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SIGN-CHANGING SOLUTIONS OF A DISCRETE FOURTH-ORDER LIDSTONE PROBLEM WITH THREE PARAMETERS

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract By combining the method of the invariant sets of descending flow with variational technique, we give a series of criteria in terms of different values of λ to ensure that a discrete fourth-order Lidstone problem with three parameters possesses at least four solutions. It is further shown that these four solutions consist of one sign-changing solution, one positive solution, one negative solution and one trivial solution. Finally, three examples are also provided to illustrate our theoretical results.

Keywords Sign-changing solutions, positive solutions, negative solutions, invariant sets of descending flow, discrete Lidstone boundary value problem.

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1. Introduction

Let a, b be two fixed integer numbers with $a < b$ and $[a+1, b+1] = \{a+1, a+2, \cdots, b+1\}$ represent a discrete segment. Consider the nonlinear discrete fourth-order Lidstone boundary value problem with three explicit parameters α, β and λ given by

$$\begin{aligned} &\Delta^4 x(n-2) + \alpha\Delta^2 x(n-1) - \beta x(n) = \lambda f(n, x(n)), \qquad n \in [a+1, b+1], \\ &x(a) = \Delta^2 x(a-1) = 0, \quad x(b+2) = \Delta^2 x(b+1) = 0. \end{aligned} \tag{1.1}$$

Here $f(n, x) : \mathbf{Z}\times\mathbf{R} \to \mathbf{R}$ is continuous with respect to x. Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$ and $\Delta^i x(n) = \Delta(\Delta^{i-1}x(n))$.

By a positive (negative) solution x of the BVP (1.1), we refer that a sequence $\{x(n)\}_{a-1}^{b+3} = x$ satisfies the BVP (1.1) with $x(n) > 0$ ($x(n) < 0$) for all $n \in [a+1, b+1]$. If there exist $i, j \in [a+1, b+1]$ such that $x(i) \cdot x(j) < 0$, then x is called a sign-changing solution.

Discrete boundary value problem emerges from real world problems. It is one of the most important topics in the qualitative theory of difference equations so that

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it has a long history of research as to the first study can retrospect as early as to 1968 [25]. Besides its theoretical interest, it also has many applications. It is widely employed as handy means to describe the processes in many fields such as computer science, mathematical biology, control systems, economics and so on, we refer the reader to [7,10,11,15,28,31,33,36,37] and the reference therein for a thoroughgoing overview.

Much interest has lately shown in discrete boundary value problems. Many authors have investigated discrete boundary value problems extensively by various techniques, such as topology method, upper and lower solution methods, fixed-point theory, critical point theory to study the existence, multiplicity, and uniqueness of solutions to boundary value problems, see [2, 3, 5, 16–19, 24, 32, 34, 38] and many works follow.

It is interesting to note that, among the numerous obtained results, there are many results that pay attention to the existence, multiplicity, and nonexistence of solutions to fourth-order difference equations derived from various discrete elastic beam problems. For example, [1] analyzed the existence, multiplicity, and nonexistence of nontrivial solutions to the BVP (1.1) by fixed point theorem and Leggett-Williams theorem. Depending on the critical point theory and monotone operator theory, [8] gave sufficient conditions for the existence and nonexistence of nontrivial solutions to the BVP (1.1). We also refer the interested reader to the papers [6,9] in which discrete fourth-order boundary problems with parameters have been investigated.

On the other hand, with the rapid development of critical point theory, it has become a powerful tool to deal with various discrete problems (see [4, 12, 20, 29, 35]). However, as mentioned to sign-changing solution, a special case of solutions as positive solution, to the best of our knowledge, it seems that there has no so many similar results. Meanwhile, our recent works [21–23, 30] have established criteria for the existence of multiple solutions, including sign-changing solution, to fourth-order difference equations with different boundary conditions via variational methods and the invariant sets of descending flow, which indicate that the method of invariant sets of descending flow plays an important role in dealing with sign-changing solutions.

Motivated by above comments, we decide to tackle the existence of sign-changing solutions as well as multiple solutions of the BVP (1.1) by variational methods together with invariant sets of descending flow. Compared to the previous results, some related works are generalized.

The brief outline of this paper is as follows: after this introduction, the variational functional, and the needed lemmas are given in Section 2. Section 3 displays the main results on the multiplicity of nontrivial solutions for the BVP (1.1) in terms of different values of λ and states their proofs in detail. Finally, several examples are provided in Section 4 to demonstrate our main results.

2. Preliminaries

In this section, we are going to construct the corresponding variational framework for the BVP (1.1) and give some basic lemmas.

First, we recall some notations from [26,27].

Let X be a real Hilbert space and $I : X \to \mathbf{R}$ be a continuously Fréchet differentiable functional, denoted by $I \in C^1(X, \mathbf{R})$. If $x_0 \in X$ such that $I'(x_0) = 0$, then x_0 is a critical point of I. We say I satisfies the Palais-Smale (PS condition) condition if any sequence $\{x^{(j)}\} \subset X$ such that $I(x^{(j)})$ is bounded and $I'(x^{(j)}) \to 0$ as $j \to \infty$, has a convergent subsequence. If any sequence $\{x^{(j)}\} \subset X$ such that $I'(x^{(j)}) \to c$ for some $c \in \mathbf{R}$ and $(1+\|x^{(j)}\|)I'(x^{(j)}) \to 0$ as $j \to \infty$ has a convergent subsequence. We say I satisfies the Cerami ($(C)_c$ for short) condition.

Now we introduce the following powerful theorem which is the main tool to prove our results.

Lemma 2.1 ([14, Theorem 3.2]). *Assume that I satisfies the PS condition on X and there exists a completely continuous operator S such that $I'(x) = x - S(x)$ for $x \in X$. Moreover, $S(\partial D_1) \subset D_1$ and $S(\partial D_2) \subset D_2$, where D_1 and D_2 are two open convex subsets of X with the properties $D_1 \bigcap D_2 \neq \emptyset$. If there exists a path $h : [0,1] \to X$ such that*

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1$$

and

$$\sup_{t\in[0,1]} I(h(t)) < \inf_{x\in \overline{D_1} \bigcap \overline{D_2}} I(x).$$

Then I has at least four critical points, $x_1 \in X \setminus (\overline{D_1} \bigcup \overline{D_2})$, $x_2 \in D_1 \setminus \overline{D_2}$, $x_3 \in D_2 \setminus \overline{D_1}$ and $x_4 \in D_1 \bigcap D_2$.

Remark 2.1. Theorem 5.1 in [13] shows that the usual PS condition in Lemma 2.1 can be substituted by the weaker $(C)_c$ condition.

In the sequel, we define a $(b-a+1)$-dimensional Hilbert space

$$X = \{x = \{x(n)\} | x(n) \in \mathbf{R}, \quad n \in [a+1, b+1]\}$$

equipped with the inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$ as

$$(x,y) = \sum_{n=a+1}^{b+1} x(n)y(n), \qquad \|x\| = \left(\sum_{n=a+1}^{b+1} |x(n)|^2\right)^{\frac{1}{2}} \qquad \forall x, y \in X,$$

respectively. For later use, given $1 \leq p < +\infty$, for any $x \in E$, define the norm

$$\|x\|_p = \left(\sum_{n=a+1}^{b+1} |x(n)|^p\right)^{\frac{1}{p}}, \qquad \forall x \in X.$$

Clearly, $\|x\| = \|x\|_2$. Moreover,

$$\|x\|_p = (b-a+1)^{\frac{2-p}{2p}} \|x\|. \tag{2.1}$$

Define another space

$$H = \{x : [a-1, b+3] \to \mathbf{R} | x(a) = \Delta^2 x(a-1) = 0, \quad x(b+2) = \Delta^2 x(b+1) = 0\}$$

equipped with the following inner product

$$\langle x, y\rangle = \sum_{n=a+1}^{b+1} [\Delta^2 x(n-1)\Delta^2 y(n-1) - \alpha \Delta x(n-1)\Delta y(n-1) - \beta x(n)y(n)], \quad x, y \in H.$$

Then the induced norm is

$$\|x\|_H = \left(\sum_{n=a+1}^{b+1} |\Delta^2 x(n-1)|^2 - \alpha|\Delta x(n-1)|^2 - \beta|x(n)|^2 \right)^{\frac{1}{2}}, \quad \forall x \in H.$$

Take account of the boundary conditions $x(a) = \Delta^2 x(a-1) = 0$ and $x(b+2) = \Delta^2 x(b+1) = 0$, that is,

$$x(a-1) = -x(a+1), \quad x(a) = 0, \quad x(b+2) = 0, \quad x(b+3) = -x(b+1), \tag{2.2}$$

it follows that $(H, \langle \cdot, \cdot \rangle)$ is also a $(b-a+1)$-dimensional Hilbert space. Thus, H is isomorphic to X, which means that $\|x\|_H$ is equivalent to $\|x\|$. Therefore, there and thereafter, we always deem $x \in H$ as an extension of $x \in E$ when it is needed.

Let $F(n,x) = \int_0^x f(n,s)ds$ for $n \in [a+1, b+1]$ and $x \in \mathbf{R}$. Consider the functional $J(x) : X \to \mathbf{R}$ defined by

$$\begin{aligned} J(x) =& \frac{1}{2} \sum_{n=a+1}^{b+1} [|\Delta^2 x(n-1)|^2 - \alpha|\Delta x(n-1)|^2 - \beta|x(n)|^2] - \lambda \sum_{n=a+1}^{b+1} F(n, x(n)) \\ =& \frac{1}{2}\|x\|_H^2 - \lambda \sum_{n=a+1}^{b+1} F(n, x(n)) \end{aligned} \tag{2.3}$$

with $x(a) = \Delta^2 x(a-1) = 0$ and $x(b+2) = \Delta^2 x(b+1) = 0$. Then the continuity of f guarantees that $J \in C^1(X, \mathbf{R})$.

To estimate J, give two $(b-a+1) \times (b-a+1)$ matrices A and B as

$$A = \begin{pmatrix} 5 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -4 & 5 \end{pmatrix}_{(b-a+1)\times(b-a+1)}$$

and

$$B = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}_{(b-a+1)\times(b-a+1)}.$$

Denote $M = A + \alpha B - \beta I$, where I is a $(b-a+1) \times (b-a+1)$ identity matrix. Then $J(x)$, defined by (2.3), can be expressed by

$$J(x) = \frac{1}{2} x^T M x - \lambda \sum_{n=a+1}^{b+1} F(n, x(n)), \quad \forall x \in X. \tag{2.4}$$

Moreover, the eigenvalues of M are

$$\omega_k = 16 \sin^4 \frac{k\pi}{2(b-a+2)} - 4\alpha \sin^2 \frac{k\pi}{2(b-a+2)} - \beta, \quad k = 1, 2, \cdots, b-a+1. \tag{2.5}$$

Denote $w = \sin \frac{k\pi}{2(b-a+2)}$, then ω_k can be rewritten by

$$\omega_k = 16w^4 - 4\alpha w^2 - \beta.$$

Simple computation gives that ω_k is strictly increasing if $8w > \alpha$. On the other hand, $16 \sin^4 \frac{\pi}{2(b-a+2)} > 4\alpha \sin^2 \frac{k\pi}{2(b-a+2)} + \beta$ ensures that

$$\omega_1 > 0.$$

Consequently, if the parameters α and β satisfy

$$\alpha < 8 \sin^2 \frac{\pi}{2(b-a+2)} \quad \text{and} \quad 16 \sin^4 \frac{\pi}{2(b-a+2)} > 4\alpha \sin^2 \frac{\pi}{2(b-a+2)} + \beta, \tag{2.6}$$

then matrix M possesses positive eigenvalues and the algebraic multiplicity of each eigenvalue ω_k, $1 \le k \le b-a+1$, is equal to 1. Therefore,

$$0 < \omega_1 < \omega_2 < \cdots < \omega_{b-a+1}. \tag{2.7}$$

Joint (2.3), (2.4), (2.7) with the definition of $\|\cdot\|_H$ and $\|\cdot\|$, we have

$$\sqrt{\omega_1}\|x\| \le \|x\|_H \le \sqrt{\omega_{b-a+1}}\|x\|, \quad \forall x \in X. \tag{2.8}$$

In the following, it is needed to show that all conditions given in Lemma 2.1 are fulfilled. So, firstly, we find a completely continuous operator S_λ such that $J'(x) = x - S_\lambda(x)$ for all $x \in X$.

Let $w(n) : [a+1, b+1] \to \mathbf{R}$, consider the following BVP

$$\begin{cases} \Delta^4 x(n-2) + \alpha \Delta^2 x(n-1) - \beta x(n) = w(n), & n \in [a+1, b+1] \\ x(a) = \Delta^2 x(a-1) = 0, \quad x(b+2) = \Delta^2 x(b+1) = 0. \end{cases} \tag{2.9}$$

Note that the BVP (2.9) can be written as linear algebra equations $Mx = w$, where $w = (w(a+1), w(a+2), \cdots, w(b+1))^T$. Thanks to (2.7), matrix M is nonsingular, then the unique solution of the BVP (2.9) is

$$x = M^{-1} w. \tag{2.10}$$

On the other hand, [8] has shown that the BVP (2.9) possesses a unique solution $x = \{x(n)\}_{a-1}^{b+3}$ in the form of

$$x(n) = \sum_{k=a+1}^{b+1} \left(\sum_{s=a+1}^{b+1} G_1(n,s) G_2(s,k) \right) w(k) = \sum_{k=a+1}^{b+1} \left(\sum_{s=a+1}^{b+1} G_2(n,s) G_1(s,k) \right) w(k) \tag{2.11}$$

and $x(a-1)=-x(a+1)$, $x(a)=0$, $x(b+2)=0$, $x(b+3)=-x(b+1)$ for $n\in[a+1,b+1]$. Here

$$G_i(n,k)=\frac{1}{\rho(1,0)\rho(b+2,a)}\begin{cases}\rho(n,a)\rho(b+2,k), & \text{if} \quad a\le n\le k\le b+1\\ \rho(k,a)\rho(b+2,n), & \text{if} \quad a\le k\le n\le b+1\end{cases} \tag{2.12}$$

with

$$\rho(n,k)=\begin{cases}\sin\varphi(n-k),\ \varphi\triangleq\arctan\dfrac{\sqrt{-r_i(r_i+4)}}{2+r_i}, & \text{if}-4\sin^2\dfrac{\pi}{2(b-a+2)}<r_i<0;\\ n-k, & \text{if} \quad r_i=0;\\ \gamma^{n-k}-\gamma^{k-n},\ \gamma\triangleq\dfrac{r_i+2+\sqrt{r_i(r_i+4)}}{2}, & \text{if} \quad r_i>0,\end{cases}$$

and $r_1=\frac{-\alpha+\sqrt{\alpha^2+4\beta}}{2}$, $r_2=\frac{-\alpha-\sqrt{\alpha^2+4\beta}}{2}$.

Write

$$G(n,k)=\sum_{k=a+1}^{b+1}G_1(n,s)G_2(s,k), \quad \forall n,k\in[a+1,b+1], \tag{2.13}$$

where $G_i(n,k)$, $i=1,2$, are given by (2.12). Then we have

Lemma 2.2 ([8, Lemma 2.2])**.** *If the parameters α and β satisfy*

$$\alpha^2+4\beta\ge 0. \tag{2.14}$$

Then $G(n,k)$, defined by (2.13), is the Green's function associated with the BVP (2.9) which satisfies $G(n,k)>0$ and $G(n,k)=G(k,n)$ for all $n,k\in[a+1,b+1]$.

Define an operator $K:X\to X$ by

$$(Kx)(n)=\sum_{n=1}^{N}G(n,k)x(k), \quad x\in X, \quad n\in[a+1,b+1]. \tag{2.15}$$

Owe to Lemma 2.2, the unique solution of the BVP (2.9) is

$$x(n)=\sum_{n=a+1}^{b+1}G(n,k)w(k),$$

together with (2.10), it follows that

$$K=M^{-1}.$$

Moreover, there holds

Lemma 2.3. *For real value parameter $\lambda\in(0,+\infty)$, define an operator $S_\lambda:X\to X$ as $S_\lambda=\lambda K\mathbf{f}$, where $\mathbf{f}x(n)=f(n,x(n))$, $x\in X$, $n\in[a+1,b+1]$. Then the operator S_λ is a completely continuous operator.*

Proof. For all $x \in X$, we have

$$(S_\lambda x)(n) = \lambda \sum_{n=a+1}^{b+1} G(n,k) f(k, x(k)), \quad x \in X, \quad n \in [a+1, b+1]. \tag{2.16}$$

Lemma 2.2 means that it is no harm in assuming $\bar{C} = \max_{n,k \in [a+1,b+1]} \{G(n,k)\} > 0$. Then for any $y, z \in X$ and $k \in [a+1, b+1]$,

$$|y(k) - z(k)| \le \|y - z\| \quad \text{and} \quad \|S_\lambda y - S_\lambda z\| \le \bar{C}\lambda \sum_{n=a+1}^{b+1} |f(k, y(k)) - f(k, z(k))|,$$

together with the continuity of $f(n,x)$ in x, which lead to the continuity of the operator S_λ.

Choose a bounded set $\tilde{X}$ such that $\tilde{X} \subset X$. Since X is a $(b-a+1)$-dimensional Hilbert space, it is enough to show that, for any $y \in \tilde{X}$, $S_\lambda(y)$ is bounded to accomplish the proof. For any $y \in \tilde{X}$, the boundedness of $\tilde{X}$ yields that there exists constant $\tilde{C} > 0$ such that $\|y\| \le \tilde{C}$, which means $|y(n)| \le \tilde{C}$. Then the continuity of $f(n,x)$ in x ensures that there exists a constant $\hat{C} > 0$ such that

$$|f(n, y(n))| \le \hat{C}, \quad \forall y \in \tilde{X}, \quad n \in [a+1, b+1].$$

Therefore, making use of (2.16), we get, for any $y \in \tilde{X}$,

$$\|S_\lambda y\| \le M\lambda \sum_{n=a+1}^{b+1} |f(s, y(s))| \le \hat{C}\lambda M(b-a+1),$$

that is, $S_\lambda(\tilde{X})$ is bounded in X. Thus the verification of Lemma 2.3 is finished. □

Remark 2.2. Since the operator equations $x = S_\lambda x$ and $K^{-1}x = \lambda \mathbf{f} x$ are equivalent, $x = \{x(n)\}_{a+1}^{b+1} \in X$ is a solution of the BVP (1.1) if and only if $x = \{x(n)\}_{a-1}^{b+3} \in H$ is a fixed point of the operator S_λ with $x(a-1) = -x(a+1)$, $x(a) = 0$, $x(b+2) = 0$, $x(b+3) = -x(b+1)$.

Now we declare that

Lemma 2.4. *Let $J(x)$ be defined by (2.3). Then $J'(x) = x - S_\lambda(x)$ for all $x \in X$.*

Proof. Notice that $\frac{\partial F(n,x)}{\partial x} = f(n,x)$. Then, for any $x, y \in X$ and every $n \in [a+1, b+1]$, applying Lagrange's mean value theorem to $F(n,x)$, it yields that there exists $\kappa(n) \in (0,1)$ such that

$$F(n, (x+y)(n)) - F(n, x(n)) = f(n, x(n) + \kappa(n) y(n)).$$

Hence

$$\begin{aligned} & J(x+y) - J(x) \\ = & \sum_{n=a+1}^{b+1} \left[|\Delta^2 x(n-1)|^2 \cdot |\Delta^2 y(n-1)|^2 - \alpha |\Delta x(n-1)| \cdot |\Delta^2 y(n-1)| - v\beta |x(n)| \cdot |y(n)| \right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{n=a+1}^{b+1}\left[|\Delta^2 y(n-1)|^2-\alpha|\Delta^2 y(n-1)|-\beta|y(n)|\right]\\
&-\lambda\sum_{n=a+1}^{b+1} f(n,x(n)+\kappa(n)y(n))y(n)\\
=&\langle x,y\rangle+\frac{1}{2}\|y\|_H^2-\lambda\sum_{n=a+1}^{b+1} f(n,x(n)+\kappa(n)y(n))y(n),
\end{aligned}
$$

together with the continuity of f, which implies that

$$
J(x+y)-J(x)-\langle x,y\rangle+\lambda\sum_{n=a+1}^{b+1} f(n,x(n))y(n)\to 0, \quad \text{as} \quad \|y\|_H\to 0.
$$

Thus,

$$
\langle J'(x),y\rangle=\langle x,y\rangle-\lambda\sum_{n=a+1}^{b+1} f(n,x(n))y(n), \quad \forall x,y\in X. \tag{2.17}
$$

On the other side, for any $x, y\in X$, make use of the boundary conditions (2.2), we have

$$
\begin{aligned}
&\sum_{n=a+1}^{b+1}\Delta^4 x(n-2)y(n)\\
=&\sum_{n=a+1}^{b+1}[\Delta^2 x(n)-2\Delta^2 x(n-1)+\Delta^2 x(n-2)]y(n)\\
=&\sum_{n=a+1}^{b+1}\Delta^2 x(n)y(n)-2\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)y(n)+\sum_{n=a+1}^{b+1}\Delta^2 x(n-2)y(n)\\
=&\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)y(n-1)-\Delta^2 x(a)y(a)+\Delta^2 x(b+1)y(b+1)-\sum_{n=a+1}^{b+1} v\Delta^2 x(n-1)y(n)\\
&-\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)y(n)+\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)y(n+1)+\Delta^2 x(b)y(b+1)\\
&-\Delta^2 x(a-1)y(a+1)\\
=&-\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)\Delta y(n-1)+\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)\Delta y(n)\\
=&\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)\Delta^2 y(n-1),
\end{aligned}
$$

and

$$
\sum_{n=a+1}^{b+1}\Delta^2 x(n-1)y(n)
$$

$$
\begin{aligned}
&=\sum_{n=a+1}^{b+1}\Delta x(n)y(n)-\sum_{n=a+1}^{b+1}\Delta x(n-1)y(n)\\
&=\sum_{n=a+1}^{b+1}\Delta x(n)y(n)+\Delta x(a)y(a)-\Delta x(b+2)y(b+2)-\sum_{n=a+1}^{b+1}\Delta x(n-1)y(n)\\
&=\sum_{n=a+1}^{b+1}\Delta x(n-1)y(n-1)-\sum_{n=a+1}^{b+1}\Delta x(n-1)y(n)\\
&=-\sum_{n=a+1}^{b+1}\Delta x(n-1)\Delta y(n-1).
\end{aligned}
$$

Thus, for any $x, y \in X$,

$$
\begin{aligned}
&\langle x-S_\lambda x, y\rangle\\
=&\langle x,y\rangle-\langle S_\lambda x,y\rangle\\
=&\langle x,y\rangle-\sum_{n=a+1}^{b+1}[\Delta^2 S_\lambda x(n-1)\Delta^2 y(n-1)-\alpha\Delta S_\lambda x(n-1)\Delta y(n-1)-\beta S_\lambda x(n)y(n)]\\
=&\langle x,y\rangle-\sum_{n=a+1}^{b+1}[\Delta^4 S_\lambda x(n-2)+\alpha\Delta^2 S_\lambda x(n-1)-\beta S_\lambda x(n)]y(n)\\
=&\langle x,y\rangle-\lambda\sum_{n=a+1}^{b+1}f(n,x(n))y(n)
\end{aligned}
$$

which leads to $J'(x) = x - S(x)$ for all $x \in X$. And the proof is completed. □

Remark 2.3. Remark 2.2 and Lemma 2.4 indicate that the critical points $x = \{x(n)\}_{a+1}^{b+1} \in X$ of $J(x)$ and the fixed points $x = \{x(n)\}_{a-1}^{b+3} \in H$ of the operator S_λ and the solutions of the BVP (1.1) are equivalent to each other. And we seek critical points of the functional $J(x)$ defined on X to obtain the solutions of the BVP (1.1).

3. Main results

With the help of above preparations, it is time for us to establish our main results and utilize Lemma 2.1 to provide their proofs in this section.

For convenience, there and thereafter, let parameters α, β always satisfy (2.6) and (2.14).

Theorem 3.1. *Assume that*

(F_1) $\max\limits_{n\in[a+1,b+1]}\limsup\limits_{x\to 0}|\frac{f(n,x)}{x}| = f_0 \in (0,+\infty)$;

(F_2) $\min\limits_{n\in[a+1,b+1]}\liminf\limits_{x\to \infty}\frac{f(n,x)}{x} = f_\infty \in (0,+\infty)$;

(F_3) *there exist constants $s > 2$ and $C > 0$ such that*

$$|f(n,x)| \le C\left(1+|x|^{s-1}\right), \quad \forall (n,x)\in[a+1,b+1]\times\mathbf{R}.$$

Let ω_1 and ω_{b-a+1} be defined by (2.5). If $\frac{\omega_1}{f_0} > \frac{\omega_{b-a+1}}{f_\infty}$, then, for every $\lambda \in \Lambda_1 \triangleq (\frac{\omega_{b-a+1}}{f_\infty}, \frac{\omega_1}{f_0})$, the BVP (1.1) has at least four distinct solutions: one is sign-changing, one is positive, one is negative and one is trivial.

Proof. We apply Lemma 2.1 to finish the proof by three steps.

Step 1 Assume (F_2) holds, then the functional $J(x)$ satisfies the PS condition.

Fix $\lambda \in \Lambda_1$. (F_2) means that there exists constant $\delta_1 > 0$ such that

$$\frac{f(n,x)}{x} \geq f_\infty, \quad |x| > \delta_1, \quad \forall n \in [a+1, b+1].$$

Choose a constant C_1 with $C_1 > \omega_{b-a+1}$. Note that $\lambda \in \Lambda_1 \triangleq (\frac{\omega_{b-a+1}}{f_\infty}, \frac{\omega_1}{f_0})$, then

$$f(n,x) \geq \frac{C_1}{\lambda} x, \quad |x| > \delta_1, \quad \forall n \in [a+1, b+1],$$

which indicates that

$$F(n,x) \geq \frac{C_1}{2\lambda} x^2, \quad |x| > \delta_1, \quad \forall n \in [a+1, b+1]. \tag{3.1}$$

Since $f(n,x)$ is continuous respect to x for all $x \in X$ and $\frac{\partial F(n,x)}{\partial x} = f(n,x)$, which ensures that $F(n,x)$ is continuous in x for all $x \in X$. Then there exists a constant $C_2 > 0$ such that

$$F(n,x) \geq \frac{C_1}{2\lambda} x^2 - C_2, \quad 0 \leq |x| \leq \delta_1, \quad \forall n \in [a+1, b+1]. \tag{3.2}$$

Combining (3.1) with (3.2), we get

$$F(n,x) \geq \frac{C_1}{2\lambda} x^2 - C_2, \quad \forall x \in \mathbf{R}, \quad \forall n \in [a+1, b+1]. \tag{3.3}$$

Hence, for sequence $\{x^{(i)}\}_{i \in \mathbf{N}} \subset X$, (2.4) and (3.3) lead to

$$\begin{aligned} J(x^{(i)}) &= \frac{1}{2} {x^{(i)}}^T M x^{(i)} - \lambda \sum_{n=a+1}^{b+1} F(n, x^{(i)}(n)) \\ &\leq \frac{\omega_{b-a+1}}{2} \|x^{(i)}\|^2 - \frac{C_1}{2} \|x^{(i)}\|^2 + C_2 \lambda (b-a+1) \\ &= \frac{1}{2} (\omega_{b-a+1} - C_1) \|x^{(i)}\|^2 + C_2 \lambda (b-a+1). \end{aligned} \tag{3.4}$$

In virtue of $C_1 > \omega_{b-a+1}$ and $J(x^{(i)})$ is bounded, (3.4) ensures the boundedness of $\{x^{(i)}\}_{i \in \mathbf{N}}$. Together with the finite-dimension of X, the PS condition is verified.

Step 2 Let

$$\Omega = \{x \in X : x \geq 0\} \quad \text{and} \quad -\Omega = \{x \in X : x \leq 0\}$$

represent the positive and the negative convex cones, respectively. Denote the distance in X with respect to $\| \cdot \|_H$ by

$$\operatorname{dist}_H(x, \pm\Omega) = \inf_{\varphi \in \pm\Omega} \|x - \varphi\|_H.$$

Then if the assumptions ($\mathbf{F_1}$) and ($\mathbf{F_3}$) are fulfilled, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$S_\lambda(\partial D_\varepsilon^-) \subset D_\varepsilon^- \quad \text{and} \quad S_\lambda(\partial D_\varepsilon^+) \subset D_\varepsilon^+,$$

where

$$D_\varepsilon^+ = \{x \in X : \mathrm{dist}_H(x, \Omega) < \varepsilon\}, \quad D_\varepsilon^- = \{x \in X : \mathrm{dist}_H(x, -\Omega) < \varepsilon\}.$$

Further, if $x \in D_\varepsilon^-$ ($x \in D_\varepsilon^+$) such that $J'(x) = 0$, then x corresponds to the negative (positive) solution of the BVP (1.1).

The proofs of the case of D_ε^- and D_ε^+ are similar, here we state that of D_ε^- in detail for brevity.

Thanks to ($\mathbf{F_1}$), ($\mathbf{F_3}$) and the continuity of $f(n,x)$, for any given $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f(n,x)| \le \delta|x| + C_\delta|x|^{s-1}, \quad \forall (n,x) \in [a+1, b+1] \times \mathbf{R}. \tag{3.5}$$

For any $x \in X$, denote $x^+ = \{x^+(n)\}_{a+1}^{b+1}$, $x^- = \{x^-(n)\}_{a+1}^{b+1}$ where $x^+(n) = \max\{x(n), 0\}$ and $x^-(n) = \min\{x(n), 0\}$. Combining (2.8) with the definition of $\mathrm{dist}_H(x, \pm\Omega)$, we have

$$\|x^+\| = \inf_{\varphi \in -\Omega} \|x - \varphi\| \le \frac{1}{\sqrt{\omega_1}} \inf_{\varphi \in -\Omega} \|x - \varphi\|_H = \frac{1}{\sqrt{\omega_1}} \mathrm{dist}_H(x, -\Omega). \tag{3.6}$$

On the other hand, let $y = S_\lambda(x) \in X$, the fact $y^+ = y - y^-$ and $y^- \in -\Omega$ yields

$$\mathrm{dist}_H(y, -\Omega) = \inf_{\varphi \in -\Omega} \|y - \varphi\|_H \le \|y - y^-\|_H = \|y^+\|_H. \tag{3.7}$$

Then, for $\lambda \in \Lambda_1$, we have

$$\begin{aligned}
&\mathrm{dist}_H(y, -\Omega)\|y^+\|_H \\
\le&\langle y^+, y^+\rangle \le \langle S_\lambda(x), y^+\rangle \\
=&\lambda \sum_{n=a+1}^{b+1} [f(n, x(n)), y^+(n)] \\
\le&\lambda \sum_{n=a+1}^{b+1} [f(n, x^+(n)), y^+(n)] \le \lambda \sum_{n=a+1}^{b+1} [(\delta|x^+(n)| + C_\delta|x^+(n)|^{s-1}, y^+(n))] \\
\le&\delta\lambda \left(\sum_{n=a+1}^{b+1} |x^+(n)|^2\right)^{\frac{1}{2}} \left(\sum_{n=a+1}^{b+1} |y^+(n)|^2\right)^{\frac{1}{2}} \\
&+ C_\delta\lambda \left(\sum_{n=a+1}^{b+1} |x^+(n)|^{\frac{(s-1)s}{s-1}}\right)^{\frac{s-1}{s}} \left(\sum_{n=a+1}^{b+1} |y^+(n)|^s\right)^{\frac{1}{s}} \\
=&\delta\lambda\|x^+\| \cdot \|y^+\| + C_\delta\lambda\|x^+\|_s^{s-1} \cdot \|y^+\|_s \\
\le&\frac{\delta\lambda}{\sqrt{\omega_1}}\|x^+\| \cdot \|y^+\|_H + \frac{C_\delta\lambda}{\sqrt{\omega_1}}(b-a+1)^{\frac{(2-s)(s-1)}{2s}}(b-a+1)^{\frac{2-s}{2s}}\|x^+\|^{s-1}\|y^+\|_H \\
=&\left(\frac{\delta\lambda}{\sqrt{\omega_1}}\|x^+\| + \frac{C_\delta\lambda}{\sqrt{\omega_1}}(b-a+1)^{\frac{2-s}{2}}\|x^+\|^{s-1}\right)\|y^+\|_H
\end{aligned}$$

$$\leq \left(\frac{\delta\lambda}{\omega_1}\mathrm{dist}_H(x,-\Omega) + \frac{C_\delta\lambda}{\sqrt{\omega_1}^s}(b-a+1)^{\frac{2-s}{2}}(\mathrm{dist}_H(x,-\Omega))^{s-1} \right) \|y^+\|_H. \tag{3.8}$$

Choose $\delta = \frac{D}{4\lambda}$ where $D = \min\{\sqrt{\omega_1}, \omega_1\}$, then (3.8) deduces

$$\mathrm{dist}_H(y,-\Omega) \leq \frac{1}{4}\mathrm{dist}_H(x,-\Omega) + C_\delta D^{-s}(b-a+1)^{\frac{2-s}{2}}(\mathrm{dist}_H(x,-\Omega))^{s-1}. \tag{3.9}$$

Make $0 < \varepsilon_0 < \left(4C_\delta D^{-s}(b-a+1)^{\frac{2-s}{2}}\right)^{\frac{1}{2-s}}$. If $\mathrm{dist}_H(x,-\Omega) \leq \varepsilon \leq \epsilon_0$, then (3.9) yields

$$\mathrm{dist}_H(S_\lambda(x),-\Omega) \leq \frac{1}{2}\mathrm{dist}_H(x,-\Omega) < \varepsilon \tag{3.10}$$

which implies that $S_\lambda(x) \in D_\varepsilon^-$ for any $x \in \partial D_\varepsilon^-$, namely, $S_\lambda(\partial D_\varepsilon^-) \subset D_\varepsilon^-$.

Moreover, if $x \in D_\varepsilon^-$ is nontrivial such that $J'(x) = 0$, then Lemma 2.4 means $0 = J'(x) = S_\lambda(x) - x$, that is, $S(x) = x$. Together with (3.10), we get $x \in -\Omega \backslash \{0\}$. Therefore, x is a negative solution of BVP (1.1).

Step 3 The last and most important thing is to find a path which meets with the assumptions in Lemma 2.1.

Since $\lambda \in \Lambda_1 \triangleq (\frac{\omega_{b-a+1}}{f_\infty}, \frac{\omega_1}{f_0})$, then $(\mathbf{F_1})$ means there exist constants $0 < \tilde{\epsilon} < 1$ and $\delta_2 > 0$ such that

$$|f(n,x)| \leq f_0(1-\tilde{\epsilon})|x| \leq \frac{\omega_1}{\lambda}(1-\tilde{\epsilon})|x|, \qquad |x| \leq \delta_2, \quad \forall n \in [a+1, b+1],$$

which arises

$$F(n,x) \leq \frac{\omega_1(1-\tilde{\epsilon})}{2\lambda}|x|^2, \qquad |x| \leq \delta_2, \quad \forall n \in [a+1, b+1].$$

For the same reason as (3.3), there exists a constant C_3 such that

$$F(n,x) \leq \frac{\omega_1(1-\tilde{\epsilon})}{2\lambda}|x|^2 + C_3, \qquad \forall x \in \mathbf{R}, \quad \forall n \in [a+1, b+1]. \tag{3.11}$$

Therefore, for any $x \in \mathbf{R}$ and all $n \in [a+1, b+1]$, there holds

$$\begin{aligned} J(x) &\geq \frac{\omega_1}{2}\|x\|^2 - \lambda \sum_{n=a+1}^{b+1} \frac{\omega_1(1-\tilde{\epsilon})}{2\lambda}|x|^2 - C_3\lambda(b-a+1) \\ &= \frac{\omega_1\tilde{\epsilon}}{2}\|x\|^2 - C_3\lambda(b-a+1). \end{aligned} \tag{3.12}$$

Note that for any $x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}$, in the same manner as (3.6), we get

$$\|x^\pm\| \leq \frac{1}{\sqrt{\omega_1}}\mathrm{dist}_H(x, \mp\Omega) \leq \frac{1}{\sqrt{\omega_1}}\varepsilon_0, \quad \forall x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}. \tag{3.13}$$

Consequently, (3.12) and (3.13) ensure $\inf\limits_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} J(x) = c_0$ for some constant $c_0 > -\infty$. Recall (3.3) and notice $C_1 > \omega_{b-a+1}$. Then for $\lambda \in \Lambda_1$,

$$\begin{aligned} J(x) &\leq \frac{\omega_{b-a+1}}{2}\|x\|^2 - \lambda \sum_{n=a+1}^{b+1} \left[\frac{C_1}{2\lambda}x^2 - C_2\right] \\ &= -\frac{1}{2}(C_1 - \omega_{b-a+1})\|x\|^2 + \lambda C_2(b-a+1) \end{aligned} \tag{3.14}$$

and $J(x) \to -\infty$ as $\|x\| \to +\infty$.

Let ν_1 and ν_2 stand for the eigenvectors associated with the eigenvalues ω_1 and ω_2 of matrix M and $X_1 = \text{span}\{\nu_1, \nu_2\}$. According to the equivalence of $\|\cdot\|$ and $\|\cdot\|_H$, (3.14) also implies that $J(x) \to -\infty$ as $\|x\|_H \to +\infty$ for $x \in X_1$. Therefore, it is not difficult to find a constant $\mu \gg \varepsilon_0$ large enough such that $J(x) < c_0 - 1$ with $\|x\|_H = \mu$. Define a path $h : [0,1] \to X_1$ as

$$h(t) = \mu \frac{(-1)^t \nu_1 + t(1-t)\nu_2}{\|(-1)^t \nu_1 + t(1-t)\nu_2\|_H}.$$

Obviously, $\|h\|_H = \mu$ and $h(t) \in X_1$. Hence,

$$\sup_{t\in[0,1]} J(h(t)) < c_0 - 1 < c_0 = \inf_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} J(x).$$

Further, simple calculation gives

$$h(0) = \mu \frac{\nu_1}{\|\nu_1\|_H} \quad \text{and} \quad h(1) = -\mu \frac{\nu_1}{\|\nu_1\|_H},$$

which means

$$h(0) \in D_\varepsilon^+ \setminus D_\varepsilon^- \quad \text{and} \quad h(1) \in D_\varepsilon^- \setminus D_\varepsilon^+.$$

Jointly with Lemma 2.3 and 2.4, all conditions in Lemma 2.1 are fulfilled. Therefore, Lemma 2.1 guarantees that J possesses at least four distinct critical points: one is trivial in $D_\varepsilon^+ \bigcap D_\varepsilon^-$, one is sign-changing in $E \setminus (\overline{D_\varepsilon^+} \bigcup \overline{D_\varepsilon^-})$, one is positive in $D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$ and one is negative in $D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$. Consequently, the BVP (1.1) admits at least one trivial solution and three distinct nontrivial solutions: one is sign-changing, one is positive and one is negative. Thus the verification of Theorem 3.1 is completed. □

Theorem 3.2. *Assume* (F_3) *holds. Further*
(F_4) $\lim_{x\to 0} \frac{f(n,x)}{x} = 0$, *uniformly for* $n \in [a+1, b+1]$;
(F_5) $\lim_{x\to +\infty} \frac{f(n,x)}{x} = +\infty$, *uniformly for* $n \in [a+1, b+1]$.
Then for all $\lambda \in \Lambda_2 \triangleq (0, +\infty)$, *the BVP (1.1) possesses at least one trivial solution and three nontrivial solutions which are composed of one sign-changing solution, one positive solution and one negative solution.*

Proof. Fix $\lambda \in \Lambda_2$. By $(\mathbf{F_5})$, there exist constant $C_4 > \frac{\omega_{b-a+1}}{2}$ and $C_5 > 0$ such that for all $x \in \mathbf{R}$ and $n \in [a+1, b+1]$

$$F(n, x(n)) \geq \frac{C_4}{\lambda} |x(n)|^2 - C_5. \tag{3.15}$$

Suppose $\{x^{(k)}\}_{k\in\mathbf{N}} \subset X$ be a PS sequence. Then $J(x^{(k)})$ is bounded for all $k \in \mathbf{N}$ and $J(x^{(k)}) \to 0$ as $k \to \infty$. On account of X is a $(b-a+1)$-dimensional space, it suffices to prove $\{x^{(k)}\}$ is bounded. By (2.4) and (3.15), it follows that

$$\begin{aligned} J(x) &= \frac{1}{2} {x^{(k)}}^T M x^{(k)} - \sum_{n=a+1}^{b+1} F(n, x^{(k)}(n)) \\ &\leq (\frac{\omega_{b-a+1}}{2} - C_4) {\|x^{(k)}\|}^2 + C_5 \lambda (b-a+1). \end{aligned} \tag{3.16}$$

Since $J(x^{(k)})$ is bounded, then (3.16) means $\{x^{(k)}\}$ is bounded. Thus $J(x)$ satisfies the PS condition.

For $\lambda \in \Lambda_2$, owing to $(\mathbf{F_4})$, there exist constants $\delta_3 > 0$ and $0 < \epsilon < \omega_1$ such that

$$F(n, x(n)) \le \frac{\epsilon}{2\lambda}|x(n)|^2, \quad \forall |x| < \delta_3, \quad n \in [a+1, b+1], \tag{3.17}$$

which leads to

$$J(x) = \frac{1}{2}x^T Mx - \sum_{n=a+1}^{b+1} F(n, x(n)) \ge \frac{\omega_1}{2}\|x\|^2 - \frac{\epsilon}{2}\|x\|^2 = \frac{\omega_1 - \epsilon}{2}\|x\|^2. \tag{3.18}$$

Similar to (3.6), we have

$$\|x^{\pm}\| \le \frac{1}{\sqrt{\omega_1}}\mathrm{dist}_H(x, \mp\Omega) \le \frac{1}{\sqrt{\omega_1}}\xi, \quad \forall x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}. \tag{3.19}$$

Therefore, by (3.18) and (3.19), we can draw a conclusion that there exists $\check{c_0} \ge 0$ such that $\inf\limits_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} J(x) = \check{c_0}$. Make use of (3.15) again, we have

$$J(x) = \frac{1}{2}x^T Mx - \sum_{n=a+1}^{b+1} F(n, x(n)) \le (\frac{\omega_{b-a+1}}{2} - C_4)\|x\|^2 + C_5\lambda(b-a+1). \tag{3.20}$$

In virtue of $\frac{\omega_{b-a+1}}{2} - C_4 < 0$, (3.20) deduces $J(x) \to -\infty$ as $\|x\| \to +\infty$.

The remain proof is similar to that of Theorem 3.1. □

Corollary 3.1. *Assume* (F_3) *and* (F_4) *hold. Replace* (F_5) *by*
(F_6) *There exist constants* $\rho > 0$ *and* $\theta > 2$ *such that*

$$0 < \theta F(n, x) \le x \cdot f(n, x), \qquad |x| \ge \rho, \quad n \in [a+1, b+1]. \tag{3.21}$$

Then conclusions in Theorem 3.2 are still true.

Proof. According to Theorem 3.2, it suffices to show (F_5) is true under the condition (F_6). In fact, if (F_6) holds, that is,

$$\frac{f(n, x)}{F(n, x)} \ge \frac{\theta}{x}, \qquad \forall n \in [a+1, b+1], \quad |x| \ge \rho. \tag{3.22}$$

Integrating both sides of (3.22), it follows that there exists constant $c > 0$ such that

$$F(n, x) \ge c|x|^\theta, \qquad \forall n \in [a+1, b+1], \quad |x| \ge \rho,$$

which implies that

$$|f(n, x)| \ge c\theta|x|^{\theta-1}, \qquad \forall n \in [a+1, b+1], \quad |x| \ge \rho.$$

Since $\theta > 2$, it yields that

$$\lim_{x \to +\infty} \frac{f(n, x)}{x} = \lim_{x \to +\infty} \frac{|f(n, x)|}{|x|} \ge \lim_{x \to +\infty} \frac{c\theta|x|^{\theta-1}}{|x|} = \lim_{x \to +\infty} c\theta|x|^{\theta-2} = +\infty.$$

Therefore, (F_5) in Theorem 3.2 is valid. And the proof is completed. □

Theorem 3.3. *Assume* (F_3) *and* (F_6) *hold. Further*
(F_7) *there exist constants* $\vartheta, \sigma > 2$ *and* $k, l > 0$ *such that*

$$k|x|^{\vartheta} \leq F(n,x) \leq l|x|^{\sigma}, \quad \forall x \in \mathbf{R}, \quad n \in [a+1, b+1].$$

Then for all $\lambda \in \Lambda_2 \triangleq (0, +\infty)$, *the BVP (1.1) admits at least four solutions which contain one sign-changing solution, one positive solution, one negative solution and one trivial solution.*

Proof. First, we show that $J(x)$ satisfies the PS condition. Recall the definition of $\|\cdot\|_H$ and $J(x) \in C^1(X, \mathbf{R})$, then for any $x \in X$

$$\begin{aligned}(J'(x), x) =& \sum_{n=a+1}^{b+1} \left[|\Delta^2 x(n-1)|^2 - \alpha |\Delta x(n-1)|^2 - \beta |x(n)|^2 \right] - \lambda \sum_{n=a+1}^{b+1} f(n, x(n)) x(n) \\ =& \|x\|_H^2 - \lambda \sum_{n=a+1}^{b+1} f(n, x(n)) x(n).\end{aligned} \tag{3.23}$$

In virtue of (F_6) and the continuity of $F(n,x) - \frac{1}{\theta} x \cdot f(n,x)$ respect to $x \in [-\rho, \rho]$, there exists $C_6 > 0$ such that

$$F(n,x) \leq \frac{1}{\theta} x \cdot f(n,x) + C_6, \quad x \in \mathbf{R}, \quad n \in [a+1, b+1]. \tag{3.24}$$

Set sequence $\{x^{(i)}\}_{i \in \mathbf{N}} \subset X$ such that $|J(x^{(i)})| \leq \bar{R}$ for some constant $\bar{R} > 0$ and $J'(x^{(i)}) \to 0$ as $i \to \infty$. For $\lambda \in \Lambda_2 \triangleq (0, +\infty)$, by (3.23) and (3.24), it is ease to get

$$\begin{aligned}\bar{R} \geq& J(x^{(i)}) = \frac{1}{2} \|x^{(i)}\|_H^2 - \lambda \sum_{n=a+1}^{b+1} F(n, x^{(i)}(n)) \\ \geq& \frac{1}{2} \|x^{(i)}\|_H^2 - \frac{\lambda}{\theta} \sum_{n=a+1}^{b+1} f(n, x^{(i)}(n)) x^{(i)}(n) - \frac{\lambda}{\theta} C_6 (b-a+1) \\ =& \frac{1}{2} \|x^{(i)}\|_H^2 - \frac{1}{\theta} (\|x^{(i)}\|_H^2 - (J'(x^{(i)}), x^{(i)})) - \frac{\lambda}{\theta} C_6 (b-a+1) \\ =& (\frac{1}{2} - \frac{1}{\theta}) \|x^{(i)}\|_H^2 + \frac{1}{\theta} (J'(x^{(i)}), x^{(i)}) - \frac{\lambda}{\theta} C_6 (b-a+1) \\ \geq& (\frac{1}{2} - \frac{1}{\theta}) \|x^{(i)}\|_H^2 - \frac{1}{\theta} \|J'(x^{(i)})\| \cdot \|x^{(i)}\|_H - \frac{\lambda}{\theta} C_6 (b-a+1).\end{aligned} \tag{3.25}$$

We claim that $\{x^{(i)}\}$ is bounded in X. Or else, without loss of generality, assume that $\lim_{i \to +\infty} \|x^{(i)}\|_H = \infty$. Then (3.25) means that

$$0 \leftarrow \frac{\bar{R}}{\|x^{(i)}\|_H^2} \geq (\frac{1}{2} - \frac{1}{\theta}) - \frac{\|J'(x^{(i)})\|}{\theta \|x^{(i)}\|_H} - \frac{\lambda C_6 (b-a+1)}{\theta \|x^{(i)}\|_H^2} \to (\frac{1}{2} - \frac{1}{\theta}), \quad \text{as } \|x^{(i)}\|_H \to \infty.$$

Meanwhile, $\theta > 2$ which contradicts $0 \geq (\frac{1}{2} - \frac{1}{\theta})$. Therefore, $\{x^{(i)}\}$ is bounded in $(b-a+1)$-dimensional space X and $J(x)$ satisfies the PS condition.

Next, to finish the similar proof to **Step 3** in Theorem 3.1, for simplicity, we only state the proofs of $J(x) \to -\infty$ as $x \to +\infty$ and there exists a constant $\tilde{c_0}$ such that $\inf_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} J(x) = \tilde{c_0}$.

By means of (F_7) and (2.1), for $\lambda \in \Lambda_2$, it yields

$$\begin{aligned} J(x) &= \frac{1}{2} x^T M x - \lambda \sum_{n=a+1}^{b+1} F(n, x(n)) \le \frac{1}{2} \omega_{b-a+1} \|x\|^2 - \lambda k \|x\|_\vartheta^\vartheta \\ &= \frac{1}{2} \omega_{b-a+1} \|x\|^2 - \lambda k (b-a+1)^{\frac{2-\vartheta}{2}} \|x\|^\vartheta \\ &\to -\infty, \quad \text{as} \quad \|x\| \to +\infty. \end{aligned} \tag{3.26}$$

On the other hand, thanks to (F_7), we have

$$\begin{aligned} J(x) &= \frac{1}{2} \|x\|_H^2 - \lambda \sum_{n=a+1}^{b+1} F(n, x(n)) \\ &\ge \frac{1}{2} \|x\|_H^2 - \lambda l \|x\|_\sigma^\sigma \\ &= \frac{1}{2} \|x\|_H^2 - \lambda l (b-a+1)^{\frac{2-\sigma}{2}} \|x\|^\sigma. \end{aligned} \tag{3.27}$$

Notice (3.6) means that $\|x^\pm\| \le \frac{1}{\sqrt{\omega_1}} \text{dist}_H(x, \mp\Omega) \le \frac{1}{\sqrt{\omega_1}} \varepsilon_0$ for all $x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}$. Thus the equivalence between $\|x\|_H$ and $\|x\|$ and (3.27) yield that there exists $\tilde{c_0} > -\infty$ such that $\inf_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} J(x) = \tilde{c_0}$.

Moreover, (F_7) can be rewritten as

$$k|x|^{\vartheta-2} \le \frac{F(n,x)}{|x|^2} \le l|x|^{\sigma-2}, \quad \forall x \in \mathbf{R} \setminus \{0\}, \quad n \in [a+1, b+1],$$

which means that

$$\lim_{x \to 0} \frac{f(n,x)}{x} = 0, \quad \forall n \in [a+1, b+1]. \tag{3.28}$$

Together with (F_3) and repeat the process of **Step 2** in Theorem 3.1, it yields the desired results similar to that in **Step 2** of Theorem 3.1.

Consequently, all conditions in Lemma 2.1 are fulfilled and Lemma 2.1 ensures that the results in Theorem 3.3 are correct. Thus the verification is completed. □

Take account of Remark 2.1, it is feasible to replace the PS condition by the $(C)_c$ condition. We have

Theorem 3.4. *Assume $f(n,x)$ satisfies the assumptions* $(\mathbf{F_1})$, $(\mathbf{F_3})$ *and*
$(\mathbf{F_8})$ *there exist constants $r, d > 0$ and $\tau > 2$ such that*

$$F(n,x) \ge r|x|^\tau - d, \quad \forall x \in \mathbf{R}, n \in [a+1, b+1].$$

Further, either
$(\mathbf{F_9})$ $\lim_{|x| \to +\infty} [xf(n,x) - 2F(n,x)] = +\infty$ *for all $n \in [a+1, b+1]$;*
or

$(\mathbf{F_{10}})$ $\lim\limits_{|x|\to+\infty}[xf(n,x)-2F(n,x)]=-\infty$ *for all* $n\in[a+1,b+1]$.

Then for $\lambda\in\Lambda_3=(0,\frac{\omega_1}{f_0})$*, the BVP (1.1) has at least four solutions including one sign-changing solution, one positive solution, one negative solution and one trivial solution.*

Proof. With the assumptions $(\mathbf{F_1})$, $(\mathbf{F_3})$, $(\mathbf{F_8})$, $(\mathbf{F_9})$ or $(\mathbf{F_1})$, $(\mathbf{F_3})$, $(\mathbf{F_8})$, $(\mathbf{F_{10}})$ the proof of Theorem 3.4 can be done by same manner. So we only state the proof under conditions $(\mathbf{F_1})$, $(\mathbf{F_3})$, $(\mathbf{F_8})$ and $(\mathbf{F_9})$.

First of all, we show $J(x)$ satisfies the $(C)_c$ condition with $(\mathbf{F_9})$.

Suppose sequence $\{x^{(j)}\}_{j\in\mathbf{N}}\subset X$ such that $J(x^{(j)})\to c$ for some $c\in\mathbf{R}$ and $(1+\|x^{(j)}\|)J'(x^{(j)})\to 0$ as $j\to\infty$. Thus there is no harm in supposing that there exists $R>2c+1$ such that

$$|J(x^{(j)})|\le\frac{R-1}{2}\quad\text{and}\quad(1+\|x^{(j)}\|)\|J'(x^{(j)})\|<1,\qquad j\in\mathbf{N}.\tag{3.29}$$

Note that X is an $(b-a+1)$-dimensional space, then the boundedness of $\{x^{(j)}\}_{j\in\mathbf{N}}$ guarantees that $J(x)$ satisfies the $(C)_c$ condition on X.

Combine (3.23) with (3.29), we have

$$\begin{aligned}&\lambda\sum_{n=1}^{N}\Big[f(n,x^{(j)}(n))x^{(j)}(n)-2F(n,x^{(j)}(n))\Big]\\&=2J(x^{(j)})-(J'(x^{(j)}),x^{(j)})\le 2|J(x^{(j)})|+\|J'(x^{(j)})\|\|x^{(j)}\|\\&\le 2|J(x^{(j)})|+(1+\|x^{(j)}\|)\|J'(x^{(j)})\|\le R.\end{aligned}\tag{3.30}$$

Now we claim that $\{x^{(j)}\}$ is bounded. Or else, suppose $\{x^{(j)}\}$ is unbounded, then there exists a subsequence of $\{x^{(j)}\}$, without loss of generality, we still denote it by $\{x^{(j)}\}$ for simplicity, and some $n_0\in[a+1,b+1]$ such that $|x^{(j)}(n_0)|\to+\infty$ as $j\to\infty$. For $\lambda\in\Lambda_3$, take account of $(\mathbf{F_9})$, it follows that

$$\lambda[f(n_0,x^{(j)}(n_0))x^{(j)}(n_0)-2F(n_0,x^{(j)}(n_0))]\to+\infty,\qquad j\to+\infty.$$

Use $(\mathbf{F_9})$ once more, the continuity of f and F means that there exists a constant $\bar{R}>0$ such that

$$f(n,x^{(j)}(n))x^{(j)}(n)-2F(n,x^{(j)}(n))\ge\bar{R},\qquad n\in[a+1,b+1],\quad x\in X.$$

Therefore,

$$\begin{aligned}&\lambda\sum_{n=a+1}^{b+1}\Big[f(n,x^{(j)}(n))x^{(j)}(n)-2F(n,x^{(j)}(n))\Big]\\&=\lambda\sum_{n=a+1}^{n_0-1}[f(n,x^{(j)}(n))x^{(j)}(n)-2F(n,x^{(j)}(n))]\\&\quad+\lambda[f(n_0,x^{(j)}(n_0))x^{(j)}(n_0)-2F(n_0,x^{(j)}(n_0))]\\&\quad+\lambda\sum_{n=n_0+1}^{b+1}\Big[f(n,x^{(j)}(n))x^{(j)}(n)+2F(n,x^{(j)}(n))\Big]\\&\ge\lambda(b-a)\bar{R}+\lambda[f(n_0,x^{(j)}(n_0))x^{(j)}(n_0)-2F(n_0,x^{(j)}(n_0))]\to+\infty.\end{aligned}\tag{3.31}$$

Obviously, (3.30) and (3.31) contradict each other. As a result, $\{x^{(j)}\}$ is bounded and $J(x)$ satisfies the $(C)_c$ condition.

In the following, owing to $(\mathbf{F_1})$, there exists constant $\delta_4 > 0$ such that

$$|f(n,x)| \leq f_0|x|, \qquad |x| \leq \delta_4, \quad \forall n \in [a+1, b+1].$$

Similar to (3.11), there exists constant C_7 such that

$$F(n,x) \leq \frac{f_0}{2}|x|^2 + C_7, \qquad \forall x \in \mathbf{R}, \quad \forall n \in [a+1, b+1]. \tag{3.32}$$

Therefore, for $\lambda \in \Lambda_3$,

$$\begin{aligned} J(x) &= \frac{1}{2}x^T Mx - \lambda \sum_{n=a+1}^{b+1} F(n, x(n)) \\ &\geq \frac{1}{2}\omega_1\|x\|^2 - \lambda \sum_{n=a+1}^{b+1} (\frac{f_0}{2}|x|^2 + C_7) \\ &= \frac{\omega_1 - \lambda f_0}{2}\|x\|^2 - \lambda(b-a+1)C_7. \end{aligned} \tag{3.33}$$

Notice $\lambda \in \Lambda_3$ guarantees $\omega_1 - \lambda f_0 > 0$ and (3.6) means

$$\|x^{\pm}\| \leq \frac{1}{\sqrt{\omega_1}}\mathrm{dist}_H(x, \mp\Omega) \leq \frac{1}{\sqrt{\omega_1}}\varepsilon_0, \quad \forall x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}.$$

Therefore, (3.33) deduces that there exists a constant $\hat{c_0}$ such that $\inf\limits_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} J(x) = \hat{c_0}$.

By $(\mathbf{F_8})$, for $\lambda \in \Lambda_3$,

$$\begin{aligned} J(x) &= \frac{1}{2}x^T Mx - \lambda \sum_{n=a+1}^{b+1} F(n, x(n)) \\ &\leq \frac{1}{2}\omega_{b-a+1}\|x\|^2 - \lambda \sum_{n=a+1}^{b+1} (r|x|^\tau - d) \\ &= \frac{1}{2}\omega_{b-a+1}\|x\|^2 - \lambda r(b-a+1)^{\frac{2-\tau}{2}}\|x\|^\tau + \lambda(b-a+1)d. \end{aligned} \tag{3.34}$$

Since $\tau > 2$, (3.34) leads to $J(x) \to -\infty$ as $\|x\| \to +\infty$.

According to the process of the proof of Theorem 3.1, the remain proof is similar so that won't be covered again here. And this completes the verification of Theorem 3.4. $\square$

4. Examples

Now we present three examples to explicate the the applications of our theoretical results.

Example 4.1. Consider the BVP (1.1) with $a = 0$, $b = 3$, $\alpha = -1$, $\beta = 0.25$ and

$$f(n,x) = p(n)\frac{x^2}{1+x^2} - \frac{16.3582x}{1+x^2} + 16.4582x, \tag{4.1}$$

where $p(n) : [a+1, b+1] \to \mathbf{R}$.

It is clear that parameters α, β always satisfy (2.6) and (2.14) and (4.1) fulfills $(\mathbf{F_3})$ with $f_0 = 0.1 \in (0, +\infty)$, $f_\infty = 16.4582 \in (0, +\infty)$. Moreover,

$$M = \begin{pmatrix} 6.75 & -5 & 1 & 0 \\ -5 & 7.75 & -5 & 1 \\ 1 & -5 & 7.75 & -5 \\ 0 & 1 & -5 & 6.75 \end{pmatrix}.$$

Then $\omega_1 = 0.2779$, $\omega_2 = 3.0418$, $\omega_3 = 9.2221$ and $\omega_4 = 16.4582$. According to Theorem 3.1, for $\lambda \in (1, 2.779)$, the BVP (1.1) admits at least four solutions: one is sign-changing, one is positive, one is negative and one is trivial.

To make the results more convenient to see, take $\lambda = 2.2 \in (1, 2.779)$ and $p(n) = n$, $n \in [1, 4]$. Then the BVP (1.1) is rewritten by

$$\begin{cases} \Delta^4 x(n-2) - \Delta^2 x(n-1) - 0.25x(n) = 2.2 \left(\dfrac{nx^2(n)}{1+x^2(n)} - \dfrac{16.3582x(n)}{1+x^2(n)} + 16.4582x(n) \right), \\ x(0) = \Delta^2 x(-1) = 0, \quad x(5) = \Delta^2 x(4) = 0, \qquad n \in [1, 4]. \end{cases} \tag{4.2}$$

With the aid of computer, we obtain the BVP (4.2) admits 13 real roots which are composed of 1 trivial solution, 1 positive solution, 1 negative solution and 10 sign-changing solutions. In detail, we list them as the following table:

Table 1. Solutions $x = \{x(n)\}_{n=-1}^{6}$ for the BVP (4.2)

$\sharp$	$x(-1)$	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$	character
1	0	0	0	0	0	0	0	0	trivial
2	-0.0067	0	0.0067	0.0110	0.0110	0.0068	0	-0.0068	positive
3	0.1155	0	-0.1155	-0.1770	-0.1563	-0.0861	0	0.0861	negative
4	-0.5928	0	0.5928	-0.5642	-0.5721	0.5112	0	-0.5112	sign-changing
5	0.3222	0	-0.3222	-0.2310	0.0654	0.2575	0	-0.2575	sign-changing
6	0.6972	0	-0.6972	0.6037	0.0847	-0.6294	0	0.6294	sign-changing
7	0.5380	0	-0.5380	0.1842	0.5816	-0.8075	0	0.8075	sign-changing
8	-0.6370	0	0.6370	-0.6799	-0.3079	0.4212	0	-0.4212	sign-changing
9	-0.4886	0	0.4886	-0.2663	-0.7088	0.5611	0	-0.5611	sign-changing
10	0.8205	0	-0.8205	0.9146	-1.0643	0.6860	0	-0.6860	sign-changing
11	-0.3605	0	0.3605	0.0489	-0.3645	-0.3728	0	0.3728	sign-changing
12	-0.7694	0	0.7694	-1.0381	0.8991	-0.9371	0	0.9371	sign-changing
13	0.6283	0	-0.6283	0.4336	0.4939	-0.7714	0	0.7714	sign-changing

Example 4.2. Let $a = 0$, $b = 4$, $\alpha = -3$, $\beta = -1$. Consider the following BVP

$$\begin{cases} \Delta^4 x(n-2) - 3\Delta^2 x(n-1) + x(n) = \lambda(1 + \sin^2 \dfrac{n\pi}{6})x^3(n), \qquad n \in [1, 5] \\ x(0) = \Delta^2 x(-1) = 0, \quad x(6) = \Delta^2 x(5) = 0. \end{cases} \tag{4.3}$$

Obviously, $f(n, x) = (1+\sin^2 \frac{n\pi}{6})x^3$ satisfies $(\mathbf{F_3})$ and $\lim\limits_{x\to 0} \frac{f(n,x)}{x} = 0$, $\lim\limits_{x\to +\infty} \frac{f(n,x)}{x} = +\infty$. Consequently, Theorem 3.2 ensures that the BVP (4.3) possesses at least four solutions: one trivial solution, one sign-changing solution, one positive solution and one negative solution.

To be more clearly, choose $\lambda = 0.5$, then the BVP (4.3) admits 21 real roots. Or, more specifically, here we list a few as following: trivial solution (0, 0, 0, 0, 0, 0, 0, 0, 0), positive solution (-0.6211, 0, 0.6211, 1.2854, 1.694, 1.354, 0.6939, 0, -0.6939), negative solution (0.6211, 0, -0.6211, -1.2854, -1.694, -1.354, -0.6939, 0, 0.6939) and sign-changing solutions (1.44E+57, 0, -1.44E+57, -4.36E+56, 2.85E+55, -3.45E+55, 3.7576, 0, -3.7576) and (-9.36E+38, 0, 9.36E+38, 2.45E+38, 1.86E+37, -3.32E+35, -2.9273, 0, 2.9273).

Example 4.3. Consider the BVP (1.1) in the form of

$$\begin{cases} \Delta^4 x(n-2) - 0.4\Delta^2 x(n-1) + 5x(n) = \lambda\left(x^3(n) + 2x(n)\right), \quad n \in [1,5] \\ x(0) = \Delta^2 x(-1) = 0, \quad x(6) = \Delta^2 x(5) = 0. \end{cases} \tag{4.4}$$

In view of (4.4), it is easy to get $\alpha = -0.4$, $\beta = -5$, $a = 0$, $b = 4$ and $f(n,x) = x^3 + 2x$ fulfills $(\mathbf{F_3})$ and $(\mathbf{F_8})$. Moreover,

$$M = \begin{pmatrix} 10.8 & -4.4 & 1 & 0 & 0 \\ -4.4 & 11.8 & -4.4 & 1 & 0 \\ 1 & -4.4 & 11.8 & -4.4 & 1 \\ 0 & 1 & -4.4 & 11.8 & -4.4 \\ 0 & 0 & 1 & -4.4 & 10.8 \end{pmatrix}$$

possesses five positive eigenvalues: $\omega_1 = 5.1790$, $\omega_2 = 6.4$, $\omega_3 = 9.8$, $\omega_4 = 15.2$ and $\omega_5 = 20.4210$. Further,

$$\max_{n\in[1,5]} \limsup_{x\to 0} |\frac{f(n,x)}{x}| = \lim_{x\to 0} \frac{x^3+2x}{x} = 2 = f_0 \in (0,+\infty)$$

and

$$\lim_{|x|\to+\infty} [xf(n,x) - 2F(n,x)] = \lim_{|x|\to+\infty} (x^4 + 2x^2 - \frac{x^4}{2} - 2x^2) = \lim_{|x|\to+\infty} \frac{x^4}{2} = +\infty.$$

Then all conditions of Theorem 3.4 are satisfied. As a result, Theorem 3.4 ensures that, for $\lambda \in \Lambda_3 = (0, \frac{5.1790}{2})$, the BVP (4.4) admits at least four solutions which contains a sign-changing solution, a positive solution, a negative solution and a trivial solution.

For more directly to see, let $\lambda = 2.4$, the BVP (4.4) has 19 real roots including 16 sign-changing solutions, 1 positive solution, 1 negative solution and 1 trivial solution. We display them by the following table.

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Table 2. Solutions $x = \{x(n)\}_{n=-1}^{7}$ for the BVP (4.4)

♯	$x(-1)$	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$	$x(7)$	character
1	0	0	0	0	0	0	0	0	0	trivial
2	-0.2131	0	0.2131	0.3911	0.4654	0.3911	0.2131	0	-0.2131	positive
3	0.2131	0	-0.2131	-0.3911	-0.4654	-0.3911	-0.2131	0	0.2131	negative
4	-3.02E+12	0	3.02E+12	-2.75E+11	5.50E+11	-2.41E+11	-1.8647	0	1.8647	sign-changing
5	3.02E+12	0	-3.02E+12	2.75E+11	-5.50E+11	2.41E+11	1.8647	0	-1.8647	sign-changing
6	-3.13E+22	0	3.13E+22	-4.75E+23	-2.20E+23	-6.40E+22	-2.2190	0	2.2190	sign-changing
7	3.13E+22	0	-3.13E+22	4.75E+23	2.20E+23	6.40E+22	2.2190	0	-2.2190	sign-changing
8	-2.0817	0	2.0817	-2.0817	0	2.0817	-2.0817	0	2.0817	sign-changing
9	2.0817	0	-2.0817	2.0817	0	-2.0817	2.0817	0	-2.0817	sign-changing
10	1.4434	0	-1.4434	0	1.4434	0	-1.4434	0	1.4434	sign-changing
11	-1.4434	0	-1.4434	0	-1.4434	0	1.4434	0	-1.4434	sign-changing
12	-2.34E+23	0	2.34E+23	-1.65E+23	-3.81E+22	2.05E+22	-2.2087	0	2.2087	sign-changing
13	2.34E+23	0	-2.34E+23	1.65E+23	3.81E+22	-2.05E+22	2.2087	0	-2.2087	sign-changing
14	-1048576	0	1048576	524288	1572864	131072	-1.6840	0	1.6839	sign-changing
15	1048576	0	-1048576	-524288	-1572864	-131072	1.6840	0	-1.6839	sign-changing
16	2.2605	0	-2.2605	2.6088	-2.6809	2.6088	-2.2605	0	2.2605	sign-changing
17	-2.2605	0	2.2605	-2.6088	2.6809	-2.6088	2.2605	0	-2.2605	sign-changing
18	-0.8165	0	0.8165	0.8165	0	-0.8165	-0.8165	0	0.8165	sign-changing
19	0.8165	0	-0.8165	-0.8165	0	0.8165	0.8165	0	-0.8165	sign-changing

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