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# ELLIPTIC SINGULAR WAVE SOLUTIONS AND THEIR LIMITS OF A SIMPLE EQUATION*

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*Dedicated to Professor Jibin Li on the occasion of his 80th birthday.*

**Abstract** In this pager, we study the elliptic singular wave solutions of the equation $u_t + 2ku_x - u_{xxt} + u^2u_x - uu_{xxx} = 0$ which has been investigated in some literatures. Firstly, for given wave speeds $c_1 = \frac{1}{2}(1 + \sqrt{1-8k})$ or $c_2 = \frac{1}{2}(1 - \sqrt{1-8k})$, we show that there exist four types of elliptic singular wave solutions, two types of elliptic sine singular wave solutions and two types of elliptic cosine singular wave solutions. Secondly, we confirm that their limits are four types of other solutions, hyperbolic smooth solitary wave solutions, hyperbolic singular wave solutions, fractional singular wave solution and trigonometric singular wave solutions. Our works extend some previous results.

**Keywords** Elliptic singular wave solutions, hyperbolic singular wave solutions, trigonometric singular wave solutions, hyperbolic smooth solitary wave solutions.

**MSC(2010)** 34A20, 34C35, 35B65, 58F05, 76B25.

## 1. Introduction

This paper is concerned with the following simple equation

$$u_t + 2ku_x - u_{xxt} + u^2u_x - uu_{xxx} = 0, \tag{1.1}$$

which is the case of $b = 0$ in the equation

$$u_t + 2ku_x - u_{xxt} + (b+1)u^2u_x - bu_xu_{xx} - uu_{xxx} = 0. \tag{1.2}$$

When $k = 0$, Eq.(1.2) changes to the following equation

$$u_t - u_{xxt} + (b+1)u^2u_x - bu_xu_{xx} - uu_{xxx} = 0. \tag{1.3}$$

When parameter $b$ and constant wave speed $c$ equal some values, for example, the traveling wave solutions of Eq.(1.3) have been studied as follows: When $b = 2$,

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Tian and Song [12] gave some peaked wave solutions. When $b = 3$, Shen and Xu [11] investigated bifurcations of smooth and non-smooth wave solutions. When $c = 1$, Khuri [4] obtained a singular wave solution composed of triangular functions. Wazwaz [14, 15] used the method of determing coefficients to get some solutions consisting of triangular functions or hyperbolic functions. When $b = 3$, He et al [3] utilized bifurcation method to build some solutions. When $b = 2$, $c = 1/3$ or $c = 3$, wang and Tang [13] constructed two solutions. Yomba [17, 18] proposed two methods to look for the exact solutions. When $b = 2$, Doros and Arruda [2] studied the instability of elliptic traveling wave solutions. When $b > 1$, Liu [9] followed Li and Liu [5, 6] to draw the bifurcation phase portraits and study the coexistence of multifarious nonliner wave solutions. When $b = 0$, Li and Liu [7, 8] gave bifurcation of smooth solutions and blow-up solutions respectively. When $b = 2$ and $k < 3/8$, Liu and Liang [10] studied the bifurcation of peakon solutions for Eq.(1.2). When $b \neq 0, -1, -2$, Chen et al. [1] studied the explicit periodic wave solutions and their limit forms for Eq.(1.2). Yang et al. [16] investigated the bifurcation of solitary waves for Eq.(1.1).

In view of previous work, we see that the singular wave solutions of Eq.(1.1) have been rarely studied. Consequently, we would like to investigate the elliptic singular wave solutions and their limits of Eq.(1.1) in this paper.

The rest part of the paper is organized as follows: In Section 2, we give the traveling wave systems and bifurcation phase portraits. In Section 3, we derive the first elliptic sine singular wave solution and its limits. In Section 4, we study the second elliptic sine singular wave solution and its limits. In Section 5, we investigate the first elliptic cosine singular wave solution and its limits. In Section 6, the second elliptic cosine singular wave solution and its limits are given.

## 2. Traveling wave system and its bifurcation phase portraits

For given parameter $k \leq \dfrac{1}{8}$, letting

$$c_0 = \frac{1}{2}(1 \pm \sqrt{1 - 8k}), \tag{2.1}$$

and substituting $u = \varphi(\xi)$ with $\xi = x - c_0 t$ into Eq.(1.1), we have

$$-c_0\varphi' + 2k\varphi' + c_0\varphi''' + \varphi^2\varphi' - \varphi\varphi''' = 0. \tag{2.2}$$

Integrating (2.2) once, we get

$$(\varphi - c_0)\varphi'' = g + (2k - c_0)\varphi + \frac{1}{3}\varphi^3 + \frac{1}{2}\varphi'^{\,2}, \tag{2.3}$$

where $g$ is an integral constant.

Putting $y = \varphi'$, we have the following traveling wave system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{g + (2k - c_0)\varphi + \frac{1}{3}\varphi^3 + \frac{1}{2}y^2}{\varphi - c_0}. \end{cases} \tag{2.4}$$

System (2.4) has a singular line $\varphi = c_0$, which brings difficult to our study. But note that when $\varphi \neq c_0$, system (2.4) equals to the following system

$$\begin{cases} \dfrac{(\varphi - c_0)\mathrm{d}\varphi}{\mathrm{d}\xi} = (\varphi - c_0)y, \\ \dfrac{(\varphi - c_0)\mathrm{d}y}{\mathrm{d}\xi} = g + (2k - c_0)\varphi + \dfrac{1}{3}\varphi^3 + \dfrac{1}{2}y^2. \end{cases} \tag{2.5}$$

Under the transformation

$$\mathrm{d}\tau = \frac{\mathrm{d}\xi}{\varphi - c_0}, \tag{2.6}$$

system (2.4) becomes

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau} = (\varphi - c_0)y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = g + (2k - c_0)\varphi + \dfrac{1}{3}\varphi^3 + \dfrac{1}{2}y^2. \end{cases} \tag{2.7}$$

We call system (2.7) is a accompany system of system (2.4). It is easy to see that systems (2.4) and (2.7) have the same first integration

$$y^2 = \frac{1}{3}\left[(\varphi - c_0)^3 + 6c_0(\varphi - c_0)^2 + h(\varphi - c_0) + \sigma_0\right], \tag{2.8}$$

where $h$ is another integral constant, and

$$\sigma_0 = 6c_0^2 - 2c_0^3 - 12kc_0 - 6g. \tag{2.9}$$

For given $k < \frac{1}{8}$, $c_0$ (see (2.1)) and arbitrary real $g$, let

$$\Delta = \sqrt{6561g^2 + 4(-9c_0 + 18k)^3}, \tag{2.10}$$

$$\alpha = \frac{3 \times \sqrt[3]{2}(c_0 - 2k)}{\sqrt[3]{-81g + \Delta}} + \frac{\sqrt[3]{-81g + \Delta}}{3 \times \sqrt[3]{2}}, \tag{2.11}$$

$$\beta = -\frac{3(1 - \mathrm{i}\sqrt{3})(c_0 - 2k)}{\sqrt[3]{2^2}\sqrt[3]{-81g + \Delta}} - \frac{(1 + \mathrm{i}\sqrt{3})\sqrt[3]{-81g + \Delta}}{6 \times \sqrt[3]{2}}, \tag{2.12}$$

$$\gamma = -\frac{3(1 + \mathrm{i}\sqrt{3})(c_0 - 2k)}{\sqrt[3]{2^2}\sqrt[3]{-81g + \Delta}} + \frac{(-1 + \mathrm{i}\sqrt{3})\sqrt[3]{-81g + \Delta}}{6 \times \sqrt[3]{2}}, \tag{2.13}$$

$$\alpha_* = -3c_0 - 2\alpha, \tag{2.14}$$

$$\beta_* = -3c_0 - 2\beta, \tag{2.15}$$

and

$$g_0 = \frac{2}{3}\sqrt[3]{(c_0 - 2k)^2}. \tag{2.16}$$

Then from (2.7)–(2.8), we obtain the bifurcation phase portraits of system (2.4) as Fig.1.

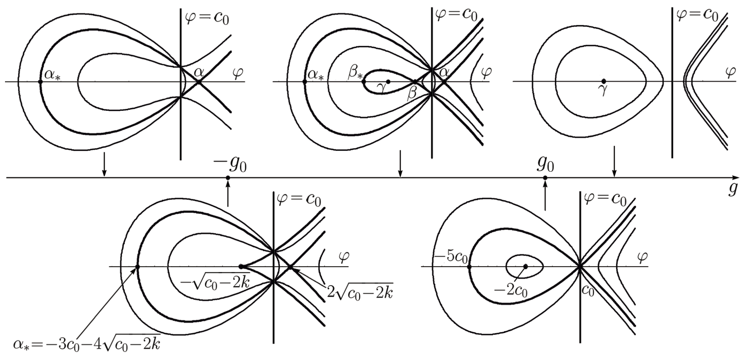


**Figure 1.** The bifurcation phase portraits of system (2.4) for given $k \leq \frac{1}{8}$, and $c_0 = \frac{1}{2}(1 \pm \sqrt{1-8k})$.

For various $g$, let

$$I = \begin{cases} (-\infty, \alpha_*), \quad \text{when } g \leq -g_0, \\ (-\infty, \alpha_*) \text{ or } (\beta_*, \gamma) \quad \text{when } |g| < g_0, \\ (-\infty, -2c_0)/(-5c_0) \quad \text{when } g = g_0, \\ (-\infty, \gamma) \quad \text{when } g > g_0. \end{cases} \tag{2.17}$$

For given $\mu \in I$, let $\mu$ be a root of the equation

$$(\varphi - c_0)^3 + 6c_0(\varphi - c_0)^2 + h_\mu(\varphi - c_0) + \sigma_0 = 0, \tag{2.18}$$

where

$$h_\mu = \frac{(\mu - c_0)^3 + 6c_0(\mu - c_0)^2 + \sigma_0}{c_0 - \mu}, \tag{2.19}$$

and $\sigma_0$ is given in (2.9).

Comparing the coefficients of the equation

$$(\varphi - c_0)^3 + 6c_0(\varphi - c_0)^2 + h_\mu(\varphi - c_0) + \delta_0 = (\varphi - \mu)(\varphi - \mu_1)(\varphi - \mu_2), \tag{2.20}$$

it follows that

$$\begin{cases} -\mu - \mu_1 - \mu_2 = 3c_0, \\ \mu\mu_1 + \mu\mu_2 + \mu_1\mu_2 = h_\mu - 9c_0^2. \end{cases} \tag{2.21}$$

Solving (2.21) for $\mu_1$ and $\mu_2$, we get

$$\mu_1 = \frac{1}{2}\left(-3c_0 - \mu + \sqrt{45c_0^2 - 6c_0\mu - 3\mu^2 - 4h_\mu}\right), \tag{2.22}$$

and

$$\mu_2 = \frac{1}{2}\left(-3c_0 - \mu - \sqrt{45c_0^2 - 6c_0\mu - 3\mu^2 - 4h_\mu}\right). \tag{2.23}$$

Next we will use above information to look for singular wave solutions of Eq.(1.1).

# 3. The first elliptic sine singular wave solution and its limits

We give the first elliptic sine singular wave solution and its limits as follows.

**Proposition 3.1.** *For arbitrary given $\mu \in I$, Eq.(1.1) has the first elliptic sine singular wave solution*

$$u_1(\xi,\mu) = \mu_2 + \frac{\mu_1 - \mu_2}{\mathrm{cn}^2(\eta_1\xi, m_1)}, \tag{3.1}$$

*where*

$$\xi = x - c_0 t, \tag{3.2}$$

$$\eta_1 = \frac{\sqrt{\mu_1 - \mu}}{2\sqrt{3}}, \tag{3.3}$$

$$m_1 = \frac{\mu_2 - \mu}{\mu_1 - \mu}, \tag{3.4}$$

*and $\mu_1$, $\mu_2$ are given in (2.22)-(2.23). For $u_1(\xi,\mu)$ there are the following limits:*

**(a)**$_1$ *If $g \le g_0$, then*

$$\lim_{\mu\to\alpha_*-0} u_1(\xi,\mu) = \alpha, \tag{3.5}$$

*where $\alpha$ is given in (2.11).*

**(a)**$_2$ *If $-g_0 < g < g_0$, then*

$$\lim_{\mu\to\beta_*+0} u_1(\xi,\mu) = \beta, \tag{3.6}$$

*where $\beta$ is given in (2.12).*

**(a)**$_3$ *If $g > -g_0$, then*

$$\lim_{\mu\to\gamma-0} u_1(\xi,\mu) = \gamma - 3(c_0+\gamma)\sec^2\eta_0\xi, \tag{3.7}$$

*which implies that Eq.(1.1) has trigonometic singular wave solution*

$$u_2(\xi) = \gamma - 3(c_0+\gamma)\sec^2\eta_0\xi, \tag{3.8}$$

*where $\gamma$ is given in (2.13) and*

$$\eta_0 = \frac{\sqrt{-c_0-\gamma}}{2}. \tag{3.9}$$

*For the evolution of wave profiles of $u_1(\xi,\mu)$ and $u_2(\xi)$, see Fig.2 (a), (b).*

**Proof.** From (2.22)-(2.23) and $\mu \in I$, we see that $\mu$, $\mu_1$, $\mu_2$ are real and satisfy inequality

$$\mu < \mu_2 < \mu_1. \tag{3.10}$$

Via Fig.1 and (3.10), we see that there are a closed orbit passing $(\mu, 0)$, $(\mu_2, 0)$ and a open orbit passing $(\mu_1, 0)$. The open orbit is of expression

$$y^2 = \frac{1}{3}(\varphi-\mu_1)(\varphi-\mu_2)(\varphi-\mu), \text{ where } \mu_1 \le \varphi. \tag{3.11}$$

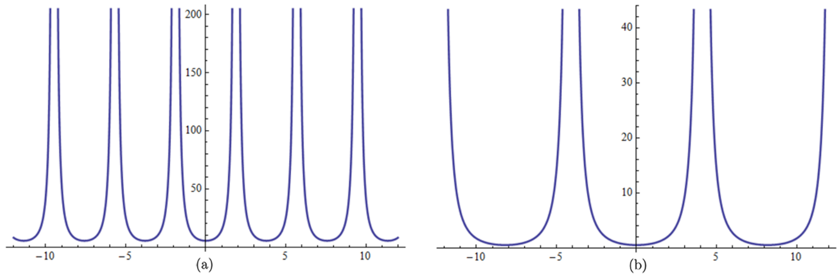


**Figure 2.** The evolution of wave profiles of $u_1(\xi,\mu)$ and $u_2(\xi)$, (a) that of $u_1(\xi,\mu)$, (b) that of $u_2(\xi)$.

Letting $\varphi(0)=\mu_1$, substituting (3.11) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y$ and integrating along the open orbit, it follows that

$$\int_{\mu_1}^{\varphi}\frac{\mathrm{d}s}{\sqrt{(s-\mu_1)(s-\mu_2)(s-\mu)}}=\frac{|\xi|}{\sqrt{3}}, \tag{3.12}$$

where $\mu<\mu_2<\mu_1<\varphi$.

Completing the integration, (3.12) changes to

$$\frac{2}{\sqrt{\mu_1-\mu}}\mathrm{sn}\left(\sqrt{\frac{\varphi-\mu_1}{\varphi-\mu_2}},\, m_1\right)=\frac{|\xi|}{\sqrt{3}}. \tag{3.13}$$

From (3.13) we obtain

$$\varphi=\frac{\mu_1-\mu_2\,\mathrm{sn}^2\eta_1\xi}{1-\mathrm{sn}^2\eta_1\xi}. \tag{3.14}$$

This implies that

$$u_1(\xi,\mu)=\mu_2+\frac{\mu_1-\mu_2}{\mathrm{cn}^2(\eta_1\xi,\, m_1)}, \tag{3.15}$$

is a elliptic singular wave solution of Eq.(1.1).

From (2.22)-(2.23) and (3.3)-(3.4), it is easy to see that the following limits.

**(a)**$_1$ When $g\le g_0$ and $\mu$ tends to $\alpha_*-0$, we have

$$\lim_{\mu\to\alpha_*-0}\mu_1=\lim_{\mu\to\alpha_*-0}\mu_2=\alpha, \tag{3.16}$$

$$\lim_{\mu\to\alpha_*-0}m_1=\frac{\alpha-\alpha_*}{\alpha-\alpha_*}=1, \tag{3.17}$$

and

$$\lim_{\mu\to\alpha_*-0}\eta_1=\frac{\sqrt{\alpha-\alpha_*}}{2\sqrt{3}}. \tag{3.18}$$

Further we have

$$\lim_{\mu\to\alpha_*-0}u_1(\xi,\mu)=\alpha. \tag{3.19}$$

**(a)**$_2$ When $|g| < g_0$ and $\mu$ tends to $\beta_* + 0$, it follows that

$$\lim_{\mu \to \beta_* + 0} \mu_1 = \lim_{\mu \to \beta_* + 0} \mu_2 = \beta, \tag{3.20}$$

$$\lim_{\mu \to \beta_* + 0} m_1 = \lim_{\mu \to \beta_* + 0} \frac{\mu_2 - \mu}{\mu_1 - \mu} = \frac{\beta - \beta_*}{\beta - \beta_*} = 1, \tag{3.21}$$

and

$$\lim_{\mu \to \beta_* + 0} \eta_1 = \lim_{\mu \to \beta_* + 0} \frac{\sqrt{\mu_1 - \mu}}{2\sqrt{3}} = \frac{\sqrt{\beta - \beta_*}}{2\sqrt{3}}. \tag{3.22}$$

Further we have

$$\lim_{\mu \to \beta_* + 0} u_1(\xi, \mu) = \beta. \tag{3.23}$$

**(a)**$_3$ When $g > -g_0$ and $\mu$ tends to $\gamma - 0$, it is seen that

$$\lim_{\mu \to \gamma - 0} \mu_2 = \gamma, \tag{3.24}$$

$$\lim_{\mu \to \gamma - 0} \mu_1 = -3c_0 - 2\gamma, \tag{3.25}$$

$$\lim_{\mu \to \gamma - 0} m_1 = 0, \tag{3.26}$$

and

$$\lim_{\mu \to \gamma - 0} \eta_1 = \frac{\sqrt{-3c_0 - 2\gamma - \gamma}}{2\sqrt{3}} = \eta_0. \tag{3.27}$$

Further we have

$$\begin{aligned} \lim_{\mu \to \gamma - 0} u_1(\xi, \mu) &= \gamma + \frac{-3c_0 - 2\gamma - \gamma}{\mathrm{cn}^2(\eta_0 \xi, 0)} \\ &= \gamma - \frac{3(c_0 + \gamma)}{\cos^2 \eta_0 \xi} \\ &= \gamma - 3(c_0 + \gamma) \sec^2 \eta_0 \xi. \end{aligned} \tag{3.28}$$

Via (3.28), we get a trigonometric singular wave solution $u_2(\xi)$ of Eq.(1.1) as (3.8). □

# 4. The second elliptic sine singular wave solution and its limits

We give the second elliptic sine singular wave solution and its limits as follows.

**Proposition 4.1.** *For arbitrary given $\mu \in I$, Eq.(1.1) has the second elliptic sine singular wave solution*

$$u_3(\xi, \mu) = \mu + \frac{\mu_1 - \mu}{\mathrm{sn}^2(\eta_1 \xi, m_1)}, \tag{4.1}$$

*where $\xi$, $\eta_1$ and $m_1$ are given in (3.2)-(3.4). For $u_3(\xi, \mu)$ there are the following limits:*

**(b)**$_1$ *If $g \le g_0$, then*

$$\lim_{\mu\to\alpha_*-0} u_3(\xi,\mu) = \alpha + 3(\alpha+c_0)\,\mathrm{csch}^2\eta_*\xi, \tag{4.2}$$

*where*

$$\eta_* = \frac{\sqrt{\alpha+c_0}}{2}. \tag{4.3}$$

**(b)**$_2$ *If $-g_0 < g < g_0$, then*

$$\lim_{\mu\to\beta_*+0} u_3(\xi,\mu) = \beta + 3(\beta+c_0)\,\mathrm{csch}^2\overline{\eta}\xi, \tag{4.4}$$

*where*

$$\overline{\eta} = \frac{\sqrt{\beta+c_0}}{2}. \tag{4.5}$$

**(b)**$_3$ *If $g > -g_0$, then*

$$\lim_{\mu\to\gamma_*-0} u_3(\xi,\mu) = \gamma - 3(c_0+\gamma)\,\csc^2\eta_0\xi, \tag{4.6}$$

*where $\eta_0$ is given in (3.9).*

*The limits (4.2) and (4.4) imply that Eq.(1.1) has two hyperbolic singular wave solutions*

$$u_4(\xi) = \alpha + 3(\alpha+c_0)\,\mathrm{csch}^2\eta_*\,\xi, \tag{4.7}$$

*and*

$$u_5(\xi) = \beta + 3(\beta+c_0)\,\mathrm{csch}^2\overline{\eta}\,\xi, \tag{4.8}$$

*where $\alpha$ and $\beta$ are given in (2.11) and (2.12).*

*The limits (4.6) imply that Eq.(1.1) has trigonometic singular wave solution*

$$u_6(\xi) = \gamma - 3(c_0+\gamma)\,\csc^2\eta_0\xi, \tag{4.9}$$

*where $\gamma$ is given in (2.13).*

*For the evolution of wave profiles of $u_3(\xi,\mu)$, $u_4(\xi)$, $u_5(\xi)$ and $u_6(\xi)$, see Fig.3 (a), (b) and Fig.4 (a), (b).*

**Proof.** Letting $\varphi(0) = +\infty$, similar to (3.12), we have

$$\int_\varphi^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s-\mu_1)(s-\mu_2)(s-\mu)}} = \frac{|\xi|}{\sqrt{3}}, \tag{4.10}$$

where $\mu < \mu_2 < \mu_1 \le \varphi < +\infty$.

In (4.10), completing the integration, it follows that

$$\frac{2}{\sqrt{\mu_1-\mu}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\mu_1-\mu}{\varphi-\mu}},\, m_1\right) = \frac{|\xi|}{\sqrt{3}}, \tag{4.11}$$

where $\mu_1$ and $m_1$ are given in (2.22) and (3.4) respectively.

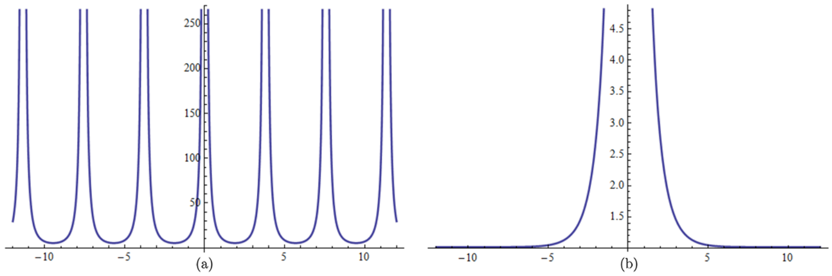


**Figure 3.** The evolution of wave profiles of $u_3(\xi,\mu)$ and $u_4(\xi)$, (a) that of $u_3(\xi,\mu)$, (b) that of $u_4(\xi)$.

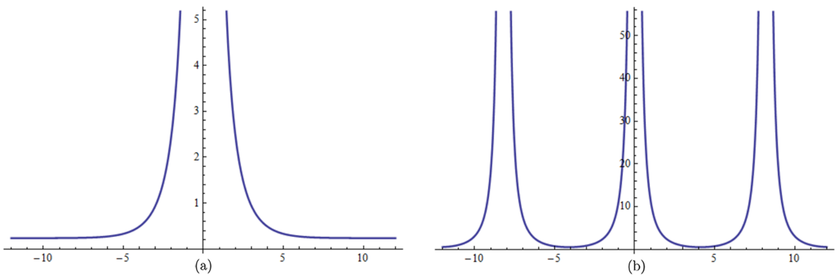


**Figure 4.** The evolution of wave profiles of $u_5(\xi)$ and $u_6(\xi)$, (a) that of $u_5(\xi)$, (b) that of $u_6(\xi)$.

Solving Eq.(4.11) for $\varphi$, we get

$$\varphi = \mu + \frac{\mu_1 - \mu}{\mathrm{sn}^2\eta_1\xi}, \tag{4.12}$$

where $\eta_1$ is given in (3.3).

From (4.12), we get another elliptic sine singular wave solution $u_3(\xi,\mu)$ as (4.1).

Similarly, there are the following limits.

**(b)**$_1$ When $g \le g_0$ and $\mu$ tends to $\alpha_* - 0$, we have limits (3.16)-(3.18).

Further we have

$$\begin{aligned}\lim_{\mu\to\alpha_*-0} u_3(\xi,\mu) =& \lim_{\mu\to\alpha_*-0}\left(\mu + \frac{\mu_1-\mu}{\mathrm{sn}^2\eta_1\xi}\right)\\ =&\alpha_* + \frac{\alpha-\alpha_*}{\mathrm{sn}^2(\eta_*\xi,1)}\\ =&-3c_0 - 2\alpha + \frac{\alpha+3c_0+2\alpha}{\tan^2\eta_*\xi}\\ =&\alpha + 3(\alpha+c_0)(\coth^2\eta_*\xi - 1)\\ =&\alpha + 3(\alpha+c_0)\,\mathrm{csch}^2\eta_*\xi.\end{aligned} \tag{4.13}$$

Via (4.13), we get a hyperbolic singular wave solution $u_4(\xi)$ of Eq.(1.1) as (4.7).

**(b)**$_2$ When $-g_0 < g < g_0$ and $\mu$ tends to $\beta_* + 0$, we have limits (3.20)-(3.22). Further we have

$$\begin{aligned}\lim_{\mu\to\beta_*+0} u_3(\xi,\mu) =& \lim_{\mu\to\beta_*+0}\left(\mu + \frac{\mu_1-\mu}{\mathrm{sn}^2\eta_1\xi}\right)\\ =&\beta_* + \frac{\beta-\beta_*}{\mathrm{sn}^2(\overline{\eta}\,\xi,1)}\end{aligned}$$

$$=\beta + 3(\beta + c_0)\,\mathrm{csch}^2\,\overline{\eta}\,\xi. \tag{4.14}$$

Via (4.14), we get another hyperbolic singular wave solution $u_5(\xi)$ of Eq.(1.1) as (4.8).

$(\mathrm{b})_3$ When $g > -g_0$ and $\mu$ tends to $\gamma - 0$, we have limits (3.24)-(3.27). Further we have

$$\begin{aligned}\lim_{\mu\to\gamma-0} u_3(\xi,\mu) &= \lim_{\mu\to\gamma-0}\left(\mu + \frac{\mu_1-\mu}{\mathrm{sn}^2\eta_1\xi}\right)\\ &=\gamma + \frac{-3c_0-2\gamma-\gamma}{\mathrm{sn}^2(\eta_0\xi,0)}\\ &=\gamma - \frac{3(c_0+\gamma)}{\sin^2\eta_0\xi}\\ &=\gamma - 3(c_0+\gamma)\csc^2\eta_0\xi. \end{aligned}\tag{4.15}$$

Thus, we obtain another trigonometric singular wave solution $u_6(\xi)$ of Eq.(1.1) as (4.9). □

# 5. The first elliptic cosine singular wave solution and its limits

We give the first elliptic cosine singular wave solution and its limits as follows.

**Proposition 5.1.** *For some $g < g_0$, let*

$$J = \begin{cases} (\alpha_*, c_0), \quad \text{when } g < -g_0,\\ (-3c_0 - 4\sqrt{c_0-2k}, -\sqrt{c_0-2k}), \quad \text{when } g = -g_0,\\ (\alpha_*, \beta_*) \quad \text{when } -g_0 < g < g_0. \end{cases}\tag{5.1}$$

*When arbitrary given $\mu \in J$, Eq.(1.1) has the first elliptic cosine singular wave solution*

$$u_7(\xi,\mu) = \frac{A+\mu-(A-\mu)\,\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)}{1+\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)},\tag{5.2}$$

*where*

$$a_1^2 = -\frac{(\mu_1-\mu_2)^2}{4},\tag{5.3}$$

$$b_1 = \frac{\mu_1+\mu_2}{2},\tag{5.4}$$

$$A = \sqrt{(b_1-\mu)^2 + a_1^2},\tag{5.5}$$

*and*

$$m_2 = \frac{A+b_1-\mu}{2A}.\tag{5.6}$$

*For $u_7(\xi,\mu)$ there are the following limits:*

**(c)**$_1$ *If $g<g_0$, then*

$$\lim_{\mu\to\alpha_*+0} u_7(\xi,\mu)=\alpha-3(\alpha+c_0)\,\mathrm{sech}^2\eta_*\,\xi. \tag{5.7}$$

**(c)**$_2$ *If $|g|<g_0$, then*

$$\lim_{\mu\to\beta_*-0} u_7(\xi,\mu)=\beta-3(\beta+c_0)\,\mathrm{sech}^2\overline{\eta}\,\xi. \tag{5.8}$$

**(c)**$_3$ *If $g=-g_0$, then*

$$\lim_{\mu\to\gamma-0} u_7(\xi,\mu)=0. \tag{5.9}$$

*The limits (5.7) and (5.8) imply that Eq.(1.1) has two smooth hyperbolic solitary wave solutions*

$$u_8(\xi)=\alpha-3(\alpha+c_0)\,\mathrm{sech}^2\eta_*\,\xi, \tag{5.10}$$

*and*

$$u_9(\xi)=\beta-3(\beta+c_0)\,\mathrm{sech}^2\overline{\eta}\,\xi. \tag{5.11}$$

*For the evolution of wave profiles of $u_7(\xi,\mu)$, $u_8(\xi)$ and $u_9(\xi)$, see Fig.5 (a), (b).*

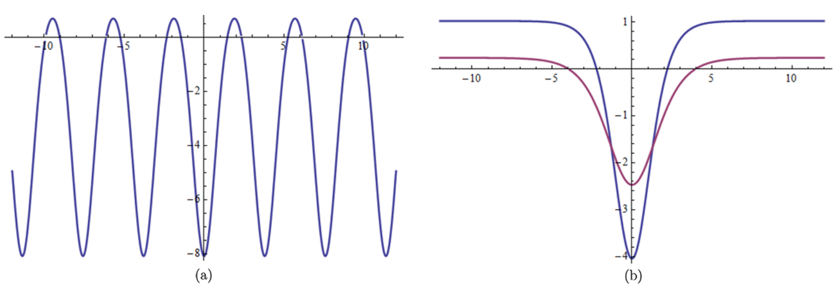


**Figure 5.** The evolution of wave profiles of $u_7(\xi,\mu)$, $u_8(\xi)$ and $u_9(\xi)$, (a) that of $u_7(\xi,\mu)$, (b) that of $u_8(\xi)$ and $u_9(\xi)$.

**Proof.** If given $\mu\in J$, then there is a open orbit passing $(\mu,0)$ (see Fig.1) and owning expression

$$y^2=\frac{1}{3}(\varphi-\mu)(\varphi-d)(\varphi-\overline{d}), \tag{5.12}$$

where $d=\mu_1$ and $\overline{d}=\mu_2$ (see (2.22)-(2.23)) are conjugate complex.

Letting $\varphi(0)=\mu$, substituting (5.12) into $\dfrac{\mathrm{d}\varphi}{y}=\mathrm{d}\xi$ and integrating along the open orbit, it follows that

$$\int_\mu^\varphi \frac{\mathrm{d}s}{\sqrt{(s-\mu)(s-\mu_1)(s-\mu_2)}}=\frac{|\xi|}{\sqrt{3}}. \tag{5.13}$$

In (5.13) completing the integration, we have

$$\frac{1}{\sqrt{A}}\,\mathrm{cn}^{-1}\left(\frac{A+\mu-\varphi}{A-\mu+\varphi},\, m_2\right)=\frac{|\xi|}{\sqrt{3}}. \tag{5.14}$$

Solving Eq.(5.14) for $\varphi$, we get

$$\varphi=\frac{A+\mu-(A-\mu)\,\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)}{1+\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)}. \tag{5.15}$$

This implies that Eq.(1.1) has a elliptic cosine singular wave solution $u_7(\xi,\mu)$ as (5.2).

Similarly, there are the following limits.

**(c)**$_1$ When $g<g_0$ and $\mu$ tends to $\alpha_*+0$, we have:

$$\lim_{\mu\to\alpha_*+0}\mu_1=\lim_{\mu\to\alpha_*+0}\mu_2=\alpha, \tag{5.16}$$

$$\lim_{\mu\to\alpha_*+0}a_1^2=0, \tag{5.17}$$

$$\lim_{\mu\to\alpha_*+0}b_1=\alpha, \tag{5.18}$$

$$\lim_{\mu\to\alpha_*+0}A=3(\alpha+c_0), \tag{5.19}$$

and

$$\lim_{\mu\to\alpha_*+0}m_2=1. \tag{5.20}$$

Further we have

$$\begin{aligned}\lim_{\mu\to\alpha_*+0}u_7(\xi,\mu)=&\frac{\alpha-(6c_0+5\alpha)\,\mathrm{cn}\,(\sqrt{\alpha+c_0}\,\xi,\,1)}{1+\mathrm{cn}\,(\sqrt{\alpha+c_0}\,\xi,\,1)}\\=&\frac{\alpha-(6c_0+5\alpha)\,\mathrm{sech}\,\sqrt{\alpha+c_0}\,\xi}{1+\mathrm{sech}\,\sqrt{\alpha+c_0}\,\xi}\\=&2\alpha-3c_0+3(\alpha+c_0)\tanh^2\eta_*\,\xi\\=&\alpha-3(\alpha+c_0)\,\mathrm{sech}^2\eta_*\,\xi.\end{aligned} \tag{5.21}$$

Via (5.21), we see that Eq.(1.1) has a hyperbolic smooth solitary wave solution $u_8(\xi)$ as (5.10).

**(c)**$_2$ When $-g_0<g<g_0$ and $\mu$ tends to $\beta_*-0$, we have:

$$\lim_{\mu\to\beta_*-0}\mu_1=\lim_{\mu\to\beta_*-0}\mu_2=\beta, \tag{5.22}$$

$$\lim_{\mu\to\beta_*-0}a_1^2=0, \tag{5.23}$$

$$\lim_{\mu\to\beta_*-0}b_1=\beta, \tag{5.24}$$

$$\lim_{\mu\to\beta_*-0}A=3(\beta+c_0), \tag{5.25}$$

and

$$\lim_{\mu\to\beta_*-0}m_2=1. \tag{5.26}$$

Further we have

$$\begin{aligned}\lim_{\mu\to\beta_*-0} u_7(\xi,\mu) =&\frac{\beta-(6c_0+5\beta)\,\mathrm{cn}\,\left(\sqrt{\beta+c_0}\,\xi,\,1\right)}{1+\mathrm{cn}\,\left(\sqrt{\beta+c_0}\,\xi,\,1\right)}\\ =&\frac{\beta-(6c_0+5\beta)\,\mathrm{sech}\,\sqrt{\beta+c_0}\,\xi}{1+\mathrm{sech}\,\sqrt{\beta+c_0}\,\xi}\\ =&\beta-(\beta+c_0)\,\mathrm{sech}^2\overline{\eta}\,\xi. \end{aligned}\tag{5.27}$$

From (5.27) we learn that Eq.(1.1) has another hyperbolic smooth solitary wave solution $u_9(\xi)$ as (5.11).

**(c)**$_3$ When $g=-g_0$, $\mu$ tends to $\gamma-0=-\sqrt{c_0-2k}$, we have:

$$\lim_{\mu\to\gamma-0}\mu_1=\lim_{\mu\to\gamma-0}\mu_2=-\sqrt{c_0-2k},\tag{5.28}$$

$$\lim_{\mu\to\gamma-0}a_1^2=0,\tag{5.29}$$

$$\lim_{\mu\to\gamma-0}b_1=-\sqrt{c_0-2k},\tag{5.30}$$

and

$$\lim_{\mu\to\gamma-0}A=0.\tag{5.31}$$

Further we have

$$\begin{aligned}\lim_{\mu\to\gamma-0}u_7(\xi,\mu)=&\lim_{\mu\to\gamma-0}\frac{A+\mu+(A-\mu)\left(1-A\,\xi^2/6+o(1)\right)}{1+(1-A\,\xi^2/6+o(1))}\\ =&\lim_{\mu\to\gamma-0}\frac{2A-A(A-\mu)\,\xi^2/6+o(1)}{2-A\,\xi^2/6+o(1)}\\ =&0.\end{aligned}\tag{5.32}$$

□

# 6. The second elliptic cosine singular wave solution and its limits

We give the second elliptic cosine singular wave solution as follows.

**Proposition 6.1.** *For arbitrary given $\mu\in J$, Eq.(1.1) has the second elliptic cosine singular wave solution*

$$u_{10}(\xi,\mu)=\frac{A+\mu+(A-\mu)\,\mathrm{cn}\left(\sqrt{A/3}\,\xi,\,m_2\right)}{1-\mathrm{cn}\left(\sqrt{A/3}\,\xi,\,m_2\right)},\tag{6.1}$$

*where $a_1^2$, $b$, $A$ and $m_2$ are given in (5.3)-(5.6).*

*For $u_{10}(\xi,\mu)$ there are the following limits:*

**(d)**$_1$ *If $g<g_0$, then*

$$\lim_{\mu\to\alpha_*+0}u_{10}(\xi,\mu)=u_4(\xi)\quad(\text{see }(4.7)).\tag{6.2}$$

**(d)$_2$** *If $|g| < g_0$, then*

$$\lim_{\mu\to\beta_*-0} u_{10}(\xi,\mu) = u_5(\xi) \quad (\text{see (4.8)}). \tag{6.3}$$

**(d)$_3$** *If $g = -g_0$, then*

$$\lim_{\mu\to\gamma-0} u_{10}(\xi,\mu) = -\sqrt{c_0 - 2k} + \frac{12}{\xi^2}. \tag{6.4}$$

*The limit (6.4) implies that Eq.(1.1) has fractional singular wave solution*

$$u_{11}(\xi) = -\sqrt{c_0 - 2k} + \frac{12}{\xi^2}. \tag{6.5}$$

*For the evolution of wave profiles of $u_{10}(\xi,\mu)$ and $u_{11}(\xi)$, see Fig.6 (a), (b).*

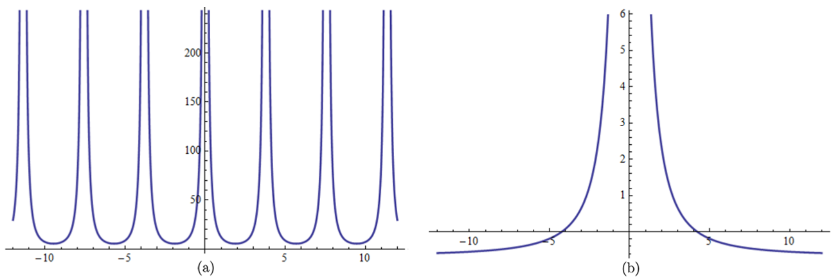


**Figure 6.** The evolution of wave profiles of $u_{10}(\xi,\mu)$ and $u_{11}(\xi)$, (a) that of $u_{10}(\xi,\mu)$, (b) that of $u_{11}(\xi)$.

**Proof.** When $\mu \in J$, let $\varphi(0) = +\infty$. Similarly, we have

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s-\mu)(s-d)(s-\bar{d})}} = \frac{|\xi|}{\sqrt{3}}. \tag{6.6}$$

Completing the integration, Eq.(6.6) becomes

$$\frac{1}{\sqrt{A}}\,\mathrm{cn}^{-1}\left(\frac{\varphi-\mu-A}{\varphi-\mu+A}, m_2\right) = \frac{|\xi|}{\sqrt{3}}, \tag{6.7}$$

which implies that

$$\frac{\varphi-\mu-A}{\varphi-\mu+A} = \mathrm{cn}\sqrt{\frac{A}{3}}\,\xi. \tag{6.8}$$

Solving Eq.(6.10) for $\varphi$, it follows that

$$\varphi = \frac{A+\mu+(A-\mu)\,\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)}{1-\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)}. \tag{6.9}$$

From (6.9), we see that Eq.(1.1) has another elliptic cosine singular wave solution $u_{10}(\xi,\mu)$ as (6.1).

Similarly, there are the following limits.

**(d)**$_1$ When $g < g_0$ and $\mu$ tends to $\alpha_* + 0$, we have limits (5.16)-(5.20). Further, we have

$$\begin{aligned}
\lim_{\mu\to\alpha_*+0} u_{10}(\xi,\mu) =& \frac{3(\alpha+c_0) - 3c_0 - 2\alpha + (6c_0+5\alpha)\,\mathrm{cn}\left(\sqrt{\alpha+c_0}\,\xi,\, 1\right)}{1-\mathrm{cn}\left(\sqrt{\alpha+c_0}\,\xi,\, 1\right)} \\
=& \frac{\alpha + (6c_0+5\alpha)\,\mathrm{sech}\,\sqrt{\alpha+c_0}\,\xi}{1-\mathrm{sech}\,\sqrt{\alpha+c_0}\,\xi} \\
=& -(2\alpha+3c_0)\frac{3(\alpha+c_0)}{\tanh^2\left[(\sqrt{\alpha+c_0})/2\right]\xi} \\
=& \alpha + 3(\alpha+c_0)\left(\coth^2\eta_*\,\xi - 1\right) \\
=& \alpha + 3(\alpha+c_0)\,\mathrm{csch}^2\eta_*\,\xi.
\end{aligned} \tag{6.10}$$

This implies that Eq.(1.1) has a hyperbolic singular wave solution $u_4(\xi)$ as (4.7).

**(d)**$_2$ When $-g_0 < g < g_0$ and $\mu$ tends to $\beta_* - 0$, we have limits (5.22)-(5.26). Further we have

$$\begin{aligned}
\lim_{\mu\to\beta_*-0} u_{10}(\xi,\mu) =& \frac{3(\beta+c_0) - 3c_0 - 2\beta + (6c_0+5\beta)\,\mathrm{cn}\left(\sqrt{\beta+c_0}\,\xi,\, 1\right)}{1-\mathrm{cn}\left(\sqrt{\beta+c_0}\,\xi,\, 1\right)} \\
=& \frac{\beta + (6c_0+5\beta)\,\mathrm{sech}\,\sqrt{\beta+c_0}\,\xi}{1-\mathrm{sech}\,\sqrt{\beta+c_0}\,\xi} \\
=& \beta + 3(\beta+c_0)\left(\coth^2\overline{\eta}\,\xi - 1\right) \\
=& \beta + 3(\beta+c_0)\,\mathrm{csch}^2\overline{\eta}\,\xi.
\end{aligned} \tag{6.11}$$

This explain that Eq.(1.1) has another hyperbolic singular wave solution $u_5(\xi)$ as (4.8).

**(d)**$_3$ When $g = -g_0$ and $\mu$ tends to $\gamma - 0 = -\sqrt{c_0 - 2k}$, we have limits (5.28)-(5.31). Further we have

$$\begin{aligned}
\lim_{\mu\to\gamma-0} u_{10}(\xi,\mu) =& \lim_{\mu\to\gamma-0} \frac{A+\mu+(A-\mu)\,\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)}{1-\mathrm{cn}\left(\sqrt{A/3}\,\xi,\, m_2\right)} \\
=& \lim_{\mu\to\gamma-0} \frac{A+\mu+(A-\mu)\left(1-A\xi^2/6+o(1)\right)}{1-(1-A\xi^2/6+o(1))} \\
=& \lim_{\mu\to\gamma-0} \frac{12+(\mu-A)\xi^2+o(1)}{\xi^2+o(1)} \\
=& -\sqrt{c_0-2k}+\frac{12}{\xi^2}.
\end{aligned} \tag{6.12}$$

Hereto, we have finished the proof of four propositions. □

# References


[1] Y. Chen, W. Ye and R. Liu, *The explicit periodic wave solutions and their limit forms for a generalized b-equation*, Acta Mathe. Appl. Sinica, 2016, 32(2), 513–528.

[2] A. Daros and L. K. Arruda, *On the instability of elliptic traveling wave solutions of the modified Camassa-Holm equation*, J. Diff. Eqs., 2019, 266, 1946–1968.

[3] B. He, W. Rui, C. Chen, et al., *Exact travelling wave solutions for a generalized Camassa-Holm equation using the integral bifurcation method*, Appl. Math. Comput., 2008, 206, 141–149.

[4] S. A. Khuri, *New ansatz for obtaining wave solutions of the generalized Camassa-Holm equation*, Chaos Solit. Fract., 2005, 25, 705–710.

[5] J. Li and Z. Liu, *Smooth and non-smooth traveling waves in a nonlinearly dispersive equation*, Applied Mathematical Modelling, 2000, 25, 41–56.

[6] J. Li and Z. Liu, *Traveling wave solutions for a class of nonlinear dispersive equations*, Chin. Ann. of Math., 2002, 23B(3), 397–418.

[7] Z. Li and R. Liu, *Bifurcations and exact solutions in a nonlinear wave equation*, Int. J. Bifurcation and Chaos, 2019, 29(7), 1950098.

[8] Z. Li and R. Liu, *Blow-up solutions for a case of b-family equations*, Acta Math. Scientia, 2020, 40(4), 910–920.

[9] R. Liu, *Coexistence of multifarious exact nonlinear wave solutions for generalized b-equation*, Int. J. Bifurcation and Chaos, 2010, 20, 3193–3208.

[10] Z. Liu and Y. Liang, *The explicit nonlinear wave solutions and their bifurcations of the generalized Camassa-Holm equation*, Int. J. Bifurcation and Chaos, 2011, 21(11), 3119–3136.

[11] J. Shen and W. Xu, *Bifurcations of smooth and non-smooth travelling wave solutions in the generalized Camassa-Holm equation*, Chaos Solit. Fract., 2005, 26, 1149–1162.

[12] L. Tian and X. Song, *New peaked solitary wave solutions of the generalized Camassa-Holm equation*, Chaos Solit. Fract., 2004, 21, 621–637.

[13] Q. Wang and M. Tang, *New exact solutions for two nonlinear equations*, Phys. Lett. A, 2008, 372, 2995–3000.

[14] A. M. Wazwaz, *Solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations*, Phys. Lett. A, 2006, 352, 500–504.

[15] A. M. Wazwaz, *New solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa-Holm equations*, Appl. Math. Comput., 2007, 186, 130–141.

[16] J. Yang, R. Liu and Y. Chen, *Bifurcations of solitary waves of a simple equation*, Int. J. Bifurcation and Chaos, 2020, 30(9), 2050138.

[17] E. Yomba, *The sub-ODE method for finding exact travelling wave solutions of generalized nonlinear Camassa-Holm, and generalized nonlinear Schrodinger equations*, Phys. Lett. A, 2008, 372, 215–222.

[18] E. Yomba, *A generalized auxiliary equation method and its application to nonlinear Klein-Gordon and generalized nonlinear Camassa-Holm equations*, Phys. Lett. A, 2008, 372, 1048–1060.