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USE SLOW-SPREAD OF ONCOLYTIC VIRUS TO DEPRESS EXPONENTIAL GROWTH OF TUMOR CELLS*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract In this paper we analyze an ODE model for oncolytic dynamics of exponential growth of tumor cells with slow-spread of virus, which was modeled by Komarova and Wodarz but not discussed yet. The involved four parameters render finding equilibria to be a difficult problem of algebraic varieties. We discuss resultants of polynomials to give complete conditions for distribution and qualitative properties of equilibria. We prove that the degenerate equilibrium is either a saddle-node or a cusp, which is of codimension infinity. Moreover, we prove that the equilibrium of center type is either a rough center or a weak center of order 1. Furthermore, analyzing equilibria at infinity, showing existence of a homoclinic orbit and giving nonexistence of limit cycles, we exhibit global phase portraits, which suggest strategies of tumor control.

Keywords Tumor control, cusp, weak center, homoclinic orbit, resultant.

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1. Introduction

In recent years, oncolytic virotherapy entered clinical trials to tumor patients and attracted increasing attention of clinicians (e.g. [12, 14, 27, 30]). The idea of this therapy is to infect the tumor cells with engineered viruses who can infect and lyse tumor cells, spread throughout the tumor, and leave healthy cells almost unharmed. In order to describe the dynamics of oncolytic viruses, a number of ODE models have been established in the past decades (e.g. [1, 7, 17, 21, 29]), one of which is of the general predator-prey type

$$\dot{x} = xF(x,y) - \beta yG(x,y), \quad \dot{y} = \beta yG(x,y) - ay \tag{1.1}$$

considered by Komarova and Wodarz ([17]), where x and y denote the population of the uninfected tumor cells and infected tumor cells respectively, the coefficient a represents the virus-infected cells death rate, and the coefficient β represents the infectivity of the virus. The function F, describing the growth of an uninfected tumor,

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and the function G, presenting the rate of infection, are both nonnegative polynomials satisfying the F-condition of 4 items (shown in [17, p.531, 532]) and the G-condition of 7 items (shown in [17, p.532]) respectively. In particular, F can be modeled in the form $F_{exp}(x,y) := 1$ for exponential growth, $F_\ell(x,y) := \eta/(\eta+x+y)$ for linear growth and $F_{lg}(x,y) := 1-(x+y)/W$ for logistic growth. G can be modeled in the form $G_1(x,y) := x/\left((x+\varepsilon_1)(y+\varepsilon_2)\right)$, $G_2(x,y) := x/\left(\sqrt{x}(y+c)+x+\varepsilon\right)$ and $G_3(x,y) := x/\left((\sqrt{xy}+\varepsilon_1)(\sqrt{x}+\sqrt{y}+\sqrt{\varepsilon_2})\right)$, all of which satisfy $\lim_{x\to+\infty} G(x,x/a) = 0$, referred to as the *slow-spread mode*.

In 2010, Komarova and Wodarz ([17]) discussed model (1.1) in the slow-spread mode with three matches, i.e., G_1 matched with F_{exp}, F_ℓ and F_{lg} separately. For $F = F_{exp}$, they found that system (1.1) has two interior equilibria for large β, one of which is a saddle, and showed that the tumor can out-run the virus infection and grow beyond control. For $F = F_\ell$, which adds saturation to the type $F = F_{exp}$, they discussed with $\varepsilon_1 = \varepsilon_2$ in G_1 and found that system (1.1) has a unique interior equilibrium $E_I : (x_I, y_I)$ for large β and both x_I and y_I tend to 0, the state of extinction, as $\beta \to +\infty$, which indicated that the tumor will be driven extinct for large β. For $F = F_{lg}$, which is limited by a carrying capacity, they found that there exists an equilibrium describing tumor growth towards carrying capacity rather than towards infinity, indicating that saturation of tumor growth at lower scales contributes to successful virus therapy. Later, Si and Zhang ([26]) further investigated the first match, i.e., system (1.1) with G_1 and F_{exp}, for its nonhyperbolic cases and showed a saddle-node bifurcation on a center manifold, a Hopf bifurcation from which exactly one limit cycle arises, and a Bogdanov-Takens bifurcation in which a homoclinic orbit arises while the limit cycle disappears. Recently, Zhang ([31]) investigated system (1.1) with G_2 and F_{exp}, discussing the distribution of equilibria and giving a saddle-node bifurcation, a degenerate Hopf bifurcation at a weak focus of multiplicity 3 and a Bogdanov-Takens bifurcation of codimension 2.

In this paper we consider the match of G_3 with F_{exp}, which was not discussed in literatures yet. With this match, system (1.1) can be presented as

$$\begin{cases} \dot{x} = x - \beta y \dfrac{x}{(\sqrt{xy}+\varepsilon_1)(\sqrt{x}+\sqrt{y}+\sqrt{\varepsilon_2})}, \\ \dot{y} = \beta y \dfrac{x}{(\sqrt{xy}+\varepsilon_1)(\sqrt{x}+\sqrt{y}+\sqrt{\varepsilon_2})} - ay \end{cases} \tag{1.2}$$

in the closure of the first quadrant $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$, where β, a, ε_1 and ε_2 are all positive constants. In section 2 we investigate its equilibria, which are determined by a cubic equation but the well-known formulae of cubic roots can hardly help determine the number of positive roots because it involves irrational expressions with four parameters. Using the resultant theory of polynomials ([10, p.398]), we prove that system (1.2) has exactly one boundary equilibrium and at most two interior equilibria, which have two non-hyperbolic cases: the degenerate case (either one zero eigenvalue or two zero eigenvalues with nilpotent linear part) and the center type case (a pair of pure imaginary eigenvalues). Section 3 is devoted to the degenerate case, in which we prove that the equilibrium is either a saddle-node or a nilpotent cusp of codimension ∞. In section 4 we discuss the case of center type and prove that the equilibrium is either a rough center or a weak center of order 1, using resultant elimination to compute period quantities. Moveover, we prove that the system has no elementary first integrals in this case. In section 5, analyzing equilibria at infinity, showing existence of a homoclinic orbit and giving nonexistence of limit cycles, we exhibit global phase portraits. We finally demonstrate the case of a stable focus with a saddle and the case of a center with a homoclinic orbit with numerical simulations in section 6, providing strategies of tumor control.

2. Analysis of equilibria

As the problems of distribution (i.e., the number and relative positions) and qualitative properties of equilibria can be reduced to real zeros of polynomials in $\mathbb{R}^4_+$, we need the following lemma on Sylvester resultant ([10]) to deal with semi-algebraic systems. For convenience, let $\mathbb{R}[x;\boldsymbol{\lambda}]$ denote the ring of real polynomials in x parameterized by $\boldsymbol{\lambda}$. Let $\mathrm{lcoeff}(f,x)$ denote the leading coefficient of polynomial f with respect to the variable x and $\mathrm{res}(f_i,f_j,x)$ denote the Sylvester resultant of polynomials f_i and f_j with respect to x.

Lemma 2.1. *Let $f_1,...,f_k\in\mathbb{R}[x;\boldsymbol{\lambda}]$, where $k\geq 1$ and $\boldsymbol{\lambda}=(\lambda_1,...,\lambda_\ell)\in\mathbb{R}^\ell_+$. If there is a region $\mathcal{U}\subset\mathbb{R}^\ell_+$ such that for all $\boldsymbol{\lambda}\in\mathcal{U}$ the conditions are true:* **(i)** $\mathrm{lcoeff}(f_i,x)\neq 0\ \forall i\in\{1,...,k\}$, **(ii)** $\mathrm{res}(f_i,(f_i)'_x,x)\neq 0$ *and* $\mathrm{res}(f_i,f_j,x)\neq 0$ $\forall i\neq j\in\{1,...,k\}$, *and* **(iii)** $f_i|_{x=0}\neq 0\ \forall i\in\{1,...,k\}$, *then the distribution of positive (and negative) zeros of $f_1,...,f_k$ does not change as $\boldsymbol{\lambda}$ varies in $\mathcal{U}$.*

Proof. First, we consider each f_i and claim that the number and relative positions of its positive (and negative) zeros never change as $\boldsymbol{\lambda}\in\mathcal{U}$ varies. Let $\deg(f_i,x)=n_i$, the degree of the polynomial in x, and $x_1(\boldsymbol{\lambda}),...,x_{n_i}(\boldsymbol{\lambda})$ be all its complex zeros. By **(iii)**, $x_1(\boldsymbol{\lambda}),...,x_{n_i}(\boldsymbol{\lambda})\neq 0$ for all $\boldsymbol{\lambda}\in\mathcal{U}$. Moreover, by **(i)** and **(ii)**, the Resultant Theorem ([10, p.398]) shows that f_i does not have a multiple zero since $\mathrm{lcoeff}(f_i,x)\neq 0$ and $\mathrm{res}(f_i,(f_i)'_x,x)\neq 0$. Note that non-real zeros of the real polynomial f_i arise in conjugate pairs as indicated in [15, p.22]. Then, for $\boldsymbol{\lambda}_*\in\mathcal{U}$ we can assume without loss of generality that

$$x_1(\boldsymbol{\lambda}_*)<\cdots<x_p(\boldsymbol{\lambda}_*)<0<x_{p+1}(\boldsymbol{\lambda}_*)<\cdots<x_q(\boldsymbol{\lambda}_*), \tag{2.1}$$
$$x_{q+s}(\boldsymbol{\lambda}_*)=\mu_s(\boldsymbol{\lambda}_*)+\mathbf{i}\nu_s(\boldsymbol{\lambda}_*),\quad x_{q+r+s}(\boldsymbol{\lambda}_*)=\mu_s(\boldsymbol{\lambda}_*)-\mathbf{i}\nu_s(\boldsymbol{\lambda}_*) \tag{2.2}$$

for all $s\in\{1,...,r\}$, where $0\leq p\leq q\leq n_i$, $\mu_s(\boldsymbol{\lambda}_*),\nu_s(\boldsymbol{\lambda}_*)\in\mathbb{R}$ and $\nu_s(\boldsymbol{\lambda}_*)\neq 0$ for all $s\in\{1,..,r\}$, $x_{q+i}(\boldsymbol{\lambda}_*)\neq x_{q+j}(\boldsymbol{\lambda}_*)$ for all $i\neq j\in\{1,...,r\}$, and $r:=(n_i-q)/2$. Then, our claim is equivalent to that (2.1) and (2.2) hold for all $\boldsymbol{\lambda}\in\mathcal{U}$. It is known in [19, Theorem 1.4] that all zeros $x_1(\boldsymbol{\lambda}),...,x_{n_i}(\boldsymbol{\lambda})$ are continuous in $\boldsymbol{\lambda}$. Thus, as $\boldsymbol{\lambda}\in\mathcal{U}$ varies, real zeros $x_1(\boldsymbol{\lambda}),...,x_q(\boldsymbol{\lambda})$ cannot become non-real; otherwise such a real zero will first become a multiple real zero and then become a pair of non-real zeros, a contradiction to the fact that f_i has no multiple zeros, which is given just before (2.1). Similarly, we also see that those non-real zeros in (2.2) cannot become real ones. For the same reason of continuity, those positive (resp. negative) zeros cannot become negative (resp. positive) ones; otherwise, such a zero will first become 0 and then become negative (resp. positive), a contradiction to **(iii)**. The continuity also implies that the order of positive (and negative) zeros in (2.1) does not change; otherwise, a multiple zero appears. Consequently, our claim is proved.

Next we consider relative positions of real zeros of different f_i and f_j, where $i\neq j$. Let $x_m(\boldsymbol{\lambda})$ and $\tilde{x}_{\tilde{m}}(\boldsymbol{\lambda})$ be positive (or negative) zeros of f_i and f_j respectively. Then $x_m(\boldsymbol{\lambda})\neq\tilde{x}_{\tilde{m}}(\boldsymbol{\lambda})$ because by the Resultant Theorem ([10, p.398]) we see from **(ii)** that f_i and f_j have no common zeros. If $x_m(\boldsymbol{\lambda}_*)<\tilde{x}_{\tilde{m}}(\boldsymbol{\lambda}_*)$ (resp. $>\tilde{x}_{\tilde{m}}(\boldsymbol{\lambda}_*)$) at a point $\boldsymbol{\lambda}_*\in\mathcal{U}$, then $x_m(\boldsymbol{\lambda})<\tilde{x}_{\tilde{m}}(\boldsymbol{\lambda})$ (resp. $>\tilde{x}_{\tilde{m}}(\boldsymbol{\lambda})$) for all $\boldsymbol{\lambda}\in\mathcal{U}$; otherwise, by the continuity of zeros, there is a point $\boldsymbol{\lambda}'_* \in \mathcal{U}$ such that $x_m(\boldsymbol{\lambda}'_*) = \tilde{x}_{\tilde{m}}(\boldsymbol{\lambda}'_*)$, a contradiction. Hence, the proof of this lemma is completed. $\square$

In order to use Lemma 2.1, we compute zeros of those leading coefficients, the two resultants and the values of $f_i|_{x=0}$ as stated in conditions **(i)-(iii)** so as to determine a region $\mathcal{U}$ mentioned in Lemma 2.1. Then, choosing a point $\boldsymbol{\lambda}_*$ in $\mathcal{U}$ arbitrarily, we implement the MAPLE command 'realroot($f_i, 10^{-n}$)' to provide a list of isolating intervals, each of which has a width $\leq 10^{-n}$, for all real roots of the polynomial $f_i|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_*}$ in x, which by Lemma 2.1 gives the distribution of positive (and negative) zeros of $f_1, ...f_k$ for $\boldsymbol{\lambda} \in \mathcal{U}$.

Theorem 2.1. *System* (1.2) *has exactly one boundary equilibrium* $O : (0, 0)$*, which is a saddle, and at most two interior equilibria. The numbers and properties of interior equilibria are listed in Table 1, where* $\beta_*(a, \varepsilon_1, \varepsilon_2)$ *is the only positive zero of the cubic polynomial*

$$\begin{aligned} B(\beta) :=& 4\sqrt{\varepsilon_2}\beta^3 + ((\sqrt{a}+1)^2\varepsilon_1 - 12\sqrt{a}\varepsilon_2)\beta^2 - 4\sqrt{a\varepsilon_2}(5(\sqrt{a}+1)^2\varepsilon_1 - 3\sqrt{a}\varepsilon_2)\beta \\ & - 4\sqrt{a}((\sqrt{a}+1)^2\varepsilon_1 + \sqrt{a}\varepsilon_2)^2. \end{aligned}$$

Table 1. Numbers and qualitative properties of interior equilibria.

Parameters		Equilibria	Number
$\beta < \beta_*(a, \varepsilon_1, \varepsilon_2)$			0
$\beta = \beta_*(a, \varepsilon_1, \varepsilon_2)$		E_* (degenerate)	1
$\beta > \beta_*(a, \varepsilon_1, \varepsilon_2)$	$a < 1$	E_1 (unstable node or focus), E_2 (saddle)	2
$\beta > \beta_*(a, \varepsilon_1, \varepsilon_2)$	$a = 1$	E_1 (center type), E_2 (saddle)	2
$\beta > \beta_*(a, \varepsilon_1, \varepsilon_2)$	$a > 1$	E_1 (stable node or focus), E_2 (saddle)	2

Proof. We first prove the existence and uniqueness of positive zero $\beta_*(a, \varepsilon_1, \varepsilon_2)$, i.e., the cubic polynomial B has a unique positive zero. In fact, by Lemma 2.1, we compute the resultant

$$\mathrm{res}(B, B'_\beta, \beta) = -64\sqrt{a\varepsilon_2}(\sqrt{a}+1)^4\varepsilon_1^2\{(\sqrt{a}+1)^2\varepsilon_1 - 27\sqrt{a}\varepsilon_2\}^3,$$

which has the only zero $\varepsilon_1 = \hat{\varepsilon}_1(a, \varepsilon_2) := 27\sqrt{a}\varepsilon_2/(\sqrt{a}+1)^2$. Note that $\mathrm{lcoeff}(B, \beta) = 4\sqrt{\varepsilon_2} > 0$ and $B(0) = -4\sqrt{a}((\sqrt{a}+1)^2\varepsilon_1 + \sqrt{a}\varepsilon_2)^2 < 0$. Fixing $(a, \varepsilon_2) = (1, 1)$ and choosing $\varepsilon_1 = 26/4 < 27/4 = \hat{\varepsilon}_1(1, 1)$, we implement the command 'realroot($B, 10^{-3}$)' with MAPLE ver.18 and find that B has only one real zero, which lies in the internal $[\frac{98743}{8192}, \frac{12343}{1024}]$. On the other hand, choosing $\varepsilon_1 = 28/4 > 27/4 = \hat{\varepsilon}_1(1, 1)$, we find that B has three zeros, covered by intervals $[-\frac{67417}{8192}, -\frac{8427}{1024}]$, $[-\frac{33641}{4096}, -\frac{67281}{8192}]$ and $[\frac{50965}{4096}, \frac{101931}{8192}]$ separately. By Lemma 2.1, the cubic polynomial B has a pair of conjugate non-real zeros and one positive zero for all $\varepsilon_1 < \hat{\varepsilon}_1(a, \varepsilon_2)$, but two negative zeros and one positive zero for all $\varepsilon_1 > \hat{\varepsilon}_1(a, \varepsilon_2)$. For $\varepsilon_1 = \hat{\varepsilon}_1(a, \varepsilon_2)$, the cubic polynomial B has a multiple zero because $\mathrm{res}(B, B'_\beta, \beta) = 0$. Moreover, since zeros of a polynomial vary continuously as a function of the coefficients and $B(0) \neq 0$, the conjugate non-real zeros of B for $\varepsilon_1 < \hat{\varepsilon}_1(a, \varepsilon_2)$ will first become one double negative zero in the critical case $\varepsilon_1 = \hat{\varepsilon}_1(a, \varepsilon_2)$ and then the two negative zeros for $\varepsilon_1 > \hat{\varepsilon}_1(a, \varepsilon_2)$; the simple positive zero of B for $\varepsilon_1 < \hat{\varepsilon}_1(a, \varepsilon_2)$ remains to be a simple positive zero for $\varepsilon_1 \geq \hat{\varepsilon}_1(a, \varepsilon_2)$. Consequently, B has only one positive zero, denoted by $\beta_*(a, \varepsilon_1, \varepsilon_2)$.

In order to find equilibria of system (1.2), we use the transformation $(x, y) \mapsto (x^2, y^2)$ and the time-rescaling $t \mapsto 2(xy + \varepsilon_1)(x + y + \sqrt{\varepsilon_2})t$ to convert (1.2) to the polynomial system

$$\dot{x} = x\{\mathcal{H}(x, y) - \beta y^2\}, \quad \dot{y} = y\{\beta x^2 - \alpha^2 \mathcal{H}(x, y)\}, \tag{2.3}$$

where $\mathcal{H}(x, y) := (xy + \epsilon_1)(x + y + \epsilon_2)$, $\alpha := \sqrt{a}$, $\epsilon_1 := \varepsilon_1$ and $\epsilon_2 := \sqrt{\varepsilon_2}$. Clearly, in the closure of the first quadrant $\mathbb{R}^2_+$, system (1.2) is topologically equivalent to system (2.3). Equilibria of system (2.3) are given by the following equations

$$x\{\mathcal{H}(x, y) - \beta y^2\} = 0, \quad y\{\beta x^2 - \alpha^2 \mathcal{H}(x, y)\} = 0. \tag{2.4}$$

On the half-axis $y \geq 0$, we see from (2.4) that $-\alpha^2 y \mathcal{H}(0, y) = -\alpha^2 y(y + \epsilon_2) = 0$, implying that the origin $O : (0, 0)$ is the only equilibria. On the half-axis $x \geq 0$, we see from (2.4) that $x\mathcal{H}(x, 0) = \epsilon_1 x(x + \epsilon_2) = 0$, implying that O is the only equilibria. Thus O is a unique boundary equilibrium.

Next, we consider interior equilibria, which are determined by the equations

$$\mathcal{H}(x, y) - \beta y^2 = 0, \quad \beta x^2 - \alpha^2 \mathcal{H}(x, y) = 0. \tag{2.5}$$

Eliminating $\mathcal{H}(x, y)$ in (2.5) shows that interior equilibria lie on the line $y = x/\alpha$. Substituting $y = x/\alpha$ in the second equation of (2.5), we obtain the equation

$$F(x) := (\alpha + 1)x^3 + (\alpha\epsilon_2 - \beta)x^2 + \alpha(\alpha + 1)\epsilon_1 x + \alpha^2 \epsilon_1 \epsilon_2 = 0, \quad x > 0, \tag{2.6}$$

called the equilibrium equation. Clearly, we cannot use the well-known formulae of cubic roots because those involved 4 parameters make the irrational expression of the formulae too complicated to discuss which of those roots is real and positive. Our strategy is to abandon determining coordinates of equilibria in convention but give distribution of equilibria with the derivative F'_x. Compute

$$F'_x(x) = 3(\alpha + 1)x^2 + 2(\alpha\epsilon_2 - \beta)x + \alpha(\alpha + 1)\epsilon_1, \tag{2.7}$$

which has the discriminant $\Delta_{F'_x} = 4S_1(\beta)$, where $S_1(\beta) := \beta^2 - 2\alpha\epsilon_2\beta - \alpha(3(\alpha + 1)^2\epsilon_1 - \alpha\epsilon_2^2)$. Clearly S_1 has two zeros $\beta_1^{\pm} := \alpha\epsilon_2 \pm (\alpha + 1)\sqrt{3\alpha\epsilon_1}$. Thus,

(C1) in the case $\Delta_{F'_x} \leq 0$, i.e., $\beta_1^- \leq \beta \leq \beta_1^+$, we have $F'_x(x) \geq 0$ for all $x \geq 0$, implying that $F(x)$ is increasing on $[0, +\infty)$. Since $F(0) = \alpha^2\epsilon_1\epsilon_2 > 0$, F has no positive zeros and therefore system (1.2) has no interior equilibria;

(C2) in the opposite case $\Delta_{F'_x} > 0$, i.e., either **(C2.1)** $\beta < \beta_1^-$ or **(C2.2)** $\beta > \beta_1^+$, derivative F'_x has two real zeros

$$x^{\pm} := \frac{\beta - \alpha\epsilon_2 \pm \sqrt{S_1(\beta)}}{3(\alpha + 1)}.$$

Note that $x^- x^+ = \alpha\epsilon_1/3 > 0$ and $x^- + x^+ = 2(\beta - \alpha\epsilon_2)/(3(\alpha + 1))$. Then $x^{\pm} > 0$ (or < 0) if $\beta > \alpha\epsilon_2$ (or $< \alpha\epsilon_2$).

In subcase **(C2.1)**, we have $x^{\pm} < 0$ since $\beta < \beta_1^- < \alpha\epsilon_2$. Then F is increasing for $x \geq 0$. Noting $F(0) > 0$, we see that F has no positive zero and therefore system (1.2) has no interior equilibria.

In subcase **(C2.2)**, we have $x^{\pm} > 0$ since $\alpha\epsilon_2 < \beta_1^+ < \beta$. Then F is increasing on the interval $(0, x^-) \cup (x^+, +\infty)$ and decreasing on the interval (x^-, x^+). Since

$F(x^-) > F(0) > 0$, the number of positive zeros of F is determined by the sign of the minimum $F(x^+)$. For the critical case that $F(x^+) = 0$, we compute the resultant

$$\mathrm{res}(F, F'_x, x) = -\alpha^2(\alpha+1)\epsilon_1 S_2(\beta), \tag{2.8}$$

where $S_2(\beta)$ is exactly the same cubic polynomial $B(\beta)$, defined in the theorem in terms of original parameters a, ε_1 and ε_2. As shown at the beginning of the proof, $S_2(\beta)$ has a unique positive root $\widetilde{\beta}_*$, which is equal to $\beta_*(\alpha^2, \epsilon_1, \epsilon_2^2)$ by the change of parameters in (2.3). By the Resultant Theorem ([10, p.398]), F and F'_x have a common zero if and only if $\beta = \widetilde{\beta}_*$ since the leading coefficient of F in x is not zero. Further we claim that

$$\widetilde{\beta}_* > \beta_1^+, \tag{2.9}$$

where β_1^+ is the positive zero of S_1 defined just below (2.7). Actually, we have

$$\mathrm{res}(S_1, S_2, \beta) = \alpha^2(\alpha+1)^4\epsilon_1^2\epsilon_2\{(\alpha+1)^2\epsilon_1 - 27\alpha\epsilon_2^2\}^2,$$

which implies by the Resultant Theorem ([10, p.398]) that S_1 and S_2 have a common zero if and only if $\epsilon_1 = \widetilde{\epsilon}_1 := 27\alpha\epsilon_2^2/(\alpha+1)^2$. Since

$$S_1(\beta)|_{\epsilon_1=\widetilde{\epsilon}_1} = (\beta + 8\alpha\epsilon_2)(\beta - 10\alpha\epsilon_2), \quad S_2(\beta)|_{\epsilon_1=\widetilde{\epsilon}_1} = \epsilon_2(\beta+8\alpha\epsilon_2)^2(4\beta - 49\alpha\epsilon_2),$$

polynomials S_1 and S_2 can only have a negative common zero, which implies that the positive zero β_1^+ is not equal to $\widetilde{\beta}_*$ for all positive α, ϵ_1 and ϵ_2. Noticing that

$$\widetilde{\beta}_*|_{\epsilon_1=\widetilde{\epsilon}_1} = 49\alpha\epsilon_2/4 > 10\alpha\epsilon_2 = \beta_1^+|_{\epsilon_1=\widetilde{\epsilon}_1},$$

we obtain $\widetilde{\beta}_* > \beta_1^+$ for all positive α, ϵ_1 and ϵ_2. Thus the claimed (2.9) is proved. Fixing $(\alpha, \epsilon_1, \epsilon_2) = (1, 27/4, 1)$ and choosing $\beta = 12 < 49/4 = \widetilde{\beta}_*(1, 27/4, 1)$, we see that F has one real zero, covered by the interval $[-\frac{3085}{8192}, -\frac{6169}{16384}]$. Moreover, choosing $\beta = 25/2 > 49/4 = \widetilde{\beta}_*(1, 27/4, 1)$, we find that F has three real zeros, covered by intervals $[-\frac{6119}{16384}, -\frac{3059}{8192}]$, $[\frac{5081}{2048}, \frac{20325}{8192}]$ and $[\frac{29839}{8192}, \frac{1865}{512}]$ separately. By Lemma 2.1, we see from (2.6) and (2.8) that F has no positive zero for all $\beta < \widetilde{\beta}_*$ and two positive zeros $x_1 < x_2$ for all $\beta > \widetilde{\beta}_*$. For $\beta = \widetilde{\beta}_*$, the cubic polynomial F has a multiple zero because $\mathrm{res}(F, F'_x, x) = 0$. Moreover, since zeros of a polynomial vary continuously as a function of the coefficients and $F(0) > 0$, the two positive zeros x_1 and x_2 of F for $\beta > \widetilde{\beta}_*$ will first become a double positive zero x_+ defined in **(C2)** at $\beta = \widetilde{\beta}_*$ and then a pair of conjugate non-real zeros for $\beta < \widetilde{\beta}_*$.

Summarily, we obtain the following distributions of interior equilibria: **(E0)** no interior equilibria if $\beta < \widetilde{\beta}_*$, **(E1)** one interior equilibrium $\widetilde{E}_* : (x_*, x_*/\alpha)$ if $\beta = \widetilde{\beta}_*$, where $x_* := x_+$, and **(E2)** two interior equilibria $\widetilde{E}_1 : (x_1, x_1/\alpha)$ and $\widetilde{E}_2 : (x_2, x_2/\alpha)$ if $\beta > \widetilde{\beta}_*$, where $x_1 \in (x_-, x_+)$ and $x_2 \in (x_+, +\infty)$. Further, we give qualitative properties for those equilibria. Let (x, y) be an equilibrium in general. Then the Jacobian matrix at (x, y) is given by

$$J(x,y) := \begin{pmatrix} \mathcal{H}(x,y) - \beta y^2 + x\mathcal{H}'_x(x,y) & x(\mathcal{H}'_y(x,y) - 2\beta y) \\ y(2\beta x - \alpha^2\mathcal{H}'_x(x,y)) & \beta x^2 - \alpha^2\mathcal{H}(x,y) - \alpha^2 y\mathcal{H}'_y(x,y) \end{pmatrix}.$$

Compute the determinant $\mathrm{Det}J(0,0) = -\alpha^2\epsilon_1^2\epsilon_2^2 < 0$, which implies that the only boundary equilibrium O is a saddle. For an interior equilibrium (x, y), which lies on the line $y = x/\alpha$ as indicated below (2.5), we compute the determinant and the trace

$$\mathrm{Det}J := \frac{-2\beta x^3 F_x'(x)}{\alpha^2}, \quad \mathrm{Tr}J := \frac{1-\alpha}{\alpha^2}\{(\alpha+1)^2x^2 + \alpha(1+\alpha)\epsilon_2 x + \alpha^2\epsilon_1\}x. \tag{2.10}$$

Clearly, $\mathrm{Det}J$ and $\mathrm{Tr}J$ have the same signs as $-F_x'(x)$ and $1-\alpha$ respectively. In case **(E1)**, since $\mathrm{Det}J|_{x=x_*} = F_x'(x_+) = 0$, equilibrium $\widetilde{E}_*$ of system (2.3) is degenerate, which implies line 2 of Table 1. In case **(E2)**, since $x_1 \in (x_-, x_+)$ and $x_2 \in (x_+, +\infty)$, we have $F_x'(x_1) < 0$ and $F_x'(x_2) > 0$ and therefore $\mathrm{Det}J|_{x=x_1} > 0$ and $\mathrm{Det}J|_{x=x_2} < 0$. Then equilibrium $\widetilde{E}_2 : (x_2, x_2/\alpha)$ is a saddle, which implies lines 3, 4 and 5 of Table 1. Moreover, since $\mathrm{Tr}J|_{x=x_1}$ has the same sign as $1-\alpha$, equilibrium $\widetilde{E}_1 : (x_1, x_1/\alpha)$ is an unstable node or focus for $\alpha < 1$, of center type for $\alpha = 1$, and a stable node or focus for $\alpha > 1$, which implies lines 3, 4 and 5 of Table 1 respectively. This completes the proof. □

In Table 1 there are a 'degenerate' case and a 'center type' case. We will further determine qualitative properties in those cases in the following sections.

3. Degenerate case

As indicated on the second line in Table 1, equilibrium E_* is degenerate for $\beta = \beta_*(a, \varepsilon_1, \varepsilon_2)$. In this section we give qualitative properties of equilibrium E_*.

Theorem 3.1. *Equilibrium E_* of system* (1.2) *is either a saddle-node when $\beta = \beta_*(a, \varepsilon_1, \varepsilon_2)$ and $a \neq 1$, or a cusp when $\beta = \beta_*(a, \varepsilon_1, \varepsilon_2)$ and $a = 1$.*

Proof. As indicated just below (2.3), system (1.2) is topologically equivalent to system (2.3). Then, we see from case **(E1)** in the proof of Theorem 2.1 that it is equivalent to investigate the degenerate equilibrium $\widetilde{E}_* : (x_*, x_*/\alpha)$ of system (2.3) for $\beta = \widetilde{\beta}_* = \beta_*(\alpha^2, \varepsilon_1, \varepsilon_2^2)$, defined before (2.9). Recall that parameters in (1.2) and (2.3) satisfy that $\varepsilon_2 = \epsilon_2^2$ and $a = \alpha^2$. For $\beta = \widetilde{\beta}_*$ and $\alpha \neq 1$, translating $\widetilde{E}_*$ to the origin and further normalizing the linear part with the transformation $x \mapsto (c_2x - \alpha c_1 y)/(c_2 - \alpha c_1)$ and $y \mapsto \alpha c_2(x-y)/(c_2 - \alpha c_1)$ together with the time-rescaling $t \mapsto (\alpha c_1 - c_2)t$, where $c_1 := x_*^3 + \epsilon_2 x_*^2 + \epsilon_1(2\alpha+1)x_* + 2\alpha\epsilon_1\epsilon_2$ and $c_2 := (\alpha+2)x_*^3 + \alpha\epsilon_2 x_*^2 + \alpha\epsilon_1 x_*$, we change system (2.3) into the following

$$\begin{cases} \dot{x} = \quad\; - \phi_{2,0}x^2 + O(xy) + O(y^2) + O(|x,y|^3) =: \Phi(x,y), \\ \dot{y} = y - \psi_{2,0}x^2 + O(xy) + O(y^2) + O(|x,y|^3) =: \Psi(x,y), \end{cases} \tag{3.1}$$

where

$$\phi_{2,0} := \frac{4\alpha^4(\alpha+1)\{(\alpha+1)x_* + 3\alpha\epsilon_2\}\{(\alpha+3)x_* + 2\alpha\epsilon_2\}}{(\alpha-1)^3\{(\alpha^2+4\alpha+1)x_* + 2\alpha\epsilon_2(\alpha+1)\}^3} \neq 0,$$

$$\psi_{2,0} := \frac{\alpha^4(\alpha+1)(\alpha^2+1)\{(\alpha+1)x_* + 3\alpha\epsilon_2\}\{(\alpha+3)x_* + 2\alpha\epsilon_2\}^2}{(\alpha-1)^3\{(\alpha+1)x_* + \alpha\epsilon_2\}\{(\alpha^2+4\alpha+1)x_* + 2\alpha\epsilon_2(\alpha+1)\}^3} \neq 0.$$

Note that $\Psi(0,0) = 0$ and $\partial\Psi(0,0)/\partial y = 1$. By the Implicit Function Theorem, there is a unique implicit function $y = h(x) := \psi_{2,0}x^2 + O(x^3)$, analytic at $x = 0$, such that $\Psi(x,h(x))\equiv 0$ near the origin $O:(0,0)$. Since $\Phi(x,h(x))=-\phi_{2,0}x^2+O(x^3)$, equilibrium O of system (3.1) is a saddle-node by Theorem II.7.1 of [32], i.e., equilibrium E_* of system (1.2) is a saddle-node for $\beta=\beta_*(a,\varepsilon_1,\varepsilon_2)$ and $a\neq 1$.

For $\beta=\widetilde{\beta}_*$ and $\alpha=1$, translating $\widetilde{E}_*$ to the origin and further normalizing the linear part with the transformation $(x,y)\mapsto(x-y,x+y)$ together with the time-rescaling $t\mapsto -2x_*^2(2x_*+\epsilon_2)^2t/(x_*+\epsilon_2)$, we change system (2.3) as

$$\dot{x}=y+yN_1(x,y^2),\quad \dot{y}=N_2(x,y^2),\tag{3.2}$$

where

$$\begin{aligned}N_1(x,y^2):=&\frac{2}{x_*}x+\frac{10x_*^2+11\epsilon_2x_*+2\epsilon_2^2}{2x_*^2(2x_*+\epsilon_2)^2}x^2+\frac{x_*+\epsilon_2}{x_*^2(2x_*+\epsilon_2)^2}x^3\\&-\frac{3x_*+2\epsilon_2}{2x_*^2(2x_*+\epsilon_2)}y^2-\frac{x_*+\epsilon_2}{x_*^2(2x_*+\epsilon_2)^2}xy^2,\\N_2(x,y^2):=&\frac{2x_*+3\epsilon_2}{2(2x_*+\epsilon_2)^2}x^2+\frac{4x_*+5\epsilon_2}{2x_*(2x_*+\epsilon_2)^2}x^3+\frac{x_*+\epsilon_2}{x_*^2(2x_*+\epsilon_2)^2}x^4\\&+\frac{1}{2(2x_*+\epsilon_2)}y^2-\frac{\epsilon_2}{2x_*(2x_*+\epsilon_2)^2}xy^2-\frac{x_*+\epsilon_2}{x_*^2(2x_*+\epsilon_2)^2}x^2y^2.\end{aligned}$$

Further, using the transformation $(x,y)\mapsto(x,y+yN_1(x,y^2))$, we reduce the above system to the following Kukles form

$$\dot{x}=y,\quad \dot{y}=\frac{2x_*+3\epsilon_2}{2(2x_*+\epsilon_2)^2}x^2+\frac{9x_*+4\epsilon_2}{2x_*(2x_*+\epsilon_2)}y^2+O(|x,y|^3).$$

By Theorem II.7.3 of [32], equilibrium O of the above system is a cusp, i.e., equilibrium E_* of system (1.2) is a cusp. Thus this theorem is proved. □

Remark that for $\beta=\beta_*(a,\varepsilon_1,\varepsilon_2)$ and $a\neq 1$ Theorem 3.1 indicates that equilibrium E_* of system (1.2) is a saddle-node, from which we easily dicsuss a saddle-node bifurcation. In the case that $\beta=\beta_*(a,\varepsilon_1,\varepsilon_2)$ and $a=1$, Theorem 3.1 indicates that E_* is a nilpotent cusp but none of known results (e.g. [2,8,28]) on Bogdanov-Takens bifurcation and its degenerate versions can be applied because the following lemma shows that the cusp is degenerate of codimension ∞.

Lemma 3.1. *Let P and Q be two analytic real functions near $O:(0,0)$ such that $P(x,y),Q(x,y)=O(|x,y|^2)$, $P(x,y)=-P(x,-y)$, $Q(x,y)=Q(x,-y)$ and $Q(x,0)=a_nx^n+O(x^{n+1})$ for an integer n and a nonzero constant a_n. Then the following system*

$$\dot{x}=y+P(x,y),\quad \dot{y}=Q(x,y)\tag{3.3}$$

has the normal form $\dot{x}=y$ and $\dot{y}=a_nx^n+O(x^{n+1})$.

Proof. As indicated in [5, Chapter 2], in order to reduce (3.3) to a Poincaré normal form, we need to perform a sequence of near-identity transformations $(x,y)\to(x+h_{i,1},y+h_{i,2})$ to eliminate those resonant terms, where $i\geq 2$ and $(h_{i,1},h_{i,2})\in H_2^i$, the vector space of homogeneous polynomials of degree i in two variables with values in $\mathbb{R}^2$. We claim that the i-th order normal form of system (3.3) is of the form

$$\dot{x}=y+P_{i+1}(x,y),\quad \dot{y}=R_{i+1}(x)+Q_{i+1}(x,y),\tag{3.4}$$

where $P_{i+1}(x,y), Q_{i+1}(x,y) = O(|x,y|^{i+1})$, $P_{i+1}(x,y) = -P_{i+1}(x,-y)$, $Q_{i+1}(x,y) = Q_{i+1}(x,-y)$, $Q_{i+1}(x,0) \equiv 0$ and $R_{i+1}(x) = a_n x^n + O(x^{n+1})$. If the claim is true then this lemma follows.

We prove the claim by induction. Clearly, it is true for $i=1$. Assume that it is true for $i = k-1$. Consider the linear operator $\mathcal{L}_A^k : H_2^k \to H_2^k$ defined by

$$\mathcal{L}_A^k h_k := \mathrm{D}h_k A(x,y)^T - A h_k, \quad h_k \in H_2^k,$$

where $\mathrm{D}h_k$ is the Jacobian matrix of h_k, matrix A is the linear part of system (3.3) at the origin and T is the transpose. Notice that the linear space H_2^k is $(2k+2)$-dimensional and has a group of bases $\{e_1^k, ..., e_{2k+2}^k\}$, where

$$\begin{array}{llll} e_1^k := (0, x^k)^T, & e_2^k := (0, x^{k-1}y)^T, & ..., & e_{k+1}^k := (0, y^k)^T \\ e_{k+2}^k := (x^k, 0)^T, & e_{k+3}^k := (x^{k-1}y, 0)^T, & ..., & e_{2k+2}^k := (y^k, 0)^T. \end{array}$$

We compute that

$$\mathcal{L}_A^k e_j^k = \begin{cases} (k+1-j)e_{j+1}^k - e_{k+1+j}^k, & j = 1, ..., k+1, \\ (2k+2-j)e_{j+1}^k, & j = k+2, ..., 2k+2. \end{cases}$$

Let $h_k(x,y) := (h_{k,1}(x,y), h_{k,2}(x,y)) \in H_2^k$, where

$$h_{k,1}(x,y) := \sum_{j=0}^{k} f_{k-j,j} x^{k-j} y^j \quad \text{and} \quad h_{k,2}(x,y) := \sum_{j=0}^{k} g_{k-j,j} x^{k-j} y^j.$$

Then we obtain that

$$L_A^k h_k = \sum_{j=0}^{k-1} (k-j) g_{k-j,j} e_{j+2}^k - g_{k,0} e_{k+2}^k + \sum_{j=0}^{k-1} \{(k-j) f_{k-j,j} - g_{k-j-1,j+1}\} e_{k+3+j}^k.$$

Note that we need to choose an appropriate $h_k \in H_2^k$ to eliminate all terms of degree k except for the term x^k in $\dot{y}$ in (3.4) with $i = k-1$. For this purpose, assume that

$$P_k(x,y) = \sum_{j=0}^{k} p_{k-j,j} x^{k-j} y^j + O(|x,y|^{k+1}), \quad Q_k(x,y) = \sum_{j=0}^{k} q_{k-j,j} x^{k-j} y^j + O(|x,y|^{k+1}).$$

Then those terms of degree k in (3.4) with $i = k-1$ are given by

$$q_{k,0} e_1^k + \sum_{j=0}^{k-1} q_{k-j-1,j+1} e_{j+2}^k + p_{k,0} e_{k+2}^k + \sum_{j=0}^{k-1} p_{k-j-1,j+1} e_{k+3+j}^k.$$

Therefore, we obtain the following equations

$$\begin{aligned} &(k-j) g_{k-j,j} = q_{k-j-1,j+1}, && j = 0, ..., k-1, \\ &-g_{k,0} = p_{k,0}, && \\ &(k-j) f_{k-j,j} - g_{k-j-1,j+1} = p_{k-j-1,j+1}, && j = 0, ..., k-1. \end{aligned}$$

Note that $p_{k-j,j} = 0$ for all even j and $q_{k-j,j} = 0$ for all odd j since $P_k(x,y) = -P_k(x,-y)$ and $Q_k(x,y) = Q_k(x,-y)$, as indicated just below (3.4). Then the above equations has a solution $g_{k,0} = 0$, $f_{1,k-1} = g_{0,k} + p_{0,k}$ and

$$g_{k-j,j} = \frac{q_{k-j-1,j+1}}{k-j}, \qquad j = 1, ..., k-1,$$

$$f_{k-j,j} = \frac{(k-j-1)p_{k-j-1,j+1} + q_{k-j-2,j+2}}{(k-j-1)(k-j)}, \qquad j = 0, ..., k-2.$$

Moreover, we choose $g_{0,k} = 0$ and $f_{0,k} = 0$. Then, $g_{k-j,j} = 0$ for all even j and $f_{k-j,j} = 0$ for all odd j. It follows that

$$h_{k,1}(x,-y) = h_{k,1}(x,y) \quad \text{and} \quad h_{k,2}(x,-y) = -h_{k,2}(x,y). \tag{3.5}$$

Under the near identity transformation $(x,y) \mapsto (x + h_{i,1}, y + h_{i,2})$, system (3.4) with $i = k-1$ becomes $\dot{x} = \mathcal{P}_k(x,y)/\mathcal{D}_k(x,y)$ and $\dot{y} = \mathcal{Q}_k(x,y)/\mathcal{D}_k(x,y)$ with

$$\begin{aligned}
\mathcal{P}_k(x,y) &:= A_{2,2}(x,y)\{y + h_{k,2}(x,y) + \widetilde{P}_k(x,y)\} - A_{1,2}(x,y)\{\widetilde{R}_k(x,y) + \widetilde{Q}_k(x,y)\}, \\
\mathcal{Q}_k(x,y) &:= -A_{2,1}(x,y)\{y + h_{k,2}(x,y) + \widetilde{P}_k(x,y)\} + A_{1,1}(x,y)\{\widetilde{R}_k(x,y) + \widetilde{Q}_k(x,y)\}, \\
\mathcal{D}_k(x,y) &:= A_{1,1}(x,y)A_{2,2}(x,y) - A_{2,1}(x,y)A_{1,2}(x,y),
\end{aligned}$$

where $A_{1,1}(x,y) := 1 + \partial h_{k,1}(x,y)/\partial x$, $A_{1,2}(x,y) := \partial h_{k,1}(x,y)/\partial y$, $A_{2,1}(x,y) := \partial h_{k,2}(x,y)/\partial x$, $A_{2,2}(x,y) := 1 + \partial h_{k,2}(x,y)/\partial y$, $\widetilde{P}_k(x,y) := P_k(x + h_{k,1}(x,y), y + h_{k,2}(x,y))$, $\widetilde{Q}_k(x,y) := Q_k(x + h_{k,1}(x,y), y + h_{k,2}(x,y))$ and $\widetilde{R}_k(x,y) := R_k(x + h_{k,1}(x,y))$. By (3.5) and properties given below (3.4), $A_{1,2}(x,y), A_{2,1}(x,y)$ and $\widetilde{P}_k(x,y)$ are all odd functions in y, and $A_{1,1}(x,y)$, $A_{2,2}(x,y)$, $\mathcal{D}_k(x,y)$, $\widetilde{Q}_k(x,y)$ and $\widetilde{R}_k(x,y)$ are all even functions in y. Note that $\mathcal{D}_k(x,y) = 1 + O(|x,y|)$ and $\mathcal{Q}_k(x,0) = a_n x^n + O(x^{n+1})$. Then we can rewritten the k-th order normal form as

$$\dot{x} = \frac{\mathcal{P}_k(x,y)}{\mathcal{D}_k(x,y)} = y + P_{k+1}(x,y), \quad \dot{y} = \frac{\mathcal{Q}_k(x,y)}{\mathcal{D}_k(x,y)} = R_{k+1}(x) + Q_{k+1}(x,y),$$

where P_{k+1}, Q_{k+1} and R_{k+1} satisfy the same properties as P_{i+1}, Q_{i+1} and R_{i+1} respectively given just below (3.4). Therefore, the claimed (3.4) is proved by induction. This completes the proof of this lemma. □

Note that system (2.3), topologically equivalent to system (1.2), can be transformed into the form (3.2), which is of the form (3.3). By Lemma 3.1, system (1.2) has the normal form

$$\dot{x} = y, \quad \dot{y} = \psi(x) := \omega x^2 + O(x^3), \tag{3.6}$$

where $\omega := (2x_* + 3\epsilon_2)/\{2(2x_* + \epsilon_2)^2\}$. Further, under the transformation $x \mapsto \omega(3\Psi(x)/\omega)^{1/3}$ and the time-rescaling $t \mapsto (3\Psi(x)/\omega)^{-2/3}\psi(x)t$, where $\Psi(x) := \int_0^x \psi(s)ds$, system (3.6) is changed as the system $\dot{x} = y$ and $\dot{y} = x^2$. As indicated in [16], a system possessing a nilpotent cusp of codimension n has the orbital normal form $\dot{x} = y$ and $\dot{y} = x^2 \pm x^\ell y$ with $\ell := [3(n-1)/2]$, the largest integer not greater than $3(n-1)/2$. Thus the cusp E_* of system (1.2) is degenerate of codimension ∞.

4. Case of center type

Theorem 4.1. *For $\beta > \beta_*(a, \varepsilon_1, \varepsilon_2)$ and $a = 1$, equilibrium E_1 of system (1.2) is a center. Moreover, it is either a rough center or a weak center of order 1.*

Proof. As indicated on line 4 of Table 1 in Theorem 2.1, equilibrium E_1 of system (1.2) is of center type for $\beta > \beta_*$ and $a = 1$. For convenience, we also apply the transformation $(x,y) \mapsto (x^2, y^2)$ to system (1.2), as done in the proof of Theorem

2.1, but do not use a time-rescaling because we will discuss the period function if the equilibrium is proved to be a center. Thus we obtain the transformed system

$$\dot{x} = \mathcal{X}(x,y) := \frac{x\{\mathcal{H}(x,y) - \beta y^2\}}{2\mathcal{H}(x,y)}, \quad \dot{y} = \mathcal{Y}(x,y) := \frac{y\{\beta x^2 - \mathcal{H}(x,y)\}}{2\mathcal{H}(x,y)}, \tag{4.1}$$

where $\mathcal{H}(x,y)$ is given just below (2.3). Clearly, the system has the same equilibria as system (2.3). Thus, we can equivalently investigate the center type equilibrium $\widetilde{E}_1 : (x_1, x_1/\alpha)$ of system (4.1) for $\beta > \widetilde{\beta}_*$ and $\alpha = 1$, as indicated in the case **(E2)** in the proof of Theorem 2.1. Since $\mathcal{H}(x,y) = \mathcal{H}(y,x)$, we have

$$\mathcal{X}(y,x) = \frac{y\{\mathcal{H}(y,x) - \beta x^2\}}{\mathcal{H}(y,x)} = -\mathcal{Y}(x,y), \quad \mathcal{Y}(y,x) = \frac{x\{\beta y^2 - \mathcal{H}(y,x)\}}{\mathcal{H}(y,x)} = -\mathcal{X}(x,y),$$

which implies that system (4.1) is time-reversible with respect to the line $y = x$. As indicated just below (2.5), equilibrium $\widetilde{E}_1$ lies on the line $y = x$ as $\alpha = 1$. By Theorem 3.5.5 of [25], equilibrium $\widetilde{E}_1$ is a center.

We further determine the order (defined in [4, p.439-440]) of the center $\widetilde{E}_1$. Translating the center equilibrium of system (4.1) to the origin and then normalizing the linear part with the transformation $x \mapsto x$ and $y \mapsto -c_3/\sqrt{4c_5c_6}x + c_4/\sqrt{4c_5c_6}x$, where $c_3 := x_1(3x_1^2 + \epsilon_2 x_1 + \epsilon_1)$, $c_4 := x_1^3 + \epsilon_2 x_1^2 + 3\epsilon_1 x_1 + 2\epsilon_1\epsilon_2$, $c_5 := (x_1^2 + \epsilon_1)(2x_1 + \epsilon_2)$ and $c_6 := -x_1^3 + \epsilon_1 x_1 + \epsilon_1\epsilon_2$, we reduce system (4.1) to the following

$$\dot{x} = -y + \sum_{i=2}^{5} X_i(x,y) + O(|x,y|^6), \quad \dot{y} = x + \sum_{i=2}^{5} Y_i(x,y) + O(|x,y|^6), \tag{4.2}$$

where X_i s and Y_i s are homogeneous polynomials of degree i. Let $T(r)$ be the minimum period of the periodic orbit around the center O through a nonzero point $(r, 0)$. As indicated in [4, Lemma 2.1], the period function $T(r)$ has the following expansion

$$T(r) = 2\pi + \sum_{k=1}^{+\infty} p_{2k} r^{2k},$$

where coefficients p_{2k} s, called periodic quantities, are polynomials in parameters of system (4.1). Using the software MAPLE ver.18, we compute the first and the second periodic quantities

$$\begin{aligned} p_2 &= \pi(2x_1 + \epsilon_2)^6 (x_1^2 + \epsilon_1)^7 F_1(x_1)/x_1^2, \\ p_4 &= 864\pi x_1^4 (2x_1 + \epsilon_2)^2 (x_1^2 + \epsilon_1)^2 c_4^4 c_6^4 F_2(x_1), \end{aligned} \tag{4.3}$$

where polynomials F_1 and F_2 are given in the Appendix. Clearly, $p_2 = 0$ (resp. $p_4 = 0$) if and only if $F_1(x_1) = 0$ (resp. $F_2(x_1) = 0$).

Let $\widehat{F} := F|_{\alpha=1}$, $\widehat{S}_2 := S_2|_{\alpha=1}$ and $\widehat{\beta}_* := \widetilde{\beta}_*|_{\alpha=1}$, where F is defined in (2.6) and S_2 and $\widetilde{\beta}_*$ are given just before (2.9). In order to determine the sign of $F_1(x_1)$ for $\beta > \widetilde{\beta}_*$ and $\alpha = 1$, we need to investigate the distribution of positive zeros of F_1 and $\widehat{F}$ for $\beta > \widehat{\beta}_*$. By Lemma 2.1, we compute resultants

$$\begin{aligned} &\mathrm{res}(\widehat{F}, \widehat{F}'_x, x) = \epsilon_1 \widehat{S}_2(\beta), \\ &\mathrm{res}(F_1, (F_1)'_x, x) = \epsilon_1^{19} \epsilon_2^{12} (4\epsilon_1 + \epsilon_2^2)^6 R_1(\epsilon_1) R_2(\epsilon_1), \\ &\mathrm{res}(\widehat{F}, F_1, x) = \epsilon_1^6 \epsilon_2^4 S_3(\beta), \end{aligned} \tag{4.4}$$

where a non-zero constant factor in each formula is omitted for convenience, $R_1(\epsilon_1) := 4\epsilon_1 - 27\epsilon_2^2$, S_3 and R_2 are given in the Appendix. Further we need to discuss the distribution of positive zeros of $\widehat{S}_2$ and S_3 for $\beta > \widehat{\beta}_*$. By Lemma 2.1, we compute resultants

$$\begin{aligned}
&\mathrm{res}(\widehat{S}_2, (\widehat{S}_2)'_\beta, \beta) = \epsilon_2\epsilon_1^2 R_1^3(\epsilon_1),\\
&\mathrm{res}(S_3, (S_3)'_\beta, \beta) = \epsilon_1^{11}\epsilon_2^{16}(4\epsilon_1 + \epsilon_2^2)^{12} R_1(\epsilon_1)R_2(\epsilon_1)R_3(\epsilon_1),\\
&\mathrm{res}(S_3, \widehat{S}_2, \beta) = \epsilon_1^6\epsilon_2^4(4\epsilon_1 + \epsilon_2^2)^6 R_1^6(\epsilon_1)R_4(\epsilon_1),
\end{aligned} \tag{4.5}$$

where a non-zero constant factor in each formula is omitted for convenience,

$$\begin{aligned}
R_4(\epsilon_1) :=& 1280\epsilon_1^6 - 1278720\epsilon_2^2\epsilon_1^5 + 149068844\epsilon_2^4\epsilon_1^4 - 3288186800\epsilon_2^6\epsilon_1^3\\
&+ 14779202661\epsilon_2^8\epsilon_1^2 - 18839179848\epsilon_2^{10}\epsilon_1 + 1124864\epsilon_2^{12}
\end{aligned}$$

and R_3 is given in the Appendix. We give the distribution of positive zeros of R_1, R_2, R_3 and R_4 in the following.

Claim 1. *R_1 has one positive zero ϵ_{10}, R_2 has four positive zeros $\epsilon_{11} < \epsilon_{12} < \epsilon_{13} < \epsilon_{14}$, R_3 has three positive zeros $\widetilde{\epsilon}_{11} < \widetilde{\epsilon}_{12} < \widetilde{\epsilon}_{13}$ and R_4 has four positive zeros $\widehat{\epsilon}_{11} < \widehat{\epsilon}_{12} < \widehat{\epsilon}_{13} < \widehat{\epsilon}_{14}$. Moreover, $\epsilon_{11} < \widehat{\epsilon}_{11} < \epsilon_{12} < \epsilon_{13} < \widetilde{\epsilon}_{11} < \epsilon_{14} < \epsilon_{10} < \widehat{\epsilon}_{12} < \widehat{\epsilon}_{13} < \widetilde{\epsilon}_{12} < \widehat{\epsilon}_{14} < \widetilde{\epsilon}_{13}$.*

In fact, one can check that $\mathrm{lcoeff}(R_i, \epsilon_1) \neq 0$, $\mathrm{res}(R_i, (R_i)'_{\epsilon_1}, \epsilon_1) \neq 0$ and $\mathrm{res}(R_i, R_j, \epsilon_1) \neq 0$ for all $i, j = 1, 2, 3, 4$ and $i \neq j$. By Lemma 2.1, the distribution of positive zeros of R_1,...,R_4 does not change as ϵ_2 varies. Hence, choosing $\epsilon_2 = 1$ and using MAPLE ver.18 command 'realroot($R_i, 10^{-3}$)' for $i = 1, 2, 3, 4$, we obtain that R_1 has a unique positive zero, covered by the interval $[\epsilon_{10}^-, \epsilon_{10}^+]$, where $\epsilon_{10}^- := \frac{269}{40}$ and $\epsilon_{10}^+ := \frac{271}{40}$; R_2 has four positive zeros, covered by the intervals $[\epsilon_{1i}^-, \epsilon_{1i}^+]$ $(i = 1, 2, 3, 4)$ separately, where $\epsilon_{11}^- := \frac{7709}{4294967296}$, $\epsilon_{11}^+ := \frac{3855}{2147483648}$, $\epsilon_{12}^- := \frac{7063}{16777216}$, $\epsilon_{12}^+ := \frac{883}{2097152}$, $\epsilon_{13}^- := \frac{5199}{4096}$, $\epsilon_{13}^+ := \frac{10399}{8192}$, $\epsilon_{14}^- := \frac{9147}{2048}$ and $\epsilon_{14}^+ := \frac{36589}{8192}$; R_3 has three positive zeros, covered by the intervals $[\tilde{\epsilon}_{1i}^-, \tilde{\epsilon}_{1i}^+]$ $(i = 1, 2, 3)$ separately, where $\tilde{\epsilon}_{11}^- := \frac{13841}{8192}$, $\tilde{\epsilon}_{11}^+ := \frac{6921}{4096}$, $\tilde{\epsilon}_{12}^- := \frac{220707}{1024}$, $\tilde{\epsilon}_{12}^+ := \frac{1765657}{8192}$, $\tilde{\epsilon}_{13}^- := \frac{53748983}{8192}$ and $\tilde{\epsilon}_{13}^+ := \frac{6718623}{1024}$; and R_4 has four positive zeros, covered by the intervals $[\hat{\epsilon}_{1i}^-, \hat{\epsilon}_{1i}^+]$ $(i = 1, 2, 3, 4)$ separately, where $\hat{\epsilon}_{11}^- := \frac{4007}{67108864}$, $\hat{\epsilon}_{11}^+ := \frac{8015}{134217728}$, $\hat{\epsilon}_{12}^- := \frac{177937}{8192}$, $\hat{\epsilon}_{12}^+ := \frac{88969}{4096}$, $\hat{\epsilon}_{13}^- := \frac{848147}{8192}$, $\hat{\epsilon}_{13}^+ := \frac{212037}{2048}$, $\hat{\epsilon}_{14}^- := \frac{7112791}{8192}$ and $\hat{\epsilon}_{14}^+ := \frac{889099}{1024}$. Note that $\epsilon_{11}^- < \epsilon_{11}^+ < \hat{\epsilon}_{11}^- < \hat{\epsilon}_{11}^+ < \epsilon_{12}^- < \epsilon_{12}^+ < \epsilon_{13}^- < \epsilon_{13}^+ < \tilde{\epsilon}_{11}^- < \tilde{\epsilon}_{11}^+ < \epsilon_{14}^- < \epsilon_{14}^+ < \epsilon_{10}^- < \epsilon_{10}^+ < \hat{\epsilon}_{12}^- < \hat{\epsilon}_{12}^+ < \hat{\epsilon}_{13}^- < \hat{\epsilon}_{13}^+ < \tilde{\epsilon}_{12}^- < \tilde{\epsilon}_{12}^+ < \hat{\epsilon}_{14}^- < \hat{\epsilon}_{14}^+ < \tilde{\epsilon}_{13}^- < \tilde{\epsilon}_{13}^+$. Then **Claim 1** follows.

Having **Claim 1**, we further determine the distribution of zeros of S_3 and $\widehat{S}_2$ for $\beta > \widehat{\beta}_*$.

Claim 2. *$\widehat{\beta}_*$ is the only positive zero of $\widehat{S}_2$. In the interval $(\widehat{\beta}_*, +\infty)$, S_3 has four zeros $\beta_{31} \leq \beta_{32} \leq \beta_{33} < \beta_{34}$ for $\epsilon_1 \leq \epsilon_{11}$, where $\beta_{31} = \beta_{32}$ if and only if $\epsilon_1 = \epsilon_{11}$, two zeros $\widetilde{\beta}_{31} < \widetilde{\beta}_{32}$ for $\epsilon_{11} < \epsilon_1 < \widehat{\epsilon}_{11}$, and one zero β_{30} for $\epsilon_1 \geq \widehat{\epsilon}_{11}$.*

In fact, it is indicated before (2.9) that $\widetilde{\beta}_*$ is the only positive zero of S_2. Then $\widehat{\beta}_*$ is the only positive zero of $\widehat{S}_2$ because $\widehat{\beta}_* = \widetilde{\beta}_*|_{\alpha=1}$ and $\widehat{S}_2 = S_2|_{\alpha=1}$. Note that $S_3(0) = -36(121\epsilon_1 + 36\epsilon_2^2)(4\epsilon_1 + \epsilon_2^2)^6 < 0$ and $\mathrm{lcoeff}(S_3, \beta) = 5120\epsilon_1^4 > 0$. Moreover, those resultants given in (4.5) are all nonzero for all positive ϵ_1 not equaling to those 12 zeros listed in **Claim 1**. By Lemma 2.1, the distribution of positive zeros of S_3 and $\widehat{S}_2$ does not change as ϵ_1 varies in each one of the intervals divided by those

12 zeros given in **Claim 1**. Thus, from the proof of **Claim 1**, we choose $(\epsilon_1, \epsilon_2) = (\epsilon_{1i}^{\pm}, 1)$, $(\tilde{\epsilon}_{1j}^{\pm}, 1)$ and $(\hat{\epsilon}_{1k}^{\pm}, 1)$ for all $i = 0, 1, 2, 3, 4$, $j = 1, 2, 3$ and $k = 1, 2, 3, 4$, and then use MAPLE ver.18 command 'realroot$(S_3, 10^{-3})$' and 'realroot$(\widehat{S}_2, 10^{-4})$' to investigate the distribution of real zeros of S_3 and $\widehat{S}_2$. Choosing $(\epsilon_1, \epsilon_2) = (\epsilon_{11}^{-}, 1)$, we obtain that $\widehat{\beta}_* \in [\frac{135889}{131072}, \frac{67945}{65536}]$ and in the interval $(\widehat{\beta}_*, +\infty)$ the polynomial S_3 has four zeros, covered by the intervals $[\beta_{3i}^{-}, \beta_{3i}^{+}]$ $(i = 1, 2, 3, 4)$ separately, where $\beta_{31}^{-} := \frac{35749}{32768} < \beta_{31}^{+} := \frac{142997}{131072} < \beta_{32}^{-} := \frac{143073}{131072} < \beta_{32}^{+} := \frac{71537}{65536} < \beta_{33}^{-} := \frac{18523}{16384} < \beta_{33}^{+} := \frac{148185}{131072} < \beta_{34}^{-} := \frac{1156409}{131072} < \beta_{34}^{+} := \frac{578205}{65536}$. Then, in the interval $(\widehat{\beta}_*, +\infty)$ the polynomial S_3 has four zeros $\beta_{31} < \beta_{32} < \beta_{33} < \beta_{34}$ for all $\epsilon_1 < \epsilon_{11}$. Similarly, choosing $(\epsilon_1, \epsilon_2) = (\epsilon_{11}^{+}, 1)$, we obtain that S_3 has two zeros in the interval $(\widehat{\beta}_*, +\infty)$. Then S_3 has two zeros $\widetilde{\beta}_{31} < \widetilde{\beta}_{32}$ in the interval $(\widehat{\beta}_*, +\infty)$ for all $\epsilon_1 \in (\epsilon_{11}, \widehat{\epsilon}_{11})$. For $\epsilon_1 = \epsilon_{11}$, the polynomial S_3 has a multiple zero because $\mathrm{res}(S_3, (S_3)'_\beta, \beta) = 0$. Moreover, since zeros of a polynomial vary continuously as a function of the coefficients and $S_3(0) < 0$, the two positive zero β_{31} and β_{32} of S_3 for all $\epsilon_1 < \epsilon_{11}$ become a multiple zero, i.e., $\beta_{31} = \beta_{32}$, in the critical case $\epsilon_1 = \epsilon_{11}$ and then a pair of conjugate non-real zeros for all $\epsilon_1 \in (\epsilon_{11}, \widehat{\epsilon}_{11})$. Further, from a similar discussion on those choices $(\epsilon_1, \epsilon_2) = (\epsilon_{1i}^{\pm}, 1)$, $(\tilde{\epsilon}_{1j}^{\pm}, 1)$ and $(\hat{\epsilon}_{1k}^{\pm}, 1)$ for all $i = 0, 2, 3, 4$, $j = 1, 2, 3$ and $k = 2, 3, 4$, we obtain that S_3 has one zero β_{30} in the interval $(\widehat{\beta}_*, +\infty)$ for all $\epsilon_1 \geq \widehat{\epsilon}_{11}$. Thus **Claim 2** is proved.

Having **Claim 2**, we further determine the distribution of positive zeros of F_1 and $\widehat{F}$ for $\beta > \widehat{\beta}_*$.

Claim 3. *In the interval $(\widehat{\beta}_*, +\infty)$, $\widehat{F}$ has two positive zeros $x_1 < x_2$, and F_1 has three positive zeros $x_{11} \leq x_{12} < x_{13}$ for all $\epsilon_1 \leq \epsilon_{11}$, where $x_{11} = x_{12}$ if and only if $\epsilon_1 = \epsilon_{11}$, and a unique positive zero x_{10} for all $\epsilon_1 > \epsilon_{11}$. The distribution of positive zeros of $\widehat{F}$ and F_1 for $\beta > \widehat{\beta}_*$ is listed in Table 2.*

Table 2. The distribution of positive zeros of $\widehat{F}$ and F_1 for $\beta > \widehat{\beta}_*$.

Parameters		distribution
$0 < \epsilon_1 < \epsilon_{11}$	$\widehat{\beta}_* < \beta < \beta_{31}$	$x_{11} < x_{12} < x_1 < x_2 < x_{13}$
	$\beta = \beta_{31}$	$x_{11} < x_{12} = x_1 < x_2 < x_{13}$
	$\beta_{31} < \beta < \beta_{32}$	$x_{11} < x_1 < x_{12} < x_2 < x_{13}$
	$\beta = \beta_{32}$	$x_{11} = x_1 < x_{12} < x_2 < x_{13}$
	$\beta_{32} < \beta < \beta_{34}$	$x_1 < x_{11} < x_{12} < x_2 < x_{13}$
	$\beta = \beta_{34}$	$x_1 < x_{11} < x_{12} < x_2 = x_{13}$
	$\beta > \beta_{34}$	$x_1 < x_{11} < x_{12} < x_{13} < x_2$
$\epsilon_1 = \epsilon_{11}$	$\widehat{\beta}_* < \beta < \beta_{31} = \beta_{32}$	$x_{11} = x_{12} < x_1 < x_2 < x_{13}$
	$\beta = \beta_{32}$	$x_{11} = x_{12} = x_1 < x_2 < x_{13}$
	$\beta_{32} < \beta < \beta_{34}$	$x_1 < x_{11} = x_{12} < x_2 < x_{13}$
	$\beta = \beta_{34}$	$x_1 < x_{11} = x_{12} < x_2 = x_{13}$
	$\beta > \beta_{34}$	$x_1 < x_{11} = x_{12} < x_{13} < x_2$
$\epsilon_{11} < \epsilon_1 < \widehat{\epsilon}_{11}$	$\widehat{\beta}_* < \beta < \widetilde{\beta}_{32}$	$x_1 < x_2 < x_{10}$
	$\beta = \widetilde{\beta}_{32}$	$x_1 < x_2 = x_{10}$
	$\beta > \widetilde{\beta}_{32}$	$x_1 < x_{10} < x_2$
$\epsilon_1 \geq \widehat{\epsilon}_{11}$	$\widehat{\beta}_* < \beta < \beta_{30}$	$x_1 < x_2 < x_{10}$
	$\beta = \beta_{30}$	$x_1 < x_2 = x_{10}$
	$\beta > \beta_{30}$	$x_1 < x_{10} < x_2$

In fact, $F_1(0) = -32\epsilon_1^2\epsilon_2^4 < 0$ and $\mathrm{lcoeff}(F_1, x) = 10 > 0$. Moreover, we see from (4.4) that $\mathrm{res}(F_1, (F_1)'_x, x) = 0$ if and only if $R_1(\epsilon_1)R_2(\epsilon_1) = 0$. From the proof of **Claim 1**, choosing $(\epsilon_1, \epsilon_2) = (\epsilon_{1i}^{\pm}, 1)$ for all $i = 0, 1, 2, 3, 4$, we can similarly obtain that F_1 has three positive zeros $x_{11} < x_{12} < x_{13}$ for all $\epsilon_1 < \epsilon_{11}$, one positive zero x_{10} for all $\epsilon_1 > \epsilon_{11}$ and, moreover, F_1 has one double positive zero $x_{11} = x_{12}$ and one simple positive zero x_{13} in the critical situation $\epsilon_1 = \epsilon_{11}$. Combined with **Claim 2**, there are four situations:

$$\epsilon_1 < \epsilon_{11}, \quad \epsilon_1 = \epsilon_{11}, \quad \epsilon_{11} < \epsilon_1 < \hat{\epsilon}_{11} \quad \text{and} \quad \epsilon_1 \geq \hat{\epsilon}_{11}.$$

In the situation $\epsilon_1 < \epsilon_{11}$, since F_1 is independent of parameter β, choosing $(\epsilon_1, \epsilon_2) = (\epsilon_{11}^-, 1)$ and using MAPLE ver.18 command 'realroot$(F_1, 10^{-4})$', we obtain that those three positive zeros x_{11}, x_{12} and x_{13} of F_1 lie in the intervals $[x_{11}^-, x_{11}^+]$, $[x_{12}^-, x_{12}^+]$ and $[x_{13}^-, x_{13}^+]$ respectively, where $x_{11}^- := \frac{78777}{16777216}$, $x_{11}^+ := \frac{39389}{8388608}$, $x_{12}^- := \frac{79079}{16777216}$, $x_{12}^+ := \frac{9885}{2097152}$, $x_{13}^- := \frac{128167}{32768}$ and $x_{13}^+ := \frac{512669}{131072}$. We see from **Claim 2** that S_3 or equivalently the resultant res$(\widehat{F}, F_1, x)$ has four real zeros $\beta_{31} < \beta_{32} < \beta_{33} < \beta_{34}$ in the interval $(\widehat{\beta}_*, +\infty)$. By Lemma 2.1, the distribution of positive zeros of F_1 and $\widehat{F}$ does not change as β varies in each one of intervals divided by β_{3i} s. From the proof of **Claim 2**, choosing $(\beta, \epsilon_1, \epsilon_2) = (\beta_{31}^-, \epsilon_{11}^-, 1)$ and using MAPLE ver.18 command 'realroot$(\widehat{F}, 10^{-4})$', we obtain that $\widehat{F}$ has two positive zeros x_1 and x_2, lying in the intervals $[x_{131}^-, x_{131}^+]$ and $[x_{231}^-, x_{231}^+]$ respectively, where $x_{131}^- := \frac{79081}{16777216}$, $x_{131}^+ := \frac{39541}{8388608}$, $x_{231}^- := \frac{94379}{2097152}$ and $x_{231}^+ := \frac{23595}{524288}$. Note that $x_{11}^- < x_{11}^+ < x_{12}^- < x_{12}^+ < x_{131}^- < x_{131}^+ < x_{231}^- < x_{231}^+ < x_{13}^- < x_{13}^+$. Then, $x_{11} < x_{12} < x_1 < x_2 < x_{13}$ for all $\epsilon_1 < \epsilon_{11}$ and all $\beta \in (\widehat{\beta}_*, \beta_{31})$, as indicated on line 1 in Table 2. Moveover, choosing $(\beta, \epsilon_1, \epsilon_2) = (\beta_{31}^+, \epsilon_{11}^-, 1)$, $(\beta_{3i}^-, \epsilon_{11}^-, 1)$ and $(\beta_{3i}^+, \epsilon_{11}^-, 1)$ for all $i = 2, 3, 4$, we similarly obtain the result stated on lines 3, 5 and 7 in Table 2. Since zeros of a polynomial vary continuously as a function of the coefficients, F_1 and $\widehat{F}$ have one common positive zero $x_{12} = x_1$ in the critical case $\beta = \beta_{31}$, one common positive zero $x_{11} = x_1$ in the critical case $\beta = \beta_{32}$ and one common positive zero $x_{13} = x_4$ in the critical case $\beta = \beta_{34}$, as indicated on lines 2, 4 and 6 in Table 2. Consequently, we obtain the distribution of positive zeros of F_1 and $\widehat{F}$ in the situation $\epsilon_1 < \epsilon_{11}$. The situation $\epsilon_{11} < \epsilon_1 < \widehat{\epsilon}_{11}$, the situation $\epsilon_1 \geq \widehat{\epsilon}_{11}$ and the critical situation $\epsilon_1 = \epsilon_{11}$ can be discussed similarly. Then **Claim 3** is proved.

By **Claim 3**, $p_2 = F_1(x_1) = 0$, i.e., equilibrium O of system (4.2) is a rough center, if and only if either $\beta = \beta_{31}$ and $\epsilon_1 \leq \epsilon_{11}$, or $\beta = \beta_{32}$ and $\epsilon_1 \leq \epsilon_{11}$.

Finally, we show that if O is a weak center then its order is at most 1, i.e., $p_4 \neq 0$ when $p_2 = 0$, which is equivalent to show that $F_2(x_1) \neq 0$ when $F_1(x_1) = 0$ by (4.3). Actually, we compute the resultant

$$\text{res}(F_1, F_2, x) = \epsilon_1^{56}\epsilon_2^{40}(4\epsilon_1 + \epsilon_2^2)^{24} R_1^4(\epsilon_1) R_5^2(\epsilon_1) R_6(\epsilon_1), \tag{4.6}$$

where a non-zero constant factor is omitted for convenience, $R_5(\epsilon_1) := 11236\epsilon_1^4 - 187731\epsilon_2^2\epsilon_1^3 + 14004\epsilon_2^4\epsilon_1^2 - 14154\epsilon_2^6\epsilon_1 - 300\epsilon_2^8$ and R_6 is given in the Appendix. One can check that lcoeff$(R_i, \epsilon_1) \neq 0$, res$(R_i, (R_i)'_{\epsilon_1}, \epsilon_1) \neq 0$ and res$(R_i, R_j, \epsilon_1) \neq 0$, where $i, j = 1, 5, 6$ and $i \neq j$. By Lemma 2.1, the distribution of positive zeros of R_1, R_2, R_5 and R_6 does not change as ϵ_2 varies. Note that $\epsilon_1 \leq \epsilon_{11}$ when $p_2 = 0$, as indicated in the last paragraph. Choosing $\epsilon_2 = 1$ and using MAPLE ver.18 command 'realroot$(R_i, 10^{-3})$' for $i = 1, 2, 5, 6$, we find that R_1, R_5 and R_6 has no real zeros in the interval $[0, \epsilon_{11}]$. It follows that $R_1(\epsilon_1), R_5(\epsilon_1), R_6(\epsilon_1) \neq 0$ for all $\epsilon_1 \leq \epsilon_{11}$ and therefore, $p_4 \neq 0$ if $p_2 = 0$. Thus the proof of this theorem is completed. □

An interesting problem about the center E_1 is: Does a change of parameters near $a = 1$ produces limit cycles from the annulus of periodic orbits around the center in system (1.2)? For this problem, an effective method is to compute zeros of the Melnikov function ([11,13,20]), which is an integral along a periodic orbit of system (1.2) with $a = 1$. This is usually completed by finding the first integral of the system in the center case and reducing to Abelian integrals ([3,6,18]). Unfortunately, we are disappointed by the following result, where a first integral is referred to as an *elementary first integral* if is expressible in terms of exponentials, logarithms and algebraic functions as indicated in [24],

Theorem 4.2. *System* (1.2) *with* $a = 1$ *has no elementary first integrals.*

Proof. As indicated just below (2.3), we only need to prove that system (2.3) with $\alpha = 1$ has no elementary first integrals. For convenience, we rewrite system (2.3) with $\alpha = 1$ as

$$\dot{x} = P(x, y) := \sum_{i=1}^{4} P_i(x, y), \quad \dot{y} = Q(x, y) := \sum_{i=1}^{4} Q_i(x, y),$$

where $P_1(x, y) := \epsilon_1\epsilon_2 x$, $P_2(x, y) := \epsilon_1 x(x+y)$, $P_3(x, y) := xy(\epsilon_2 x-\beta y)$, $P_4(x, y) := x^2 y(x + y)$, $Q_1(x, y) := -\epsilon_1\epsilon_2 y$, $Q_2(x, y) := -\epsilon_1 y(x + y)$, $Q_3(x, y) := xy(\beta x - \epsilon_2 y)$ and $Q_4(x, y) := -xy^2(x+y)$. For an indirect proof, we assume that system (2.3) has an elementary first integral. Then, by Propositions 1 and 2 of [24] or Proposition 2.4 of [22], system (2.3) has an invariant algebraic curve $f(x, y) = 0$ such that

$$P(x, y)f'_x(x, y) + Q(x, y)f'_y(x, y) = K(x, y)f(x, y), \tag{4.7}$$

where $K(x, y) := m(P'_x+Q'_y) = m(x-y)(xy+\beta(x+y)+\epsilon_1)$ for an integer m. Direct computation shows that neither $f(x, y) = x$ nor $f(x, y) = y$ is a solution of equation (4.7). Except for the lines $x = 0$ and $y = 0$, there are no orbits connecting with the equilibrium O since it is a saddle as indicated before (2.10). Then the invariant curve $f(x, y) = 0$ in (4.7) does not pass through the origin O and therefore we assume without loss of generality that

$$f(x, y) = 1 + \sum_{i=1}^{n} f_i(x, y)$$

for an integer $n \geq 1$, where each f_i is a homogeneous polynomial of degree i and $f_n \neq 0$. We also rewrite K as $K(x, y) = K_1(x, y) + K_2(x, y) + K_3(x, y)$, where $K_1(x, y) := m\epsilon_1(x - y)$, $K_2(x, y) := m\beta(x^2 - y^2)$ and $K_3(x, y) := mxy(x - y)$.

In the case $n \leq 2$, direct computation shows that equation (4.7) has no polynomial solutions. In the oppositive case $n \geq 3$, substituting expansions of P, Q, f and K in (4.7) and equaling the homogeneous polynomials of the same degree, we obtain that

$$P_4(x, y)\frac{\partial f_i(x, y)}{\partial x} + Q_4(x, y)\frac{\partial f_i(x, y)}{\partial y} - K_3(x, y)f_i(x, y) = \Lambda_i(x, y) \tag{4.8}$$

for all $i = 1, ..., n - 3$, where $\Lambda_n(x, y) := 0$,

$$\begin{aligned}
\Lambda_{n-1}(x, y) :=& K_2 f_n - P_3\partial f_n/\partial x - Q_3\partial f_n/\partial y \\
\Lambda_{n-2}(x, y) :=& K_2 f_{n-1} + K_1 f_n - P_3\frac{\partial f_{n-1}}{\partial x} - P_2\frac{\partial f_n}{\partial x} - Q_3\frac{\partial f_{n-1}}{\partial y} - Q_2\frac{\partial f_n}{\partial y},
\end{aligned}$$

$$\begin{aligned}\Lambda_i(x,y) :=& K_2 f_{i+1} + K_1 f_{i+2} - P_3\frac{\partial f_{i+1}}{\partial x} - P_2\frac{\partial f_{i+2}}{\partial x} - P_1\frac{\partial f_{i+3}}{\partial x}\\ &- Q_3\frac{\partial f_{i+1}}{\partial y} - Q_2\frac{\partial f_{i+2}}{\partial y} - Q_1\frac{\partial f_{i+3}}{\partial y}.\end{aligned}$$

As done in [9], we make the change $w = x$ and $u = y/x$ in the above homogeneous polynomial equations. Considering the homogeneity of the involved polynomials, we define polynomials $P_i^*(u) := P_i(w, uw)/w^i$, $Q_i^*(u) := Q_i(w, uw)/w^i$, $f_i^*(u) := f_i(w, uw)/w^i$, $K_i^*(u) := K_i(w, uw)/w^i$ and $\Lambda_i^*(u) := \Lambda_i(w, uw)/w^{i+3}$. Note that

$$\begin{aligned}\frac{\partial f_i(w,uw)}{\partial x} &= \frac{\partial f_i(w,uw)}{\partial w} - \frac{u}{w}\frac{\partial f_i(w,uw)}{\partial u} = iw^{i-1}f_i^*(u) - uw^{i-1}\frac{df_i^*(u)}{du},\\ \frac{\partial f_i(w,uw)}{\partial y} &= \frac{1}{w}\frac{\partial f_i(w,uw)}{\partial u} = w^{i-1}\frac{df_i^*(u)}{du}.\end{aligned}$$

Then equation (4.8) becomes

$$\Gamma(u)\frac{df_i^*(u)}{du} + \Gamma_i(u)f_i^*(u) = \Lambda_i^*(u),$$

where $\Gamma(u) := Q_4^*(u) - uP_4^*(u)$ and $\Gamma_i(u) := iP_4^*(u) - K_3^*(u)$. If we obtain the polynomial solution $f_i^*(u)$ of the above equation, then $f_i(x, y) = x^i f_i^*(y/x)$. By the variation of constants formula,

$$f_i^*(u) = \mathcal{A}_i(u)\left(C_i + \int \mathcal{B}_i(u)du\right), \tag{4.9}$$

where C_i is a constant, $\mathcal{A}_i(u) := \exp\left(-\int \frac{\Gamma_i(u)}{\Gamma(u)}du\right)$ and $\mathcal{B}_i(u) := \frac{\Lambda_i^*(u)}{\Gamma(u)\mathcal{A}_i(u)}$. Solving recursively for polynomials f_n^*, f_{n-1}^* and f_{n-2}^*, we obtain that $n = 3m$ and

$$\begin{aligned}&f_n^*(u) = (1+u)^m u^m C_n,\\ &f_{n-1}^*(u) = (1+u)^{m-1}u^m C_n m(\beta+\epsilon_2),\\ &f_{n-2}^*(u) = (1+u)^{m-2}u^{m-1}\{C_{n-2} + \cdots + C_{n-2}u^2\},\end{aligned}$$

where $C_n \neq 0$ since we assumed that $f_n(x, y) = x^n f_n^*(y/x) \neq 0$. Then, the inequality $n = 3m$ implies that equation (4.7) has no polynomial solutions in the subcase $n \geq 3$ and $n \neq 3m$.

In the opposite subcase $n \geq 3$ and $n = 3m$, we have either $m = 1$ or $m \geq 2$. In the first situation $m = 1$, we have $n = 3$. Direct computation shows that equation (4.7) has no polynomial solutions in this situation. In the opposite situation $m \geq 2$, we further solve f_{n-3}^* and f_{n-4}^* and obtain that

$$\begin{aligned}f_{n-3}^*(u) =& (1+u)^{m-3}u^{m-2}\{C_{n-2}\beta + \cdots + C_{n-2}\beta u^4\},\\ f_{n-4}^*(u) =& (1+u)^{m-4}u^{m-3}\{C_{n-2}\beta^2 + \cdots + C_{n-2}\beta^2 u^6\}\end{aligned}$$

with

$$C_{n-2} = -\frac{m\beta\epsilon_1 C_n}{\beta+\epsilon_2} \neq 0. \tag{4.10}$$

Moreover, we claim that for all $k = 2, ..., m$,

$$f_{n-k}^*(u) = (1+u)^{m-k}u^{m-k+1}\Xi_k(u), \tag{4.11}$$

where $\Xi_k(u) := C_{n-2}\beta^{k-2} + \cdots + C_{n-2}\beta^{k-2}u^{2k-2}$, a polynomial in u of degree $2k-2$. In fact, (4.11) holds obviously for $k = 2$, 3 and 4. Suppose that (4.11) holds for $k = 2, 3, ..., \ell - 1$ with $\ell \geq 5$. By (4.9), we compute that $\Gamma(u) = -2(1+u)u^2$, $\Gamma_{n-\ell}(u) = (2m-\ell)u + (4m-\ell)u^2$ and

$$\mathcal{A}_{n-\ell}(u) = \exp\left(\int -\frac{\Gamma_{3m-\ell}(u)}{\Gamma(u)}du\right) = (1+u)^m u^{m-\ell/2}.$$

Moreover, since $f^*_{n-\ell+1}$, $f^*_{n-\ell+2}$ and $f^*_{n-\ell+3}$ are all of form (4.11), by expression of $\Lambda_i(x,y)$ given just below (4.8), we obtain that

$$\Lambda^*_{n-\ell}(u) = (1+u)^{m-\ell}u^{m-\ell+2}\Upsilon_\ell(u),$$

where $\Upsilon_\ell(u) := (\ell-2)C_{n-2}\beta^{\ell-2} + \cdots + (2-\ell)C_{n-2}\beta^{\ell-2}u^{2\ell-1}$, a polynomial in u of degree $2\ell - 1$. Then

$$\mathcal{B}_{n-\ell}(u) = \frac{\Lambda^*_i(u)}{\Gamma(u)\mathcal{A}_i(u)} = \frac{\Upsilon_\ell(u)}{-2(1+u)^{\ell+1}u^{\ell/2}}.$$

When ℓ is even, $\mathcal{B}_{n-\ell}(u)$ is a rational function and can be decomposed as

$$\mathcal{B}_{n-\ell}(u) = \phi_\ell(u) + \sum_{i=1}^{\ell/2}\frac{A_i}{u^i} + \sum_{i=1}^{\ell+1}\frac{B_i}{(1+u)^i},$$

where ϕ_ℓ is a polynomial of degree $(\ell-4)/2$ with leading coefficient $\mathrm{lcoeff}(\phi_\ell, u) := (\ell-2)C_{n-2}\beta^{\ell-2}/2$, A_is and B_is are polynomials in β, ϵ_1 and ϵ_2, and $A_{\ell/2} = (2-\ell)C_{n-2}\beta^{\ell-2}/2$. Let $\Phi_\ell(u) := \int_0^u \phi_\ell(s)ds$. We see from (4.9) that

$$\begin{aligned}
f^*_{n-\ell}(u) =&(1+u)^m u^{m-\ell/2}\left\{C_{3m-\ell} + \Phi_\ell(u) + A_1\ln|u| + \sum_{i=2}^{\ell/2}\frac{A_i}{(1-i)u^{i-1}}\right.\\
&\left.+B_1\ln|1+u| + \sum_{i=2}^{\ell+1}\frac{B_i}{(1-i)(1+u)^{i-1}}\right\}\\
=&\frac{(1+u)^m u^{m-\ell/2}}{(1+u)^\ell u^{\ell/2-1}}\left\{\frac{2A_{\ell/2}}{2-\ell} + \cdots + \frac{2\mathrm{lcoeff}(\phi_\ell,u)}{\ell-2}u^{2\ell-1}\right\}\\
=&(1+u)^{m-\ell}u^{m-\ell+1}\{C_{n-2}\beta^{\ell-2} + \cdots + C_{n-2}\beta^{\ell-2}u^{2\ell-1}\},
\end{aligned}$$

the same form as (4.11), where $A_1 = B_1 = 0$ since $f^*_{n-\ell}$ needs to be a polynomial. It follows that the claimed (4.11) holds for even ℓ.

When ℓ is odd, changing variable $u = \nu^2$, we obtain that

$$\begin{aligned}
\int \mathcal{B}_{n-\ell}(u)du =& \int \frac{-\Upsilon_\ell(\nu^2)}{(1+\nu^2)^{\ell+1}\nu^{\ell-1}}d\nu\\
=& \int\left\{\tilde{\phi}_\ell(\nu) + \sum_{i=1}^{\ell-1}\frac{\tilde{A}_i}{\nu^i} + \sum_{i=1}^{\ell+1}\frac{\tilde{B}_i\nu + \tilde{C}_i}{(1+\nu^2)^i}\right\}d\nu\\
=&\tilde{\Phi}_\ell(\nu) + \tilde{A}_1\ln|\nu| + \sum_{i=2}^{\ell-1}\frac{\tilde{A}_i}{(1-i)\nu^{i-1}} + \frac{\tilde{B}_1}{2}\ln(1+\nu^2) + \tilde{C}_1\arctan\nu\\
&+\sum_{i=2}^{\ell+1}\left\{\frac{\tilde{B}_i}{2(1-i)(1+\nu^2)^{i-1}} + \mu_{i,0}\tilde{C}_i\arctan\nu + \sum_{j=1}^{i-1}\frac{\mu_{i,j}\tilde{C}_i\nu}{(1+\nu^2)^j}\right\},
\end{aligned}$$

where $\tilde{\phi}_\ell(\nu)$ is a polynomial in ν of degree $\ell-3$ with leading coefficient lcoeff$(\tilde{\phi}_\ell,\nu) := (\ell-2)\beta^{\ell-2}C_{n-2}$, $\tilde{\Phi}_\ell(\nu) := \int_0^\nu \tilde{\phi}_\ell(s)ds$, $\tilde{A}_i$ s, $\tilde{B}_i$ s and $\tilde{C}_i$ s are all polynomials in β, ϵ_1 and ϵ_2, $\mu_{i,j}$ s are constants and $\tilde{A}_{\ell-1} = (2-\ell)\beta^{\ell-2}C_{n-2}$. Further, since $f^*_{n-\ell}(\nu^2)$ needs to be a polynomial, we see from (4.9) that $\tilde{A}_1 = \tilde{B}_1 = \tilde{C}_1 + \sum_{i=2}^{\ell+1}\mu_{i,0}\tilde{C}_i = 0$ and therefore

$$\begin{aligned}f^*_{n-\ell}(\nu^2) =&(1+\nu^2)^m\nu^{2m-\ell}\left\{\tilde{\Phi}_\ell(\nu) + \sum_{i=2}^{\ell-1}\frac{\tilde{A}_i}{(1-i)\nu^{i-1}}\right.\\&\left.+\sum_{i=2}^{\ell+1}\left(\frac{\tilde{B}_i}{2(1-i)(1+\nu^2)^{i-1}} + \sum_{j=1}^{i-1}\frac{\mu_{i,j}\tilde{C}_i\nu}{(1+\nu^2)^j}\right)\right\}\\=&\frac{(1+\nu^2)^m\nu^{2m-\ell}}{(1+\nu^2)^\ell\nu^{\ell-2}}\left\{\frac{\tilde{A}_{\ell-1}}{2-\ell} + \cdots + \frac{\mathrm{lcoeff}(\tilde{\phi}_\ell)}{\ell-2}\nu^{4\ell-2}\right\}\\=&(1+\nu^2)^{m-\ell}\nu^{2(m-\ell)+2}\{C_{n-2}\beta^{\ell-2} + \cdots + C_{n-2}\beta^{\ell-2}\nu^{4\ell-2}\}.\end{aligned}$$

Then $f^*_{n-\ell}(u)$ is of the form (4.11) when ℓ is odd and therefore the claimed (4.11) is proved. By claim (4.11), computing similarly to the above for $i = m+1$, we get

$$f^*_{n-m-1}(u) = (m-1)C_{n-2}\beta^{m-1} + \cdots + \frac{m-1}{2m-3}C_{n-2}\beta^{m-1}u^{2m-1}.$$

Similar to the above computation,

$$f^*_{n-m-2}(u) = u^{-1}\{m(m-1)C_{n-2}\beta^m/2 + O(u)\}.$$

Since f^*_{n-m-2} needs to be a polynomial and $m(m-1)\beta^m > 0$, we have $C_{n-2} = 0$, a contradiction to (4.10). Hence equation (4.7) has no polynomial solutions in the subcase $n = 3m$ and $m \geq 2$.

As above, equation (4.7) has no polynomial solutions, a contradiction to our assumption given before (4.7). Hence, system (2.3) with $\alpha = 1$, i.e., system (1.2) with $a = 1$, has no elementary first integrals. Thus, Theorem 4.2 is proved. □

5. Global dynamics

In order to obtain the global phase portraits of system (1.2) in the closure of the first quadrant, we first investigate equilibria at infinity.

Theorem 5.1. *System* (1.2) *has exactly two equilibria* I_x *and* I_y *at infinity, which lie on the positive* x*-axis and the positive* y*-axis at infinity respectively. Moreover,* I_x *is asymptotically stable and* I_y *is unstable but asymptotically stable if the time is reversed.*

Proof. As indicated just below (2.3), system (1.2) is topologically equivalent to system (2.3). Hence we only need to consider equilibria of system (2.3) at infinity. Under the Poincaré transformation ([23, p.248]) $x = 1/z$ and $y = \vartheta/z$ and the time-rescaling $t \mapsto z^3t$, system (2.3) becomes

$$\begin{cases}\dot{\vartheta} = -(\alpha^2+1)\vartheta^2 + \beta\vartheta z - (\alpha^2+1)\vartheta^3 - (\alpha^2+1)\epsilon_2\vartheta^2 z - (\alpha^2+1)\epsilon_1\vartheta z^2\\ \qquad + \beta\vartheta^3 z - (\alpha^2+1)\epsilon_1\vartheta^2 z^2 - (\alpha^2+1)\epsilon_1\epsilon_2\vartheta z^3 := U(\vartheta, z),\\ \dot{z} = -\vartheta z - \vartheta^2 z - \epsilon_2\vartheta z^2 - \epsilon_1 z^3 + \beta\vartheta^2 z^2 - \epsilon_1\vartheta z^3 - \epsilon_1\epsilon_2 z^4 := Z(\vartheta, z).\end{cases} \tag{5.1}$$

System (5.1) has a unique non-negative equilibrium $O_1 : (0, 0)$ on the ϑ-axis, which corresponds to the equilibrium $\widetilde{I}_x$ of system (2.3) on the positive x-axis at infinity, i.e., the equilibrium I_x of system (1.2) on the positive x-axis at infinity. Obviously, equilibrium O_1 is degenerate. Using the Briot-Bouquet transformation ([32, Chapter II]) $z = \vartheta\omega$ together with the time-rescaling $t \mapsto \vartheta t$ to desingularize the degenerate equilibrium O_1, we reduce system (5.1) to the form

$$\begin{cases} \dot{\vartheta} = -(\alpha^2+1)\vartheta - (\alpha^2+1)\vartheta^2 + \beta\vartheta\omega - (\alpha^2+1)\epsilon_2\vartheta^2\omega + \beta\vartheta^3\omega - (\alpha^2+1) \\ \qquad \times \epsilon_1\vartheta^2\omega^2 - (\alpha^2+1)\epsilon_1\vartheta^3\omega^2 - (\alpha^2+1)\epsilon_1\epsilon_2\vartheta^3\omega^3 := \mathcal{U}(\vartheta,\omega), \\ \dot{\omega} = \alpha^2\omega + \alpha^2\vartheta\omega - \beta\omega^2 + \alpha^2\epsilon_2\vartheta\omega^2 + \alpha^2\epsilon_1\vartheta\omega^3 + \alpha^2\epsilon_1\vartheta^2\omega^3 \\ \qquad + \alpha^2\epsilon_1\epsilon_2\vartheta^2\omega^4 := \mathcal{W}(\vartheta,\omega). \end{cases} \tag{5.2}$$

System (5.2) has equilibria $O_1^* : (0, 0)$ and $I_1^* : (0, \alpha^2/\beta)$ on the ω-axis. Equilibrium O_1^* has eigenvalues α^2 and $-(\alpha^2 + 1)$ with eigenvectors $(0, 1)^T$ and $(1, 0)^T$ respectively, where T is the transpose, implying that O_1^* is a saddle. Equilibrium I_1^* has eigenvalues -1 and $-\alpha^2$ with eigenvectors $((\alpha^2-1)\beta^3, \alpha^4(\beta^2+\alpha^2\beta\epsilon_2+\alpha^4\epsilon_1))^T$ and $(0, 1)^T$ respectively, implying that I_1^* is a stable node, as shown in Figure 1(b). By the geometric property of the Briot-Bouquet transformation, system (5.1) has a unique orbit approaching to O_1 in the direction of the ϑ-axis and infinitely many orbits approaching to O_1 in the direction of $\theta = \arctan\big(\alpha^2/\beta\big)$. In order to determine whether there is an orbit connecting to O_1 in the direction of the positive z-axis, we use another Briot-Bouquet transformation $v = \nu z$ and the time-rescaling $t \mapsto zt$, which rewrites system (5.4) as

$$\begin{cases} \dot{\nu} = \beta\nu - \alpha^2\nu^2 - \alpha^2\epsilon_1\nu z - \alpha^2\epsilon_2\nu^2 z - \alpha^2\epsilon_1\epsilon_2\nu z^2 - \alpha^2\nu^3 z - \alpha^2\epsilon_1\nu^2 z^2 := \mathcal{V}(\nu, z), \\ \dot{z} = -\nu z - \epsilon_1 z^2 - \epsilon_2\nu z^2 - \epsilon_1\epsilon_2 z^3 - \nu^2 z^2 - \epsilon_1\nu z^3 + \beta\nu^3 z^3 := \mathcal{Z}(\nu, z). \end{cases} \tag{5.3}$$

Clearly, the equilibrium $\widetilde{O}_1^* : (0, 0)$ of the above system has exactly one zero eigenvalue, and $\nu = 0$ is a center manifold since the first equation of (5.2) has a common factor ν. Restricted to the center manifold, system (5.3) becomes the equation $\dot{z} = -\epsilon_1 z^2 - \epsilon_1\epsilon_2 z^3$, implying that the equilibrium $\widetilde{O}_1^*$ is a saddle-node, as shown in Figure 1(c). By the geometric property of the Briot-Bouquet transformation, system (5.1) has a unique orbit approaching to O_1 in the direction of the z-axis. Then equilibrium $O_1 : (0, 0)$ of system (5.1) is asymptotically stable, as shown in Figure 1(a), i.e., the equilibrium I_x of system (1.2) is asymptotically stable.

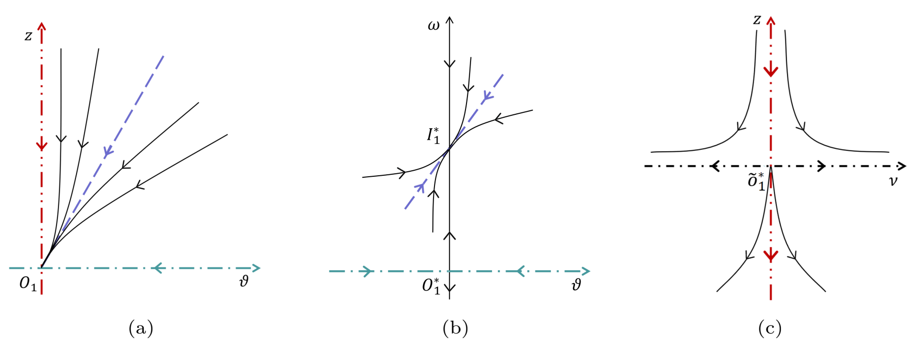


Figure 1. (a) Phase portrait of system (5.1) near O_1. (b) Phase portrait of system (5.2) near the ω-axis. (c) Phase portrait of system (5.3) near $\widetilde{O}_1^*$.

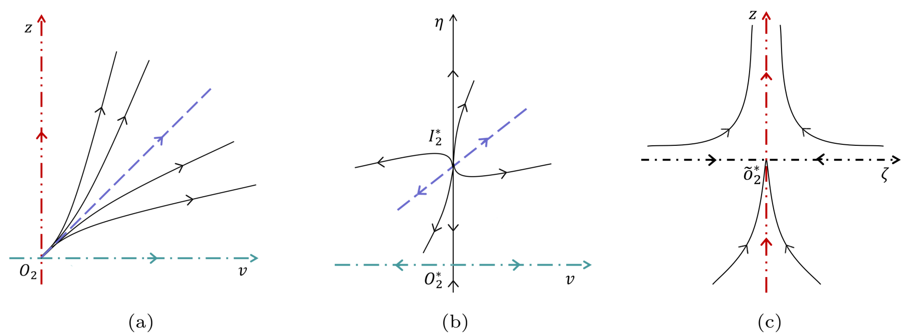


Figure 2. (a) Phase portrait of system (5.4) near O_2. (b) Phase portrait of system (5.5) near the η-axis. (c) Phase portrait of system (5.6) near $\widetilde{O}_2^*$.

Applying another Poincaré transformation $x = v/z$ and $y = 1/z$ and the time-rescaling $t \mapsto z^3 t$, we change system (2.3) into the form

$$\dot{v} = -U(v, z), \quad \dot{z} = -\alpha^2 Z(v, z) + (\alpha^2 - 1)\beta v^2 z^2. \tag{5.4}$$

It suffices to discuss the equilibrium $O_2 : (0, 0)$ of system (5.4), which corresponds to the equilibrium $\widetilde{I}_y$ of system (2.3) on the positive y-axis at infinity, i.e., the equilibrium I_y of system (1.2) on the positive y-axis at infinity. Note that the equilibrium O_2 is degenerate. Using the Briot-Bouquet transformation $z = v\eta$ and the time-rescaling $t \mapsto vt$ to desingularize the degenerate equilibrium O_2, we reduce system (5.4) to the following

$$\dot{v} = -\mathcal{U}(v, \eta), \quad \dot{\eta} = -\alpha^2 \mathcal{W}(v, \eta) + (\alpha^2 - 1)\beta \eta^2. \tag{5.5}$$

Similar to system (5.2), system (5.5) has two equilibria on the positive η-axis: saddle $O_2^* : (0, 0)$ and unstable node $I_2^* : (0, 1/\beta)$, as shown in Figure 2(b). Then system (5.4) has a unique orbit leaving from O_2 in the direction of the v-axis and infinitely many orbits leaving from O_2 in the direction of $\theta = \arctan(1/\beta)$. Using another Briot-Bouquet transformation $v = \zeta z$ and the time-rescaling $t \mapsto zt$, we rewrite system (5.4) as

$$\dot{\zeta} = -\mathcal{V}(\zeta, z)/\alpha^2 + (\alpha^2 - 1)\beta\zeta/\alpha^2, \quad \dot{z} = \alpha^2 \mathcal{Z}(\zeta, z) + (\alpha^2 - 1)\beta\zeta^2 z^3. \tag{5.6}$$

Similarly to system (5.3), we reduce system (5.6) to the center manifold $\zeta = 0$ and see that the origin $\widetilde{O}_2^*$ is a saddle-node and the phase portrait is given in Figure 2(c). Then system (5.4) has a unique orbit leaving from O_2 in the direction of the z-axis. It follows that the equilibrium O_2 of system (5.4) is unstable and all orbits nearby approach to O_2 as the time tends to $-\infty$, as shown in Figure 2(a), i.e., the equilibrium I_y of system (1.2) is unstable but asymptotically stable if the time is reversed. Thus the proof of this theorem is completed. $\square$

For each $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}_+^2$, we define the following curves and regions

$$\Upsilon_0 := \{(\beta, a) \in \mathbb{R}_+^2 : \beta > \beta_*(a, \varepsilon_1, \varepsilon_2), a = 1\},$$
$$\Upsilon_- := \{(\beta, a) \in \mathbb{R}_+^2 : \beta = \beta_*(a, \varepsilon_1, \varepsilon_2), a < 1\},$$
$$\Upsilon_+ := \{(\beta, a) \in \mathbb{R}_+^2 : \beta = \beta_*(a, \varepsilon_1, \varepsilon_2), a > 1\},$$

$$\mathcal{B}_1 := \{(\beta, a) \in \mathbb{R}_+^2 : \beta < \beta_*(a, \varepsilon_1, \varepsilon_2)\},$$
$$\mathcal{B}_2 := \{(\beta, a) \in \mathbb{R}_+^2 : \beta > \beta_*(a, \varepsilon_1, \varepsilon_2), a < 1\},$$
$$\mathcal{B}_3 := \{(\beta, a) \in \mathbb{R}_+^2 : \beta > \beta_*(a, \varepsilon_1, \varepsilon_2), a > 1\},$$

and moreover let M denote the point $(\beta, a) = (\beta_*(1, \varepsilon_1, \varepsilon_2), 1)$. Clearly, the first quadrant of the (β, a)-plane is divided as the union $\Upsilon_0 \cup \Upsilon_\pm \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup M$.

Theorem 5.2. *For all $(\beta, a) \in \Upsilon_0 \cup \Upsilon_\pm \cup \mathcal{B}_1 \cup M$ and all positive ε_1 and ε_2, system* (1.2) *has no limit cycles and its global phase portraits are given in Figure 3. Moreover, for all $(\beta, a) \in \Upsilon_0$ and all positive ε_1 and ε_2, system* (1.2) *has an orbit homoclinic to the saddle E_2 and the open region inside the homoclinic orbit is fully filled with a continuous family of periodic orbits around the center E_1.*

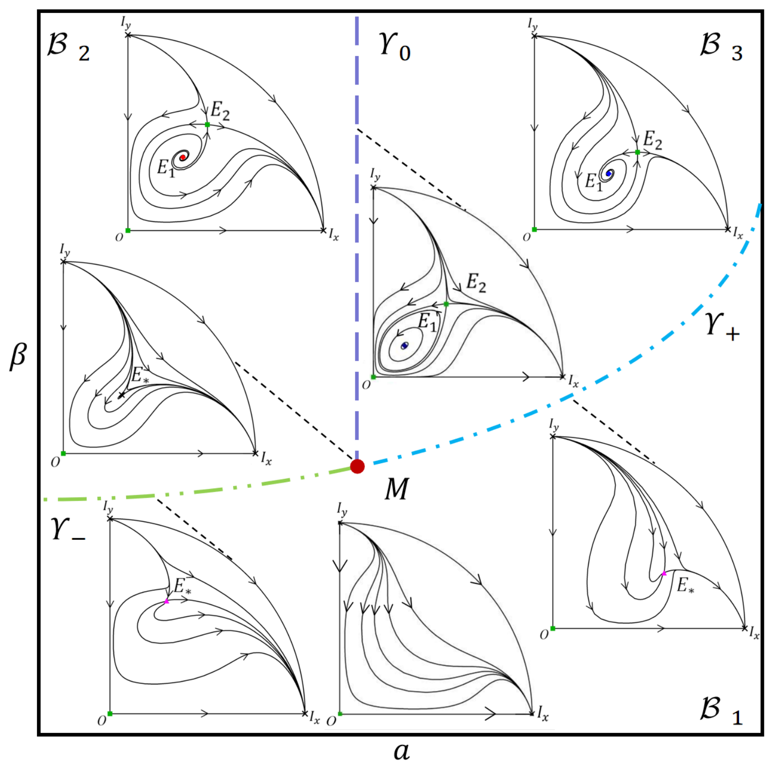


Figure 3. Global phase portraits of system (1.2).

Proof. By Theorem 5.1, system (1.2) has only two equilibria at infinity, i.e., I_x and I_y, where I_x is asymptotically stable and I_y is unstable but asymptotically stable if the time is reversed. As indicated in Theorem 2.1, system (1.2) has only one boundary equilibrium, i.e., the origin $O : (0, 0)$, which is a hyperbolic saddle and its stable manifold and unstable manifold are the y-axis and the x-axis respectively. However, the distribution of interior equilibria changes as $(\beta, a, \varepsilon_1, \varepsilon_2)$ varies.

For all $(\beta, a) \in \mathcal{B}_1$ and all positive ε_1 and ε_2, there are no interior equilibria as indicated in Theorem 2.1. By Property 2 of [32, p.148], which says that the open region inside a limit cycle contains an equilibrium, system (1.2) has no limit cycles.

It follows that each orbit in the interior of the first quadrant approaches to I_x as the time tends to $+\infty$ and approaches to I_y as the time tends to $-\infty$.

For all $(\beta, a) = M$ and all positive ε_1 and ε_2, system (1.2) has a unique interior equilibrium, i.e., the cusp E_*, by Theorems 2.1 and 3.1. As indicated in [32, Property 2, p.148], the sum of indices of equilibria of system (1.2) in the region enclosed by a limit cycle is exact 1. In order to show the nonexistence of limit cycles, we employ the Bendixson's formula (see [32, Chapter III, Section 6]) $\mathcal{I}(E_*) = 1+(e-h)/2$, where $\mathcal{I}(E_*)$ denotes the Poincaré index of the equilibrium E_*, e is the number of elliptic sectors and h is the number of hyperbolic sectors adjacent to the equilibrium E_*. Thus $\mathcal{I}(E_*) = 0$ since the cusp E_* has only two hyperbolic sectors and no other sectors. It follows that there are no limit cycles. On the other hand, the stable manifold and unstable manifold of the cusp cannot coincide and form an orbit homoclinic to the cusp E_*; otherwise, the open region inside the homoclinic orbit contains an equilibrium by Property 2 of [32, p.148], a contradiction. Therefore, the stable manifold of the cusp E_* connects I_y and the unstable manifold connects I_x and, moreover, other orbits in the first quadrant all approach to I_x as the time tends to $+\infty$ and approach to I_y as the time tends to $-\infty$.

In order to investigate global phase portraits of system (1.2) for all $(\beta, a) \in \Upsilon_\pm \cup \Upsilon_0$ and all positive ε_1 and ε_2, we consider its topological equivalent system (2.3) for all $(\beta, \alpha) \in \widetilde{\Upsilon}_\pm \cup \widetilde{\Upsilon}_0$ and all positive ϵ_1 and ϵ_2, where $\widetilde{\Upsilon}_- := \{(\beta, \alpha) \in \mathbb{R}^2_+ : \beta = \widetilde{\beta}_*, \alpha < 1\}$, $\widetilde{\Upsilon}_+ := \{(\beta, \alpha) \in \mathbb{R}^2_+ : \beta = \widetilde{\beta}_*, \alpha > 1\}$, $\widetilde{\Upsilon}_0 := \{(\beta, \alpha) \in \mathbb{R}^2_+ : \beta > \widetilde{\beta}_*, \alpha = 1\}$ and $\widetilde{\beta}_*$ is defined just before (2.9). Note that the two equilibria $\widetilde{I}_x$ and $\widetilde{I}_y$ of system (2.3) at infinity correspond to equilibria I_x and I_y of system (1.2) at infinity respectively, as indicated in the proof of Theorem 5.1. For $(\beta, \alpha) \in \widetilde{\Upsilon}_\pm$, there is a unique interior equilibria $\widetilde{E}_*$, which is a saddle-node, by Theorem 3.1. Since the index of a saddle-node is 0, there are no limit cycles similar to the above situation. We see from (2.10) that the trace of the Jacobian matrix of system (2.3) at the saddle-node $\widetilde{E}_*$ has the same sign as $1 - \alpha$. It follows that the nonzero eigenvalue is positive (resp. negative) as $\alpha < 1$ (resp. > 1). Then for $\alpha < 1$ (resp. $\alpha > 1$) orbits in the parabolic sector of the saddle-node $\widetilde{E}_*$ all approach to $\widetilde{E}_*$ as the time tends to $-\infty$ (resp. $+\infty$) and approach to $\widetilde{I}_x$ (resp. $\widetilde{I}_y$) as the time tends to $+\infty$ (resp. $-\infty$); the orbit that separates the two hyperbolic sectors of the saddle-node $\widetilde{E}_*$ approaches to $\widetilde{E}_*$ as the time tends to $+\infty$ (resp. $-\infty$) and approaches to $\widetilde{I}_y$ (resp. $\widetilde{I}_x$) as the time tends to $-\infty$ (resp. $+\infty$); other orbits all approach to $\widetilde{I}_x$ as the time tends to $+\infty$ and approach $\widetilde{I}_y$ as the time tends to $-\infty$.

For all $(\beta, \alpha) \in \widetilde{\Upsilon}_0$ and all positive ε_1 and ε_2, we see from the proof of Theorem 2.1 and Theorem 4.1 that the equivalent system (2.3) has two interior equilibria $\widetilde{E}_1 : (x_1, x_1)$ and $\widetilde{E}_2 : (x_2, x_2)$, which are center and saddle respectively. We see from (2.10) that the Jacobian matrix of system (2.3) at the saddle $\widetilde{E}_2$ has eigenvalues $\lambda_\pm = \pm\sqrt{c_7^2 - c_8^2}$ with eigenvectors $v_\pm = (-c_8, -c_7 \pm \sqrt{c_7^2 - c_8^2})$ respectively, where $c_7 := x_2(3x_2^2 + \epsilon_2 x_2 + \epsilon_1)$ and $c_8 := x_2^3 + \epsilon_2 x_2^2 + 3\epsilon_1 x_2 + 2\epsilon_2\epsilon_1$. We claim that the orbit Γ_1 (resp. Γ_2) approaching to $\widetilde{E}_2$ as the time tends to $+\infty$ (resp. $-\infty$) in Ω_1 (resp. Ω_2) along the eigenvector v_- (resp. v_+) approaches to the equilibria $\widetilde{I}_y$ (resp. $\widetilde{I}_x$) as the time tends to $-\infty$ (resp. $+\infty$) in the region Ω_1 (resp. Ω_2), where

$$\Omega_1 := \{(x, y) \in \mathbb{R}^2_+ : 0 < x < y\} \quad \text{and} \quad \Omega_2 := \{(x, y) \in \mathbb{R}^2_+ : 0 < y < x\}.$$

Actually, on the line $y = x$, we have

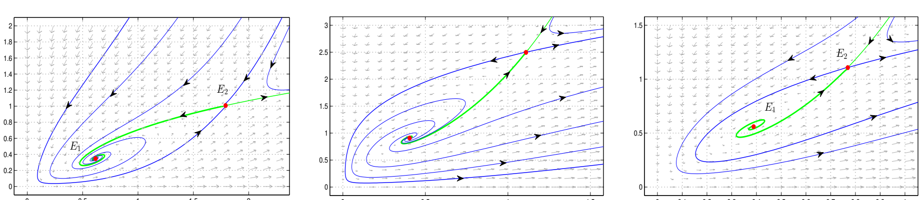
$$\dot{x}|_{y=x} = xF(x)|_{\alpha=1} \quad \text{and} \quad \dot{y}|_{y=x} = -xF(x)|_{\alpha=1},$$

where F is defined in (2.6). Note that $F(x)|_{\alpha=1} > 0$ (resp. < 0) for $x \in (0, x_1) \cup (x_2, +\infty)$ (resp. $x \in (x_1, x_2)$) because x_1 and x_2 are the only two positive zero of cubic $F|_{\alpha=1}$ and $F(0) > 0$. It follows that orbit starting from the point (x, x) leaves (resp. enters) the region Ω_1 and enters (resp. leaves) the region Ω_2 for all $x \in (0, x_1) \cup (x_2, +\infty)$ (resp. $x \in (x_1, x_2)$). If the claim is not true, then, according to the direction of orbits on the line $y = x$, the orbit Γ_1 enters Ω_1 from the linear segment $\mathcal{S} := \{(x, x) \in \mathbb{R}_+^2 : x_1 < x < x_2\}$. As indicated just below (4.1), the phase portrait of system (2.3) is symmetric with respect to the line $y = x$. Hence, the orbit Γ_2 leaves Ω_2 from $\mathcal{S}$. Because of the symmetry of the phase portrait, orbits Γ_1 and Γ_2 coincide and form an orbit homoclinic to the saddle $\widetilde{E}_2$. However, as indicated in [32, Property 2, p.148], the open region inside the homoclinic orbit contains an equilibrium, a contradiction. Thus our above claim is proved. By the claim, the orbit Γ_3 (resp. Γ_4) approaching to $\widetilde{E}_2$ as the time tends to $-\infty$ (resp. $+\infty$) in Ω_1 (resp, Ω_2) along the eigenvector v_+ (resp. v_-) leaves (resp. enters) Ω_1 (resp. Ω_2) from the segment $\{(x, x) \in \mathbb{R}^2 : 0 < x < x_1\}$. Therefore, Γ_3 and Γ_4 coincide because of the symmetry and form an orbit homoclinic to the saddle $\widetilde{E}_2$, and the open region inside the homoclinic orbit is fully filled with a continuous family of periodic orbits around the center $\widetilde{E}_1$. Thus, there are no limit cycles.

As above, we obtain the global phase portraits of system (1.2) for all $(\beta, a) \in \Upsilon_0 \cup \Upsilon_\pm \cup \mathcal{B}_1 \cup M$ and all positive ε_1 and ε_2, as illustrated in Figure 3. Thus, this theorem is proved. $\square$

6. Simulations and conclusions

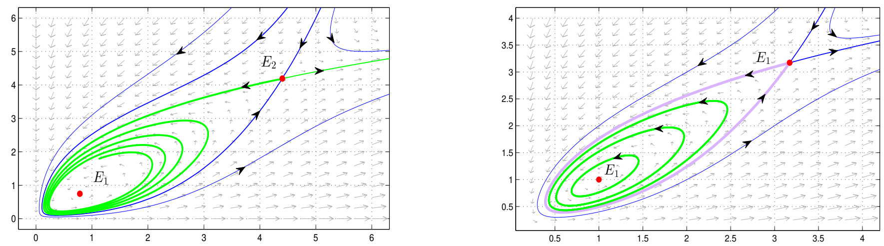
Remark that it is still unknown whether there exists a limit cycle in the case that $(\beta, a) \in \mathcal{B}_2 \cup \mathcal{B}_3$, i.e., $\beta > \beta_*(a, \varepsilon_1, \varepsilon_2)$ and $a \neq 1$. Many simulations (see Figures 4(a)-4(c)) to the phase portraits of system (1.2) suggest nonexistence of limit cycles in this case, but we fail to prove the nonexistence by the well-known Bendixson-Dulac Criterion ([23, p.264] or [32, Theorem 1.7, p195]) with Dulac functions of the forms $x^m y^n$ and $\exp(mx + ny)$.



(a) (b) (c)

Figure 4. Simulations of system (1.2). (a) No cycles for $(\beta, a, \varepsilon_1, \varepsilon_2) = (26/5, 16/9, 1/2, 1/4)$. (b) No cycles for $(\beta, a, \varepsilon_1, \varepsilon_2) = (315/100, 4/9, 1/2, 1)$. (c) No cycles for $(\beta, a, \varepsilon_1, \varepsilon_2) = (14/5, 25/36, 1/2, 1/16)$.

Our Theorem 5.2 indicates two interesting phenomena: a stable focus or node and a saddle for all $(\beta, a) \in \mathcal{B}_3$ and all positive ε_1 and ε_2; a center surrounded by a homoclinic orbit for all $(\beta, a) \in \Upsilon_0$ and all positive ε_1 and ε_2. In what follows, we demonstrate the two phenomena with numerical simulations. With the choice $(\beta, a, \varepsilon_1, \varepsilon_2) = (13/2, 21/20, 1, 1)$, which lies in the region $\mathcal{B}_3$, we use MATLAB ver.12 to plot the phase portrait of system (1.2) in Figure 5(a), which shows that system (1.2) has a stable focus and a saddle, the same as displayed in Figure 3. With the choice $(\beta, a, \varepsilon_1, \varepsilon_2) = (6, 1, 1, 1)$ lying on the curve Υ_0, similar simulation produces Figure 5(b), showing that system (1.2) has both a homoclinic orbit and a center, the same as displayed in Figure 3.



(a) (b)

Figure 5. Simulations of system (1.2). (a) One saddle and one stable focus when $(\beta, a, \varepsilon_1, \varepsilon_2) = (13/2, 21/20, 1, 1)$. (b) Coexistence of homoclinic orbit and center when $(\beta, a, \varepsilon_1, \varepsilon_2) = (6, 1, 1, 1)$.

Our results of this paper provide thresholds to control the exponential growth of tumor cells with slow spread of oncolytic virus. More concretely,

- For all $(\beta, a) \in \mathcal{B}_3$ and all positive ε_1 and ε_2, Theorem 5.2 gives the original system (1.2) a threshold for the appearance of a stable node or focus E_1, giving a method to control tumor cells: if the death rate a of infected tumor cells is not equal to 1, the viral replication rate β is beyond a definite quantity β_*, the ratio of the initial value of uninfected tumor cells to infected tumor cells, i.e., x/y, is equal to a definite quantity $\sqrt{a}$, and the initial number of infected tumor cells x lies in a definite interval $(\sqrt{x_{10}}, \sqrt{x_{20}})$, where $x_{10} := x_1(\beta, \varepsilon_1, \sqrt{\varepsilon_2}, \sqrt{a})$ and $x_{20} := x_2(\beta, \varepsilon_1, \sqrt{\varepsilon_2}, \sqrt{a})$, then the population of infected tumor cells is controlled within a bounded range.
- For all $(\beta, a) \in \Upsilon_0$ and all positive ε_1 and ε_2, Theorem 5.2 gives the original system (1.2) a threshold for the appearance of a center and a homoclinic orbit, which suggests that if the viral replication rate β is beyond a definite quantity β_*, the initial value of uninfected tumor cells is equal to the initial value of infected tumor cells, and the initial value of infected tumor cells x lies in a definite interval $(\sqrt{x_{10}}, \sqrt{x_{20}})$, then the population of infected tumor cells is controlled within a bounded range. In this case the death rate of infected tumor cells is 1, and the population of uninfected tumor cells oscillates periodically and coexists together with the infected tumor cells.

Appendix. Some complicated formulae

The function F_1, F_2 in (4.3) have the form

$$\begin{aligned}F_1(x) :=\, & 10x^{10} - 34\epsilon_2 x^9 + (-20\epsilon_2^2 - 282\epsilon_1)x^8 - 520\epsilon_1\epsilon_2 x^7 + 3\epsilon_1(58\epsilon_1 - 83\epsilon_2^2)x^6 \\ & + \epsilon_1\epsilon_2(-29\epsilon_2^2 + 394\epsilon_1)x^5 - 2\epsilon_1^2(31\epsilon_1 - 167\epsilon_2^2)x^4 - \epsilon_1^2\epsilon_2(200\epsilon_1 - 99\epsilon_2^2)x^3 \\ & - \epsilon_1^2\epsilon_2^2(265\epsilon_1 - 4\epsilon_2^2)x^2 - 154\epsilon_1^3\epsilon_2^3 x - 32\epsilon_1^3\epsilon_2^4,\end{aligned}$$

$$
\begin{aligned}
F_2(x):=\,&336x^{26}+6048\epsilon_2x^{25}+(6816\epsilon_2^2+55480\epsilon_1)x^{24}+4\epsilon_2(-1612\epsilon_2^2+34217\epsilon_1)x^{23}\\
&-(13152\epsilon_2^4+22176\epsilon_2^2\epsilon_1+432096\epsilon_1^2)x^{22}-2\epsilon_2(1362500\epsilon_1^2+160009\epsilon_2^2\epsilon_1\\
&+3360\epsilon_2^4)x^{21}-(1120\epsilon_2^6+341304\epsilon_2^4\epsilon_1+6713278\epsilon_2^2\epsilon_1^2+1469480\epsilon_1^3)x^{20}\\
&-\epsilon_1\epsilon_2(6996436\epsilon_1^2+8481797\epsilon_2^2\epsilon_1+149484\epsilon_2^4)x^{19}+2\epsilon_1(326352\epsilon_1^3-6825162\epsilon_2^2\epsilon_1^2\\
&-3011135\epsilon_2^4\epsilon_1-13524\epsilon_2^6)x^{18}+2\epsilon_1\epsilon_2(1902192\epsilon_1^3-6824920\epsilon_2^2\epsilon_1^2-1226639\epsilon_2^4\epsilon_1\\
&-607\epsilon_2^6)x^{17}+\epsilon_1^2(229608\epsilon_1^3+10091550\epsilon_2^2\epsilon_1^2-7087167\epsilon_2^4\epsilon_1-557948\epsilon_2^6)x^{16}\\
&+\epsilon_1^2\epsilon_2(962428\epsilon_1^3+15469077\epsilon_2^2\epsilon_1^2-1508130\epsilon_2^4\epsilon_1-66165\epsilon_2^6)x^{15}-\epsilon_1^2(130208\epsilon_1^4\\
&-1302952\epsilon_2^2\epsilon_1^3-14398847\epsilon_2^4\epsilon_1^2-150134\epsilon_2^6\epsilon_1+3624\epsilon_2^8)x^{14}-2\epsilon_1^3\epsilon_2(420020\epsilon_1^3\\
&+242285\epsilon_2^2\epsilon_1^2-3999861\epsilon_2^4\epsilon_1-59186\epsilon_2^6)x^{13}-\epsilon_1^3(84856\epsilon_1^4+2732322\epsilon_2^2\epsilon_1^3\\
&+3724878\epsilon_2^4\epsilon_1^2-2434750\epsilon_2^6\epsilon_1-14397\epsilon_2^8)x^{12}-\epsilon_1^3\epsilon_2(632396\epsilon_1^4+5486187\epsilon_2^2\epsilon_1^3\\
&+5127276\epsilon_2^4\epsilon_1^2-294941\epsilon_2^6\epsilon_1-250\epsilon_2^8)x^{11}+\epsilon_1^4(31632\epsilon_1^4-2005620\epsilon_2^2\epsilon_1^3\\
&-7002664\epsilon_2^4\epsilon_1^2-3592638\epsilon_2^6\epsilon_1-19347\epsilon_2^8)x^{10}+2\epsilon_1^4\epsilon_2(131512\epsilon_1^4-1732338\epsilon_2^2\epsilon_1^3\\
&-2826957\epsilon_2^4\epsilon_1^2-698119\epsilon_2^6\epsilon_1-2730\epsilon_2^8)x^9+\epsilon_1^4\epsilon_2^2(991010\epsilon_1^4-3467483\epsilon_2^2\epsilon_1^3\\
&-2783084\epsilon_2^4\epsilon_1^2-281648\epsilon_2^6\epsilon_1+38\epsilon_2^8)x^8+\epsilon_1^5\epsilon_2^3(-21802\epsilon_2^6-742086\epsilon_2^4\epsilon_1\\
&-1885326\epsilon_2^2\epsilon_1^2-2194699\epsilon_1^3)x^7+\epsilon_1^5\epsilon_2^4(3148903\epsilon_1^3-296764\epsilon_2^2\epsilon_1^2-59820\epsilon_2^4\epsilon_1\\
&+276\epsilon_2^6)x^6+2\epsilon_1^6\epsilon_2^5(1529411\epsilon_1^2+134859\epsilon_2^2\epsilon_1+7672\epsilon_2^4)x^5+2\epsilon_1^6\epsilon_2^6(1019311\epsilon_1^2\\
&+92256\epsilon_2^2\epsilon_1+1420\epsilon_2^4)x^4+4\epsilon_1^7\epsilon_2^7(11758\epsilon_2^2+230035\epsilon_1)x^3\\
&+16\epsilon_1^7\epsilon_2^8(283\epsilon_2^2+16801\epsilon_1)x^2+45824\epsilon_1^8\epsilon_2^9x+3456\epsilon_2^{10}\epsilon_1^8.
\end{aligned}
$$

The function R_2 in (4.4) is of the form

$$
\begin{aligned}
R_2(\epsilon_1):=\,&873050466396275993566080000\epsilon_1^{13}-21128904182076535793267712000\epsilon_2^2\epsilon_1^{12}\\
&+188584067953032786234517161600\epsilon_2^4\epsilon_1^{11}-111915790087631757377044411936\epsilon_2^6\epsilon_1^{10}\\
&+986605067653275612361634934414\epsilon_2^8\epsilon_1^9-238056906330159212236466362600 2\epsilon_2^{10}\epsilon_1^8\\
&-452390174055524097400049464819 6\epsilon_2^{12}\epsilon_1^7+344415533822502428356274113274 7\epsilon_2^{14}\epsilon_1^6\\
&+433238041933217696633950174593 6\epsilon_2^{16}\epsilon_1^5+137576554551897362375711843156 4\epsilon_2^{18}\epsilon_1^4\\
&+170130580137317580811182175536\epsilon_2^{20}\epsilon_1^3+6840397093522452481603652544\epsilon_2^{22}\epsilon_1^2\\
&-29226368862664998727979 52\epsilon_2^{24}\epsilon_1+5224132890685440000\epsilon_2^{26}.
\end{aligned}
$$

The function S_3 in (4.4) is of the form

$$
\begin{aligned}
S_3(\beta):=\,&5120\epsilon_2^4\beta^{10}+124\epsilon_2^3(125\epsilon_1-704\epsilon_2^2)\beta^9+\epsilon_2^2(16465\epsilon_1^2-1034416\epsilon_2^2\epsilon_1+523088\epsilon_2^4)\beta^8\\
&+\epsilon_2(6580\epsilon_1^3-2599992\epsilon_2^2\epsilon_1^2+6707007\epsilon_2^4\epsilon_1-1646752\epsilon_2^6)\beta^7\\
&+(620\epsilon_1^4-2563756\epsilon_2^2\epsilon_1^3+38008266\epsilon_2^4\epsilon_1^2-18046753\epsilon_2^6\epsilon_1+3137824\epsilon_2^8)\beta^6\\
&-2\epsilon_2(493040\epsilon_1^4-39261668\epsilon_2^2\epsilon_1^3+46753436\epsilon_2^4\epsilon_1^2-12020293\epsilon_2^6\epsilon_1+1914512\epsilon_2^8)\beta^5\\
&-(91776\epsilon_1^5-70229520\epsilon_2^2\epsilon_1^4+377711968\epsilon_2^4\epsilon_1^3-53223957\epsilon_2^6\epsilon_1^2+14458038\epsilon_2^8\epsilon_1\\
&-3022016\epsilon_2^{10})\beta^4+\epsilon_2(25467648\epsilon_1^5-697696000\epsilon_2^2\epsilon_1^4-231417440\epsilon_2^4\epsilon_1^3\\
&+22743888\epsilon_2^6\epsilon_1^2+317219\epsilon_2^8\epsilon_1-1492448\epsilon_2^{10})\beta^3+(145152\epsilon_1^4-36466896\epsilon_2^2\epsilon_1^3\\
&-14636920\epsilon_2^4\epsilon_1^2+324811\epsilon_2^6\epsilon_1+415712\epsilon_2^8)(\epsilon_2^2+4\epsilon_1)^2\beta^2-24\epsilon_2(32532\epsilon_1^2\\
&+16877\epsilon_2^2\epsilon_1+1956\epsilon_2^4)(\epsilon_2^2+4\epsilon_1)^4\beta-36(36\epsilon_2^2+121\epsilon_1)(\epsilon_2^2+4\epsilon_1)^6.
\end{aligned}
$$

The function R_3 in (4.5) is of the form

$$
\begin{aligned}
R_3(\epsilon_1) :=\, & 2795443200000000000\epsilon_1^{18} - 239651462161133568000\epsilon_2^2\epsilon_1^{17} \\
& + 4087394570165184479232000\epsilon_2^4\epsilon_1^{16} - 2715095392745924451391488000\epsilon_2^6\epsilon_1^{15} \\
& + 713914839434115240963632240640\epsilon_2^8\epsilon_1^{14} - 84419030845670732042645968314880\epsilon_2^{10}\epsilon_1^{13} \\
& + 5144476828691196370710907684957952\epsilon_2^{12}\epsilon_1^{12} - 153080132913998051765831487300057536\epsilon_2^{14}\epsilon_1^{11} \\
& + 2262218457076920275374225156485173492\epsilon_2^{16}\epsilon_1^{10} - 15995771185621919181518213748037651007\epsilon_2^{18}\epsilon_1^{9} \\
& + 59093324654412993538992764747233114841\epsilon_2^{20}\epsilon_1^{8} - 113356650394881241911639806230394233751\epsilon_2^{22}\epsilon_1^{7} \\
& + 91114191063083555448417933778722674375\epsilon_2^{24}\epsilon_1^{6} + 13302968728887868565243683016634749314\epsilon_2^{26}\epsilon_1^{5} \\
& - 38390893414533393312710308157668476800\epsilon_2^{28}\epsilon_1^{4} - 9644248889312296744954553617399829688\epsilon_2^{30}\epsilon_1^{3} \\
& - 2745651784830864129554944737382560\epsilon_2^{32}\epsilon_1^{2} - 1672011850514556312771089338368\epsilon_2^{34}\epsilon_1 \\
& - 685002530024337050739671 04\epsilon_2^{36}.
\end{aligned}
$$

The function R_6 in (4.6) is of the form

$$
\begin{aligned}
R_6(\epsilon_1) :=\, & 808718173470720000000000\epsilon_2^{36} - 806566798329855621480358871 04\epsilon_2^{34}\epsilon_1 \\
& + 3159785909712974275125715990192128\epsilon_2^{32}\epsilon_1^{2} - 8151682564969199321739242792040474624\epsilon_2^{30}\epsilon_1^{3} \\
& - 22186932887759248473868937411383298918 4\epsilon_2^{28}\epsilon_1^{4} - 196795251456348296805933986567426548396 8\epsilon_2^{26}\epsilon_1^{5} \\
& - 61051391948091135987520613429157938522 88\epsilon_2^{24}\epsilon_1^{6} + 41774574373746593381120146573313037920 16\epsilon_2^{22}\epsilon_1^{7} \\
& + 53218838410790533500956343094277170239168\epsilon_2^{20}\epsilon_1^{8} + 50827698752204231253879525237335263215372\epsilon_2^{18}\epsilon_1^{9} \\
& - 64463042496757485172047694412983256411784\epsilon_2^{16}\epsilon_1^{10} - 19408209429659740313884009147172106131589\epsilon_2^{14}\epsilon_1^{11} \\
& + 17248106288306129146925269877022027990078\epsilon_2^{12}\epsilon_1^{12} - 5608594318156630502136818025178342245609\epsilon_2^{10}\epsilon_1^{13} \\
& + 1026011484832250013784739977191430519068\epsilon_2^{8}\epsilon_1^{14} - 124479914915802026733172682260510714528\epsilon_2^{6}\epsilon_1^{15} \\
& + 1216377200138355896660635251640771 2128\epsilon_2^{4}\epsilon_1^{16} - 806251185268088034678395165219196160\epsilon_2^{2}\epsilon_1^{17} \\
& + 26942653115498102927949434654796800\epsilon_1^{18}.
\end{aligned}
$$

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