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AN UNBOUNDED CRITICAL POINT THEORY FOR A CLASS OF NON-DIFFERENTIABLE FUNCTIONALS AND ITS APPLICATION*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract In this paper, a nonsmooth version of multiple critical point theorem is established by adopting the framework of nonsmooth analysis theory. Then an application of this theorem to a discontinuous quasilinear Schrödinger equation is presented. Some continuous results are extended to discontinuous cases.

Keywords Nonsmooth analysis, quasilinear Schrödinger equation, differential inclusion.

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1. Introduction

Many practical problems can be transformed into ordinary or partial differential equations with discontinuous nonlinearities. The problem of mosquito population suppression by releasing sterile males can be changed to piecewise continuous ordinary differential equations [1, 15–17, 20–22]. Some obstacle problems and free boundary problems may be reduced to Dirichlet boundary value problems with discontinuous nonlinearities which have been studied in recent years. The area of nonsmooth analysis is closely related with the development of critical points theory for non-differentiable functionals, in particular, for locally Lipschitz continuous functionals based on Clarke's generalized gradient [4]. In 1981, Chang [3] extended the variational methods to a class of non-differentiable functionals, and directly applied the variational methods for non-differentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. It provides an appropriate mathematical framework to extend classic critical points theory for C^1-functionals in a natural way, and to meet specific needs in applications such as

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in nonsmooth mechanics and engineering. For a comprehensive understanding, we refer to references [7,18,19] and monographs [12,13].

The main purpose of the present paper is to establish a nonsmooth version of multiple critical points theorem by adopting the framework of nonsmooth analysis theory. Let E be a real Banach space and let $B_r = \{x \in E : \|x\| < r\}(r > 0)$ and $S_r = \partial B_r$. Assume $h \in C(E, \mathbb{R}^1)$, and set $h^{(0)} = \{x \in E : h(x) \geq 0\}$. We make the following hypotheses on h:

$(H1)$ $h(0) = 0$ and there exist $\rho, \alpha > 0$ such that $\bar{B}_\rho \subset h^{(0)}$ and $h(x) \geq \alpha,\ \ \forall x \in S_\rho$;

$(H2)$ For any finite dimension space $E_0 \subset E,\ E_0 \cap h^{(0)}$ is bounded.

Our main result is the following theorem:

Theorem 1.1. *Assume that E is an infinite dimensional real Banach space, K is compact and symmetric, $K \subset E$, $h : E \to \mathbb{R}^1$ is an even and locally Lipschitz functional, and satisfies the nonsmooth C-condition. Hypotheses $(H1)$ and $(H2)$ are satisfied. For any positive integer m, set*

$$b_m = \inf_{K \in \Gamma_m} \max_{x \in K} h(x). \tag{1.1}$$

Then

(i) $0 < \alpha \leq b_m < +\infty$ *and* b_m *is a critical value of* h*(m=1,2,...);*

(ii) $b_m = b_{m+1} = \cdots = b_{m+r-1} = b(r \geq 1) \Rightarrow \gamma(K_b) \geq r$*, where* $K_b = \{x \in E : h(x) = b, 0 \in \partial h(x)\}$*;*

(iii) $b_m \leq b_{m+1}(m = 1, 2, ...)$*, and* $b_m \to +\infty(m \to \infty)$*;*

(iv) *h has infinitely many critical points and infinitely many critical values.*

Remark 1.1. In [4] Marano also obtained an infinitely many critical points theory for non-differentiable functions. Compared to their conditions in Theorem 1.1, our conditions in this theorem are very simple, and it is very easy to verify.

We next present an application of the following discontinuous quasilinear Schrödinger equation and extend the corresponding result of [5] into the discontinuous case.

$$\begin{cases} -\,div(g^2(u)(\nabla u)) + g(u)g'(u)|\nabla u|^2 + a(x)u \in \partial F(x, u) \text{ a.e. in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $N \geq 3$, $g : \mathbb{R} \to \mathbb{R}^+$ is an even differential function, Ω is a bounded region with smooth boundary in $\mathbb{R}^N$, $\partial F(x, u)$ is the partial generalized gradient of $F(x, \cdot)$ at the point u, $g'(t) \geq 0$ for all $t \geq 0$, $a(x)$ is a continuous function in Ω. Such equations arise in various branches of mathematical physics and are related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$i\partial_t z = -\Delta z + W(x)z - k(x, z) - \Delta l(|z|^2)l(|z|^2)z, \tag{1.3}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. (1.3) is used to describe various physical phenomena corresponding to various types of nonlinear term l. Such as plasma physics, condensed matter theory and fluid mechanics. Some related results can be found in [2,8,10,11] and references therein.

Since the energy functional of problem (1.2) is non-differentiable, it will raise some essential difficulties. In order to state our results, we give the following hypotheses:

($F0$) For all $u \in \mathbb{R}$, $\Omega \ni x \mapsto F(x,u) \in \mathbb{R}$ is measurable and for all $x \in \Omega$, $\mathbb{R} \ni u \mapsto F(x,u)$ is locally Lipschitz;

($F1$) $g \in C^1(\mathbb{R})$ is positive and even, $g'(u) \geq 0$ for all $u \geq 0$ and $\lim_{u\to\infty} g(u) = A$;

($F2$) $0 \leq a(x) \in C^1(\Omega)$, $a_1 = \inf_{x\in\Omega} a(x)$ and $a_2 = \sup_{x\in\Omega} a(x)$;

($F3$) There exists $2 < q < 2^* - 1 = \frac{N+2}{N-2}$ such that for all $\xi(u) \in \partial F(x,u)$, $|\xi(u)| \leq a + b|u|^q$, $a > 0$, $b > 0$;

($F4$) There exist $2 < \beta$ and $M > 0$ such that for all $\xi(u) \in \partial F(x,u)$, $\beta g(u) F(x,u) \leq G(u)\xi(u), \quad \forall u \geq M$;

($F5$) (i)$\lim_{u\to 0} \frac{F(x,u)}{u^2} = 0$ for a.e. $x \in \Omega$, (ii)$\lim_{u\to+\infty} \frac{F(x,u)}{u^2} = +\infty$ for a.e. $x \in \Omega$;

($F6$) $F(x,-u) = F(x,u), \quad \forall x \in \Omega,\ u \in \mathbb{R}$.

Based on Theorem 1.1 and the above hypotheses, we have the following two theorems.

Theorem 1.2. *If hypotheses $(F0)-(F4)$ and $(F5)(i)$ hold, then problem (1.2) has at least one nontrivial solution in $H_0^1(\Omega)$.*

Theorem 1.3. *If hypotheses (F0)-(F6) hold, then problem (1.2) has infinitely many solutions in $H_0^1(\Omega)$.*

This paper is organized as follows. In section 2, we present some necessary preliminary knowledge and use the genus theory, nonsmooth deformation lemma and minimax theory to prove Theorem 1.1. In section 3, Theorems 1.2, 1.3 are proved.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we firstly give some preliminaries. $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ denotes its topological dual. c, c_i, C, C_i denote estimated constants(c, C may be different from line to line). θ represents the origin of coordinates.

Definition 2.1 ([6]). A function I: $X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every ν, $\eta \in U$

$$|I(\nu) - I(\eta)| \leq L\|\nu - \eta\|.$$

Definition 2.2 ([6]). Let $I : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized derivative of I in u along the direction ν is defined by

$$I^0(u;\nu) = \limsup_{\eta\to u, \tau\to 0^+} \frac{I(\eta + \tau\nu) - I(\eta)}{\tau},$$

where u, $\nu \in X$.

It is easy to see that the function $\nu \mapsto I^0(u;\nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \{u^* \in X^* : \langle u^*, \nu\rangle_X \leq I^0(u;\nu) \text{ for all } v \in X\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \{I'(u)\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

Definition 2.3 ([6]). I satisfies the nonsmooth C-condition if every sequence $\{u_n\} \subset X$, satisfying $I(u_n) \to c$ and $(1 + \|u_n\|)m^I(u_n) \to 0$ as $n \to \infty$, has a strongly convergent subsequence, where $m^I(u_n) = \inf_{u_n^* \in \partial I(u_n)} \|u_n^*\|_{X^*}$.

Definition 2.4. Let E be a Banach space and let $\mathcal{A}_{cs}(E) = \{A \subset E : A$ is closed and $A = -A\}$ (i.e., $\mathcal{A}_{cs}(E)$ is the family of all closed symmetric subsets of E). A nonempty subset $A \in \mathcal{A}_{cs}(E)$ is said to have Krasnoselskii's genus k (write $\gamma(A) = k$), if k is the smallest integer with the property that there exists an odd continuous map $h : A \to \mathbb{R}^k \setminus \{0\}$. If no such k exists we set $\gamma(A) = +\infty$ and if $A = \emptyset$, we set $\gamma(A) = 0$.

The following deformation lemma plays a very important role to obtain minimax characterizations of critical points for locally Lipschitz functionals.

Lemma 2.1 ([6]). *If $I : X \to \mathbb{R}$ satisfies the nonsmooth C-condition, then for any $\varepsilon_0 > 0$ and for any neighborhood U of K_c^I (if $K_c^I = \varnothing$), there exist $\varepsilon \in (0, \varepsilon_0)$ and a continuous map $\eta : [0,1] \times X \to X$ such that for all $(t, x) \in [0,1] \times X$, we have*

- (i) $\|\eta(t,x) - x\|_X \leq (1 + e)(1 + \|x\|_X)t$;
- (ii) *if* $|I(x) - c| \geq \varepsilon_0$ *or* $m^I(x) = 0$, *then* $\eta(t,x) = x$;
- (iii) $\eta(\{1\} \times I^{c+\varepsilon}) \subset I^{c-\varepsilon} \cap U$;
- (iv) $I(\eta(t,x)) \leq I(x)$;
- (v) *if* $\eta(t,x) \neq x$, *then* $I(\eta(t,x)) < I(x)$;
- (vi) η *satisfies the semigroup property, i.e.,*

$$\eta(s,.) \circ \eta(t,.) = \eta(s + t,.) \quad \forall s,t \in [0,1],\ s + t \leq 1;$$

- (vii) *for any* $t \in [0,1]$, $\eta(t,.)$ *is a homeomorphism of* X;
- (viii) *if* I *is even, then for any* $t \in [0,1]$, $\eta(t,.)$ *is odd.*

Lemma 2.2 ([6]). *Let $A, B \in \mathcal{A}_{cs}$ and $h \in C(Y;Y)$ be an odd map. The following hold:*

- (i) $\gamma(A) \geq 0$ *and* $\gamma(A) = 0$ *if and only if* $A = \varnothing$;
- (ii) *if* $h(A) \subset B$, *then* $\gamma(A) \leq \gamma(B)$;
- (iii) *if* $A \subset B$, *then* $\gamma(A) \leq \gamma(B)$ *(monotonicity)*;
- (iv) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$ *and if* $\gamma(B) < +\infty$, *then* $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$ *(subadditivity)*;
- (v) $\gamma(A) \leq \gamma(\overline{h(A)})$ *(supervariance)*;

(vi) *if* $A \in \mathcal{A}_{cs}$ *is compact, then* $\gamma(A) < +\infty$ *and there exists* $\delta > 0$ *such that* $\gamma(A) = \gamma(\overline{A_\delta})$*, where* $A_\delta = \{y \in Y : d_Y(y, A) < \delta\}$ *(continuity);*

(vii) *for any* $t \in [0, 1]$*,* $\eta(t, .)$ *is a homeomorphism of* X*;*

(viii) *if* u *is a bounded symmetric neighborhood of the origin in* $\mathbb{R}^k$*, then* $\gamma(\partial U) = k$*;*

(ix) *if there exists an old homeomorphism mapping between* A *and* B*, then* $\gamma(A) = \gamma(B)$*.*

Let $\Gamma = \{f | f : E \to E$ is an old homeomorphism mapping, $f(\bar{B}_1) \subset h^{(0)}\}$. It is easy to see that if h satisfies $(H1)$, then $\Gamma \neq \varnothing$. Set $\Gamma_m = \{K \subset E | K$ is compact and symmetric on θ, and for each $f \in \Gamma, \gamma(K \cap f(S_1)) \geq m\}$, where m is an integer for $f \in \Gamma$. Since f is an old homeomorphism mapping, we have $f(0) = 0$. Then the closed set $f(S_1)$ is symmetric on θ and $0 \notin f(S_1)$.

Lemma 2.3. *Let* $dimE \geq m$ *and the functional* $h : E \to \mathbb{R}^1$ *satisfies* $(H1)$ *and* $(H2)$*. Then*

(i) $\Gamma_m \neq \varnothing$*;*

(ii) $\Gamma_{m+1} \subset \Gamma_m$*;*

(iii) $K \in \Gamma_m,\ A \in \mathcal{A}_{cs}(E),\ \gamma(A) \leq r < m \Rightarrow \overline{K - A} \in \Gamma_{m-r}$*;*

(iv) *If* $\varphi : E \to E$ *is an odd homeomorphism mapping and satisfies* $\varphi^{-1}(h^{(0)}) \subset h^{(0)}$*, then* $\varphi(K) \in \Gamma_m$*,* $\forall K \in \Gamma_m$*.*

Proof. (i) Take a m dimensional subspace E_0 satisfying $E_0 \subset E$. Since h satisfies hypothesis $(H2)$, we can choose sufficiently large R such that $E_0 \cap \overline{B}_R \supset E_0 \cap h^{(0)}$. Set $K_R = E_0 \cap \overline{B}_R$. Obviously, K_R is compact and symmetrical on θ. For any $f \in \Gamma$, there exist $E_0 \supset K_R \supset E_0 \cap h^{(0)} \supset E_0 \cap f(S_1)$. Consequently, $K_R \cap f(S_1) = E_0 \cap f(S_1)$. Noting that f is an odd homeomorphism mapping and $\theta \in f(B_1)$, $f(B_1)$ is an open set in E, and is symmetric on θ. Thus $E_0 \cap f(B_1)$, containing θ, is a symmetric open set on θ. It follows from $\partial(f(B_1)) = f(\partial B_1) = f(S_1)$, $\partial(E_0 \cap f(B_1)) \subset E_0 \cap \partial(f(B_1)) = E_0 \cap f(S_1)$ and Lemma 2.2 (vii) that

$$\begin{aligned} m = dimE_0 &= \gamma(\partial(E_0 \cap f(B_1))) \\ &\leq \gamma(E_0 \cap f(S_1)) = \gamma(K_R \cap f(S_1)), \end{aligned}$$

i.e., $K_R \in \Gamma_m$. Hence $\Gamma_m \neq \varnothing$.

(ii) is obvious.

(iii) It is easy to see that $\overline{K \setminus A}$ is a compact set and is symmetric on θ. For $f \in \Gamma$, one has

$$\begin{aligned} \overline{K \setminus A} \cap f(S_1) &\supset \overline{(K \cap f(S_1)) \setminus (A \cap f(S_1))} \\ &= \overline{(K \cap f(S_1)) \setminus A}. \end{aligned}$$

Since $\gamma(K \cap f(S_1)) \geq m$, Lemma 2.2(iv) deduces that

$$\begin{aligned} \gamma(\overline{K \setminus A} \cap f(S_1)) &\geq \overline{\gamma(K \cap f(S_1)) \setminus A} \\ &\geq \gamma(K \cap f(S_1)) - \gamma(A) \geq m - r. \end{aligned}$$

Thus $\overline{K \setminus A} \in \Gamma_{m-r}$.

(iv) It is obvious that $\varphi(K)$ is a compact set and is symmetric on θ. Since $\varphi : E \to E$ is a homeomorphism mapping,

$$\varphi(A_1 \cap A_2) = \varphi(A_1) \cap \varphi(A_2), \quad \forall A_1, A_2 \subset E.$$

For any $f \in \Gamma$ we have

$$\varphi(K \cap \varphi^{-1}(f(S_1))) = \varphi(K) \cap f(S_1), \tag{2.1}$$

which implies $\varphi^{-1}(f(S_1)) \in \mathcal{A}_{cs}(E)$, thus $K \cap \varphi^{-1}(f(S_1)) \in \mathcal{A}_{cs}(E)$. It follows from Lemma 2.2 (ix) that

$$\gamma(K \cap \varphi^{-1}(f(S_1))) = \gamma(\varphi(K \cap \varphi^{-1}(f(S_1)))). \tag{2.2}$$

Recalling that $\varphi^{-1}(h^{(0)}) \subset h^{(0)}$ and $f(\overline{B}_1) \subset h^{(0)}$, one has $\varphi^{-1}(f(\overline{B}_1)) \subset h^{(0)}$ and $\varphi^{-1}f \in \Gamma$. Consequently,

$$\gamma(K \cap \varphi^{-1}(f(S_1))) \geq m. \tag{2.3}$$

From (2.1)-(2.3) we derive $\gamma(\varphi(K) \cap f(S_1)) \geq m$, which means $\varphi(K) \in \Gamma_m$. Thus the proof is completed. □

Proof of Theorem 1.1. If $K \in \Gamma_m$, then K is a compact set. Hence $h(x)$ can attain its maximum value on K, which implies $b_m < +\infty$. On the other hand, setting $f_0(x) = \rho x \ \forall x \in E$, we have $f_0 \in \Gamma$ and $f_0(S_1) = S_\rho$. Consequently, for $K \in \Gamma_m$, $\gamma(K \cap S_\rho) \geq m$, and so $K \cap S_\rho \neq \emptyset$. From hypothesis $(H1)$, one has $b_m \geq \alpha$. If (ii) is true, then b_m is a critical value of h. Therefore (i) is proved. (In (ii) setting $r = 1$, we have $\gamma(K_{b_m}) \geq m$, whence $K_{b_m} \neq \emptyset$).

In the following, we prove (ii). Proceeding by contradiction, assume $\gamma(K_b) < r$. Since K_b is a compact set (h satisfies the nonsmooth C-condition), from Lemma 2.2 (vi), there exists $\delta > 0$ such that

$$\gamma(\overline{N_\delta(K_b)}) = \gamma(K_b) < r. \tag{2.4}$$

Hence, by Lemma 2.1, there exist $\epsilon > 0$ and an old homeomorphism mapping $\eta_1 : E \to E$ such that

$$\eta_1(h_{b+\epsilon} \setminus N_\delta(K_b)) \subset h_{b-\epsilon}. \tag{2.5}$$

From the definition of b_{m+r-1}, there is $K^* \in \Gamma_{m+r-1}$ such that

$$\max_{x \in K^*} h(x) < b_{m+r-1} + \epsilon = b + \epsilon. \tag{2.6}$$

It follows from Lemma 2.3 (iii) that $\overline{K^* \setminus \overline{N_\delta(K_b)}} \in \Gamma_m$. But $\overline{K^* \setminus \overline{N_\delta(K_b)}} = K^* \setminus N_\delta(K_b)$, then $K^* \setminus N_\delta(K_b) \in \Gamma_m$. (2.6) deduces $K^* \subset h_{b+\epsilon}$. By Lemma 2.1 (iv) and Lemma 2.3(iv), one has $\eta_1^{-1}(h^{(0)}) \subset h^{(0)}$ and $\eta_1(K^* \setminus N_\delta(K_b)) \in \Gamma_m$. Thus

$$\max_{x \in \eta_1(K^* \setminus N_\delta(K_b))} h(x) \geq b_m = b. \tag{2.7}$$

On the other hand, since $K^* \setminus N_\delta(K_b) \subset h_{b+\epsilon} \setminus N_\delta(K_b)$, from (2.5), we have

$$\eta_1(K^* \setminus N_\delta(K_b)) \subset \eta_1(h_{b+\epsilon} \setminus N_\delta(K_b)) \subset h_{b-\epsilon}.$$

Therefore

$$h(x) \leq b - \epsilon, \quad \forall x \in \eta_1(K^* \setminus N_\delta(K_b)),$$

which contradicts to (2.7). Then (ii) is proved.

We now prove (iii). By virtue of Lemma 2.3(ii), we can directly deduce that $b_m \leq b_{m+1}(m = 1, 2, ...)$, from which it follows that $\{b_m\}$ is an increase sequence.

Next we show $b_m \to +\infty$ as $m \to +\infty$. Suppose, by contradiction, that $b_m \to b^* < +\infty$. Since K_{b^*} is compact and $K_{b^*} \in \mathcal{A}_{cs}(E)$, it follows from Lemma 2.2(vi) that $\gamma(K_{b^*}) = s < +\infty$, and there exists $\delta > 0$ such that

$$\gamma(\overline{N_\delta(K_{b^*})}) = \gamma(K_{b^*}) = s. \tag{2.8}$$

According to Lemma 2.1, there exist $\epsilon > 0$ and an old homeomorphism mapping $\eta_1 : E \to E$ such that

$$\eta_1(h_{b^*+\epsilon} \setminus N_\delta(K_{b^*})) \subset h_{b^*-\epsilon}. \tag{2.9}$$

Since $b_m \to b^*$ we can find an integer n such that $b_n > b^* - \epsilon$. Recalling that $\{b_m\}$ is increasing, we have $b_{n+s} \leq b^* < b^* + \epsilon$. So there exists $K_* \in \Gamma_{n+s}$ such that

$$\max_{x \in K_*} h(x) < b^* + \epsilon,$$

i.e.,

$$K_* \subset h_{b^*+\epsilon}. \tag{2.10}$$

Furthermore, by Lemma 2.3 (iii) one has

$$K_* \setminus N_\delta(K_{b^*}) = \overline{K_* \setminus \overline{N_\delta(K_{b^*})}} \in \Gamma_m. \tag{2.11}$$

Lemma 2.1 (iv) deduces that $\eta_1^{-1}(h^{(0)}) \subset h^{(0)}$. Then $\eta_1(K_* \setminus N_\delta(K_{b^*})) \in \Gamma_n$,

$$\max_{x \in \eta_1(K_* \setminus N_\delta(K_{b^*}))} h(x) \geq b_n > b^* - \epsilon. \tag{2.12}$$

On the other hand, from (2.9) and (2.10) one has

$$\eta_1(K_* \setminus N_\delta(K_{b^*})) \subset \eta_1(h_{b^*+\epsilon} \setminus N_\delta(K_{b^*})) \subset h_{b^*-\epsilon}.$$

Thus

$$h(x) \leq b^* - \epsilon, \qquad \forall x \in \eta_1(K_* \setminus N_\delta(K_{b^*})),$$

which contradicts to (2.12). This means that $b_m \to +\infty$.

(iv) is a direct conclusion of (iii). Since $b_m \to +\infty$, f has infinitely many different critical values, and all are the critical values of h. Of course, the corresponding critical points of b_m are different. The proof is completed. □

3. Applications

In this section, we use Theorem 1.1 to prove that the following quasilinear Schrödinger differential inclusion has infinitely many solutions.

$$\begin{cases} -\,div(g^2(u)(\nabla u)) + g(u)g'(u)|\nabla u|^2 + a(x)u \in \partial F(x,u) \text{ a.e. in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $N \geq 3$, $g : \mathbb{R} \to \mathbb{R}^+$ is an even and differential function, $g'(t) \geq 0$ for all $t \geq 0$, $a(x)$ is a continuous function in Ω, Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$.

It is easy to see that the energy functional of problem (3.1) is

$$I(u) = \frac{1}{2}\int_\Omega g^2(u)|\nabla u|^2 dx + \frac{1}{2}\int_\Omega a(x)|u|^2 dx - \int_\Omega F(x,u)dx. \tag{3.2}$$

From (3.2), $I(u)$ may not be well defined in $H_0^1(\Omega)$ as the appearance of $g(u)$. In order to overcome this difficulty, we make a change of variable constructed by Shen and Wang [14]. Set

$$v = G(u) = \int_0^u g(t)dt.$$

Then

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega a(x)|G^{-1}(v)|^2 dx - \int_\Omega F(x, G^{-1}(v))dx.$$

Noting that g is nondecreasing and positive, we derive $|G^{-1}(v)| \leq \frac{1}{g(0)}|v|$. For this reason, J is well defined in $H_0^1(\Omega)$. We endow $H_0^1(\Omega)$ with the norm $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)dx$ for $u \in H_0^1(\Omega)$.

If u is a nontrivial solution of problem (3.1), then it should satisfy

$$\int_\Omega [g^2(u)\nabla u \nabla \varphi + g(u)g'(u)|\nabla u|^2\varphi + a(x)u\varphi - \xi(u)\varphi]dx = 0 \tag{3.3}$$

for some $\xi(u) \in \partial F(x,u)$ and all $\varphi \in C_0^\infty(\Omega)$. Set $\varphi = \frac{1}{g(u)}\psi$; then (3.3) is equivalent to

$$\langle v_n^*, \psi\rangle = \int_\Omega \left[\nabla v \nabla\varphi + a(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi - \frac{\xi(G^{-1}(v))}{g(G^{-1}(v))}\psi\right]dx = 0 \tag{3.4}$$

for some $v_n^* \in \partial J(v)$, $\xi(G^{-1}(v)) \in \partial F(x, G^{-1}(v))$, and all $\psi \in C_0^\infty(\Omega)$.

Thus, in order to find the nontrivial solutions of problem (3.1), it suffices to deal with the nontrivial solutions of the following differential inclusion

$$-\Delta v + a(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} \in \partial F(x, G^{-1}(v)).$$

From hypothesis $(F1)$ we obtain the following lemma.

Lemma 3.1. *The functions $g(t)$ and $G(t) = \int_0^t g(\rho)d\rho$ satisfy the following properties*

1. *$G(t)$ and $G^{-1}(s)$ are odd;*
2. *For all $t \geq 0$, $s \geq 0$, $G(t) \leq g(t)t$, $G^{-1}(s) \leq \frac{s}{g(0)}$;*
3. *For all $s \geq 0$, $\frac{G^{-1}(s)}{s}$ is nonincreasing, $\lim_{s\to 0}\frac{G^{-1}(s)}{s} = \frac{1}{g(0)}$, and $\lim_{s\to\infty}\frac{G^{-1}(s)}{s} = \frac{1}{A}$.*

Proof. (1) and (2) are immediately deduced by the definition of $G(t)$ and the differential mean value theorem. We now prove (3). By virtue of the rule of L'Hosptial rule we derive

$$\left(\frac{G^{-1}(s)}{s}\right)_s' = \left(\frac{G^{-1}(s)}{s}\right)_t' \frac{1}{g(t)} = \frac{1}{g(t)}\left(\frac{t}{G(t)}\right)_t' = \frac{1}{g(t)G^2(t)}(G(t) - g(t)t) \leq 0$$

for all $t \geq 0$, which shows that $\frac{s}{A} \leq G^{-1}(s) \leq \frac{s}{g(0)}$ from hypothesis $(F1)$. □

Using the nonsmooth mountain pass theorem [6], we can show that problem (3.1) has at least one nontrivial solution.

Proof of Theorem 1.2. We firstly claim that the functional J satisfies the mountain pass geometry. Indeed, from hypothesis $(F5)(i)$ there exists $\delta > 0$ such that

$$|F(x,u)| \leq \epsilon u^2 \quad \forall \epsilon > 0, \ |u| \leq \delta. \tag{3.5}$$

By hypothesis $(F3)$, for all $z \in \Omega \setminus D$ with $|D|_N = 0$(where $|.|_N$ denotes the Lebesgue measure on Ω), the function $u \mapsto F(x,u)$ is locally Lipschitz, and so, from Rademacher's theorem, it is almost everywhere differentiable. Moreover, at any such point $\rho \in \mathbb{R}$ of differentiability, we derive

$$\frac{d}{d\rho} F(x,\rho) \in \partial F(x,\rho)$$

(see Clarke [4, p.32]). Hence, from $(F3)$

$$\frac{d}{d\rho} F(x,\rho) \leq a + b|\rho|^q \quad \text{for } a.e. \ x \in \Omega.$$

Integrating this inequality on $[0,x]$ (without loss of generality, here we assume $x > 0$), one has

$$F(x,u) \leq a|u| + \frac{b}{q+1}|u|^{q+1} \quad \text{for } a.e. \ x \in \Omega.$$

This combining (3.5) deduces that

$$|F(x,u)| \leq \epsilon |u|^2 + c_\epsilon |u|^{q+1} \quad \text{for } a.e. \ x \in \Omega. \tag{3.6}$$

It follows from (3.6) and Lemma 3.1 that

$$\begin{aligned} J(v) &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega a(x)|G^{-1}(v)|^2 dx - \epsilon \int_\Omega (G^{-1}(v))^2 dx - c_\epsilon \int_\Omega |G^{-1}(v)|^{q+1} dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega a(x)\frac{v^2}{A^2} dx - \epsilon c_1 \|v\|^2 - c_\epsilon c_2 \|v\|^{q+1} \\ &\geq \frac{c_3}{2}\|v\|^2 - \epsilon c_1 \|v\|^2 - c_\epsilon c_2 \|v\|^{q+1}, \end{aligned}$$

where $c_1, c_2 > 0$, $c_3 = \min\left\{1, \frac{a_1}{2A^2}\right\}$. If we choose $\epsilon < \frac{c_3}{2c_1}$, then there exist $\alpha, \rho > 0$ such that $J(v) \geq \alpha$ for all $\|u\| = \rho$.

It follows from $(F4)$ that

$$\begin{aligned} G(t)^{\beta+1}\partial(G(t)^{-\beta}F(x,t)) &= -\beta g(t)F(x,t) + G(t)\partial F(x,t) \\ &\geq 0 \end{aligned}$$

for $t > M$, which implies $\partial(G(t)^{-\beta}F(x,t)) \geq 0$ for $t > M$. Because of hypothesis $(F0)$ and Rademacher's theorem, for a.e. $x \in \Omega$, the function $t \mapsto G(t)^{-\beta}F(x,t)$ is differentiable at a.e. $x \in \mathbb{R}^N$. Moreover at any differentiable point, we have

$$\frac{d}{dt}(G(t)^{-\beta}F(x,t)) \in \partial(G(t)^{-\beta}F(x,t)).$$

Hence

$$\frac{d}{dt}(G(t)^{-\beta}F(x,t)) \geq 0 \quad \text{for } a.e. \ x \in \Omega, \ t > M.$$

Integrating for M to t, we obtain

$$F(x,t) \geq c(G(t))^{\beta} \quad \text{for } a.e.\ x \in \Omega,\ t > M.$$

Therefore, for a.e. $x \in \Omega$ and all $s \geq 0$ we have $F(x, G^{-1}(s)) \geq c_4|s|^{\beta} - c_5$ for some $c_4, c_5 > 0$. So for each $t > 0$, choosing $v_0 \in H_0^1(\Omega)$ such that $\|v_0\| = 1$ and $v_0(x) > 0$, one has

$$\begin{aligned}
J(tv_0) &= \frac{t^2}{2}\int_{\Omega}|\nabla v_0|^2dx + \frac{1}{2}\int_{\Omega}a(x)|G^{-1}(tv_0)|^2dx - \int_{\Omega}F(x,tv_0)dx \\
&\leq \frac{t^2}{2}\int_{\Omega}|\nabla v_0|^2dx + \frac{1}{2}\int_{\Omega}a_2\frac{t^2v_0^2}{g^2(0)}dx - c_4\int_{\Omega}|tv_0|^{\beta}dx + c_5|\Omega| \\
&\leq \frac{c_6}{2}t^2\|v_0\|^2 - c_4\left(\int_{\Omega}|v_0|^{\beta}dx\right)t^{\beta} + c_5|\Omega| \\
&\leq \frac{c_6}{2}t^2 - c_4\left(\int_{\Omega}|v_0|^{\beta}dx\right)t^{\beta} + c_5|\Omega| \\
&\to -\infty \quad \text{as} \quad t \to +\infty \quad (\text{since} \quad \beta > 2),
\end{aligned}$$

where $c_6 = \max\left\{1, \frac{a_2}{g^2(0)}\right\}$, from which there exists $w \in H_0^1(\Omega)$ such that $\|w\| > \rho$ and $J(w) < 0$.

In the following, we will show that J satisfies the nonsmooth C-condition. Assume that $\{v_n\} \subset H_0^1(\Omega)$, $|J(v_n)| \to c$, $(1+\|v_n\|)m^J(v_n) \to 0(n = 1, 2, ...)$. Then for any $\psi \in C_0^{\infty}(\Omega)$ there exists some $v_n^* \in \partial J(v_n)$ and $\xi(G^{-1}(v_n)) \in \partial F(x, G^{-1}(v_n))$ such that

$$\begin{aligned}
\langle v_n^*, \psi\rangle &= \int_{\Omega}\left[\nabla v_n\nabla\psi + a(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi - \frac{\xi(G^{-1}(v_n))}{g(G^{-1}(v_n))}\psi\right]dx \\
&\leq o_n(1)\|\psi\| \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Since $C_0^{\infty}(\Omega)$ is dense in $H_0^1(\Omega)$, choosing $\psi = v_n$, we only need to prove to that $\{v_n\}$ is bounded.

$$\begin{aligned}
\langle v_n^*, v_n\rangle &= \int_{\Omega}\left[|\nabla v_n|^2 + a(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n - \frac{\xi(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\right]dx \\
&\leq o_n(1)\|v_n\| \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus, it follows from Lemma 3.1(2) and $(F4)$ that

$$\begin{aligned}
\beta c - \langle v_n^*, v_n\rangle &= \beta J(v_n) - \langle v_n^*, v_n\rangle \\
&= \frac{\beta-2}{2}\int_{\Omega}|\nabla v_n|^2dx + \int_{\Omega}a(x)G^{-1}(v_n)\left[\frac{\beta}{2}G^{-1}(v_n) - \frac{v_n}{g(G^{-1}(v_n))}\right]dx \\
&\quad + \int_{\Omega}\left[\frac{\xi(G^{-1}(v_n))v_n}{g(G^{-1}(v_n))} - \beta F(x, G^{-1}(v_n))\right]dx \\
&\geq \frac{\beta-2}{2}\left[\int_{\Omega}|\nabla v_n|^2dx + \int_{\Omega}a(x)|G^{-1}(v_n)|^2dx\right] \\
&\geq \frac{\beta-2}{2}\min\left\{1, \frac{a_1}{A}\right\}\|v_n\|^2,
\end{aligned}$$

which means that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Therefore J satisfies the nonsmooth C-condition. It follows from the nonsmooth mountain pass theorem that problem (3.1) has at least one nontrivial solution. Thus the proof is completed. □

Proof of Theorem 1.3. From Theorem 1.2 we already know that J satisfies the nonsmooth C-condition. So we only need to verify conditions $(H1)$ and $(H2)$. Since $G^{-1}(v)$ is odd, by hypothesis $(F6)$ we have $F(x, G^{-1}(-v)) = F(x, G^{-1}(v))$. Then $J(v)$ is an even function. We now prove that $J(v)$ satisfies $(H1)$ and $(H2)$. From Theorem 1.2 we already know that $(H1)$ is satisfied. So we only need to verify $(H2)$. It follows from $(F5)$ and $(F6)$ that

$$\lim_{u\to+\infty}\frac{F(x,-u)}{(-u)^2}=\lim_{u\to+\infty}\frac{F(x,u)}{u^2}=+\infty \text{ for } a.e. \ x\in\Omega, \tag{3.7}$$

which means that

$$\lim_{u\to-\infty}\frac{F(x,u)}{u^2}=+\infty \text{ for } a.e. \ x\in\Omega. \tag{3.8}$$

Suppose, by contradiction, that $(H2)$ is not satisfied. Then there exists a finite subspace E in $H_0^1(\Omega)$ such that $E\cap J^{(0)}$ is an unbounded set, where

$$J^{(0)}=\{v\in H_0^1(\Omega): J(v)\geq 0\}.$$

Therefore, there exists $v_n\in E$ satisfying $\|v_n\|\to+\infty$ such that

$$J(v_n)\geq 0 (n=1,2,...). \tag{3.9}$$

Let $t_n=\|v_n\|$, $y_n=\frac{1}{t_n}v_n\in X$. Then

$$v_n=t_ny_n,\ \|y_n\|=1\ (n=1,2,...).$$

Since E is finite, the unit sphere in E is compact. This means that $\{v_n\}$ has a convergent subsequence, still denoted by itself, $y_n\to y_0$ ($\|y_n-y_0\|\to 0$), $y_0\in X$, $\|y_0\|=1$. By Friedrichs inequality, we have $\|y_n-y_0\|_2\to 0$. Thus, there exists a subsequence of $\{y_n\}$ such that $y_n\to y_0$ a.e. in Ω.

Set $\Omega_0=\{x\in\Omega: y_0(x)\neq 0 \text{ and } y_n(x)\to y_0(x)\}$. Then $meas(\Omega_0)>0$. Set $a_0=(\int_{\Omega_0}(y_0(x))^2dx)^{\frac{1}{2}}$, then $a_0>0$. From (3.7) and (3.8) there exists $M_0>0$ such that

$$F(x,u)\geq cu^2, \qquad \forall |u|\geq M_0, \quad x\in\Omega. \tag{3.10}$$

Let $D_n=\{x\in\Omega: |G^{-1}(t_ny_n)|\geq M_0\}$, then $\Omega\setminus D_n=\{x\in\Omega: |G^{-1}(t_ny_n)|<M_0\}$. $D_n=D_n^{(1)}\cup D_n^{(2)}$, $D_n^{(1)}\cap D_n^{(2)}=\emptyset$, where $D_n^{(1)}=\{x\in\Omega: G^{-1}(t_ny_n)\geq M_0\}$ and $D_n^{(2)}=\{x\in\Omega: G^{-1}(t_ny_n)\leq -M_0\}$. By virtue of hypothesis $(F3)$ and Lemma 3.1, we have

$$\begin{aligned}\int_\Omega F(x,G^{-1}(t_ny_n))dx &= \int_{D_n}F(x,G^{-1}(t_ny_n))dx+\int_{\Omega\setminus D_n}F(x,G^{-1}(t_ny_n))dx\\ &\geq C_1\int_{D_n}|G^{-1}(t_ny_n)|^2dx-C_2\\ &\geq \frac{C_1}{A}t_n^2\int_{D_n}|y_n|^2dx-C_2,\end{aligned} \tag{3.11}$$

where $C_1 > 0$, $C_2 = 2\left(aM_0 + \frac{b}{q+1}M_0^{q+1} + C_1 M_0^2\right)|\Omega|$. Since

$$
\begin{aligned}
&\left|\left(\int_{D_n}(y_n(x))^2dx\right)^{\frac{1}{2}} - \left(\int_{D_n}(y_0(x))^2dx\right)^{\frac{1}{2}}\right| \\
\le&\left(\int_{D_n}(y_n(x)-y_0(x))^2dx\right)^{\frac{1}{2}} \\
\le&\left(\int_{\Omega}(y_n(x)-y_0(x))^2dx\right)^{\frac{1}{2}} \\
=&\|y_n - y_0\|_2 \to 0 \ (n\to\infty),
\end{aligned}
$$

there exists $N_1 > 0$ such that

$$
\left(\int_{D_n}(y_n(x))^2dx\right)^{\frac{1}{2}} > \left(\int_{D_n}(y_0(x))^2dx\right)^{\frac{1}{2}} - \frac{a_0}{4} \quad \forall n > N_1. \tag{3.12}
$$

Set $D_n^* = \cap_{k=n}^{\infty} D_k$ and $D^* = \cup_{n=1}^{\infty} D_n^*$, then $D_1^* \subset D_2^* \subset D_3^* \subset \cdots$ and $D_n \supset D_n^*$, $D_n^* \subset D^*$, $|D_n^*| \to |D^*|(n\to\infty)$. Thus there exists $N_2 > 0$ such that

$$
\begin{aligned}
\left(\int_{D_n}(y_0(x))^2dx\right)^{\frac{1}{2}} \ge& \left(\int_{D_n^*}(y_0(x))^2dx\right)^{\frac{1}{2}} \\
>& \left(\int_{D^*}(y_0(x))^2dx\right)^{\frac{1}{2}} - \frac{a_0}{4}, \quad \forall n > N_2.
\end{aligned}
$$

Applying the definition of Ω_0, we obtain $\Omega_0 \subset D^*$, which shows that

$$
\left(\int_{D^*}(y_0(x))^2dx\right)^{\frac{1}{2}} \ge \left(\int_{\Omega_0}(y_0(x))^2dx\right)^{\frac{1}{2}} = a_0. \tag{3.13}
$$

The above inequality deduces that $\left(\int_{D_n}(y_n(x))^2dx\right)^{\frac{1}{2}} > \frac{a_0}{2}$, $\forall n > N = \max\{N_1, N_2\}$. Then

$$
\int_{\Omega} F(x, G^{-1}(t_n y_n))dx \ge \frac{C_1 a_0^2}{4A}t_n^2 - C_4.
$$

So

$$
\begin{aligned}
J(v_n) =& J(t_n y_n) \\
=& \frac{t_n^2}{2}\int_{\Omega}|\nabla y_n|^2dx + \frac{1}{2}\int_{\Omega}a(x)|G^{-1}(t_n y_n)|^2dx - \int_{\Omega}F(x, G^{-1}(t_n y_n))dx \\
\le& \frac{t_n^2}{2}\int_{\Omega}|\nabla y_n|^2dx + \frac{1}{2}\int_{\Omega}a(x)|y_n|^2dx - \int_{\Omega}F(x, G^{-1}(t_n y_n))dx \\
\le& \frac{B}{2}t_n^2 - \frac{C_1 a_0^2}{4A}t_n^2 + C_2 \\
\to& -\infty \quad \text{as } t \to +\infty,
\end{aligned}
$$

where $B = \max\{1, a_2\}$ and $C_1 > \frac{2BA}{a_0}$, which means $\lim_{n\to\infty} J(v_n) = -\infty$, contradicting to (3.9). Thus the proof is finished. $\square$

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