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ON THE GLOBAL CENTER OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS AND THE RELATED PROBLEMS*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract In the paper we characterize planar polynomial differential systems with a global center, that is, every orbit of the system is a periodic orbit in $\mathbb{R}^2$. Further, we give algebraic sufficient and necessary conditions for potential systems and Liénard systems which have a global center, respectively. Last we discuss some related problems.

Keywords Planar polynomials, differential systems, a global center, sufficient and necessary conditions, integrable.

MSC(2010) 34C05, 34C08, 34C25.

1. Introduction

Consider planar polynomial differential systems

$$\dot{x} = P(x, y),\ \dot{y} = Q(x, y),\ (x, y) \in \mathbb{R}^2, \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are relatively prime real polynomials of degree m and n, respectively. Without loss of generality, we always assume that $n \geq m$.

A classical and difficult problem in the qualitative theory of planar polynomial differential systems is to give the global phase portraits of the systems having some center. The notion of center goes back to Poincaré and Dulac. A *center* is an equilibrium point p of system (1.1) in $\mathbb{R}^2$, which has a neighborhood U such that p is the unique equilibrium point in U and $U \setminus \{p\}$ is filled by nontrivial periodic orbits (closed orbits) of system (1.1) enclosing p. The center p is *global* if $\mathbb{R}^2 \setminus \{p\}$ is entirely filled by nontrivial periodic orbits. There are three kinds of centers: linear type, nilpotent and degenerate, see for instance [12]. More precisely, after moving the center to the origin of coordinates, and making a linear change of variables and a scaling of the time variable (if necessary), the planar polynomial differential system having a center at the origin can be written in one of the following three forms with polynomials $X_2(x, y)$ and $Y_2(x, y)$ starting at least with terms of second order:

$$\dot{x} = -y + X_2(x, y), \qquad \dot{y} = x + Y_2(x, y),$$

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which is called a *linear type center* or an *elementary center*;

$$\dot{x} = y + X_2(x, y), \qquad \dot{y} = Y_2(x, y),$$

which is called a *nilpotent center*;

$$\dot{x} = X_2(x, y), \qquad \dot{y} = Y_2(x, y),$$

which is called a *degenerate center*.

In [7] we characterized planar polynomial Hamiltonian systems with a global center, and showed that the global center of the systems with degree $n \geq 3$ can exhibit any one type of centers, respectively. The aim of this paper is to characterize planar polynomial differential systems with a global center. In particular, our goal is to give algebraic sufficient and necessary conditions for some polynomial systems with a global center. The algebraic sufficient and necessary conditions have been obtained for cubic homogeneous polynomial Hamiltonian systems with a global center in [7] and general quasi-homogeneous polynomial Hamiltonian systems with a global center in [2]. We here focus on potential systems and Liénard systems. In this study, the Poincaré compactification and the index theory of a planar polynomial vector field play a key role. From the results in [7] one can see that local dynamics of the system at infinity can determine if the unique equilibrium point p in $\mathbb{R}^2$ is a global center of the planar polynomial Hamiltonian system. And there are several kinds of global phase portraits in the Poincaré disc $\mathbb{D}$ for planar polynomial Hamiltonian systems with a global center, where $\partial\mathbb{D} = \mathbb{S}^1$ corresponds to the infinity of $\mathbb{R}^2$. Precisely speaking, the only difference between the corresponding global phase portraits of two Hamiltonian systems with a global center is the difference in their orbits at infinity, that is, the equator $\mathbb{S}^1$ is the closed orbit of the Hamiltonian system or not. If the equator $\mathbb{S}^1$ is not a closed orbit, then there are finitely many isolated infinite equilibrium points in $\mathbb{S}^1$ such that $\mathbb{S}^1$ is a *monodromic polycycle* (cf. [4]), in other words, $\mathbb{S}^1$ is divided into several open trajectories by finitely many infinite equilibrium points of system (1.1), where each infinite equilibrium point is the ω limit set of an open trajectory and the α limit set of another adjacent open trajectory, and every infinite equilibrium point has *only hyperbolic sectors*. If we *ignore the exact number* of isolated infinite equilibrium points in $\mathbb{S}^1$, then *there are only two kinds* of global phase portraits in the Poincaré disc $\mathbb{D}$ for planar polynomial Hamiltonian systems with a global center: one has a closed orbit $\mathbb{S}^1$ and the other has a monodromic polycycle $\mathbb{S}^1$. We now assume (possibly reverting t into $-t$) that, along $\mathbb{S}^1$, the orbit is oriented clockwise and there are two pairs of isolated infinite equilibrium points in $\mathbb{S}^1$. Accordingly, we can sketch the two kinds of global phase portraits in Figure 1 for planar polynomial Hamiltonian systems with a global center.

In this paper we prove that the conclusion about the two kinds of global phase portraits is also true for planar polynomial non-Hamiltonian systems with a global center, see Figure 1. Note that any one of the two kinds of local dynamics on the equator $\mathbb{S}^1$ can determine that the unique equilibrium point p in $\mathbb{R}^2$ is a center of a planar polynomial Hamiltonian system. However, the local dynamics on the equator $\mathbb{S}^1$ cannot determine whether the unique equilibrium point p in $\mathbb{R}^2$ is a center of a planar polynomial non-Hamiltonian system. For example, the equator $\mathbb{S}^1$ is a closed orbit of linear differential systems with a focus at p, but the unique equilibrium point p is not center. Hence, to characterize planar polynomial differential systems with

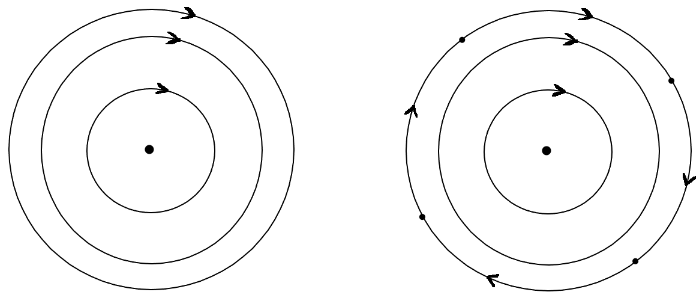

Figure 1. Two kinds of global phase portraits for planar polynomial systems with a global center.

a global center, we need to constrain the local dynamics of the unique equilibrium point p except the local dynamics of the system at the equator $\mathbb{S}^1$. Further, we study the algebraic sufficient and necessary conditions of the global center for the general potential systems and cubic generalized Liénard systems, respectively, and prove that the classic Liénard system can not have a global center.

This paper is organized as follows. In Section 2, we characterize planar polynomial differential systems with a global center. In Section 3, we investigate the existence of a global center for the potential systems and Liénard systems, obtain the algebraic sufficient and necessary conditions for the two kinds of polynomial differential systems, and prove that the cubic generalized Liénard systems are analytically integrable if this system has a global center. We end the paper with a brief discussion in last section, and some related problems are proposed.

2. The polynomial differential systems with a global center

In the section we suppose that system (1.1) has an equilibrium point at p in $\mathbb{R}^2$. Without loss of generality, we always assume that p is the origin $O = (0, 0)$. Then system (1.1) becomes

$$\dot{x} = P(x, y) = \sum_{i=1}^{m} p_i(x, y),\ \dot{y} = Q(x, y) = \sum_{j=1}^{n} q_j(x, y), \tag{2.1}$$

where $p_i(x, y)$ and $q_j(x, y)$ are the ith order and jth order homogeneous parts of polynomials $P(x, y)$ and $Q(x, y)$, respectively, and $n \geq m$.

To characterize system (2.1) with a global center, we have to study the dynamical behaviour of system (2.1) in a neighbourhood at infinity. The most natural way is to use the compactification by Poincaré sphere such that the equator $\mathbb{S}^1$ of Poincaré sphere corresponds to the infinity of $\mathbb{R}^2$. Using the Poincaré transformations

$$u = \frac{y}{x},\ z = \frac{1}{x};\quad \text{or } v = \frac{x}{y},\ z = \frac{1}{y} \tag{2.2}$$

and a scaling of the time variable by multiplying z^{n-1}, we can obtain the induced differential system for calculating infinite equilibrium points of system (2.1). The infinite equilibrium points are determined by real linear factors of $(n+1)$-th order homogeneous polynomial

$$yp_n(x,y) - xq_n(x,y), \text{ if } n = m, \tag{2.3}$$

or

$$-xq_n(x,y), \text{ if } n > m. \tag{2.4}$$

These infinite equilibrium points appear in pairs on opposite diametrically points of $\mathbb{S}^1$, the boundary of Poincaré disc $\mathbb{D}$.

The main result of this section is as follows.

Theorem 2.1. *System* (2.1) *has a global center if and only if the following two conditions hold.*

(i) System (2.1) *has a unique equilibrium point at the origin* $O = (0,0)$ *in* $\mathbb{R}^2$ *and it is a center.*

(ii) Either system (2.1) *has no infinite equilibrium points, that is the equator* $\mathbb{S}^1$ *is a closed orbit (see the left figure in Figure 1), or system* (2.1) *has finitely many infinite equilibrium points and the equator* $\mathbb{S}^1$ *is a monodromic polycycle (see the right figure in Figure 1).*

Proof. If system (2.1) has a global center, then from the definition of global center, we know that system (2.1) has no orbits in $\mathbb{R}^2$ whose α or ω limit set contains the infinite equilibrium points. Therefore, the equator $\mathbb{S}^1$ of Poincaré sphere is invariant under the flow of the induced system on the Poincaré disc. Note that system (2.1) is analytically equivalent to the induced system on the Poincaré disc. We denote the number of infinite equilibrium points of system (2.1) by $N(\infty)$. Then only three cases can occur: $N(\infty) = 0$, $N(\infty) = 2k$ with $1 \le k < +\infty$ and $N(\infty) = +\infty$.

If $N(\infty) = 0$, then $\mathbb{S}^1$ is a closed orbit. And if $N(\infty) = 2k$ then $\mathbb{S}^1$ is a monodromic polycycle by the continuous dependence of the orbit on the initial points in $\mathbb{D}$. This leads that condition (ii) holds. To prove the necessity, we only need to prove that $N(\infty) \neq +\infty$.

Assume that $N(\infty) = +\infty$. Then $n = m$, and every point at $\mathbb{S}^1$ is an infinite equilibrium point of system (2.1), that is, $\mathbb{S}^1$ is entirely filled by infinite equilibrium points of system (2.1). By Poincaré transformations (2.2) and $d\tau/dt = z^{1-n}$, system (2.1) can be transformed to

$$\begin{aligned} \frac{du}{d\tau} &= q_n(1,u) - up_n(1,u) + z\left(\sum_{i=1}^{n-1} z^{n-1-i}(q_i(1,u) - up_i(1,u))\right), \\ \frac{dz}{d\tau} &= -z\left(\sum_{i=1}^{n} z^{n-i}p_i(1,u)\right), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \frac{dv}{d\tau} &= p_n(v,1) - vq_n(v,1) + z\left(\sum_{i=1}^{n-1} z^{n-1-i}(p_i(v,1) - vq_i(v,1))\right), \\ \frac{dz}{d\tau} &= -z\left(\sum_{j=1}^{n} z^{n-j}q_j(v,1)\right). \end{aligned} \tag{2.6}$$

Since $\mathbb{S}^1$ is entirely filled by infinite equilibrium points of system (2.1),

$$q_n(1,u) - up_n(1,u) \equiv 0, \quad \text{and } p_n(v,1) - vq_n(v,1) \equiv 0.$$

This implies that system (2.5) is orbitally equivalent to the following system in the half-plane $z > 0$

$$\begin{aligned} \frac{du}{d\tilde{\tau}} &= \sum_{i=1}^{n-1} z^{n-1-i}(q_i(1,u) - up_i(1,u)), \\ \frac{dz}{d\tilde{\tau}} &= -\sum_{i=1}^{n} z^{n-i} p_i(1,u) = -p_n(1,u) - z\sum_{i=1}^{n-1} z^{n-i} p_i(1,u), \end{aligned} \tag{2.7}$$

Note that $\frac{dz}{d\tilde{\tau}}|_{z=0} = -p_n(1,u) \not\equiv 0$. This leads that the equator $\mathbb{S}^1$ is not invariant, which contradicts the definition of the global center. Thus, $N(\infty) \neq +\infty$, and the assertion of necessity follows.

We now prove the sufficiency. Since system (2.1) has a unique center at the origin O, there exists a neighborhood U surrounding the origin in Poincaré disc $\mathbb{D}$ such that every orbit passing through a point in $U \setminus \{O\}$ is a closed orbit of system (2.1) in U. We call the neighborhood U as *periodic annulus*. Let Ω be *the maximum periodic annulus* surrounding the origin in $\mathbb{D}$. Notice that system (2.1) has a unique equilibrium point in $\mathbb{R}^2$. Then

$$O \in \Omega \subseteq \mathbb{D},$$

and the boundary $\partial\Omega$ is either a closed orbit in the interior of $\mathbb{D}$, or $\partial\Omega = \mathbb{S}^1 = \partial\mathbb{D}$.

We claim it is impossible that $\partial\Omega$ is a closed orbit in the interior of $\mathbb{D}$. In fact, if $\partial\Omega$ is a closed orbit, we consider the Poincaré map $\mathcal{P}$ of system (2.1) associated to a transversal section to $\partial\Omega$. Then the map $\mathcal{P}$ is an analytic function with one argument because (2.1) is polynomial system. Note that $\mathcal{P}$ is an identical mapping defining in the part of the transversal section contained in the interior of Ω. Hence, there exists a tubular neighbourhood of $\partial\Omega$ such that every orbit through any a point in the tubular neighbourhood is a closed orbit of system (2.1). This is a contradiction to the definition of Ω, the maximum periodic annulus surrounding the origin.

Hence, $\partial\Omega = \mathbb{S}^1$. This implies that $\Omega = \mathbb{D}$. By the condition (ii), the origin O is a global center of system (2.1). □

Let us recall the characterization of planar polynomial Hamiltonian systems with global center in [7].

Proposition 2.1. *Assume that system* (2.1) *is a Hamiltonian system. Then it has a global center if and only if the following two conditions hold.*

- $(\tilde{i})$ *The Hamiltonian system has a unique equilibrium point at the origin* $O = (0,0)$ *in* $\mathbb{R}^2$.
- *(ii) Either the equator* $\mathbb{S}^1$ *is a closed orbit, or the equator* $\mathbb{S}^1$ *is a monodromic polycycle.*

Comparing Theorem 2.1 and Proposition 2.1, we can see the difference between Hamiltonian polynomial systems and non-Hamiltonian polynomial systems. The constraint on the local dynamics of the unique equilibrium point in the condition (i) is removed for Hamiltonian polynomial systems. The condition $(\tilde{i})$ in Proposition

2.1 is an algebraic condition. You may be wondering why Hamiltonian systems can remove this constraint. The reason is that the sum of indices at the infinite equilibrium points is equal to zero by condition (ii), which leads that the index of the unique equilibrium point is equal to one by Poincaré-Hopf theorem. From Proposition 2.1 in [1], an equilibrium point of Hamiltonian system is a center if the index of the equilibrium point is one.

For condition (ii), if the equator $\mathbb{S}^1$ is a closed orbit, we can obtain the algebraic condition: the $(n+1)$-th order homogeneous polynomial (2.3) does not have real linear factors. If the equator $\mathbb{S}^1$ is a monodromic polycycle, then the $(n+1)$-th order homogeneous polynomial (2.3) or (2.4) has and only has real linear factors with even multiplicity by calculation, but the complete algebraic conditions cannot be obtained for the existence of the monodromic polycycle. Hence, a necessary condition for the existence of a global center is the $(n+1)$-th order homogeneous polynomial (2.3) or (2.4) cannot have real linear factors with odd multiplicity. Note that the $(n+1)$-th order homogeneous polynomial (2.3) or (2.4) must have real linear factors with odd multiplicity if n is even. Therefore, the following corollary is obvious.

Corollary 2.1. *If* $n = \max\{m, n\}$ *which is even, then system* (2.1) *cannot have a global center.*

3. The algebraic sufficient and necessary conditions of the global center

In last section we characterize the polynomial differential system (2.1) with a global center, which satisfies the conditions (i) and (ii) in Theorem 2.1. Using the conditions (i) and (ii), we should study the local dynamics of system (2.1) at the unique equilibrium point O, and infinite equilibrium points, respectively. A natural question is asked whether we can give some conditions from the expressions of $P(x, y)$ and $Q(x, y)$ which discriminate if system (2.1) has a global center. Such conditions are called *algebraic condition* of the global center. In this section we apply Theorem 2.1 to study the algebraic sufficient and necessary conditions for two kinds of planar polynomial differential systems with a global center. One is the general potential system and the other is generalized Liénard system, and discuss the integrability of generalized Liénard system if the system has a global center.

3.1. The potential systems with a global center

Consider a general potential system with an equilibrium point at the origin $O = (0, 0)$

$$\begin{aligned} \dot{x} &= U(y), \\ \dot{y} &= -W(x), \end{aligned} \tag{3.1}$$

where $U(y)$ and $W(x)$ are real polynomials of degree m and n, respectively. Without loss of generality, we assume that $n \geq m$ and

$$U(y) = \sum_{i=m_0}^{m} u_i y^i, \quad W(x) = \sum_{j=n_0}^{n} w_j x^j, \ m_0, n_0 \in \mathbb{N},$$

where u_i and w_j are real coefficients, $m_0 \geq 1$, $n_0 \geq 1$, and $w_{n_0} \neq 0$ and $u_{m_0} > 0$. Our main result in this section is as follow.

Theorem 3.1. *System* (3.1) *has a global center at the origin* O *if and only if the following two conditions hold.*

(A) $U(y) = 0$ *and* $W(x) = 0$ *have a unique real root* $y = 0$ *and* $x = 0$*, respectively;*

(B) both n *and* m *are odd and* $w_n > 0$.

Proof. It is clear that system (3.1) is Hamiltonian system with Hamiltonian function

$$H(x,y) = \int_0^y U(s)ds + \int_0^x W(s)ds = \sum_{i=m_0}^{m} \frac{u_i}{i+1} y^{i+1} + \sum_{j=n_0}^{n} \frac{w_j}{j+1} x^{j+1}.$$

By Proposition 2.1, we need to check whether conditions (A) and (B) hold if O is a global center of system (3.1). Note that the condition $(\tilde{i})$ in Proposition 2.1 is exactly condition (A). Hence, to prove the necessity, we only need to show that the condition (ii) in Proposition 2.1 implies condition (B).

Assume the condition (ii) holds. Then either system (3.1) has no infinite equilibrium points or the indices at every infinite equilibrium points are equal to zero in the monodromic polycycle. What are infinite equilibrium points of system (3.1)? By equalities (2.3) and (2.4), we let

$$G(x,y) = \begin{cases} y \cdot u_n y^n - x \cdot (-w_n x^n) = w_n x^{n+1} + u_n y^{n+1}, & n = m, \\ -x \cdot (-w_n x^n) = w_n x^{n+1}, & n > m. \end{cases}$$

Then the infinite equilibrium points of system (3.1) correspond to the intersection points of $\mathbb{S}^1$ and the lines determined by the real linear factors of $G(x,y)$.

To calculate the indices at the infinite equilibrium points, we use Poincaré transformation $v = x/y, z = 1/y$ and time scaling $d\tau/dt = z^{1-n}$ for system (3.1) to induce a vector field as follows.

$$\begin{aligned} \frac{dv}{d\tau} &= z^n \Big(U\big(\frac{1}{z}\big) + vW\big(\frac{v}{z}\big)\Big) = \sum_{i=1}^{n} z^{n-i}(\tilde{u}_i + \tilde{w}_i v^{i+1}), \\ \frac{dz}{d\tau} &= z^{n+1} W\big(\frac{v}{z}\big) = \sum_{i=1}^{n} \tilde{w}_i z^{n+1-i} v^i, \end{aligned} \tag{3.2}$$

where

$$\tilde{u}_i = \begin{cases} u_i, & m_0 \leq i \leq m, \\ 0, & \text{others}, \end{cases}$$

and

$$\tilde{w}_i = \begin{cases} w_i, & n_0 \leq i \leq n, \\ 0, & \text{others}. \end{cases}$$

Thus, we discuss the indices of infinite equilibrium points in the two cases: $m = n$ and $n > m$, separately.

Case 1: $m = n$. Note that $m = n$ means $w_n u_n \neq 0$.

Obviously, if $G(v,1) = w_n v^{n+1} + u_n = 0$ has a nonzero real root v_0, then the equilibrium point $(v_0, 0)$ of system (3.2) is an infinite equilibrium point of system

(3.1). Note that $w_n v^{n+1} + u_n = 0$ has only simple nonzero roots when $w_n u_n \neq 0$ if it has. Assume that $(v_0, 0)$ with $v_0 \neq 0$ is an infinite equilibrium point of system (3.1). We now calculate the index of $(v_0, 0)$. Consider Jacobian matrix of system (3.2) at the point $(v, z) = (v_0, 0)$

$$\begin{bmatrix} (n+1)w_n v_0^n & * \\ 0 & w_n v_0^n \end{bmatrix},$$

which has two real eigenvalues with the same sign. This leads that $(v_0, 0)$ is a hyperbolic node of system (3.2). This implies the index of $(v_0, 0)$ is equal to one, which contradicts condition (ii). Hence, equation $G(v, 1) = w_n v^{n+1} + u_n = 0$ has no real roots. Note that $u_n > 0$ since $U(y)$ has a unique real zero $y = 0$ and $u_{m_0} > 0$. Thus, $w_n > 0$ and $m = n$ is odd. This is condition (B).

Case 2: $n > m$. When $n > m$, system (3.1) has only a pair of infinite equilibrium points which correspond to the intersection points of $\mathbb{S}^1$ and the line $x = 0$. And the infinite equilibrium point of system (3.1) corresponds to an equilibrium point $(v, z) = (0, 0)$ of system (3.2). Using the computing method of the index at an equilibrium point defined by Poincaré, we calculate the index of system (3.2) at $(v, z) = (0, 0)$.

Taking a small circle $S_\varepsilon = \{(v, z) : v^2 + z^2 = \varepsilon^2,\ 0 < \varepsilon \ll 1\}$ around $(v, z) = (0, 0)$ and choosing a direction vector $V = (1, 0)$, we consider the points at which the vector field of (3.2) is parallel to V, that is the points lie on the real algebraic curve

$$z^{n+1} W(\frac{v}{z}) = \sum_{i=1}^{n} \tilde{w}_i z^{n+1-i} v^i = \sum_{i=n_0}^{n} w_i z^{n+1-i} v^i = 0.$$

Notes that $W(x) = 0$ has the unique solution $x = 0$ in $\mathbb{R}$. Thus $z^{n+1} W(\frac{v}{z}) = 0$ on $\mathbb{R}^2$ is equivalent to $z = 0$ or $v = 0$. Therefore, near the intersection points of $z = 0$ and S_ε, the direction vector of system (3.2) on S_ε can be approximated by

$$(\frac{dv}{d\tau}, \frac{dz}{d\tau}) \approx (w_n v^{n+1}, w_n v^n z).$$

Near the intersection points of $v = 0$ and S_ε, the direction vector of system (3.2) on S_ε can be approximated by

$$(\frac{dv}{d\tau}, \frac{dz}{d\tau}) \approx (u_m z^{n-m}, w_{n_0} v^{n_0} z^{n+1-n_0}).$$

Note that $u_m > 0, w_{n_0} w_n > 0$ and $n - n_0$ is even since $U(y) = W(x) = 0$ has only the solution $(x, y) = (0, 0)$ on $\mathbb{R}^2$. Further, since the boundary $\mathbb{S}^1$ of Poincaré disc is a monodromic polycycle, system (3.2) on $z = 0$ satisfies that $\frac{dv}{d\tau} = w_n v^{n+1}$ dose not change its sign, so n and n_0 are both odd.

By discussing the sign of w_n and the parity of m, we easily obtain the four cases on direction vectors at the intersection points of curve $vz = 0$ and S_ε in counterclockwise sense, see Figure 2. And we obtain that the index at point $(v, z) = (0, 0)$ is

$$ind(0,0) = \begin{cases} 0, & m \text{ is odd and } w_n > 0, \\ +2, & m \text{ is odd and } w_n < 0, \\ +1, & \text{others.} \end{cases}$$

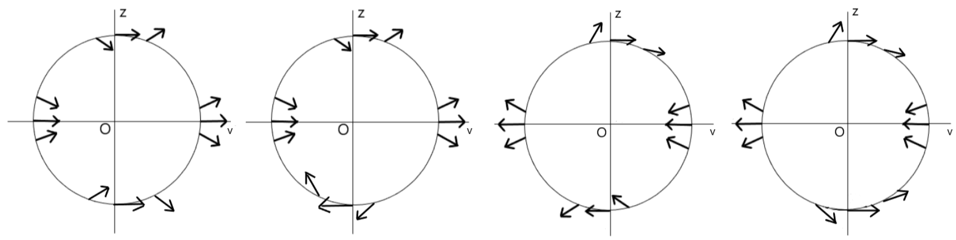

Figure 2. The changes on the direction of vector field (3.2) near the intersection points of S_ε and $vz = 0$.

Hence, both m and n are odd and $w_n > 0$ if the index of $(v, z) = (0, 0)$ is zero. This implies condition (B) holds.

Next, we prove the sufficiency. When $n = m$, the sufficiency can be obtained by Proposition 2.1 immediately since system (3.1) has no infinite equilibrium points if n is odd and $u_n > 0, w_n > 0$.

When $n > m$, system (3.1) has only a pair of infinite equilibrium points corresponding the equilibrium $(v, z) = (0, 0)$ of system (3.2). For system (3.2), we claim that there exists a neighborhood of $(v, z) = (0, 0)$ such that the ω or α limit set of any orbits except $z = 0$ in this neighborhood is not $(v, z) = (0, 0)$. That is for any point $p = (v_p, z_p)$ near $(v, z) = (0, 0)$ and $z_p \neq 0$, the integral curve $t \mapsto \gamma(t, p)$ which passes through p can not be the characteristic orbit of system (3.2).

If the claim is not true, then there exists a point $p = (v_p, z_p)$ near $(v, z) = (0, 0)$ such that $\gamma(t, p)$ is a characteristic orbit, i.e.

$$\gamma(t, p) = (v_\gamma(t, p), z_\gamma(t, p)) \to (0, 0), \quad \text{when } t \to +\infty$$

and

$$\frac{v_\gamma(t, p)}{z_\gamma(t, p)} \to \lambda, \quad \text{for some } \lambda \in [-\infty, +\infty].$$

Note that system (3.2) has a first integral

$$F(v, z) = \int_0^{\frac{1}{z}} U(s) ds + \int_0^{\frac{v}{z}} W(s) ds = \sum_{i=m_0}^{m} \frac{u_i}{i+1} \frac{1}{z^{i+1}} + \sum_{i=n_0}^{n} \frac{w_i}{i+1} \frac{v^{i+1}}{z^{i+1}}.$$

Let $K = F(v_p, z_p)$. Then

$$(F \circ \gamma)(t, p) = F(v_\gamma(t, p), z_\gamma(t, p)) = K < +\infty.$$

On the other hand, $u_m > 0$ since $u_{m_0} > 0$ and $U(y) = 0$ has a unique solution $y = 0$ on $\mathbb{R}$. Recall n, m are both odd and $u_m, w_n > 0$, hence we have

$$\begin{aligned} F(v_\gamma(t, p), z_\gamma(t, p)) &= \frac{u_m}{m+1} \frac{1}{z_\gamma(t, p)^{m+1}} + o\Big(\frac{1}{z_\gamma(t, p)^{m+1}}\Big) + \sum_{i=n_0}^{n} \frac{w_i}{i+1} \Big(\frac{v_\gamma(t, p)}{z_\gamma(t, p)}\Big)^{i+1} \\ &\to +\infty, \quad \text{when } t \to +\infty. \end{aligned}$$

It is a contradiction. In summary, the claim is true.

Hence, the equilibrium point $(v, z) = (0, 0)$ has only elliptic sectors or hyperbolic sectors and $z = 0$ is exactly a separatrix. Suppose the number of elliptic sectors and hyperbolic sectors of $(v, z) = (0, 0)$ are e and h, respectively. Thus $e + h = 2$. On the other hand, by Bendixson formula, we have

$$ind(0, 0) = 1 + \frac{e - h}{2}.$$

The index $ind(0, 0)$ is calculated in the proof of the necessity, and it is equal to 0. Hence, we have $e = 0, h = 2$, i.e. the neighbourhood of point $(x, z) = (0, 0)$ is exactly composed of two hyperbolic sectors. By Proposition 2.1, O is the global center of system (3.1). The proof is complete. □

Remark 3.1. The condition (A) is usually checked by some methods about symbolic computation, see [14] for detail.

3.2. Liénard systems with a global center

Consider generalized Liénard system

$$\begin{aligned} \dot{x} &= y - F(x) = y - \int_0^x f(s)ds, \\ \dot{y} &= -g(x), \end{aligned} \tag{3.3}$$

where $f(x)$ and $g(x)$ are real polynomials of variable x with degree m and n, respectively, and $f(0) = g(0) = 0$. Then

$$f(s) = \sum_{i=1}^{m} a_i s^i, \quad g(x) = \sum_{j=1}^{n} b_j x^j, \quad F(x) = \sum_{i=1}^{m} \frac{a_i}{i+1} x^{i+1},$$

where $a_i, b_i \in \mathbb{R}$, n and m are positive integers.

If $g(x) = x$, then system (3.3) becomes the classical Liénard system

$$\begin{aligned} \dot{x} &= y - F(x) = y - \int_0^x f(s)ds, \\ \dot{y} &= -x. \end{aligned} \tag{3.4}$$

In the subsection, we prove that classical Liénard system (3.4) cannot have a global center, and study the algebraic sufficient and necessary conditions of cubic generalized Liénard system (3.3) having a global center. And we show the cubic generalized Liénard system (3.3) is integrable if it has a global center. Our main results are as follows.

Theorem 3.2. *System* (3.4) *does not have a global center if* $f(s) \not\equiv 0$.

Theorem 3.3. *Assume that* $f(s) = a_1 s + a_2 s^2$ *and* $g(x) = b_1 x + b_2 x^2 + b_3 x^3$ *in system* (3.3). *Then the cubic system* (3.3) *has a global center at the origin* $O = (0, 0)$ *if and only if one of following conditions holds.*

- *(I)* $a_1 = a_2 = 0$, *and* b_i, $i = 1, 2, 3$, *satisfies one of the following conditions.*
  - *(I.1)* $b_1 > 0$, $b_2 = b_3 = 0$;
  - *(I.2)* $b_1 > 0, b_3 > 0$ *and* $b_2^2 < 4b_1 b_3$;

(I.3) $b_3 > 0$ *and* $b_1 = b_2 = 0$.

(II) $0 \neq a_1^2 < 8b_3$, $b_1 \geq 0$ *and* $a_2 = b_2 = 0$.

Moreover, the global center O *is elementary (nilpotent) if* $b_1 > 0$ *(*$b_1 = 0$*, resp.).*

Corollary 3.1. *If the cubic system* (3.3) *has a global center at the origin* $O = (0,0)$*, then there exists a first integral of the system in* $\mathbb{R}^2$ *or in* $\mathbb{R}^2 \setminus \{(0,0)\}$*. Furthermore, the first integral is analytic in* $\mathbb{R}^2$ *if either condition (I) holds or condition (II) with* $b_1 > 0$*. And the first integral is analytic in* $\mathbb{R}^2 \setminus \{(0,0)\}$ *if condition (II) with* $b_1 = 0$ *holds.*

Proof of Theorem 3.2. It is clear that system (3.4) has a unique equilibrium point at the origin $O = (0,0)$ in $\mathbb{R}^2$. If $f(s) \not\equiv 0$, then system (3.4) has a pair of infinite equilibrium points which corresponds to the y-axis direction, that is $x = 0$. Using the technique of calculation of index at an infinite equilibrium point in the proof of Theorem 3.1, we can find that the index at the infinite equilibrium point is equal to -1 by standard calculation. To save space, we omit the detail here. Therefore, the origin O is not a global center by Theorem 2.1. The proof is complete. $\square$

Remark 3.2. In [3], authors studied the perturbation of classical Liénard system $\dot{x} = y - \frac{1}{2}x^2$, $\dot{y} = -x$ which has a unique center at the origin in $\mathbb{R}^2$. This center is not a global center since the limit set of orbit $y - \frac{1}{2}x^2 + 1 = 0$ is an infinite equilibrium point. Proposition 16 in [3] shown that the origin of system $\dot{x} = y - \frac{1}{2}x^2 + \varepsilon\tilde{\alpha}(x^2)$, $\dot{y} = -x$ is still a center, where $0 < |\varepsilon| \ll 1$. By Theorem 3.2, we know that this center is not yet a global center.

To prove Theorem 3.3, we first state two propositions.

Proposition 3.1. *Assume that* $f(s) = a_1 s + a_2 s^2$ *and* $g(x) = b_1 x + b_2 x^2 + b_3 x^3$ *in system* (3.3)*. Then the cubic system* (3.3) *has an equilibrium point at the origin* $O = (0,0)$*, which is a center if and only if one of following conditions holds.*

(i) $b_1 > 0$, $a_2 = b_2 = 0$;

(ii) $b_1 > 0$, $a_1 b_2 = a_2 b_1$, $a_1 b_3 = 0$;

(iii) $a_1^2 < 8b_3$, $a_2 = b_1 = b_2 = 0$.

Proof. This result can be obtained from Theorem 3.8 in [5] by some transformations. Notice that Theorem 3.8 in [5] was given under assumption either $b_1 = 1$ or $b_1 = b_2 = 0$ and $b_3 = 1$. In order to apply Theorem 3.8, we make the following transformation if $0 < b_1 \neq 1$

$$x \mapsto \sqrt{b_1}x, \quad y \mapsto y, \quad t \mapsto \sqrt{b_1}t.$$

Then system (3.3) is transformed into

$$\begin{aligned} \dot{x} &= y - (\frac{a_1}{2b_1}x^2 + \frac{a_2}{3b_1^{\frac{3}{2}}}x^3), \\ \dot{y} &= -(x + \frac{b_2}{b_1^{\frac{3}{2}}}x^2 + \frac{b_3}{b_1^2}x^3). \end{aligned} \tag{3.5}$$

If $b_1 = b_2 = 0$ and $b_3 > 0$, then using the transformation

$$x \mapsto \sqrt[4]{b_3}x, \quad y \mapsto y, \quad t \mapsto \sqrt[4]{b_3}t,$$

we can make system (3.3) become

$$\begin{aligned} \dot{x} &= y - (\frac{a_1}{2\sqrt{b_3}}x^2 + \frac{a_2}{3\sqrt[4]{b_3^3}}x^3), \\ \dot{y} &= -(\frac{b_1}{\sqrt{b_3}}x + \frac{b_2}{\sqrt[4]{b_3^3}}x^2 + x^3). \end{aligned} \tag{3.6}$$

Apply Theorem 3.8 in [5] to system (3.5) and (3.6), we obtain the conclusion. The proof is finished. □

The follow proposition give the necessary conditions for cubic system (3.3) having a global center.

Proposition 3.2. *If the origin O is a global center of the cubic system* (3.3)*, then* $a_2 = 0$.

Proof. If $a_2 \neq 0$, then the cubic system (3.3) has a pair of infinite equilibrium points which corresponds to the infinity of y-axis, that is the line $x = 0$. By Poincaré transformation and time scaling, the cubic system (3.3) can be transformed to the following form

$$\begin{aligned} \frac{dv}{d\tau} &= (b_3 v - \frac{a_2}{3})v^3 + (b_2 x - \frac{a_1}{2})v^2 z + (1 + b_1 v^2)z^2, \\ \frac{dz}{d\tau} &= vz(b_3 v^2 + b_2 vz + b_1 z^2). \end{aligned} \tag{3.7}$$

On the invariant line $z = 0$, near the equilibrium $(v, z) = (0, 0)$, we have

$$\frac{dv}{d\tau} = -\frac{a_2}{3}v^3 + o(v^3).$$

Since the sign of $-\frac{a_2}{3}v^3 + o(v^3)$ near $v = 0$ can change on the line $z = 0$, it is impossible that the equator $\mathbb{S}^1$ is a monodromic polycycle. This contradicts to Theorem 2.1. Hence, we have $a_2 = 0$. The proof is complete. □

Proof of Theorem 3.3. From Proposition 3.1 and Proposition 3.2, we divide the parameters into the following three sets of conditions for verifying the conclusion.

C1. $b_1 > 0$, $a_1 = a_2 = 0$.

C2. $b_1 > 0$, $a_1 \neq 0$, $a_2 = b_2 = 0$.

C3. $0 \neq a_1^2 < 8b_3$ and $a_2 = b_1 = b_2 = 0$.

If condition $C1$ holds, then the cubic system (3.3) becomes a potential system as follows

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(b_1 + b_2 x + b_3 x^2). \end{aligned} \tag{3.8}$$

By Theorem 3.1, we know that the condition C1 is the sufficient and necessary conditions for system (3.8) having a global center. Hence, the cubic system (3.3) has a global center under the condition C1.

If condition C2 hold, then $b_3 \neq 0$ by Theorem 3.2. Since $g(x) = 0$ has a unique real solution $x = 0$, we have $b_3 > 0$. The cubic system (3.3) has exactly a pair of infinite equilibrium points, which correspond to the infinity along the line $x = 0$

under the conditions C2 or C3. When $a_2 = b_2 = 0$, $b_1 \geq 0$, $b_3 > 0$ and $a_1 \neq 0$, the cubic system (3.3) becomes

$$\begin{aligned} \dot{x} &= y - \frac{a_1}{2}x^2, \\ \dot{y} &= -(b_1 x + b_3 x^3). \end{aligned} \tag{3.9}$$

To obtain the conclusion, we need to prove that the neighbourhoods of system (3.9) at the infinite equilibrium point are exactly composed of two hyperbolic sectors if and only if $a_1^2 < 8b_3$.

In fact, by Poincaré transformation and time scaling, system (3.9) can be transformed to

$$\begin{aligned} \frac{dv}{d\tau} &= b_3 v^4 - \frac{a_1}{2}v^2 z + (1 + b_1 v^2)z^2, \\ \frac{dz}{d\tau} &= vz(b_3 v^2 + b_1 z^2). \end{aligned} \tag{3.10}$$

By the transformation $(u_1, z) \mapsto (v, z) := (u_1 z, z)$ and time scaling $d\tau \mapsto zd\tau$, we blow up the equilibrium $(v, z) = (0, 0)$ of system (3.10) in the z-direction and obtain

$$\begin{aligned} \frac{du_1}{d\tau} &= -\frac{a_1}{2}zu_1^2 + 1, \\ \frac{dz}{d\tau} &= z^3 u_1(b_3 u_1^2 + b_1), \end{aligned} \tag{3.11}$$

which has no equilibrium points on line $z = 0$. and it means there is no equilibrium under blowing up on direction $\theta \neq 0$ in polar blowing up.

Next we blow up the equilibrium point $(v, z) = (0, 0)$ in v-direction by map $(v, v_1) \mapsto (v, z) := (v, vv_1)$ and time scaling $d\tau \mapsto vd\tau$, system (3.10) can be transformed into

$$\begin{aligned} \frac{dv}{d\tau} &= v(v_1^2 - \frac{a_1}{2}vv_1 + b_3 v^2 + b_1 v^2 v_1^2), \\ \frac{dv_1}{d\tau} &= v_1^2(\frac{a_1}{2}v - v_1). \end{aligned} \tag{3.12}$$

$(v, v_1) = (0, 0)$ is the unique equilibrium of system (3.12) on $v = 0$ and it is still non-elementary. Then we shall blow up this equilibrium twice. It is easily calculated that there are two elementary saddles on direction $\theta = \frac{\pi}{2}, -\frac{\pi}{2}$ by blowing up $(v, v_1) = (0, 0)$ of system (3.12) in v_1-direction. And blowing up point $(v, v_1) = (0, 0)$ in v-direction again (the process of blowing up see Figure 3), we have

$$\begin{aligned} \frac{dv}{d\tau} &= v(b_3 - \frac{a_1}{2}v_2 + v_2^2 + b_1 v^2 v_2^2), \\ \frac{dv_2}{d\tau} &= v_2(-b_3 + a_1 v_2 - 2v_2^2 - b_1 v^2 v_2^2). \end{aligned} \tag{3.13}$$

System (3.13) has isolated equilibrium point $(v, v_2) = (0, v_2^*)$ on $v = 0$, where $v = v_2^*$ is the real root of equation

$$v_2(2v_2^2 - a_1 v_2 + b_3) = 0.$$

If $v_2^* = 0$, it is no hard to see the equilibrium $(v, v_2) = (0, 0)$ is an elementary saddle. If $\Delta = a_1^2 - 8b_3 \geq 0$ and $v_2 = v_2^*$ is a real root of equation $h(v_2) = 2v_2^2 - a_1 v_2 + b_3 = 0$,

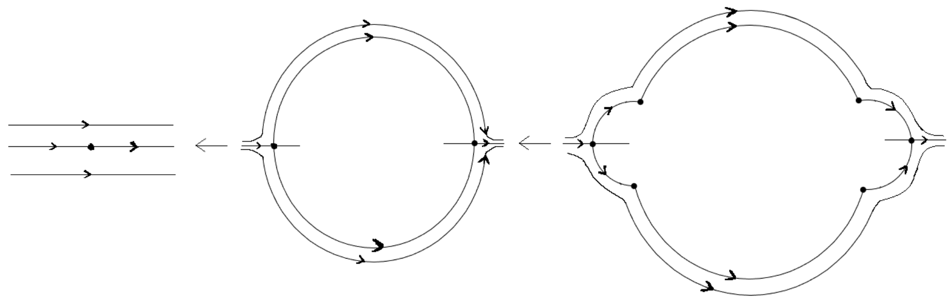

Figure 3. The process of blowing up for the equilibrium point of system (3.12).

we claim Jacobian matrix of system (3.13) at the equilibrium point $(v, v_2) = (0, v_2^*)$ has two real eigenvalues and at least one of them is not zero, i.e. the equilibrium point $(v, v_2) = (0, v_2^*)$ is either a saddle or a node or a saddle-node.

Indeed, Jacobian matrix of system (3.13) at $(v, v_2) = (0, v_2^*)$ has the form

$$\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} (v_2^*)^2 - \frac{a_1}{2} v_2^* + b_3 & * \\ 0 & v_2^*(a_1 - 4v_2^*) \end{bmatrix}. \tag{3.14}$$

If $\Delta = a_1^2 - 8b_3 > 0$, $v = v_2^*$ is a simple real root for equation $h(v_2) = 0$ since $\Delta > 0$, then we have $\lambda_2 = -h'(v_2^*) \neq 0$. Thus the claim is true.

If $\Delta = a_1^2 - 8b_3 = 0$, i.e. $b_3 = \frac{a_1^2}{8}$, it is easily calculated that $v_2^* = \frac{a_1}{4}$. Thus the other eigenvalue of the Jacobian matrix

$$\lambda_1 = (v_2^*)^2 - \frac{a_1}{2} v_2^* + b_3 = \frac{a_1^2}{16} \neq 0.$$

Hence, the claim still holds in $\Delta = 0$.

The claim implies $\Delta = a^2 - 8b_3 < 0$ if and only if the neighbourhood of equilibrium $(x, z) = (0, 0)$ of system (3.10) is exactly composed of two hyperbolic sectors. Therefore, the conclusions in Theorem 3.3 are true.

Last, From the linearized system of system (3.3) at $O = (0, 0)$, we can see that the global center $O = (0, 0)$ is elementary if $b_1 > 0$ in condition (I) and condition (II), and the global center $O = (0, 0)$ is nilpotent if $b_1 = 0$ in condition (I) and condition (II). The proof is complete. □

Proof of Corollary 3.1. By Theorem 3.3, we know that one of condition (I) and condition (II) holds if the cubic Liénard system (3.3) has a global center.

If condition (I) holds, then the cubic Liénard system (3.3) is a potential system, which is Hamiltonian system with Hamiltonian function

$$H(x, y) = \frac{1}{2} y^2 + \frac{b_1}{2} x^2 + \frac{b_2}{3} x^3 + \frac{b_3}{4} x^4.$$

Hence, the first integral $H(x, y)$ of system (3.3) exists in $\mathbb{R}^2$, and it is analytic in whole plane $\mathbb{R}^2$ whether the global center O is an elementary center or a nilpotent center.

If condition (II) holds, then by direct calculation we obtain a first integral of the cubic Liénard system (3.3) as follows

$$F(x,y)=\frac{e^{\frac{a_1\arctan\left(\frac{\frac{2b_3y+a_1b_1}{2b_3x^2+2b_1}-\frac{a_1}{4}}{\sqrt{\frac{b_3}{2}-\frac{a_1^2}{16}}}\right)}{4\cdot\sqrt{\frac{b_3}{2}-\frac{a_1^2}{16}}}}}{(x^2+\frac{b_1}{b_3})\sqrt{\frac{b_3}{2}-\frac{a_1^2}{16}+(\frac{2b_3y+a_1b_1}{2b_3x^2+2b_1}-\frac{a_1}{4})^2}}.$$

It is clear that $F(x,y)$ is analytic in $\mathbb{R}^2$ if $b_1>0$ in condition (II) holds. In this case the global center O is elementary.

If $b_1=0$ in condition (II) holds, then the global center O is nilpotent and the first integral is

$$F_0(x,y)=\frac{e^{\frac{a_1}{\sqrt{8b_3-a_1^2}}\arctan\left(\frac{4y-a_1x^2}{x^2\sqrt{8b_3-a_1^2}}\right)}}{\sqrt{(8b_3-a_1^2)x^4+(4y-a_1x^2)^2}}.$$

It can be checked that $F_0(x,y)$ is analytic in $\mathbb{R}^2\setminus\{(0,0)\}$ by using the following transformation

$$X=x^2\sqrt{8b_3-a_1^2},\ Y=4y-a_1x^2$$

and $X=r\cos\theta, Y=r\sin\theta$. The proof is finished. □

4. Discussion

The existence on periodic orbits of planar polynomial differential systems (1.1) with degree n is the most fundamental object of ordinary differential equations. This is related to the Hilbert's 16th problem and the infinitesimal Hilbert's 16th Problem, see [9, 10]. In this paper, we characterize planar polynomial differential systems (1.1) with degree n whose every orbit is a periodic orbit. In other words, the period annulus of the system is whole plane $\mathbb{R}^2$, it is said the existence of a global center. Our result shows the existence of a global center depends on parity of n. This reveals a big difference between planar polynomial differential systems of degree $2n$ and degree $2n-1$, where n is a positive integer. More precisely, there is not planar polynomial differential systems of degree $2n$ whose every orbit is a periodic orbit, and for any positive integer n, there exists planar polynomial differential systems of degree $2n-1$ which has a global center. In [7] it has shown that the global center can become any one type of elementary center, nilpotent center and degenerated center for Hamiltonian systems of degree $2n+1$ with $n\geq 1$. From qualitative analysis of viewpoint, we give algebraic sufficient and necessary conditions for the existence of global center of several polynomial differential systems, such as potential systems, cubic Liénard systems. Interestingly these polynomial differential systems have the first integral if each of them has a global center. It is well known there is a close relationship between the existence of a center and the integrability of the system, see [12, 13, 15]. Hence, a natural question is proposed as follows.

Problem 1: Does there exist the first integral of system (1.1) in $\mathbb{R}^2$ or $\mathbb{R}^2\setminus\{(0,0)\}$ if the system has a global center? What conditions can guarantee that the system with a nilpotent (or degenerate) global center has an analytical first integral in $\mathbb{R}^2$?

It is well known that if system (1.1) has a global center, then every solution of the system is a nontrivial periodic solution except the center point in $\mathbb{R}^2$. It is interesting to know what property the period of these periodic solutions has. Hence, we ask whether the period function of planar polynomial differential system (1.1) with a global center has a finite number of critical points.

Problem 2: How many critical points has the period function of system (1.1) with a global center if the number of critical points is finite? Can you give sufficient and necessary conditions for system (1.1) having an isochronous global center?

On the other hand, a challenge problem is the number of limit cycles bifurcating from the period annulus, see [6, 9, 10]. For a global center, we can consider limit cycle bifurcation in any bounded period annulus of $\mathbb{R}^2$. There have been some works on the number of limit cycles bifurcating from any bounded period annulus of $\mathbb{R}^2$ for quasi-homogeneous polynomial systems, see [2, 8, 11]. Under the light of quasi-homogeneous polynomial systems, we are wonder if one can

Problem 3: Find a new class of planar polynomial differential systems with a global center such that the number of limit cycles bifurcating from any bounded period annulus in $\mathbb{R}^2$ of the system can be estimated by the degree n of the small polynomial perturbation and the order k of Melnikov functions of the perturbation.

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