$$\begin{aligned} &\langle q^*(s),\, q(\theta)\rangle \\ =&\bar{D}(1,\overline{\beta^*},\overline{\gamma^*})(1,\beta,\gamma)^T \\ &-\int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1,\overline{\beta^*},\overline{\gamma^*})e^{-i(\xi-\theta)\omega_0\tau_{10}}d\eta(\theta,0)(1,\beta,\gamma)^\top e^{i\xi\omega_0\tau_{10}}d\xi \\ =&\bar{D}(1+\beta\overline{\beta^*}+\gamma\overline{\gamma^*} - \int_{-1}^{0}(1,\overline{\beta^*},\overline{\gamma^*})\theta e^{i\theta\omega_0\tau_{10}}d\eta(\theta,0)(1,\beta,\gamma)^\top) \\ =&\bar{D}(1+\beta\overline{\beta^*}+\gamma\overline{\gamma^*}+\tau_{10}e^{-i\omega_0\tau_{10}}\overline{\beta^*}\frac{a_1y_*}{(1+b_1x_*)^2}+\tau_{20}e^{-i\omega_0\tau_{20}}\beta\overline{\gamma^*}\frac{a_2z_*}{(1+b_2y_*)^2}). \end{aligned}$$

Thus we choose

$$\bar{D} = \frac{1}{1+\beta\bar{\beta}^*+\gamma\bar{\gamma}^*+\tau_{10}e^{-i\omega_0\tau_{10}}\overline{\beta^*}\frac{a_1y_*}{(1+b_1x_*)^2}+\tau_{20}e^{-i\omega_0\tau_{20}}\beta\overline{\gamma^*}\frac{a_2z_*}{(1+b_2y_*)^2}}.$$

such that $< q^*(s), q(\theta) >= 1$, $< q^*(s), \bar{q}(\theta) >= 0$.

Let u_t be the solution of Eq.(3.5) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t,\theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}, \tag{3.7}$$

then we can write (3.5) as

$$\begin{cases} \dot{z} = i\omega_0 z + \dfrac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \dfrac{g_{02}}{2}\bar{z}^2 + \dfrac{g_{21}}{2}z^2\bar{z} + \cdots, \\ \dot{W} = \mathcal{A}W + H(z,\bar{z},0). \end{cases} \tag{3.8}$$

Using Hassard's method, we obtain following values:

$$\mu_2 = -\frac{Re\{k_{11}(0)\}}{Re\{\lambda'(\tau_{10})\}},\ \nu_2 = 2Re\{k_{11}(0)\},$$

where $k_{11}(0) = \frac{i}{2\omega_0\tau_{10}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}$.

$$\begin{aligned} g_{20} &= 2\bar{D}\tau_{10}\{l_1 + \overline{\beta^*}l_2 + \overline{\gamma^*}l_3\}, \\ g_{11} &= \bar{D}\tau_{10}\{l_4 + \overline{\beta^*}l_5 + \overline{\gamma^*}l_6\}, \\ g_{02} &= 2\bar{D}\tau_{10}\{l_7 + \overline{\beta^*}l_8 + \overline{\gamma^*}l_9\}, \\ g_{21} &= 2\bar{D}\tau_{10}\{l_{10} + \overline{\beta^*}l_{11} + \overline{\gamma^*}l_{12}\}, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} l_1 =& m_1 - m_2\beta - 1, \\ l_2 =& -m_1e^{-2i\omega_0\tau_{10}} + m_2\beta e^{-i\omega_0\tau_{10}} + m_3\beta^2 - m_4\beta\gamma, \\ l_3 =& -m_3\beta^2e^{-2i\omega_0\tau_{20}} + m_4\beta\gamma e^{-i\omega_0\tau_{20}}, \\ l_4 =& 2m_1 - m_2(\beta + \bar{\beta}) - 2, \\ l_5 =& -2m_1 + m_2(\beta e^{i\omega_0\tau_{10}} + \bar{\beta}e^{-i\omega_0\tau_{10}}) + 2m_3\beta\bar{\beta} - m_4(\beta\bar{\gamma} + \bar{\beta}\gamma) \\ l_6 =& -2m_3\beta\bar{\beta} + m_4(\beta\bar{\gamma}e^{-i\omega_0\tau_{20}} + \bar{\beta}\gamma e^{i\omega_0\tau_{20}}), \\ l_7 =& m_1 - m_2\bar{\beta} - 1, \\ l_8 =& -m_1e^{2i\omega_0\tau_{10}} + m_2\bar{\beta}e^{i\omega_0\tau_{10}} + m_3\bar{\beta}^2 - m_4\bar{\beta}\bar{\gamma} \\ l_9 =& -m_3\bar{\beta}^2e^{2i\omega_0\tau_{20}} + m_4\bar{\beta}\bar{\gamma}e^{i\omega_0\tau_{20}}, \\ l_{10} =&(m_1 - 1)(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) - m_2(W_{11}^{(2)}(0) + \beta W_{11}^{(1)}(0) \\ &+ \frac{W_{20}^{(2)}(0) + W_{20}^{(1)}(0)\bar{\beta}}{2}) - 3m_5 + (2\beta + \bar{\beta})m_6, \\ l_{11} =& - m_1(2W_{11}^{(1)}(-1)e^{-i\omega_0\tau_{10}} + W_{20}^{(1)}(-1)e^{i\omega_0\tau_{10}}) + m_2(W_{11}^{(2)}(0)e^{-i\omega_0\tau_{10}} \\ &+ \beta W_{11}^{(1)}(-1) + \frac{W_{20}^{(2)}(0)e^{i\omega_0\tau_{10}} + W_{20}^{(1)}(-1)\bar{\beta}}{2}) + m_3(2\beta W_{11}^{(2)}(0) + \bar{\beta}W_{20}^{(2)}(0)) \\ &- m_4(\beta W_{11}^{(3)}(0) + \gamma W_{11}^{(2)}(0) + \frac{\bar{\beta}W_{20}^{(3)}(0) + \bar{\gamma}W_{20}^{(2)}(0)}{2}) + 3m_5e^{-i\omega_0\tau_{10}} \\ &- m_6(2\beta + \bar{\beta}e^{-2i\omega_0\tau_{10}}) - 3m_7\beta^2\bar{\beta} + m_8(\beta^2\bar{\gamma} + 2\beta\gamma\bar{\beta}), \\ l_{12} =& - m_3(2\beta W_{11}^{(2)}(-\tau_{20}/\tau_{10})e^{-i\omega_0\tau_{20}} + \bar{\beta}W_{20}^{(2)}(-\tau_{20}/\tau_{10})e^{i\omega_0\tau_{20}}) \\ &+ m_4(\beta W_{11}^{(3)}(0)e^{-i\omega_0\tau_{20}} + \gamma W_{11}^{(2)}(-\tau_{20}/\tau_{10}) \\ &+ \frac{\bar{\beta}W_{20}^{(3)}(0)e^{i\omega_0\tau_{20}} + \bar{\gamma}W_{20}^{(2)}(-\tau_{20}/\tau_{10})}{2}) + 3m_7\beta^2\bar{\beta}e^{-i\omega_0\tau_{20}} \\ &- m_8(\beta^2\bar{\gamma}e^{-2i\omega_0\tau_{20}} + 2\beta\bar{\beta}\gamma). \end{aligned} \tag{3.10}$$

$$\begin{aligned} m_1 &= \frac{a_1b_1y_*}{(1 + b_1x_*)^3}, \quad m_2 = \frac{a_1}{(1 + b_1x_*)^2}, \quad m_3 = \frac{a_2b_2z_*}{(1 + b_2y_*)^3}, \quad m_4 = \frac{a_2}{(1 + b_2y_*)^2}, \\ m_5 &= \frac{a_1b_1^2y_*}{(1 + b_1x_*)^4}, \quad m_6 = \frac{a_1b_1}{(1 + b_1x_*)^3}, \quad m_7 = \frac{a_2b_2^2z_*}{(1 + b_2y_*)^4}, \quad m_8 = \frac{a_2b_2}{(1 + b_2y_*)^3}. \end{aligned}$$

$$
\begin{aligned}
&W_{20}(\theta)=\frac{ig_{20}}{\omega_0\tau_{10}}q(0)e^{i\theta\omega_0\tau_{10}}+\frac{i\bar{g}_{02}}{3\omega_0\tau_{10}}\bar{q}(0)e^{-i\theta\omega_0\tau_{10}}+E_1e^{2i\theta\omega_0\tau_{10}},\\
&W_{11}(\theta)=-\frac{ig_{11}}{\omega_0\tau_{10}}q(0)e^{i\theta\omega_0\tau_{10}}+\frac{i\bar{g}_{11}}{\omega_0\tau_{10}}\bar{q}(0)e^{-i\theta\omega_0\tau_{10}}+E_2,\\
&E_1=(E_1^{(1)},E_1^{(2)},E_1^{(3)})^\top, E_2=(E_2^{(1)},E_2^{(2)},E_2^{(3)})^\top,
\end{aligned}
$$

and satisfy

$$
\begin{pmatrix}
2i\omega_0-p_1 & \frac{a_1x_*}{1+b_1x_*} & 0\\
-\frac{a_1y_*e^{-2i\omega_0\tau_{10}}}{(1+b_1x_*)^2} & 2i\omega_0-p_2 & \frac{a_2y_*}{1+b_2y_*}\\
0 & -\frac{a_2z_*e^{-2i\omega_0\tau_{20}}}{(1+b_2y_*)^2} & 2i\omega_0-p_3
\end{pmatrix}
\begin{pmatrix} E_1^{(1)}\\ E_1^{(2)}\\ E_1^{(3)}\end{pmatrix}
=2\begin{pmatrix} l_1\\ l_2\\ l_3\end{pmatrix}, \tag{3.11}
$$

$$
\begin{pmatrix}
p_1 & -\frac{a_1x_*}{1+b_1x_*} & 0\\
\frac{a_1y_*e^{-2i\omega_0\tau_{10}}}{(1+b_1x_*)^2} & p_2 & -\frac{a_2y_*}{1+b_2y_*}\\
0 & \frac{a_2z_*e^{-2i\omega_0\tau_{20}}}{(1+b_2y_*)^2} & p_3
\end{pmatrix}
\begin{pmatrix} E_2^{(1)}\\ E_2^{(2)}\\ E_2^{(3)}\end{pmatrix}
=-\begin{pmatrix} l_4\\ l_5\\ l_6\end{pmatrix}. \tag{3.12}
$$

μ_2 determines the directions of the Hopf bifurcation: If $Re\{\lambda'(\tau_{10})\}>0$, $\mu_2>0(resp.\ \mu_2<0)$, then the Hopf bifurcation is supercritical (resp. subcritical) and the periodic solution exist for $\tau_1>\tau_{10}(\tau_1<\tau_{10})$. If $Re\{\lambda'(\tau_{10})\}<0$, however, the bifurcating periodic solution are on the opposite direction. ν_2 determines the stability of the bifurcation periodic solutions: The bifurcating periodic solutions are stable(unstable) if $\nu_2<0(\nu_2>0)$.

(ii) $0<\tau_{10}<\tau_{20}$.

Let $\mu=\tau_2-\tau_{20}$, $\mu=0$ is the Hopf bifurcation value of system (1.3). System (1.3) can be rewritten as

$$
\dot{u}(t)=L_{2\mu}u_t+f_2(\mu,u_t), \tag{3.13}
$$

where

$$
\begin{aligned}
&L_{2\mu}\phi=(\tau_{20}+\mu)[A\phi(0)+B\phi(-1)+C\phi(-\tau_{10}/\tau_2)]. \qquad (3.14)\\
&f_2(\mu,\phi)=(\tau_{20}+\mu)(f_{21},f_{22},f_{23})^\top,
\end{aligned}
$$

where

$$
\begin{aligned}
f_{21}=&[\frac{a_1b_1y_*}{(1+b_1x_*)^3}-1]\phi_1^2(0)-\frac{a_1}{(1+b_1x_*)^2}\phi_1(0)\phi_2(0)-\frac{a_1b_1^2y_*}{(1+b_1x_*)^4}\phi_1^3(0)\\
&+\frac{a_1b_1}{(1+b_1x_*)^3}\phi_1^2(0)\phi_2(0)+\cdots,\\
f_{22}=&-\frac{a_1b_1y_*}{(1+b_1x_*)^3}\phi_1^2(-\tau_{10}/\tau_2)+\frac{a_1}{(1+b_1x_*)^2}\phi_1(-\tau_{10}/\tau_2)\phi_2(0)\\
&+\frac{a_1b_1^2y_*}{(1+b_1x_*)^4}\phi_1^3(-\tau_{10}/\tau_2)-\frac{a_1b_1}{(1+b_1x_*)^3}\phi_1^2(-\tau_{10}/\tau_2)\phi_2(0)+\frac{a_2b_2z_*}{(1+b_2y_*)^3}\phi_2^2(0)\\
&-\frac{a_2}{(1+b_2y_*)^2}\phi_2(0)\phi_3(0)-\frac{a_2b_2^2z_*}{(1+b_2y_*)^4}\phi_2^3(0)+\frac{a_2b_2}{(1+b_2y_*)^3}\phi_2^2(0)\phi_3(0)+\cdots,\\
f_{23}=&-\frac{a_2b_2z_*}{(1+b_2y_*)^3}\phi_2^2(-1)+\frac{a_2}{(1+b_2y_*)^2}\phi_2(-1)\phi_3(0)+\frac{a_2b_2^2z_*}{(1+b_2y_*)^4}\phi_2^3(-1)
\end{aligned}
$$

$$- \frac{a_2 b_2}{(1+b_2 y_*)^3}\phi_2^2(-1)\phi_3(0) + \cdots .$$

Using Hassard's method, we obtain following values:

$$\begin{aligned} g_{20} &= 2\bar{D}\tau_{20}\{l_1 + \overline{\beta^*} l_2 + \overline{\gamma^*} l_3\}, \\ g_{11} &= \bar{D}\tau_{20}\{l_4 + \overline{\beta^*} l_5 + \overline{\gamma^*} l_6\}, \\ g_{02} &= 2\bar{D}\tau_{20}\{l_7 + \overline{\beta^*} l_8 + \overline{\gamma^*} l_9\}, \\ g_{21} &= 2\bar{D}\tau_{20}\{l_{10} + \overline{\beta^*} h_{11} + \overline{\gamma^*} h_{12}\}. \end{aligned} \tag{3.15}$$

Where $l_1 - l_{10}$ are the same as in (3.10), and

$$\begin{aligned} h_{11} =& - m_1(2W_{11}^{(1)}(-\tau_{10}/\tau_{20})e^{-i\omega_0\tau_{10}} + W_{20}^{(1)}(-\tau_{10}/\tau_{20})e^{i\omega_0\tau_{10}}) \\ & + m_2(W_{11}^{(2)}(0)e^{-i\omega_0\tau_{10}} + \beta W_{11}^{(1)}(-\tau_{10}/\tau_{20}) \\ & + \frac{W_{20}^{(2)}(0)e^{i\omega_0\tau_{10}} + W_{20}^{(1)}(-\tau_{10}/\tau_{20})\bar{\beta}}{2}) + m_3(2\beta W_{11}^{(2)}(0) + \bar{\beta} W_{20}^{(2)}(0)) \\ & - m_4(\beta W_{11}^{(3)}(0) + \gamma W_{11}^{(2)}(0) + \frac{\bar{\beta} W_{20}^{(3)}(0) + \bar{\gamma} W_{20}^{(2)}(0)}{2}) + 3m_5 e^{-i\omega_0\tau_{10}} \\ & - m_6(2\beta + \bar{\beta} e^{-2i\omega_0\tau_{10}}) - 3m_7\beta^2\bar{\beta} + m_8(\beta^2\bar{\gamma} + 2\beta\gamma\bar{\beta}), \\ h_{12} =& - m_3(2\beta W_{11}^{(2)}(-1)e^{-i\omega_0\tau_{20}} + \bar{\beta} W_{20}^{(2)}(-1)e^{i\omega_0\tau_{20}}) + m_4(\beta W_{11}^{(3)}(0)e^{-i\omega_0\tau_{20}} \\ & + \gamma W_{11}^{(2)}(-1) + \frac{\bar{\beta} W_{20}^{(3)}(0)e^{i\omega_0\tau_{20}} + \bar{\gamma} W_{20}^{(2)}(-1)}{2}) \\ & + 3m_7\beta^2\bar{\beta} e^{-i\omega_0\tau_{20}} - m_8(\beta^2\bar{\gamma} e^{-2i\omega_0\tau_{20}} + 2\beta\bar{\beta}\gamma). \end{aligned} \tag{3.16}$$

$$\begin{aligned} W_{20}(\theta) &= \frac{i g_{20}}{\omega_0\tau_{20}} q(0)e^{i\theta\omega_0\tau_{20}} + \frac{i\bar{g}_{02}}{3\omega_0\tau_{20}}\bar{q}(0)e^{-i\theta\omega_0\tau_{20}} + E_1 e^{2i\theta\omega_0\tau_{20}}, \\ W_{11}(\theta) &= -\frac{i g_{11}}{\omega_0\tau_{20}} q(0)e^{i\theta\omega_0\tau_{20}} + \frac{i\bar{g}_{11}}{\omega_0\tau_{20}}\bar{q}(0)e^{-i\theta\omega_0\tau_{20}} + E_2, \end{aligned}$$

$E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^\top, E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^\top$ are the same as in (3.11), (3.12).

Then we obtain

$$\mu_2 = -\frac{Re\{k_{21}(0)\}}{Re\{\lambda'(\tau_{20})\}},\ \nu_2 = 2Re\{k_{21}(0)\},$$

where

$$k_{21}(0) = \frac{i}{2\omega_0\tau_{20}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}.$$

μ_2 determines the directions of the Hopf bifurcation: If $Re\{\lambda'(\tau_{20})\} > 0$, $\mu_2 > 0$($resp.$ $\mu_2 < 0$), then the Hopf bifurcation is supercritical (resp. subcritical) and the periodic solution exist for $\tau_2 > \tau_{20}(\tau < \tau_{20})$. If $Re\{\lambda'(\tau_{20})\} < 0$, however, the bifurcating periodic solution are on the opposite direction. ν_2 determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable(unstable) if $\nu_2 < 0(\nu_2 > 0)$.

4. Numerical simulation of switching curve and periodic solution

In this section, we consider (1.3) as the following system:

$$\begin{cases} \dot{x}(t) = x(1-x) - \dfrac{5xy}{1+2x}, \\ \dot{y}(t) = \dfrac{5x(t-\tau_1)y}{1+2x(t-\tau_1)} - \dfrac{0.2yz}{1+0.3y} - 0.6y, \\ \dot{z}(t) = \dfrac{0.2y(t-\tau_2)z}{1+0.3y(t-\tau_2)} - 0.02z, \end{cases} \tag{4.1}$$

where $a_1 = 5, b_1 = 2, d_1 = 0.6, a_2 = 0.2, b_2 = 0.3, d_2 = 0.02$. System (4.1) has a positive equilibrium $E_* = (0.8021, 0.1031, 4.8453)$. The corresponding characteristic equation of system (4.1) at E_* is given by

$$\lambda^3 + 0.6519\lambda^2 - 0.0192\lambda + 0.1171\lambda e^{-\lambda\tau_1} + (0.0182\lambda + 0.0124)e^{-\lambda\tau_2} = 0. \tag{4.2}$$

When $\tau_1 = \tau_2 = 0$, the roots of Eq.(4.2) are $-0.457, -0.0973-0.1329i, -0.0973+0.1329i$. Thus, the positive equilibrium E_* is asymptotically stable. From Lemma 2.1, we obtain that $\lambda = i\omega$ can be a solution of $Q(\lambda, \tau_1, \tau_2) = 0$ if and only if $\omega \in (0.0738, 0.2371)$ (see Fig.1).

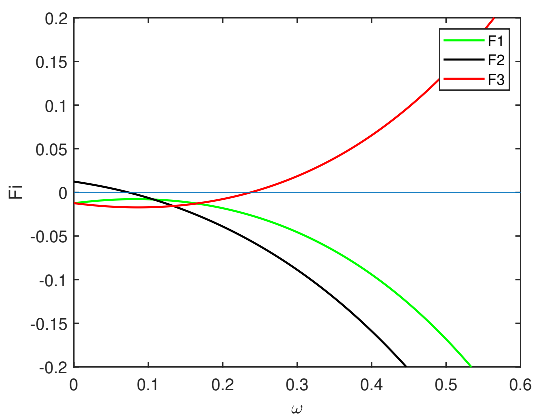

Figure 1. For $a_1 = 5, b_1 = 2, d_1 = 0.6, a_2 = 0.2, b_2 = 0.3, d_2 = 0.02$, $F1, F2, F3$ are all less then 0 if $\omega \in (0.0738, 0.2371)$.

Considering $\tau_1(\omega), \tau_2(\omega)$ on $\omega \in (0.0738, 0.2371)$, we obtain a continuous curve $T = (\tau_{10}^{1-}(\omega), \tau_{20}^{0-}(\omega))$. From Theorem 2.2, E_* is stable in the internal region T° surrounded by $\tau_1 = 0, \tau_2 = 0$ and T (see Fig. 2). For example, we choose $\tau_1 = 2.91, \tau_2 = 3.44$ which is in the area of stable, so we can see E_* is stable (see Fig.3).

When $(\tau_1, \tau_2) = (\tau_{10}, \tau_{20}) \in T$ and (H3) holds , then system (1.3) undergoes Hopf bifurcation at E_*. For example, we choose $(\tau_{10}, \tau_{20}) = (3.5105, 3.4364)$ which is $\tau_{20} < \tau_{10}$, the corresponding ω_0 is 0.2073. By the analysis in Section 3 , it is obtained that at (3.5105,3.4364), $Re\{\lambda'(\tau_{10})\} = 0.0255 > 0$, $Re\{k_{11}(0)\} = -3.2608e{+}01 < 0$, $\mu_2 = 1.2811e + 03 > 0$, and $\nu_2 = -65.2153 < 0$, Therefore, it is known that

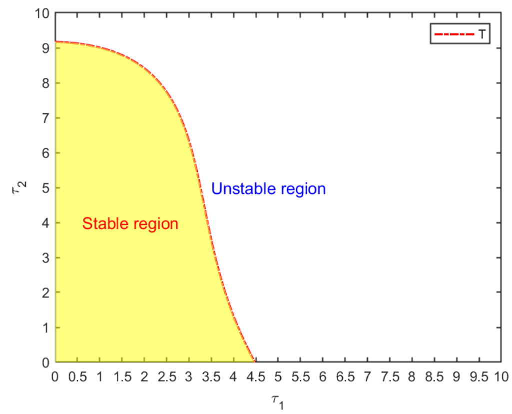

Figure 2. For $a_1 = 5, b_1 = 2, d_1 = 0.6, a_2 = 0.2, b_2 = 0.3, d_2 = 0.02$, T is a switching curve on $\tau_1 - \tau_2$ plane with $\omega \in (0.0738, 0.2371).E_*$ is stable in the region surrounded by $\tau_1 = 0, \tau_2 = 0$ and curve T .

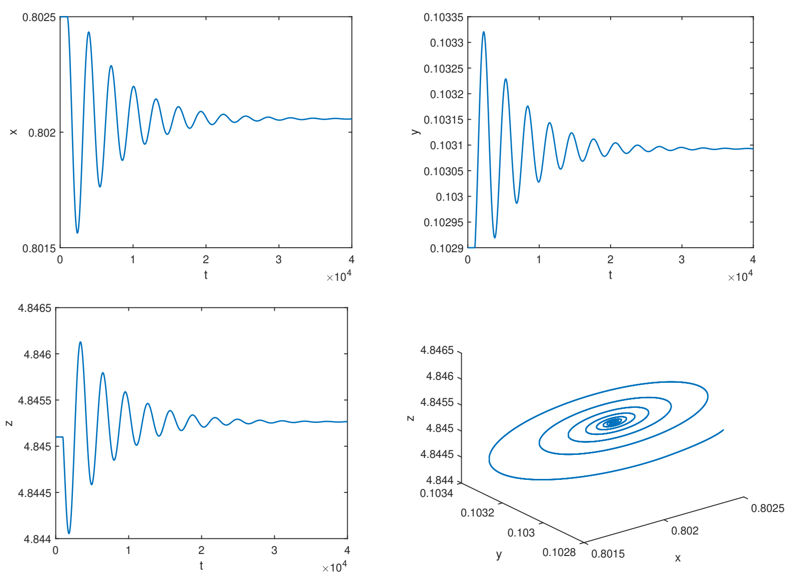

Figure 3. When $\tau_1 = 2.91, \tau_2 = 3.44$, E_* is asymptotically stable.

at the point (3.5105,3.4364), when (τ_1, τ_2) increases crossing T, the bifurcation is supercritical, E_* loses its stability, a stabile periodic solution bifurcate from positive equilibrium. (see Fig.4).

If we choose $(\tau_{10}, \tau_{20}) = (2.9105, 6.6782)$ which is $\tau_{20} < \tau_{10}$, the corresponding ω_0 is 0.1617. By the analysis in Section 3 , it is obtained that at (2.9105, 6.6782), $Re\{\lambda'(\tau_{20})\} = 0.0098 > 0$, $Re\{k_{21}(0)\} = -7.7077e+02 < 0$, $\mu_2 = 7.8954e+04 > 0$, and $\nu_2 = -1.5415e+03 < 0$, Therefore, it is known that at the point (2.9105, 6.6782), when (τ_1, τ_2) increases crossing T, the bifurcation is supercritical, E_* loses its stability, a stabile periodic solution bifurcate from positive equilibrium. (see Fig.5).

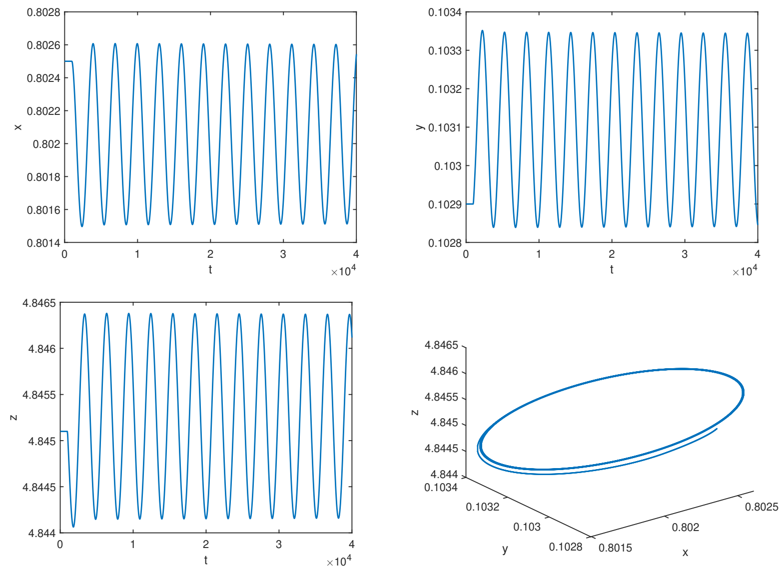

Figure 4. When $\tau_1 = 3.51, \tau_2 = 3.44$, E_* is unstable, and there is a stable periodic solution surrounding E_*.

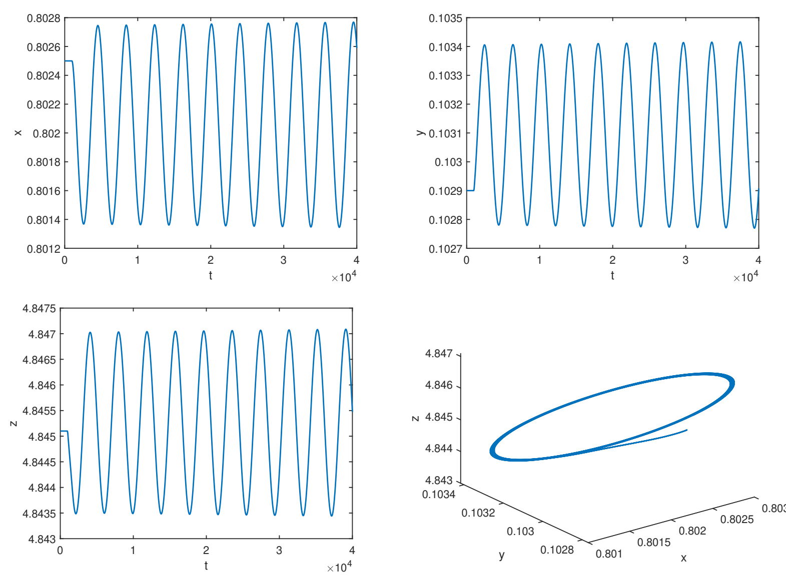

Figure 5. When $\tau_1 = 2.91, \tau_2 = 6.68$, E_* is unstable and there is a stable periodic solution surrounding E_*.

5. Conclusions

In this paper, the method of stability switching curves is applied to a two delayed three species food chain system. By analyzing the distribution of the delays, the stability of positive equilibrium is determined in an area surrounding by coordinate axis and stability switching curves. The conditions of Hopf bifurcation is obtained.

By using Hassard's method, the direction and stability of the Hopf bifurcation periodic solution are determined. Numerical simulations are employed to show the switching curve, explain the analytical results.

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