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LINEAR RECURSION FORMULAS OF GENERALIZED FOCUS QUANTITIES AND LOCAL INTEGRABILITY FOR A CLASS OF THREE-DIMENSIONAL SYSTEMS*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract In this paper, the local integrability of a class of three-dimensional systems is studied. The recursive formulas to compute the generalized focus quantities of the system are deduced firstly, then they are applied to a Lotka-Volterra system. The integrable conditions of the system are obtained and the local integrability is solved completely. The algorithm corresponding to the above formulas is an extension and development of the power series method for the planar differential systems with $p : -q$ arbitrary resonant saddle point and also readily done with using computer algebra system such as Mathematica or Maple.

Keywords Three-dimensional system, integrability, power series method, generalized focus quantity.

MSC(2010) 34C.

1. Introduction

In this paper, we investigate the local integrability of the origin for the following three-dimensional system with linear part of $p : -q : r$ resonant singular point type

$$\frac{dx}{dt} = px + P(x, y, z),\ \frac{dy}{dt} = -qy + Q(x, y, z),\ \frac{dz}{dt} = rz + R(x, y, z) \tag{1.1}$$

where $p, q, r \in \mathbb{Z}^+$, $x, y, z, t \in \mathbb{R}$, P, Q and R are polynomials.

Restricting it to $z = 0, R = 0$, system (1.1) becomes the following planar polynomial vector field in $\mathbb{C}^2$ with $p : -q$ resonant elementary singular point

$$\frac{dx}{dt} = px + P(x, y),\ \frac{dy}{dt} = -qy + Q(x, y) \tag{1.2}$$

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where $p, q \in \mathbb{Z}^+$, P and Q are polynomials. We can see the above problem for (1.2) as a natural generalization of the classical center problem was proposed by Dulac [11] as early as 1908, see also [26] and Definition 3.3.7 of [20], and more if $\mathrm{GCD}(p,q) = 1$, one can calculate the supposed first integral $H(x,y) = x^q y^p + \cdots$, and the $p : -q$ resonant focus number g_k such that $\dot{H} = \sum g_k (x^q y^p)^{k+1}$, and if all $g_k = 0$, there exists a local analytic first integral $H(x,y)$, thus the $p : -q$ resonant elementary singular point is called a generalized center. This problem for system (1.2) was considered by many authors, especially, for classic Lotka-Volterra systems, i.e. P and Q are polynomials of degree 2 with respective factor x or y. All the integrable conditions for $1 : -q$ or $2 : -q$ arbitrary resonant cases were given respectively by Fronville et al. in [12] and Gravel et al. in [14]. But for $3 : -q$ arbitrary resonance case, only part problems were solved [18, 21, 23]. As for the natural mechanisms which lead to the origin being linearizable, integrable or normalizable for classic L-V systems, Christopher et at. gave some valuable results in [9]. More work on the integrability of planar differential systems with $p : -q$ resonance can be found in [8, 19, 26] for general quadratic systems, in [6, 15] for cubic systems, in [13] for quintic systems, in [7, 10] for more general systems. Recently the authors of [16, 17] also discussed the complex integrability and linearizability of cubic Z_2 systems with two $1 : q$ resonant singular points.

For the three-dimensional system (1.1) with $p : -q : r$ resonant elementary singular point, the results about the local integrability are not many. Though there exist some researchers such as the authors of [3–5] who considered the integrability of the 3D L-V systems, the resonant singular points at the origin aren't belong to the above classification. Only recently the authors of [1] considered this class of problem, the necessary and sufficient conditions of both integrability and inheritability are obtained for $(1 : -1 : 1)$, $(2 : -1 : 1)$ and $(1 : -2 : 1)$ resonance cases of 3D L-V systems. Further, they also pointed out that the problems for integrability were much harder in this three-dimensional case. In fact, the complicated things exist in two asides, namely determining the necessary conditions and proving the sufficient conditions. For the latter, there exist all kinds of methods such as verifying an algebraic symmetry, figuring out a Darboux integrating factor, blow-down to a node, reduction to a Riccati equation and so on, but these still can not guarantee to prove the sufficiency of the obtained integrable conditions for system (1.2). For the former, it is also a challenging issue how to find a suitable algorithm to computing generalized focus quantities. In this paper, we generalize and develop the algorithms in [22, 25] for the planar system (1.2), from which the obtained formulas are linear and also readily done with using computer algebra systems such as Mathematica and Maple.

Similar to the case of the planar system (1.2), we have the following lemmas and definitions about the local integrability of the three-dimensional system (1.1), and some of these have been proposed for the three-dimensional L-V systems in [1], here we will also extend theirs to the general case for system (1.1).

Lemma 1.1. *There exists a variables substitution tangent to identity,*

$$\xi = x + \sum_{m+n+i=2}^{\infty} A_{mni} x^m y^n z^i,$$

$$\eta = y + \sum_{m+n+i=2}^{\infty} B_{mni} x^m y^n z^i, \tag{1.3}$$

$$\zeta = z + \sum_{m+n+i=2}^{\infty} C_{mni} x^m y^n z^i$$

where A_{mni}, $B_{mni}, C_{mni} \in \mathbb{R}$ *and* $m, n, i \in \mathbb{N}$, *such that system (1.1) can be reduced to the normal form*

$$\begin{aligned} \frac{d\xi}{dt} &= p\xi + \sum\nolimits_{(k,j,l)\in D_1} p_{kjl} U^{(kjl)}, \\ \frac{d\eta}{dt} &= -q\eta + \sum\nolimits_{(k,j,l)\in D_2} q_{kjl} U^{(kjl)}, \\ \frac{d\zeta}{dt} &= r\zeta + \sum\nolimits_{(k,j,l)\in D_3} r_{kjl} U^{(kjl)} \end{aligned} \tag{1.4}$$

where p_{kjl}, q_{kjl} *and* r_{kjl} *are polynomials of the coefficients of system (1.1) with rational coefficients, and* $U^{(kjl)} = \xi^k \eta^j \zeta^l$ *with the respective exponential sets as follows:*

$$\begin{aligned} D_1 &= \{(k,j,l)\,|p(k-1) - qj + rl = 0, \quad k+j+l \geq 2, \quad k,j,l \in \mathbb{N}\}, \\ D_2 &= \{(k,j,l)\,|pk - q(j-1) + rl = 0, \quad k+j+l \geq 2, \quad k,j,l \in \mathbb{N}\}, \\ D_3 &= \{(k,j,l)\,|pk - qj + r(l-1) = 0, \quad k+j+l \geq 2, \quad k,j,l \in \mathbb{N}\}. \end{aligned} \tag{1.5}$$

Remark 1.1. In fact, if letting all resonant terms in substitution (1.3) vanish, namely the coefficients $A_{mni} = B_{mni} = C_{mni} = 0$ for $(m, n, i) \in D$, then the substitution (1.3) can be determined uniquely, where $D = \{(m,n,i)\big|pm - qn + ri = 0, m + n + i \geq 2\}$.

As for the conclusion of the lemma 1.1, the similar statement can be seen in the early references, such as [2, 20]. Since the strict proof of the lemma is almost the same as the one of the planar system (see [24]), we omit it here.

Lemma 1.2. *The system (1.1) is integrable at the origin if the above normal forms in (1.4) satisfy*

$$\begin{aligned} \frac{d\xi}{dt} &= p\xi(1 + M(\xi,\eta,\zeta)), \\ \frac{d\eta}{dt} &= -q\eta(1 + M(\xi,\eta,\zeta)), \\ \frac{d\zeta}{dt} &= r\zeta(1 + M(\xi,\eta,\zeta)) \end{aligned} \tag{1.6}$$

where $M(\xi, \eta, \zeta)$ *is a formal series in* (ξ, η, ζ).

Furthermore, from the Lemma 1.1 and the Lemma 1.2, we have the following conclusion.

Lemma 1.3. *The system (1.1) is integrable at the origin if and only if there exist two first integrals which have the following respective forms*

$$F(x,y,z) = x^q y^p + \sum_{\alpha+\beta+\gamma=p+q+1}^{\infty} c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma, \tag{1.7}$$

$$H(x,y,z) = y^r z^q + \sum_{\alpha+\beta+\gamma=r+q+1}^{\infty} d_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma. \tag{1.8}$$

The paper is organized as follows: In section 2, by using the power formal series method, we research the algorithm of generalized focus quantities for the general system (1.1) and deduce the linear recursion formulas. In section 3, by applying the algorithm, we discuss integrable conditions for a class of three-dimensional L-V systems, which is a special case of systems (1.1) with $1 : -1 : r$ resonance, $r \in \mathbb{Z}^+$, then the integrability of the systems is solved completely.

2. Linear recursion formulas of generalized focus quantities

We discuss the method of computing generalized focus quantities here. A good computational method of the generalized focus quantities for the two-dimensional system (1.2) has been introduced in [22, 25]. The algorithm corresponding to the above formulas will be developed from the planar differential systems with $p : -q$ arbitrary resonance to the three-dimensional differential systems with $p : -q : r$ arbitrary resonance.

Theorem 2.1. *For the system (1.1), using the program of term by term calculations, we can determine two formal power series*

$$F(x,y,z) = x^q y^p + \sum_{\alpha+\beta+\gamma=p+q+1}^{\infty} c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma, \tag{2.1}$$

$$H(x,y,z) = y^r z^q + \sum_{\alpha+\beta+\gamma=r+q+1}^{\infty} d_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma \tag{2.2}$$

such that

$$\left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(1.1)} = \sum_{(k,j,l)\in D_1} \mu_{k,j,l} x^k y^j z^l, \tag{2.3}$$

$$\left.\frac{\mathrm{d}H}{dt}\right|_{(1.1)} = \sum_{(k,j,l)\in D_2} \lambda_{k,j,l} x^k y^j z^l \tag{2.4}$$

where

$$D_1 = \{(k,j,l)\,|pk - qj + rl = 0, \quad k+j+l > p+q, \quad k,j,l \in N\},$$
$$D_2 = \{(k,j,l)\,|pk - qj + rl = 0, \quad k+j+l > q+r, \quad k,j,l \in N\},$$

and when $(k,j,l) \in D_1$, let $c_{k,j,l} = 0$, when $(k,j,l) \in D_2$, let $d_{k,j,l} = 0$, and more, $\mu_{k,j,l}$ for $(k,j,l) \in D_1$ and $\lambda_{k,j,l}$ for $(k,j,l) \in D_2$ are called the generalized focus quantities of origin of the system (1.1).

Theorem 2.1 gives the method to find the generalized focus quantities $\mu_{k,j,l}$ and $\lambda_{k,j,l}$, which are the key to determine the integrability of system (1.1). Obviously, system (1.1) is integrable at the origin if and only if $\mu_{k,j,l} = \lambda_{k,j,l} = 0$ from Lemma 1.2.

Considering a concrete kind of system (1), we can obtain the recursive formulas to compute generalized focus quantities $\mu_{k,j,l}$ and $\lambda_{k,j,l}$ from Theorem 2.1. Lotka-Volterra system is a very classic and common model. As an application of our method, we consider the following three-dimensional Lotka-Volterra system with $p : -q :\ r$ resonance

$$\begin{aligned}\dot{x} &= x(p + a_1x + b_1y + c_1z),\\ \dot{y} &= y(-q + a_2x + b_2y + c_2z),\\ \dot{z} &= z(r + a_3x + b_3y + c_3z).\end{aligned} \tag{2.5}$$

Without loss of generality, we can let $q = 1$ in system (2.5), and have the following conclusion.

Theorem 2.2. *For the system (2.5) with $q = 1$, using the program of term by term calculations, we can determine two formal power series*

$$F(x, y, z) = xy^p + \sum_{\alpha+\beta+\gamma=p+2}^{\infty} c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma, \tag{2.6}$$

$$H(x, y, z) = y^r z + \sum_{\alpha+\beta+\gamma=r+2}^{\infty} d_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma \tag{2.7}$$

such that

$$\left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(2.5)} = \sum_{m=2}^{\infty}\sum_{n=0}^{m} \mu_{n,m-n} x^n y^{np+(m-n)r} z^{m-n}, \tag{2.8}$$

$$\left.\frac{\mathrm{d}H}{\mathrm{d}t}\right|_{(2.5)} = \sum_{m=2}^{\infty}\sum_{n=0}^{m} \lambda_{n,m-n} x^n y^{np+(m-n)r} z^{m-n} \tag{2.9}$$

where when $\beta = \alpha p + \gamma r$, let $c_{1,p,0} = 1, c_{n,np+mr-nr,m-n} = 0$ and $d_{0,r,1} = 1, d_{n,np+mr-nr,m-n} = 0$ for $0 \le n \le m$, $m = 2, 3, \cdots$, and when $\beta \ne \alpha p + \gamma r$, $c_{\alpha\beta\gamma}$ and $d_{\alpha\beta\gamma}$ are determined uniquely by the following recursive formula respectively:

$$\begin{aligned}c_{\alpha\beta\gamma} = {} & [(a_1(\alpha-1) + a_2\beta + a_3\gamma)c_{\alpha-1,\beta,\gamma} + (b_1\alpha + b_2(\beta-1) + b_3\gamma)c_{\alpha,\beta-1,\gamma}\\ & +(c_1\alpha + c_2\beta + c_3(\gamma-1))c_{\alpha,\beta,\gamma-1}]/(\beta - \alpha p - \gamma r),\end{aligned} \tag{2.10}$$

$$\begin{aligned}d_{\alpha\beta\gamma} = {} & [(a_1(\alpha-1) + a_2\beta + a_3\gamma)d_{\alpha-1,\beta,\gamma} + (b_1\alpha + b_2(\beta-1) + b_3\gamma)d_{\alpha,\beta-1,\gamma}\\ & +(c_1\alpha + c_2\beta + c_3(\gamma-1))d_{\alpha,\beta,\gamma-1}]/(\beta - \alpha p - \gamma r),\end{aligned} \tag{2.11}$$

and for any positive integer $0 \le n \le m$, $m = 2, 3, \cdots$, $\mu_{n,m-n}$ and $\lambda_{n,m-n}$ are determined uniquely by the following recursive formula respectively:

$$\begin{aligned}\mu_{n,m-n} = {} & [(a_1(\alpha-1) + a_2\beta + a_3\gamma)c_{\alpha-1,\beta,\gamma} + (b_1\alpha + b_2(\beta-1) + b_3\gamma)c_{\alpha,\beta-1,\gamma}\\ & +(c_1\alpha + c_2\beta + c_3(\gamma-1))c_{\alpha,\beta,\gamma-1}],\end{aligned} \tag{2.12}$$

$$\lambda_{n,m-n} = [(a_1(\alpha-1)+a_2\beta+a_3\gamma)d_{\alpha-1,\beta,\gamma} + (b_1\alpha+b_2(\beta-1)+b_3\gamma)d_{\alpha,\beta-1,\gamma} + (c_1\alpha+c_2\beta+c_3(\gamma-1))d_{\alpha,\beta,\gamma-1}] \tag{2.13}$$

where $\alpha = n, \gamma = m-n, \beta = np+(m-n)r$, and in the above all expressions, if $\alpha < 0$ or $\beta < 0$ or $\gamma < 0$, $c_{\alpha\beta\gamma} = 0$, then $\mu_{n,m-n}$ and $\lambda_{n,m-n}$ are called the generalized focus quantities of origin of the system (2.5) with $q = 1$.

3. Integrability for a class of Lotka-Volterra system

The method of computing generalized focus quantities has been discussed in this section. Thus by applying the formulas of generalized focus quantities to investigate a class of three-dimensional Lotka-Volterra systems (2.5) with $p = q = 1, c_1 = c_2 = c_3 = 0$, namely the corresponding form as follows

$$\begin{aligned} \dot{x} &= x(1+a_1x+b_1y), \\ \dot{y} &= y(-1+a_2x+b_2y), \\ \dot{z} &= z(r+a_3x+b_3y). \end{aligned} \tag{3.1}$$

For the system (3.1), by using the formulas in Theorem 2.2, we can compute the first enough generalized focus quantities for any given values of r, for example when $r = 1, 2, \cdots, 20$, we can obtain the first corresponding generalized focus quantities for $m = 9$ as follows:

$$\begin{aligned} &\mu_{20} = a_2b_2 - a_1b_1, \mu_{11} = \mu_{02} = 0, \\ &\mu_{30} = \mu_{12} = \mu_{21} = \mu_{03} = 0, \cdots, \mu_{90} = \mu_{81} = \cdots = \mu_{09} = 0, \\ &\lambda_{20} = ra_2(b_2-b_1) + a_2b_3 - a_3b_1, \lambda_{11} = \lambda_{02} = 0, \\ &\lambda_{30} = \lambda_{12} = \lambda_{21} = \lambda_{03} = 0, \cdots, \lambda_{90} = \lambda_{81} = \cdots = \lambda_{09} = 0 \end{aligned} \tag{3.2}$$

where for each $\mu_{i,l-i}\,(0 \le i \le l)$ in the above expression, we have already let $\mu_{i,l-1-i} = 0\,(0 \le i \le l-1)$, for each $\lambda_{i,l-i}\,(0 \le i \le l)$, we have already let $\mu_{i,l-1-i} = 0 = \lambda_{i,l-1-i} = 0\,(0 \le i \le l-1)$, $l = 2, 3, \cdots, 9$.

According to the above calculating results, we can find the law, then we have

Theorem 3.1. *System (3.1) is integrable at the origin if and only if the following conditions is satisfied:*

$$a_2b_2 - a_1b_1 = 0,\ ra_2(b_2-b_1) + a_2b_3 - a_3b_1 = 0. \tag{3.3}$$

We first consider the sufficient conditions of the theorem 3.1.

Lemma 3.1. *System (3.1) is integrable at the origin if one of the following conditions is satisfied:*

$$\text{(i) } a_2 = b_1 = 0, \tag{3.4}$$

$$\text{(ii) } a_1 = pa_2, b_2 = pb_1, a_2b_3 = a_3b_1 - r(p-1)a_2b_1, \tag{3.5}$$

where for the condition (3.5), $a_2^2 + b_1^2 \neq 0$ holds and p is a real number.

Proof. In fact, the conditions (3.4) and (3.5) are completely from the conditions (3.3). When the condition (3.4) holds, the conditions (3.3) holds. But when the condition (3.4) doesn't holds, namely $a_2^2 + b_1^2 \neq 0$, from $a_2 b_2 - a_1 b_1 = 0$, there must exist certain real number p such that $a_1 = pa_2, b_2 = pb_1$, then $a_2 b_3 = a_3 b_1 - r(p-1)a_2 b_1$ in the conditions (3.3) holds, and more we may as well assume that $a_2 \neq 0$, thus we have

$$b_3 = \frac{a_3 b_1 - r(p-1)a_2 b_1}{a_2}. \tag{3.6}$$

Now we prove this lemma. Case (i): if the condition (3.4) holds, the system (3.1) has the following form

$$\dot{x} = x(1 + a_1 x), \quad \dot{y} = y(-1 + b_2 y), \quad \dot{z} = z(r + a_3 x + b_3 y). \tag{3.7}$$

Obviously, $f_1 = x, f_2 = y$ and $f_3 = z$ are three invariant planes of this system with cofactor $1 + a_1 x$, $1 - b_2 y$ and $r + a_3 x + b_3 y$, respectively. Moreover, the system has other two invariant algebraic surfaces $f_4 = 1 + a_1 x$ and $f_5 = 1 - b_2 y$ with the cofactor $a_1 x$ and $b_2 y$ respectively. Thus we obtain two Darboux first integral as follows

$$H_1 = xy f_4^{-1} f_5^{-1},$$
$$H_2 = y^r z f_4^{-\frac{a_3}{a_1}} f_5^{-\frac{b_2 r + b_3}{b_2}}.$$

So system (3.7) is integrable at the origin.

Case (ii): if the condition (3.5) holds, and from (3.6), obviously, the system (3.1) has the following form

$$\begin{aligned} \dot{x} &= x(1 + pa_2 x + b_1 y), \\ \dot{y} &= y(-1 + a_2 x + pb_1 y), \\ \dot{z} &= z(r + a_3 x + \tfrac{a_3 b_1 - r(p-1)a_2 b_1}{a_2} y). \end{aligned} \tag{3.8}$$

Obviously, $f_1 = x, f_2 = y$ and $f_3 = z$ are three invariant planes of this system with cofactor $1 + pa_2 x + b_1 y$, $1 - a_2 x - pb_1 y$ and $r + a_3 x + \frac{a_3 b_1 - r(p-1)a_2 b_1}{a_2} y$, respectively. Moreover, the system has another invariant algebraic surface $f_4 = 1 + pa_2 x - pb_1 y$ with the cofactor $p(a_2 x + b_1 y)$. Thus we obtain two Darboux first integral as follows

$$H_1 = xy f_4^{-\frac{1+p}{p}},$$
$$H_2 = y^r z f_4^{-\frac{ra_2 + a_3}{pa_2}}.$$

So system (3.8) is integrable at the origin. We complete the proof of this lemma. □

Next we consider the necessary conditions of the theorem 3.1.

Lemma 3.2. *If system (3.1) is integrable at the origin, then the conditions (3.3) hold necessarily.*

Proof. Firstly, it is easy to see that the first and second equations of the system (3.1) are independent with respect to z, that is, the two equations forms a planar system, for which the integrability at the origin can be determined completely by calculating generalized focus quantities via the formal power series with the following form

$$F(x,y) = xy + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta} x^{\alpha} y^{\beta} \tag{3.9}$$

and more we can figure out the necessary condition: $\mu_{20} = a_2b_2 - a_1b_1 = 0$ for the existence of the first integral with the same form as (3.9), it's the H_1 obtained in the proof of Lemma 3.1

Then we figure out the necessary conditions for the existence of the first integral with the same form as (2.7) for any $r \in \mathbb{Z}^+$, namely the obtained H_2 in the proof of Lemma 3.1. In fact, by a simple application of induction, we can obtain the first generalized focus quantities $\lambda_{20} = \lambda_{20}(a_1, a_2, b_1, b_2, a_3, b_3)$ is a homogeneous quadrati polynomial in $a_1, a_2, b_1, b_2, a_3, b_3$. On the other hand, if $\mu_{20} = a_2b_2 - a_1b_1 = 0$, and $ra_2(b_2-b_1) + a_2b_3 - a_3b_1 = 0$, then system (3.1) is integrable, hence $ra_2(b_2 - b_1) + a_2b_3 - a_3b_1$ is a factor of λ_{20}. So there exists a polynomial $R_1(a_1, a_2, b_1, b_2)$ so that $\lambda_{20}(a_1, a_2, b_1, b_2, a_3, b_3) = R_1(a_1, a_2, b_1, b_2)(ra_2(b_2 - b_1) + a_2b_3 - a_3b_1)$. Since $ra_2(b_2 - b_1) + a_2b_3 - a_3b_1$ is a homogeneous polynomial of degree 2, so R_1 must be non-zero constant. Thus we complete the proof of the lemma. □

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