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HOPF BIFURCATION OF A FRACTIONAL-ORDER PREY-PREDATOR-SCAVENGER SYSTEM WITH HUNTING DELAY AND COMPETITION DELAY*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract This paper deals with Hopf bifurcation of a fractional-order prey-predator-scavenger system (FPSS in short) with hunting delay and two-predator competition delay. We introduce the notion of Hopf bifurcation of fractional-order system with double delays. The characteristic equation of the linearized system of FPSS is obtained by using the method of linearization and Laplace transform. Choosing the hunting delay and the competition delay as bifurcation parameters, respectively, we obtain the stability switch conditions and the critical delay values of emergence of Hopf bifurcation by analyzing the characteristic equation of the linearized system around a coexistence equilibrium. Especially, the delay bifurcation curve of emergence of Hopf bifurcation for FPSS with nonzero double delays is determined. Numerical simulations are performed to further illustrate our theoretical results.

Keywords Hopf bifurcation, fractional-order, prey-predator-scavenger system, hunting delay, competition delay.

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1. Introduction

Predation is one of the common ecological interactions among populations in nature, such as the relationship between hoverflies and aphids [13]. The prey-predator model has been well studied in various forms [1, 6–8] and many generalized models were obtained based on real situations, for example, the prey-predator model for Holling's type [2, 15, 21], and the prey-predator-scavenger model [29]. Remarkably, the dynamical properties of those models are analyzed by a variety of methods [9, 10, 29, 32].

With the development of fractional calculus theory and its applications [24, 33], researchers found that integer-order differential equations have limitations in describing complex dynamical behaviors in many real problems, while fractional-order

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differential equations are more suitable for describing some actual systems in the real world because of their memory effect. For biological models, the existence of memory of biology affects the current and future status of the system potentially. This motivated researchers to explore the biological models by introducing fractional calculus to biological differential systems [3, 12, 27]. Many results reveal that fractional-order models are more suitable to describe the biological evolution process according to the memory and the global nature of fractional derivative [4, 25].

In biological system, the reaction of some species to external information and stimuli is inevitably lagging, rather than immediately. It means delay exists generally in various periods and interactions during the growth of species, such as pursuit delay, competition delay, pregnancy delay, etc. Some scholars merged delay into fractional-order predator-prey models and achieved a few of valuable results [16, 28, 30]. Furthermore, with the in-depth study of biological models, double-delay systems have attracted the attention of researchers. Matsumoto and Szidarovszky [22] considered a competitive Lotka-Volterra system with two discrete delays in population biology, and strictly determined the stability conditions according to the stability transformation curve. Li et al. [19] studied the stability and bifurcation of a fractional-order predation models with two different delays, and found that adjusting the fractional-order or delay term can destroy or improve the stability of the model.

Recently, Satar and Naji [31] proposed a prey-predator-scavenger model

$$
\begin{aligned}
\frac{dX}{dT} &= r_0 X(T)\left(1-\frac{X(T)}{K}\right)-\alpha_1 X(T)Y(T)-\alpha_2 X(T)Z(T)-c_1 E X(T)-\gamma_1 X^3(T),\\
\frac{dY}{dT} &= e_1\alpha_1 X(T)Y(T)-d_1 Y(T)-c_2 E Y(T)-\gamma_2 Y^2(T),\\
\frac{dZ}{dT} &= e_2\alpha_2 X(T)Z(T)+\alpha_3 Y(T)Z(T)-d_2 Z(T)-c_3 E Z(T)-\gamma_3 Z(T)^2,
\end{aligned}
\tag{1.1}
$$

where $X(T), Y(T)$ and $Z(T)$ represent the population density of prey, predator and scavenger, respectively, at the time T. Omitting the predation, harvesting and toxic substances, the inherent growth rate and carrying capacity of the prey are $0 < r_0 < 1$ and $K > 0$. All species have the same harvest rate $0 < E < 1$, and predation harvest coefficient of prey, predator and scavenger is $c_i \geq 0$ $(i = 1, 2, 3)$. $\alpha_i > 0$ $(i = 1, 2, 3)$ refers to the maximum attack of each species. $e_1 \geq 0$ and $e_2 \geq 0$ describe the conversion rates of prey to predators and scavengers, respectively. The natural mortality rates of predators and scavengers are $d_1 > 0$ and $d_2 > 0$. γ_i $(i = 1, 2, 3) \geq 0$ refers to the toxicity coefficients of each species respectively. For brevity, we perform the following transformations on (1.1):

$$
\begin{aligned}
&x=\frac{X}{K}, y=\frac{\alpha_1}{r_0}Y, z=\frac{\alpha_2}{r_0}Z, t=r_0T, a=\frac{c_1E}{r_0}, b=\frac{\gamma_1K^2}{r_0}, c=\frac{c_3E}{r_0}+\frac{d_2}{r_0},\\
&m=\frac{\gamma_3}{\alpha_2}, n=\frac{d_1}{r_0}+\frac{c_2E}{r_0}, r=\frac{e_1\alpha_1K}{r_0}, u=\frac{\gamma_2}{\alpha_1}, v=\frac{e_2\alpha_2K}{r_0}, w=\frac{\alpha_3}{\alpha_1}.
\end{aligned}
$$

Then (1.1) can be reduced to

$$
\frac{dx}{dt} = x(t)\left(-x(t)-y(t)-z(t)+1-a-bx^2(t)\right),
$$

$$\frac{dy}{dt} = y(t)(rx(t) - uy(t) - n), \tag{1.2}$$
$$\frac{dz}{dt} = z(t)(vx(t) + wy(t) - mz(t) - c).$$

Enlightened by the predecessors' researches, fractional-order is introduced to (1.2), and we get the following system

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} x &= x(t)\left(-x(t) - y(t) - z(t) + 1 - a - bx^2(t)\right), \\ {}_{t_0}^{C}D_t^{\alpha} y &= y(t)(rx(t) - uy(t) - n), \\ {}_{t_0}^{C}D_t^{\alpha} z &= z(t)(vx(t) + wy(t) - mz(t) - c), \end{aligned} \tag{1.3}$$

where $0 < \alpha < 1$. ${}_{t_0}^{c}\mathcal{D}_t^{\alpha}$ denotes the Caputo fractional derivative and $t_0 \geq 0$ denotes the initial time.

In reality, in the process of catching the prey, the feeding ability of predator is usually stronger in the adult stage, that is to say, the damage to the prey of adult predators is greater than younger one and older one. While some younger and older predators are slow in the predation process and it means the phenomenon of pursuit delay occurs. On the other hand, there is competition among predators. Those with strong feeding ability always get the bait first and the income will increase. It leads to the lag phenomenon in the predation process among predators. According to the above analysis, it is reasonable to add the hunting delay τ_1 into the first equation and the competition delay τ_2 into the second equation of system (1.3). We obtain the following fractional-order prey-predator-scavenger system with double delays

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} x(t) &= x(t)\left(-x(t) - y\left(t - \tau_1\right) - z(t) + 1 - a - bx^2(t)\right), \\ {}_{t_0}^{C}D_t^{\alpha} y(t) &= y(t)\left(rx(t) - uy\left(t - \tau_2\right) - n\right), \\ {}_{t_0}^{C}D_t^{\alpha} z(t) &= z(t)(vx(t) + wy(t) - mz(t) - c), \end{aligned} \tag{1.4}$$

where $\tau_1, \tau_2 > 0$, and the other parameters a, b, c, r, m, n, u, v and w have the same meanings as system (1.2). Initial conditions are $x(0) > 0$, $z(0) > 0$ and $y(t) = \phi(t) > 0$ $(t \in [-\tau, 0], \tau = \max\{\tau_1, \tau_2\})$, where $\phi(t)$ is a smooth function.

In this paper, Hopf bifurcation of system (1.4) is studied by using the method of linearization. The rest of this paper is organized as follows. In Section 2, basic definitions are stated. In Section 3, Hopf bifurcation of single delay system and stability switch of double-delay system corresponding to system (1.4) are analyzed in details. In Section 4, numerical simulations for different situations are performed to further illustrate our theorical results. Finally, conclusions and discussions are provided in Section 5.

2. Preliminaries

Some basic definitions are stated in this section.

Definition 2.1 ([24, 26])**.** The Caputo fractional derivative of order α is defined as

$${}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n - 1 < \alpha \leq n, \tag{2.1}$$

where $f(t) \in C^n([t_0,\infty),R)$. In particular, if $0<\alpha\le 1$ and $t_0=0$, (2.1) can be written as

$$ {}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad 0<\alpha\le 1, t>0. \tag{2.2} $$

Definition 2.2 ([24]). The Laplace transform of Caputo fractional derivative of order α $(n-1<\alpha\le n)$ for the function $f(t) \in C^n\ ([t_0,\infty),R)$ is

$$ L\left\{ {}_{t_0}^{C}\mathcal{D}_t^{\alpha} f(t); s \right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(t_0), \tag{2.3} $$

where $F(s)$ is the Laplace transform of $f(t)$, and $f^k(t_0)$ $(k=0,1,...,n-1)$ are the initial conditions. Obviously, if $f^k(t_0)=0$ for $k=0,1,...,n-1$, (2.3) can be written as

$$ L\left\{ {}_{t_0}^{C}\mathcal{D}_t^{\alpha} f(t); s \right\} = s^{\alpha} F(s). $$

Definition 2.3 ([23]). Consider an n-dimensional fractional-order system with single delay

$$ {}_{t_0}^{C}D_t^{\alpha} x_i(t) = f_i\left(x_1(t),\cdots,x_n(t);\tau\right), \quad i=1,2,\cdots,n, \tag{2.4} $$

where $0<\alpha\le 1$ and the delay $\tau\ge 0$. System (2.4) undergos a Hopf bifurcation at the equilibrium $x^*=(x_1^*,x_2^*,...,x_n^*)$ when $\tau=\tau_0$ if the following three conditions are satisfied:

(C_1) All the eigenvalues λ_j $(j=1,2,...,n)$ of the coefficient matrix J of the linearized system of (2.4) with $\tau=0$ satisfy $|arg(\lambda_j)|>\frac{\alpha\pi}{2}$.

(C_2) The characteristic equation of the linearized system of (2.4) has a pair of pure imaginary roots $s(\tau)=\pm i\omega_0$ when $\tau=\tau_0$.

(C_3) $Re\left[\frac{ds(\tau)}{d\tau}\right]|_{\omega=\omega_0,\tau=\tau_0}>0$, where $Re[\cdot]$ denotes the real part of the complex number.

We further introduce the notion of Hopf bifurcation of fractional-order system with double delays.

Definition 2.4. Consider an n-dimensional fractional-order system with double delays

$$ {}_{t_0}^{C}D_t^{\alpha} x_i(t) = f_i\left(x_1(t),\cdots,x_n(t);\tau_1,\tau_2\right), \quad i=1,2,\cdots,n, \tag{2.5} $$

where $0<\alpha\le 1$ and delays $\tau_j\ge 0$ $(j=1,2)$.

Assume that the following three conditions are satisfied:

(D_1) All the eigenvalues λ_j $(j=1,2,...,n)$ of the coefficient matrix J of the linearized system of (2.5) with $\tau_1=\tau_2=0$ satisfy $|arg(\lambda_j)|>\frac{\alpha\pi}{2}$.

(D_2) The characteristic equation of the linearized system of (2.5) has a pair of pure imaginary roots $\pm i\bar{\omega}_j$ when $\tau_j=\bar{\tau}_j$ and $\tau_k=0, j,k\in\{1,2\}$ $(j\ne k)$;

(D_3) For any fixed $\tau_j\in[0,\bar{\tau}_j)$ $(j\ne k)$, there exists a nonnegative number $\bar{\tau}_k(\tau_j)$ such that the root $s(\tau_k)=\gamma(\tau_k)+i\omega(\tau_k)$ of the characteristic equation of the linearized system of (2.5) satisfies

$$ \gamma(\bar{\tau}_k(\tau_j))=0,\ \omega(\bar{\tau}_k(\tau_j))=\bar{\omega}_k>0 $$

and

$$ Re\left[\frac{ds(\tau_k)}{d\tau_k}\right]|_{\tau_k=\bar{\tau}_k(\tau_j)}>0, $$

where $Re[\cdot]$ denotes the real part of the complex number.

Then System (2.5) is said to undergo a Hopf bifurcation at the equilibrium $x^* = (x_1^*, x_2^*, ..., x_n^*)$ when (τ_j, τ_k) is on the curve $\{(\tau_j, \tau_k) \mid \tau_k = \bar{\tau}_k(\tau_j), \tau_j \in [0, \bar{\tau}_j)\}$ with $j, k \in \{1, 2\}$ and $j \neq k$.

Remark 2.1. (C_3) in Definition 2.3 and (D_3) in Definition 2.4 are the so-called transversality conditions.

Remark 2.2. For clarity, we call the curve $\{(\tau_j, \tau_k) \mid \tau_k = \bar{\tau}_k(\tau_j), \tau_j \in [0, \bar{\tau}_j)\}$ the delay bifurcation curve of Hopf bifurcation.

3. Hopf bifurcation, existence and uniqueness of solutions for system (1.4)

In this section, we establish sufficient conditions of the occurrence of Hopf bifurcation by analyzing the characteristic equation of the linearized system around the coexistence equilibrium for system (1.4).

3.1. Existence and uniqueness of solutions of system (1.4)

We first provide a criterion for the existence and uniqueness of solutions of system (1.4) for the initial value problem

$$\begin{aligned} &{}_{t_0}^{C}D_t^{\alpha} x(t) = x(t)\left(-x(t) - y\left(t - \tau_1\right) - z(t) + 1 - a - bx^2(t)\right), \\ &{}_{t_0}^{C}D_t^{\alpha} y(t) = y(t)\left(rx(t) - uy\left(t - \tau_2\right) - n\right), \\ &{}_{t_0}^{C}D_t^{\alpha} z(t) = z(t)(vx(t) + wy(t) - mz(t) - c), \quad t \in [t_0, t_0 + H], \\ &(x(t), y(t), z(t)) = \Phi(t) := (\phi_1(t), \phi_2(t), \phi_3(t)), \quad t \in [t_0 - \tau_0, t_0], \end{aligned} \tag{3.1}$$

where $0 < \alpha \leq 1$, $t_0 \geq 0$, $\tau_0 = max(\tau_1, \tau_2)$, $\tau_1, \tau_2 > 0$, $H > 0$, and the initial value function $\Phi(t) \in C([t_0 - \tau_0, t_0], \mathbb{R}^3)$.

Denote

$$X(t) = (x(t), y(t), z(t)),\ \ F(X(t)) = (f_1(X(t)), f_2(X(t)), f_3(X(t))),$$

where

$$\begin{aligned} &f_1(X(t)) = x(t)\left(-x(t) - y\left(t - \tau_1\right) - z(t) + 1 - a - bx^2(t)\right), \\ &f_2(X(t)) = y(t)\left(rx(t) - uy\left(t - \tau_2\right) - n\right), \\ &f_3(X(t)) = z(t)(vx(t) + wy(t) - mz(t) - c). \end{aligned}$$

For $X = (x, y, z) \in \mathbb{R}^3$, we take the norm $\|X\| = |x| + |y| + |z|$. Take $\mathcal{X} = C([t_0 - \tau, t_0 + H], \mathbb{R}^3)$, and define the norm $\|X\|_{\mathcal{X}} = \max\limits_{t \in [t_0 - \tau_0, t_0 + H]} \|X(t)\|$ for $X(t) = (x(t), y(t), z(t)) \in \mathcal{X}$.

Let

$$\mathcal{D} = \{X \in \mathcal{X} : X(t) = \Phi(t) \text{ for } t \in [t_0 - \tau_0, t_0], \text{ and } \max_{t \in [t_0, t_0 + H]} \|X(t) - \Phi(t_0)\| \leq G, G > 0\}.$$

Clearly, for any $X(t) \in \mathcal{D}$, we have

$$\|X\|_{\mathcal{X}} \leq M := \max\{\max_{t \in [t_0 - \tau, t_0]} \|\Phi(t)\|, \|\Phi(t_0)\| + G\}.$$

Set

$$L := \max\{1-a+(4+r+v)M+3bM^2, (r+u+m)M+n, M, uM, (1+v+w+2m)M+c)\}.$$

Following the same technique as that in the proof of Theorem 1 in [20], by the Banach contraction principle, one can prove the following theorem:

Theorem 3.1. *If* $H < \min\{(\frac{\Gamma(\alpha+1)G}{LM})^{1/\alpha}, (\frac{\Gamma(\alpha+1)}{3L})^{1/\alpha}\}$, *then the initial value problem* (3.1) *has a unique solution.*

3.2. Hopf bifurcation analysis of system (1.4)

According to Definition 2.4, in order to study Hopf bifurcation of the system (1.4) with nonzero double delays, we first need to consider the single delay system and get the critical values of delay when the system emerges Hopf bifurcation. Secondly, by fixing any nonnegative delay less than the obtained critical value of delay, we can further calculate the critical value of the other delay for the occurrence of the stability switch of the linearized system of the double-delay system. We now work on the system (1.4) along this line.

With Maple, one is easy to get the following eight equilibria of the system (1.4):
(i) Zero equilibrium: $E_0(0,0,0)$.
(ii) Boundary equilibria:

$$\begin{aligned}
&E_1\left(0,0,-\frac{c}{m}\right), E_2\left(0,-\frac{n}{u},0\right), E_3\left(0,-\frac{n}{u},-\frac{cu+wn}{mu}\right),\\
&E_4\left(\frac{-1+\sqrt{-4ab+4b+1}}{2b},0,0\right), E_5\left(R_3,0,\frac{vR_3-c}{m}\right), E_6\left(R_1,\frac{rR_1-n}{u},0\right).
\end{aligned}$$

(iii) Coexistence equilibrium: $E_7\left(x_0,y_0,z_0\right)$ with

$$x_0 = R_2, y_0 = \frac{R_2 r - n}{u}, z_0 = \frac{1}{mu}\left(rwR_2 + \frac{vR_2}{u} - cu - nw\right),$$

where R_1, R_2 and R_3 are defined as follows

$$\begin{aligned}
R_1 &= \frac{-r-u+\sqrt{r^2+2ru-4abu^2+4bu^2+u^2+4bun}}{2bu},\\
R_2 &= \frac{-mr-mu-rw-uv+\sqrt{b_1}}{2bmu},\\
R_3 &= \frac{-m-v+\sqrt{v^2+2mv-4abm^2+4bm^2+u^2+4bcm}}{2bm},\\
b_1 &= -4a\beta m^2u^2+4\beta cmu^2+4\beta m^2nu+4\beta m^2u^2+4\beta mnuw+m^2r^2+2m^2ru\\
&\quad +m^2u^2+2mr^2w+2mruv+2mruw+2mu^2v+r^2+2ruvw+u^2v^2.
\end{aligned}$$

For the biological system, we only concern non-negative coexistence equilibrium point E_7. Thus the parameters of system (1.4) must satisfy the following condition

$$(H_0)\ R_2 \geq \max\left\{\frac{n}{r}, \frac{cu^2+umw}{rwu+v}\right\}.$$

Firstly, it is necessary to obtain the linearized system of (1.4) at the equilibrium E_7. Let $u_1(t) = x(t) - x_0$, $u_2(t) = y(t) - y_0$, $u_3(t) = z(t) - z_0$ $(x_0 \geq 0, y_0 \geq 0, z_0 \geq 0)$, the linearized system of (1.4) at E_7 is given by

$$\begin{aligned} D_t^\alpha u_1(t) &= a_{11}u_1(t) + a_{12}u_2(t-\tau_1) + a_{13}u_3(t), \\ D_t^\alpha u_2(t) &= a_{21}u_1(t) + a_{22}u_2(t) + a_{23}u_2(t-\tau_2), \\ D_t^\alpha u_3(t) &= a_{31}u_1(t) + a_{32}u_2(t) + a_{33}u_3(t), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &a_{11} = 1 - 2x_0 - y_0 - z_0 - a - 3bx_0^2, a_{12} = -x_0, a_{13} = -x_0, \\ &a_{21} = ry_0, a_{22} = rx_0 - n - uy_0, a_{23} = -uy_0, \\ &a_{31} = vz_0, a_{32} = wz_0, a_{33} = vx_0 + wy_0 - c - 2mz_0. \end{aligned}$$

Secondly, the Laplace transform is performed to the linearized delay system (3.2):

$$\begin{aligned} s^\alpha U_1(s) - s^{\alpha-1}u_1(t_0) =& a_{11}U_1(s) + a_{13}U_3(s) \\ &+ a_{12}e^{-s\tau_1}\left(U_2(s) + \int_{t_0-\tau}^{t_0} e^{-st}\varnothing_1(t)dt\right), \\ s^\alpha U_2(s) - s^{\alpha-1}u_2(t_0) =& a_{21}U_1(s) + a_{22}U_2(s) \\ &+ a_{23}e^{-s\tau_2}\left(U_2(s) + \int_{t_0-\tau}^{t_0} e^{-st}\varnothing_1(t)dt\right), \\ s^\alpha U_3(s) - s^{\alpha-1}u_3(t_0) =& a_{31}U_1(s) + a_{32}U_2(s) + a_{33}U_3(s), \end{aligned} \tag{3.3}$$

where $U_1(s)$, $U_2(s)$ and $U_3(s)$ are Laplace transforms of $u_1(t)$, $u_2(t)$ and $u_3(t)$, respectively.

Taking the initial values $u_1(t_0) = 0$, $u_2(t) = \varnothing_1(t) > 0$, $u_3(t_0) = 0$, $t \in [t_0 - \tau, t_0]$, then the characteristic equation of system (3.2) is given by

$$\det(\Delta s) = \begin{vmatrix} s^\alpha - a_{11} & -a_{12}e^{-s\tau_1} & -a_{13} \\ -a_{21} & s^\alpha - a_{22} - a_{23}e^{-s\tau_2} & 0 \\ -a_{31} & -a_{32} & s^\alpha - a_{33} \end{vmatrix} = 0.$$

That is

$$P_1(s)e^{-s\tau_1} + P_2(s)e^{-s\tau_2} + P_3(s) = 0, \tag{3.4}$$

where

$$\begin{aligned} P_1(s) =& a_{12}a_{21}\left(a_{33} - s^\alpha\right), \\ P_2(s) =& a_{23}\left(-s^{2\alpha} + \left(a_{33} + a_{11}\right)s^\alpha + a_{31}a_{13} - a_{11}a_{33}\right), \\ P_3(s) =& \left(-a_{33} - a_{11} - a_{22}\right)s^{2\alpha} + s^{3\alpha} + \left(\left(a_{11} + a_{22}\right)a_{33} - a_{31}a_{21} + a_{11}a_{22}\right)s^\alpha \\ &+ a_{13}a_{22}a_{31} - a_{13}a_{21}a_{32} - a_{11}a_{22}a_{33}. \end{aligned}$$

Obviously, each delay affects the stability of the co-existence equilibrium point for system (1.4). According to the three conditions in Definition 2.3 and 2.4, the stability switch conditions of the linearized system (3.2) of system (1.4) at the coexistence equilibrium point can be derived by selecting different delay parameters. There are six cases to consider.

In the following, we always make the following assumptions:

(G_1) $|P_1(0) + P_2(0)| > |P_3(0)|$.

(G_2) $|P_2(0) + P_3(0)| < |P_1(0)|$.

(G_3) $|P_1(0) + P_3(0)| < |P_2(0)|$.

Case I. $\tau_1 = \tau_2 = 0$.

In this case, both of the hunting delay and the competition delay are zero, the linearized system (3.2) is reduced to

$$\begin{aligned} D_t^\alpha u_1(t) &= a_{11}u_1(t) + a_{12}u_2(t) + a_{13}u_3(t), \\ D_t^\alpha u_2(t) &= a_{21}u_1(t) + (a_{22} + a_{23})u_2(t), \\ D_t^\alpha u_3(t) &= a_{31}u_1(t) + a_{32}u_2(t) + a_{33}u_3(t). \end{aligned} \tag{3.5}$$

The coefficient matrix of system (3.5) is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} + a_{23} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Thus, the characteristic equation of A is

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{3.6}$$

where

$$\begin{aligned} a_1 =& -a_{33} - a_{11} - a_{22} - a_{23}, \\ a_2 =& \, a_{11}a_{22} + a_{11}a_{23} + a_{11}a_{33} - a_{13}a_{31} - a_{12}a_{21} + a_{22}a_{33} + a_{23}a_{33}, \\ a_3 =& -a_{11}a_{22}a_{33} - a_{11}a_{23}a_{33} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} + a_{13}a_{23}a_{31} + a_{21}a_{12}a_{33}. \end{aligned}$$

According to Routh-Hurwitz criterion [5], the discriminant of the characteristic equation (3.6) can be expressed as

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2.$$

By this, one then has

Lemma 3.1. *Assume that* (H_0) *and the following assumption* (H_1) *hold:*

(H_1) $D(P) > 0,\ a_1 > 0,\ a_3 > 0,\ a_1a_2 - a_3 > 0.$

Then all the eigenvalues of the coefficient matrix A of the linearized system (3.5) *have negative real parts.*

Case II. $\tau_1 = \tau_2 = \tau \neq 0$.

In this case, the hunting delay and the competition delay are considered to be equal. system (1.4) is reduced to the following single delay system

$$\begin{aligned} {}_{t_0}^{C}D_t^\alpha x(t) &= x(t)\left(-x(t) - y\left(t - \tau\right) - z(t) + 1 - a - bx^2(t)\right), \\ {}_{t_0}^{C}D_t^\alpha y(t) &= y(t)\left(rx(t) - uy\left(t - \tau\right) - n\right), \\ {}_{t_0}^{C}D_t^\alpha z(t) &= z(t)(vx(t) + wy(t) - mz(t) - c). \end{aligned} \tag{3.7}$$

Basing on the Definition 2.3, we analyze the characteristic equation of the linearized system of system (3.7) and deduce the critical value of τ that system (3.7) emerges Hopf bifurcation.

Due to $\tau_1 = \tau_2 = \tau \neq 0$, the characteristic equation (3.4) is reduced to

$$(P_1(s) + P_2(s))\, e^{-s\tau} + P_3(s) = 0. \tag{3.8}$$

Assume that the characteristic equation (3.8) has a pair of pure imaginary roots: $\pm i\omega = \pm\omega\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$, where $\omega > 0$ and i is the imaginary unit. Substituting $s = i\omega$ into the characteristic equation (3.8), separating the real and imaginary parts and simplifying them, one can get

$$\begin{aligned} A_1 + A_2\cos(\omega\tau) + B_2\sin(\omega\tau) = 0,\\ B_1 + B_2\cos(\omega\tau) - A_2\sin(\omega\tau) = 0, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} &A_1 = \mathrm{Re}(P_3(i\omega)),\ B_1 = \mathrm{Im}((P_3(i\omega)),\\ &A_2 = \mathrm{Re}(P_1(i\omega) + P_2(i\omega)),\ B_2 = \mathrm{Im}(P_1(i\omega) + P_2(i\omega)), \end{aligned}$$

and the expressions of A_i, B_i $(i = 1, 2)$ are given in Appendix I.

From system (3.9), one can get

$$\begin{aligned} \sin(\omega\tau) &= \frac{A_2B_1 - A_1B_2}{A_2^2 + B_2^2} = \frac{\mathrm{Im}\left(P_3(i\omega)\cdot\overline{P_1(i\omega) + P_2(i\omega)}\right)}{|P_1(i\omega) + P_2(i\omega)|^2},\\ \cos(\omega\tau) &= -\frac{B_1B_2 + A_1A_2}{A_2^2 + B_2^2} = -\frac{\mathrm{Re}\left(P_3(i\omega)\cdot\overline{P_1(i\omega) + P_2(i\omega)}\right)}{|P_1(i\omega) + P_2(i\omega)|^2}. \end{aligned} \tag{3.10}$$

It follows from (3.8) that

$$|P_1(i\omega) + P_2(i\omega)| = |P_3(i\omega)|.$$

It is easy to see that

$$\begin{aligned} |P_1(i\omega) + P_2(i\omega)| - |P_3(i\omega)| &\le |P_1(i\omega) + P_2(i\omega)| - \left(|(i\omega)^{3\alpha}| - |P_3(i\omega) - (i\omega)^{3\alpha}|\right)\\ &= -\omega^{3\alpha} + |P_1(i\omega) + P_2(i\omega)| + |P_3(i\omega) - (i\omega)^{3\alpha}|. \end{aligned}$$

Therefore

$$\lim_{\omega\to+\infty}(|P_1(i\omega) + P_2(i\omega)| - |P_3(i\omega)|) = -\infty.$$

From this and the assumption (G_1), it follows that the equation $|P_1(i\omega)+P_2(i\omega)| = |P_3(i\omega)|$ has at least a positive root. Note that $|P_1(i\omega) + P_2(i\omega)| = |P_3(i\omega)|$ is a higher order polynomial equation with respect to ω^α, without loss of generality, we can assume that all positive roots are ω_k $(k = 1, 2, ..., K)$. By substituting each ω_k into (3.10) and the corresponding critical value of $\hat{\tau}_k$ can be obtained (For the exact mathematical expressions, please refer to [20]). Since we mainly focus on the minimum positive value which relates to Hopf bifurcation, the critical bifurcation value is chosen as follows:

$$\tau_0 = \min_{k\in\{1,2,...,K\}}\{\hat{\tau}_k\}. \tag{3.11}$$

According to Definition 2.3, it is necessary to find the transversality condition. If $s(\tau)=\gamma(\tau)+i\omega(\tau)$ is the root of the characteristic equation (3.8), then we have

$$\frac{ds}{d\tau}=\frac{\left(P_1(s)+P_2(s)\right)se^{-s\tau}}{\left(P_1'(s)+P_2'(s)\right)e^{-s\tau}-\left(P_1(s)+P_2(s)\right)\tau e^{-s\tau}+P_3'(s)}=\frac{M_0(s)}{N_0(s)}, \tag{3.12}$$

where

$$\begin{aligned} P_1'(s)=&\frac{d\left(P_1(s)\right)}{ds}=-a_{31}a_{21}\alpha s^{\alpha-1},\\ P_2'(s)=&\frac{dP_2(s)}{ds}=a_{23}\left(-2\alpha s^{2\alpha-1}+\left(a_{33}+a_{11}\right)\alpha s^{\alpha-1}\right),\\ P_3'(s)=&\frac{dP_3(s)}{ds}=2\alpha\left(-a_{33}-a_{11}-a_{22}\right)s^{2\alpha-1}+3\alpha s^{3\alpha-1}+\left(\left(a_{11}+a_{22}\right)a_{33}\right.\\ &\left.-a_{31}a_{12}+a_{11}a_{22}\right)\alpha s^{\alpha-1}. \end{aligned}$$

Taking ω_0 as some ω_j which is corresponding to $\tau_0=\min\limits_{k\in\{1,2,\ldots,K\}}\{\hat{\tau}_k\}$, then we have $\gamma(\tau_0)=0$, $\omega_0=\omega(\tau_0)$, and the characteristic equation (3.8) has a pair of pure imaginary roots $\pm i\omega_0$. Substituting $\tau=\tau_0$ into (3.12), we have

$$\mathrm{Re}\left[\frac{ds}{d\tau}\right]\Bigg|_{\tau=\tau_0}=\frac{M_{01}N_{01}+M_{02}N_{02}}{N_{01}^2+N_{02}^2}=\frac{\mathrm{Re}\left(M_0(i\omega_0)\cdot\overline{N_0(i\omega_0)}\right)}{|N_0(i\omega_0)|^2}, \tag{3.13}$$

where M_{0k}, N_{0k} $(k=1,2)$ are the real and imaginary parts of $M_0(i\omega_0)$ and $N_0(i\omega_0)$, respectively, $\overline{N_0(i\omega_0)}$ is the conjugate number of $N_0(i\omega_0)$, and the exact expressions are shown in Appendix I.

We make the following assumption:

(H_2) $\mathrm{Re}\left(M_0(i\omega_0)\cdot\overline{N_0(i\omega_0)}\right)=M_{01}N_{01}+M_{02}N_{02}>0$.

Based on the previous analysis, one then has

Lemma 3.2. *Assume that* (H_2) *and* (G_1) *hold. Let* $s(\tau)=\gamma(\tau)+i\omega(\tau)$ *be the root of characteristic equation* (3.8)*, then* $\gamma(\tau_0)=0,\omega(\tau_0)=\omega_0$*, and the transversality condition holds, that is*

$$\mathrm{Re}\left[\frac{ds}{d\tau}\right]\Bigg|_{\tau=\tau_0}>0,$$

where τ_0 *is given by* (3.11)*.*

By Definition 2.3, Lemma 3.1 and Lemma 3.2, one then has

Theorem 3.2. *Assume that* (H_0)-(H_2) *and* (G_1) *hold. Then system* (3.7) *undergoes Hopf bifurcation at the coexistence equilibrium point* E_7 *when* $\tau=\tau_0$*.*

Case III. $\tau_1\neq 0$ **and** $\tau_2=0$.

In this case, system (1.4) is reduced to fractional-order system only with the hunting delay τ_1

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}x(t)&=x(t)\left(-x(t)-y\left(t-\tau_1\right)-z(t)+1-a-bx^2(t)\right),\\ {}_{t_0}^{C}D_t^{\alpha}y(t)&=y(t)\left(rx(t)-uy(t)-n\right),\\ {}_{t_0}^{C}D_t^{\alpha}z(t)&=z(t)(vx(t)+wy(t)-mz(t)-c). \end{aligned} \tag{3.14}$$

It is obvious that the characteristic equation (3.4) has the following simplified form

$$P_1(s)e^{-s\tau_1} + P_2(s) + P_3(s) = 0. \tag{3.15}$$

Let $s(\tau_1) = \gamma(\tau_1) + i\omega(\tau_1)$ be the root of the characteristic equation (3.15). Under the assumption (G_2), similar to Case II, one can obtain the critical value $\bar{\tau}_1$ and the corresponding ω_1 such that $\gamma(\bar{\tau}_1) = 0$, $\omega(\bar{\tau}_1) = \bar{\omega}_1 > 0$, and the characteristic equation (3.15) has a pair of pure imaginary roots $\pm i\bar{\omega}_1$.

Differentiating both sides of the characteristic equation (3.15), one can obtain

$$\frac{ds}{d\tau_1} = \frac{P_1(s)se^{-s\tau_1}}{P_1'(s)e^{-s\tau_1} - P_1(s)\tau_1 e^{-s\tau_1} + P_2'(s) + P_3'(s)} = \frac{M_1(s)}{N_1(s)}. \tag{3.16}$$

Substituting $\tau_1 = \bar{\tau}_1$ into (3.16), we have

$$\mathrm{Re}\left[\frac{ds}{d\tau_1}\right]\bigg|_{\tau_1=\bar{\tau}_1} = \frac{M_{11}N_{11} + M_{12}N_{12}}{N_{11}^2 + N_{12}^2} = \frac{\mathrm{Re}\left(M_1(i\bar{\omega}_1)\cdot\overline{N_1(i\bar{\omega}_1)}\right)}{|N_1(i\bar{\omega}_1)|^2}, \tag{3.17}$$

where M_{1i}, N_{1i} $(i = 1, 2)$ are the real and imaginary parts of $M_1(i\bar{\omega}_1)$ and $N_1(i\bar{\omega}_1)$, respectively, and the exact expressions are omitted.

We make the following assumption

(H_3) $\mathrm{Re}\left(M_1(i\bar{\omega}_1)\cdot\overline{N_1(i\bar{\omega}_1)}\right) = M_{11}N_{11} + M_{12}N_{12} > 0.$

Based on the previous analysis, one then has

Lemma 3.3. *Assume that (H_3) and (G_2) hold. Let $s(\tau_1) = \gamma(\tau_1) + i\omega(\tau_1)$ be the root of the characteristic equation* (3.15)*, then $\gamma(\bar{\tau}_1) = 0$, $\omega(\bar{\tau}_1) = \bar{\omega}_1$, and the transversality condition holds, that is*

$$\mathrm{Re}\left[\frac{ds}{d\tau_1}\right]\bigg|_{\tau_1=\bar{\tau}_1} > 0.$$

By Definition 2.3, Lemma 3.1 and Lemma 3.3, one then has

Theorem 3.3. *Assume that $(H_0), (H_1), (H_3)$ and (G_2) hold. Then system* (3.14) *undergoes Hopf bifurcation at the coexistence equilibrium point E_7 when $\tau_1 = \bar{\tau}_1$.*

Case IV. $\tau_1 \in [0, \bar{\tau}_1)$ and $\tau_2 \neq 0$, where $\bar{\tau}_1$ is given in Case III.

In this case, for any fixed $\tau_1 \in [0, \bar{\tau}_1)$, regarding τ_2 as a parameter, we can analyze the stability switch of the linearized system of the double-delay system (1.4).

Let $s(\tau_2) = \gamma(\tau_2) + i\omega(\tau_2)$ be the root of the characteristic equation (3.4). Under the assumption (G_3), similar to Case II, for any fixed $\tau_1 \in [0, \bar{\tau}_1)$, one can obtain the critical value $\bar{\tau}_2$ such that $\gamma(\bar{\tau}_2) = 0$, $\omega(\bar{\tau}_2) = \bar{\omega}$, and the characteristic equation (3.4) has a pair of pure imaginary roots $\pm i\bar{\omega}$. Since $\bar{\tau}_2$ dependents on the delay τ_1, we denote $\bar{\tau}_2$ as $\bar{\tau}_2(\tau_1)$ below.

Differentiating both sides of the characteristic equation (3.4), one can obtain

$$\begin{aligned}\frac{ds}{d\tau_2} &= \frac{P_2(s)se^{-s\tau_2}}{P_1'(s)e^{-s\tau_1} - P_1(s)\tau_1 e^{-s\tau_1} - P_2(s)\tau_2 e^{-s\tau_2} + P_2'(s)e^{-s\tau_2} + P_3'(s)} \\ &= \frac{M_2(s)}{N_2(s)}.\end{aligned} \tag{3.18}$$

Substituting $\tau_2 = \bar{\tau}_2(\tau_1)$ into (3.18), we have

$$\operatorname{Re}\left[\frac{ds}{d\tau_2}\right]_{\tau_2=\bar{\tau}_2(\tau_1)} = \frac{M_{21}N_{21} + M_{22}N_{22}}{N_{21}^2 + N_{22}^2} = \frac{\operatorname{Re}\left(M_2(i\bar{\omega}) \cdot \overline{N_2(i\bar{\omega})}\right)}{|N_2(i\bar{\omega})|^2}, \tag{3.19}$$

where M_{2k}, N_{2k} $(k = 1, 2)$ are the real and imaginary parts of $M_2(i\bar{\omega})$, $N_2(i\bar{\omega})$, respectively, and the exact expressions are omitted.

We make the following assumption

(H_4) $\operatorname{Re}\left(M_2(i\bar{\omega}) \cdot \overline{N_2(i\bar{\omega})}\right) = M_{21}N_{21} + M_{22}N_{22} > 0.$

Based on the previous analysis, one then has

Lemma 3.4. *Assume that* (H_4) *and* (G_3) *hold. For any fixed* $\tau_1 \in [0, \bar{\tau}_1)$, *let* $s(\tau_2) = \gamma(\tau_2) + i\omega(\tau_2)$ *be the root of the characteristic equation* (3.4), *then* $\gamma(\bar{\tau}_2(\tau_1)) = 0$, $\omega(\bar{\tau}_2(\tau_1)) = \bar{\omega}$, *and the transversality condition holds, that is*

$$\operatorname{Re}\left[\frac{ds}{d\tau_2}\right]\bigg|_{\tau_2=\bar{\tau}_2(\tau_1)} > 0.$$

By Definition 2.4, Lemma 3.1 and Lemma 3.4, one then has

Theorem 3.4. *Assume that* $(H_0), (H_1), (H_4)$ *and* (G_3) *hold. Then system* (1.4) *undergoes Hopf bifurcation at the coexistence equilibrium point* E_7 *when* (τ_1, τ_2) *is on the curve* $\{(\tau_1, \tau_2) \mid \tau_2 = \bar{\tau}_2(\tau_1), \tau_1 \in [0, \bar{\tau}_1)\}$.

Case V. $\tau_1 = 0$ **and** $\tau_2 \neq 0$**.**

In this case, system (1.4) is reduced to a fractional-order system only with the competition delay τ_2

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}x(t) &= x(t)\left(-x(t) - y(t) - z(t) + 1 - a - bx^2(t)\right), \\ {}_{t_0}^{C}D_t^{\alpha}y(t) &= y(t)\left(rx(t) - uy(t-\tau_2) - n\right), \\ {}_{t_0}^{C}D_t^{\alpha}z(t) &= z(t)(vx(t) + wy(t) - mz(t) - c). \end{aligned} \tag{3.20}$$

The characteristic equation (3.4) has the following simplified form:

$$P_1(s) + P_2(s)e^{-s\tau_2} + P_3(s) = 0. \tag{3.21}$$

Let $s(\tau_2) = \gamma(\tau_2) + i\omega(\tau_2)$ be the root of the characteristic equation (3.21). Under the assumption (G_3), similar to Case II, one can obtain the critical value $\bar{\tau}_2$ and the corresponding ω_2 such that $\gamma(\bar{\tau}_2) = 0$, $\omega(\bar{\tau}_2) = \bar{\omega}_2 > 0$, and the characteristic equation (3.21) has a pair of pure imaginary roots $\pm i\bar{\omega}_2$.

Differentiating both sides of the characteristic equation (3.21), one can obtain

$$\frac{ds}{d\tau_2} = \frac{P_2(s)se^{-s\tau_2}}{P_2'(s)e^{-s\tau_2} - P_2(s)\tau_2 e^{-s\tau_2} + P_1'(s) + P_3'(s)} = \frac{M_3(s)}{N_3(s)}. \tag{3.22}$$

Substituting $\tau_2 = \bar{\tau}_2$ into (3.22), we have

$$\operatorname{Re}\left[\frac{ds}{d\tau_2}\right]\bigg|_{\tau=\bar{\tau}_2} = \frac{M_{31}N_{31} + M_{32}N_{32}}{N_{31}^2 + N_{32}^2} = \frac{\operatorname{Re}\left(M_3(i\bar{\omega}_2) \cdot \overline{N_3(i\bar{\omega}_2)}\right)}{|N_3(i\bar{\omega}_2)|^2}, \tag{3.23}$$

where M_{3i}, N_{3i} $(i=1,2)$ are the real and imaginary parts of $M_3(i\bar{\omega}_2)$, $N_3(i\bar{\omega}_2)$, respectively, and the exact expressions are omitted.

We make the following assumption

(H_5) $\mathrm{Re}\left(M_3(i\bar{\omega}_2)\cdot\overline{N_3(i\bar{\omega}_2)}\right)=M_{31}N_{11}+M_{32}N_{32}>0.$

Based on the previous analysis, one then has

Lemma 3.5. *Assume that* (H_5) *and* (G_3) *hold. Let* $s(\tau_2)=\gamma(\tau_2)+i\omega(\tau_2)$ *be the root of the characteristic equation* (3.21)*, then* $\gamma(\bar{\tau}_2)=0$, $\omega(\bar{\tau}_2)=\bar{\omega}_2$, *and the transversality condition holds, that is*

$$\mathrm{Re}\left[\frac{ds}{d\tau_2}\right]\Bigg|_{\tau_2=\bar{\tau}_2}>0.$$

By Definition 2.3, Lemma 3.1 and Lemma 3.5, one then has

Theorem 3.5. *Assume that* $(H_0),(H_1),(H_5)$ *and* (G_3) *hold. Then system* (3.20) *undergoes Hopf bifurcation at the coexistence equilibrium point* E_7 *when* $\tau_2=\bar{\tau}_2$.

Case VI. $\tau_1\neq 0$ **and** $\tau_2\in[0,\bar{\tau}_2)$**, where** $\bar{\tau}_2$ **is given in Case V.**

In this case, for any fixed $\tau_2\in[0,\bar{\tau}_2)$, regarding τ_1 as a parameter, we can analyze the stability switch of the linearized system of the double-delay system (1.4).

Let $s(\tau_1)=\gamma(\tau_1)+i\omega(\tau_1)$ be the root of the characteristic equation (3.4). Under the assumption (G_2), similar to Case II, for any fixed $\tau_2\in[0,\bar{\tau}_2)$, one can obtain the critical value $\bar{\tau}_1$ such that $\gamma(\bar{\tau}_1)=0$, $\omega(\bar{\tau}_1)=\hat{\omega}$, and the characteristic equation (3.4) has a pair of pure imaginary roots $\pm i\hat{\omega}$. Since $\bar{\tau}_1$ dependents on the delay τ_2, we denote $\bar{\tau}_1$ as $\bar{\tau}_1(\tau_2)$ below.

Differentiating both sides of the characteristic equation (3.4), one can obtain

$$\begin{aligned}\frac{ds}{d\tau_1}=&\frac{P_1(s)se^{-s\tau_1}}{P_1'(s)e^{-s\tau_1}-P_1(s)\tau_1e^{-s\tau_1}+P_2'(s)e^{-s\tau_{22}}-P_2(s)\tau_{22}e^{-s\tau_{22}}+P_3'(s)}\\=&\frac{M_4(s)}{N_4(s)}.\end{aligned}\tag{3.24}$$

Substituting $\tau_1=\bar{\tau}_1(\tau_2)$ into (3.24), the transversality condition is

$$\mathrm{Re}\left[\frac{ds}{d\tau_1}\right]_{\tau_1=\bar{\tau}_1(\tau_2)}=\frac{M_{41}N_{41}+M_{42}N_{42}}{N_{41}^2+N_{42}^2}=\frac{\mathrm{Re}\left(M_4(i\hat{\omega})\cdot\overline{N_4(i\hat{\omega})}\right)}{|N_4(i\hat{\omega})|^2},\tag{3.25}$$

where M_{4i}, N_{4i} $(i=1,2)$ are the real and imaginary parts of $M_4(i\hat{\omega}))$, $N_4(i\hat{\omega}))$, respectively, and the exact expressions are omitted.

We make the following assumption

(H_6) $\mathrm{Re}\left(M_4(i\hat{\omega})\cdot\overline{N_4(i\hat{\omega})}\right)=M_{41}N_{41}+M_{42}N_{42}>0.$

Similarly, one has

Lemma 3.6. *Assume that* (H_6) *and* (G_2) *hold. For any fixed* $\tau_2\in[0,\bar{\tau}_2)$, *let* $s(\tau_1)=\gamma(\tau_1)+i\omega(\tau_1)$ *be the root of the characteristic equation* (3.4)*, then* $\gamma(\bar{\tau}_1(\tau_2))=0$, $\omega(\bar{\tau}_1(\tau_2))=\hat{\omega}$, *and the transversality condition holds, that is*

$$\mathrm{Re}\left[\frac{ds}{d\tau_1}\right]\Bigg|_{\tau_1=\bar{\tau}_1(\tau_2)}>0.$$

It follows from Definition 2.4, Lemma 3.1 and Lemma 3.6 that

Theorem 3.6. *Assume that* $(H_0), (H_1), (H_6)$ *and* (G_2) *hold. Then system* (1.4) *undergoes Hopf bifurcation at the coexistence equilibrium point* E_7 *when* (τ_2, τ_1) *is on the curve* $\{(\tau_2, \tau_1) \mid \tau_1 = \bar{\tau}_1(\tau_2), \tau_2 \in [0, \bar{\tau}_2)\}$.

4. Numerical simulations

To further illustrate our analytical results, we employ the Adams-Bashforth-Moulton predictor-corrector scheme [18] to perform numerical simulations. The system parameters involved in the following examples are from [31].

Example 4.1. The occurrences of Hopf bifurcations are illustrated for Cases II, V and VI. We take

$$a = 0.2, b = 0.3, c = 0.1, m = 0.2, n = 0.1, r = 0.7, u = 0.5, v = 0.5, w = 0.2, \alpha = 0.9,$$

then system (1.4) is reduced to

$$\begin{aligned} D_t^{0.9} x(t) &= x(t) \left(-0.3x^2(t) - x(t) - y\left(t - \tau_1\right) - z(t) + 1 - 0.2\right), \\ D_t^{0.9} y(t) &= y(t) \left(-0.5y\left(t - \tau_2\right) - 0.1 + 0.7x(t)\right), \\ D_t^{0.9} z(t) &= z(t)(0.2y(t) + 0.5x(t) - 0.2z(t) - 0.1). \end{aligned} \tag{4.1}$$

By Maple, the coexistence equilibrium point is $E_7(0.266, 0.173, 0.339)$, and (H_1) is reduced to

$$D(P) = 0.017 > 0, \ a_1 = 0.463 > 0, \ a_3 = 0.010 > 0, \ a_1a_2 - a_3 = 0.05 > 0.$$

In the following numerical simulations, the initial values are chosen as (0.24, 0.14, 0.38), and the step size as $h = 0.2$.

Case II. $\tau_1 = \tau_2 = \tau \neq 0$**.**

In this case, the critical values are $\omega_0 = 0.14562$, $\tau_0 = 12.7408$, and the transversality condition is reduced to $\mathrm{Re}\left[\frac{ds}{d\tau}\right]\Big|_{\tau=\tau_0=12.7408} = 0.00272 > 0$. According to Theorem 3.2, system (4.1) undergoes Hopf bifurcation at the coexistence equilibrium point E_7 when $\tau = \tau_0 = 12.7408$. The occurrence of Hopf bifurcation of system (4.1) is illustrated in Figures 1-2.

Case V. $\tau_1 = 0$**,** $\tau_2 \neq 0$.

In this case, the critical values are $\bar{\omega}_2 = 0.02788$, $\bar{\tau}_2 = 94.22664$, and the transversality condition is reduced to $\mathrm{Re}\left[\frac{ds}{d\tau_2}\right]\Big|_{\tau_2=\bar{\tau}_2=94.22664} = 0.00003 > 0$. According to Theorem 3.5, system (4.1) undergoes Hopf bifurcation at the coexistence equilibrium point E_7 when $\tau_2 = \bar{\tau}_2 = 94.22664$. The occurrence of Hopf bifurcation of system (4.1) is illustrated in Figures 3-4.

Case VI. $\tau_1 > 0$, $\tau_2 \in [0, \bar{\tau}_2)$, where $\bar{\tau}_2 = 94.22664$ is given in Case V.

In this case, selecting $\tau_2 = 12 \in [0, \bar{\tau}_2)$, the critical values are $\hat{\omega} = 0.14473$, $\bar{\tau}_1 = 13.84928$, and the transversality condition is reduced to $\mathrm{Re}\left[\frac{ds}{d\tau_2}\right]\Big|_{\tau_1=\bar{\tau}_1=13.84928} = 0.00086 > 0$. According to Theorem 3.6, system (4.1) undergoes Hopf bifurcation at the coexistence equilibrium point E_7 when (τ_2, τ_1) is on the curve $S_1 : \{(\tau_2, \tau_1) \mid$

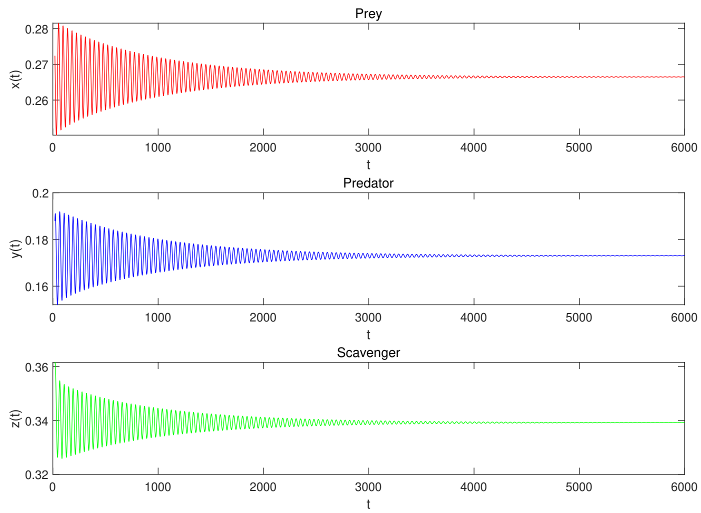

Figure 1. Waveform plots of Case II with $\tau_1 = \tau_2 = 12.5 < \tau_0 = 12.7408$. The coexistence equilibrium point E_7 of system (4.1) is asymptotically stable.

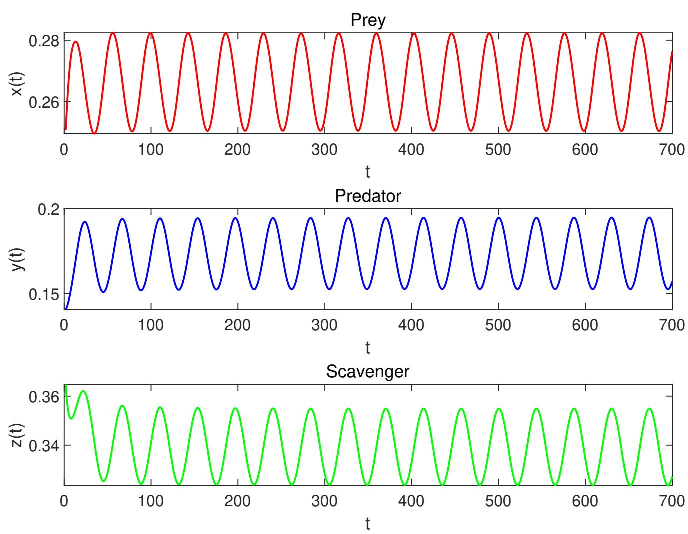

Figure 2. Waveform plots of Case II with $\tau_1 = \tau_2 = 12.8 > \tau_0 = 12.7408$. System (4.1) undergoes periodic oscillation.

$\tau_1 = \bar{\tau}_1(\tau_2), \tau_2 \in [0, \bar{\tau}_2)\}$. Figures 5-6 display the occurrence of Hopf bifurcation of system (4.1) when $(\tau_2, \tau_1) = (12, 13.84928)$ is on the curve S_1.

The change of τ_1 with respect to τ_2 is illustrated in Table 1, and the delay bifurcation curve S_1 on the (τ_2, τ_1)-plane is displayed in Figure 7.

Example 4.2. The occurrences of Hopf bifurcations are illustrated for Case III

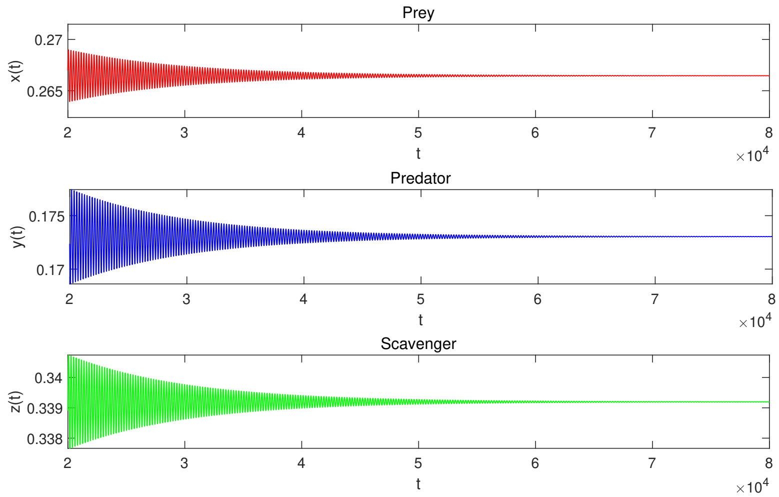

Figure 3. Waveform plots of Case V with $\tau_1 = 0, \tau_2 = 93.5 < \bar{\tau}_2 = 94.22664$. The coexistence equilibrium point E_7 of (4.1) is asymptotically stable.

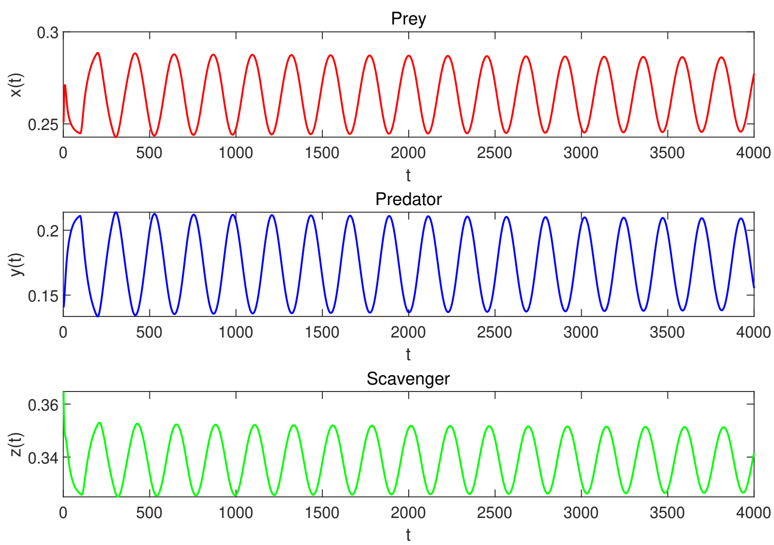

Figure 4. Waveform plots of Case V with $\tau_1 = 0, \tau_2 = 94.6 > \bar{\tau}_2 = 94.22664$. System (4.1) undergoes periodic oscillation.

and Case IV. We take

$$a = 0.2, b = 0.1, c = 0.1, m = 0.4, n = 0.1, r = 0.7, u = 0.1, v = 0.5, w = 0.2, \alpha = 0.9,$$

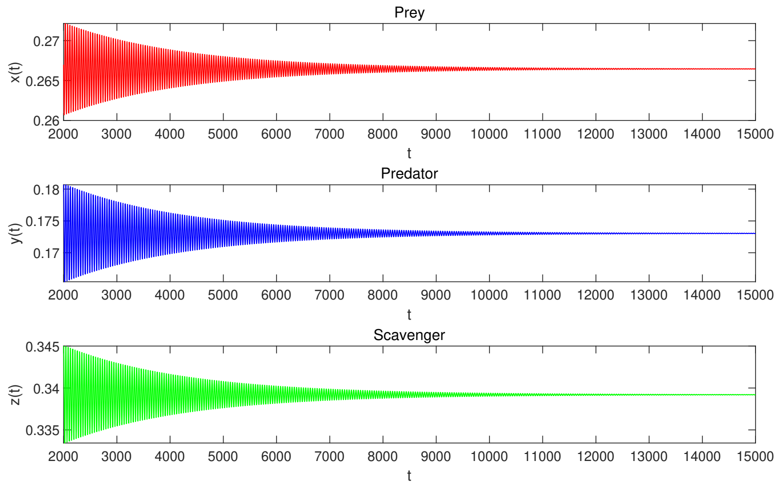

Figure 5. Waveform plots of Case VI with $\tau_1 = 13.4 < 13.84928, \tau_2 = 12$. The coexistence equilibrium point E_7 of system (4.1) is asymptotically stable.

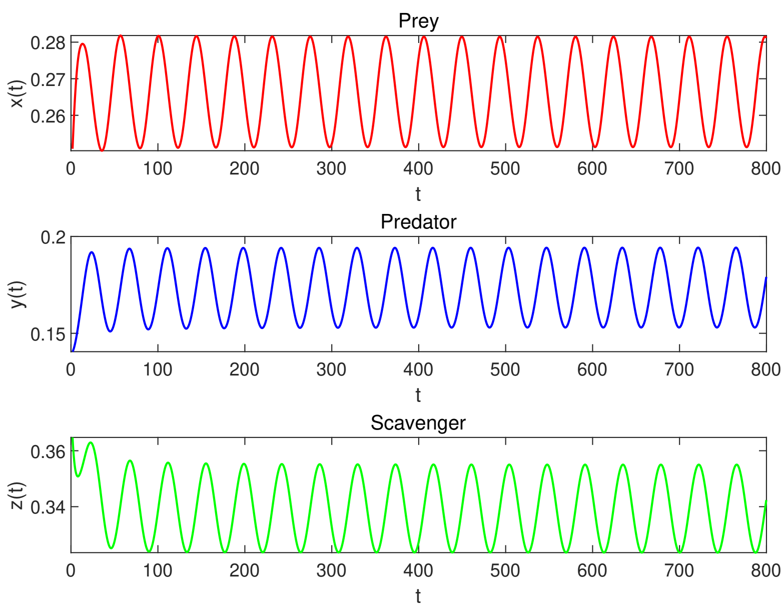

Figure 6. Waveform plots of Case VI with $\tau_1 = 14 > 13.84928, \tau_2 = 12$. System (4.1) undergoes periodic oscillation.

then the system (1.4) is reduced to

$$\begin{aligned} D_t^{0.9}x(t) &= x(t)\left(-0.1x^2(t) - x(t) - y\left(t - \tau_1\right) - z(t) + 1 - 0.2\right), \\ D_t^{0.9}y(t) &= y(t)\left(-0.1y\left(t - \tau_2\right) - 0.1 + 0.7x(t)\right), \\ D_t^{0.9}z(t) &= z(t)(0.2y(t) + 0.5x(t) - 0.4z(t) - 0.1). \end{aligned} \tag{4.2}$$

Table 1. The values of τ_1, τ_2 , $\hat{\omega}$ and the transversality conditions.

τ_2	τ_1	$\hat{\omega}$	Transversality Condition
11.4	17.13892315	0.13546182	0.0000556202
11.5	15.74769240	0.13984196	0.0003160627
12.0	13.84927979	0.14473313	0.0008612570
12.5	13.01394144	0.14568015	0.0012130626
13.0	12.50428645	0.14531888	0.0014831960
13.5	12.16651370	0.14426219	0.0016983606
14.0	11.93690010	0.14278476	0.0018717442

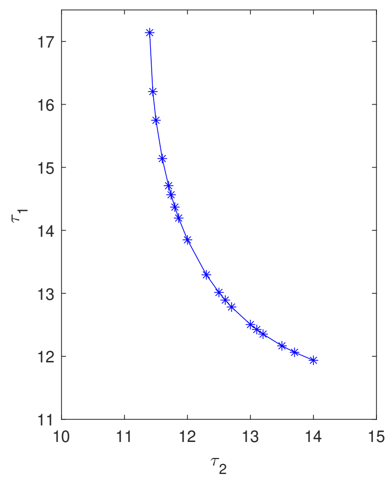

Figure 7. The delay bifurcation curve S_1 on the (τ_2, τ_1)-plane in Case VI.

By Maple, the coexistence equilibrium point is $E_7(0.199, 0.398, 0.198)$, and (H_1) is reduced to

$$D(P) = 0.008 > 0,\ a_1 = 0.0.327 > 0,\ a_3 => 0008,\ a_1a_2 - a_3 = 0.026 > 0.$$

In the following numerical simulations, the initial values are chosen as (0.21, 0.39, 0.18), and the step size as $h = 0.2$.

Case III. $\tau_1 \neq 0$**,** $\tau_2 = 0$**.**

In this case, the critical value are $\bar{\omega}_1 = 0.16686$, $\bar{\tau}_1 = 9.52225$, and the transversality condition is reduced to $\mathrm{Re}\left[\frac{ds}{d\tau_2}\right]\Big|_{\tau_1=\bar{\tau}_1=9.52225} = 0.00369 > 0$. According to Theorem 3.3, system (4.2) undergoes Hopf bifurcation at the coexistence equilibrium point E_7 when $\tau_1 = \bar{\tau}_1 = 9.52225$. The occurrence of Hopf bifurcation of system (4.2) is illustrated in Figures 8-9.

Case IV. $\tau_1 \in [0, \bar{\tau}_1)$, $\tau_2 \neq 0$, where $\bar{\tau}_1 = 9.52225$ is given in Case III.

In this case, selecting $\tau_1 = 6 \in [0, \bar{\tau}_1)$, the critical values are $\bar{\omega} = 0.20203$, $\bar{\tau}_2 = 8.33174$, and the transversality condition is reduced to $\mathrm{Re}\left[\frac{ds}{d\tau_2}\right]\Big|_{\tau_2=\bar{\tau}_2=8.33174} = 0.00438 > 0$. According to Theorem 3.4, system (4.2) undergoes Hopf bifurcation

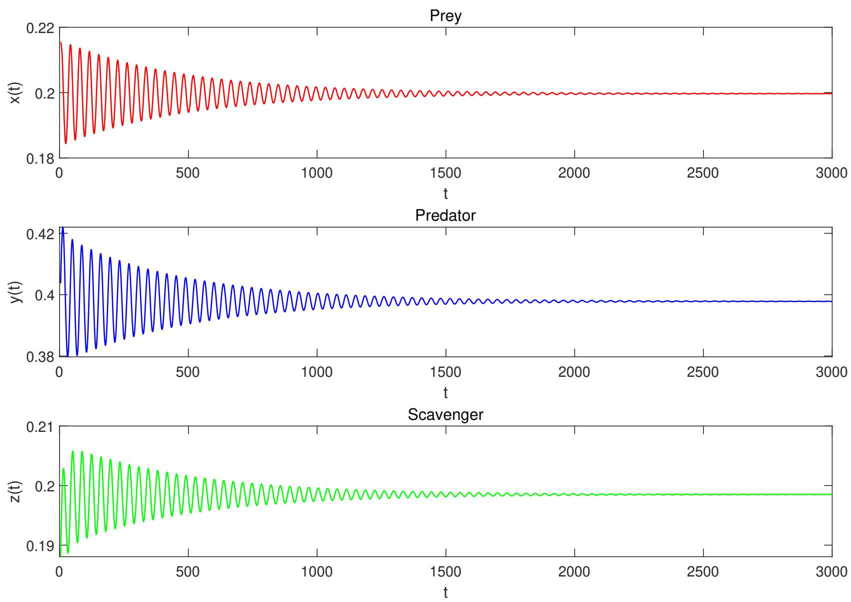

Figure 8. Waveform plots of Case III with $\tau_1 = 9.1 < \bar{\tau}_1 = 9.52225$, $\tau_2 = 0$. The coexistence equilibrium point E_7 of system (4.2) is asymptotically stable.

at the coexistence equilibrium point E_7 when (τ_1, τ_2) is on the curve $S_2 : \{(\tau_1, \tau_2) \mid \tau_2 = \bar{\tau}_2(\tau_1), \tau_1 \in [0, \bar{\tau}_1)\}$. Figures 10-11 display the occurrence of Hopf bifurcation of system (4.2) when $(\tau_1, \tau_2) = (6, 8.33174)$ is on the curve S_2.

The change of τ_2 with respect to τ_1 is illustrated in Table 2, and the delay bifurcation curve S_2 on the (τ_1, τ_2)-plane is displayed in Figure 12.

Table 2. The values of τ_1, τ_2 , $\bar{\omega}$ and the transversality conditions.

τ_1	τ_2	$\bar{\omega}$	Transversality Condition
5.6	13.42690163	0.19185477	0.0054934306
6.0	8.33173728	0.20202675	0.0043819845
6.4	7.02400978	0.20031727	0.0035324924
6.7	6.24448264	0.19810407	0.0029115262
7.0	5.54873432	0.19544265	0.0022937126
7.4	4.69100859	0.19145294	0.0014652912
8.0	3.46117387	0.18490866	0.0001936822

5. Conclusions and discussions

In order to make it more adjustable to the reality, fractional order, hunting delay and competition delay are introduced into a prey-predator-scavenger model to build

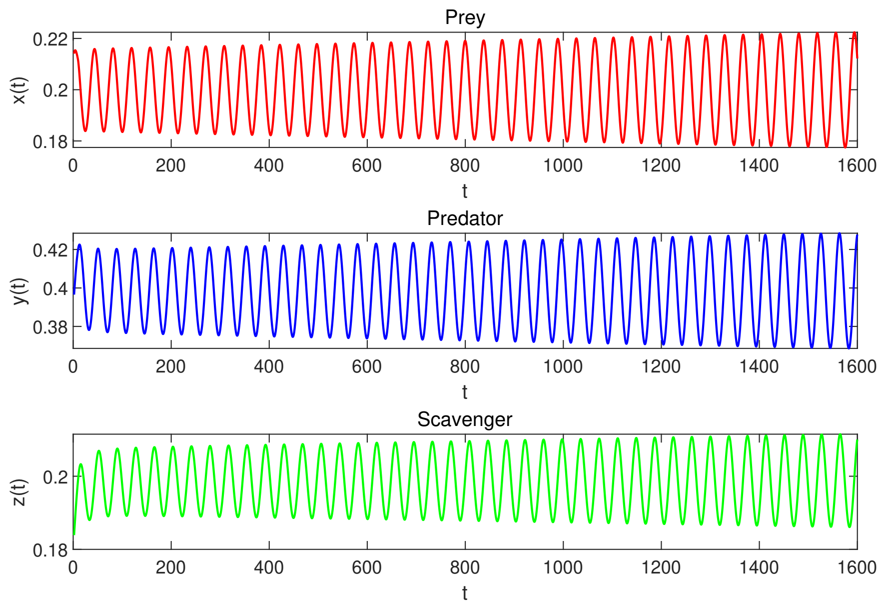

Figure 9. Waveform plots of Case III with $\tau_1 = 9.7 > \bar{\tau}_1 = 9.52225, \tau_2 = 0$. System (4.2) undergoes periodic oscillation.

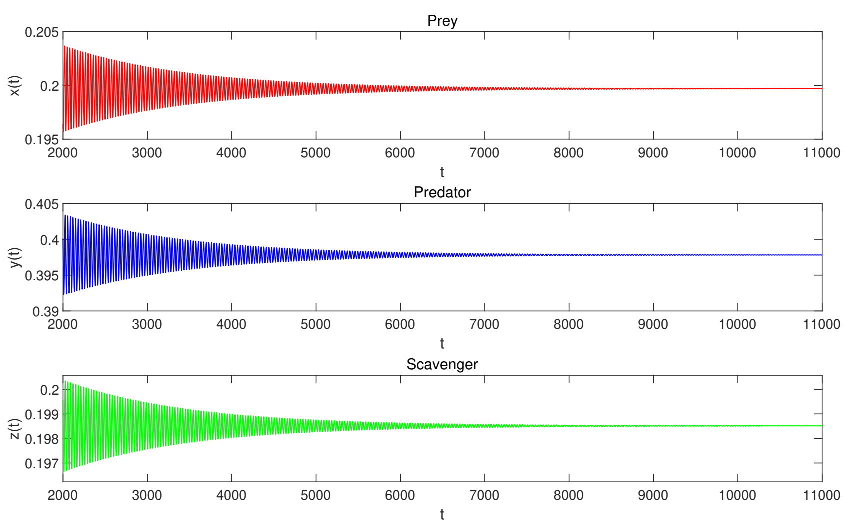

Figure 10. Waveform plots of Case IV with $\tau_1 = 6, \tau_2 = 8 < \bar{\tau}_2 = 8.33174$. The coexistence equilibrium point E_7 of system (4.2) is asymptotically stable.

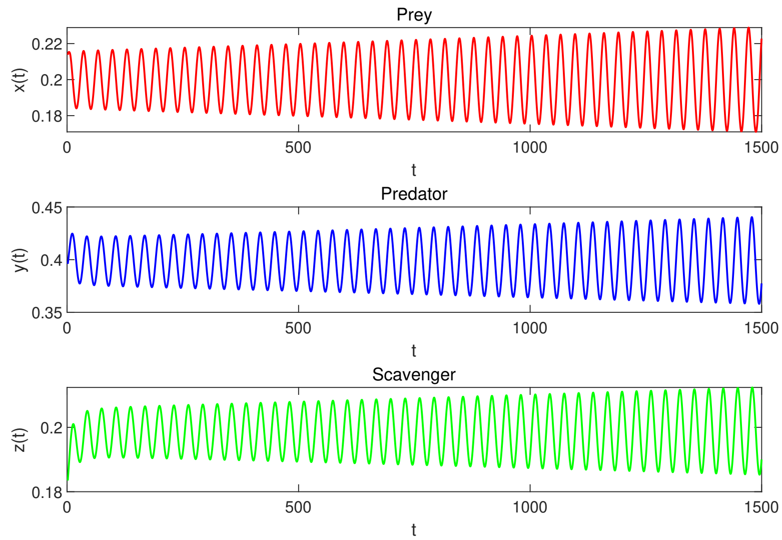

Figure 11. Waveform plots of Case VI with $\tau_1 = 6, \tau_2 = 8.5 > \bar{\tau}_2 = 8.33174$. System (4.2) undergoes periodic oscillation.

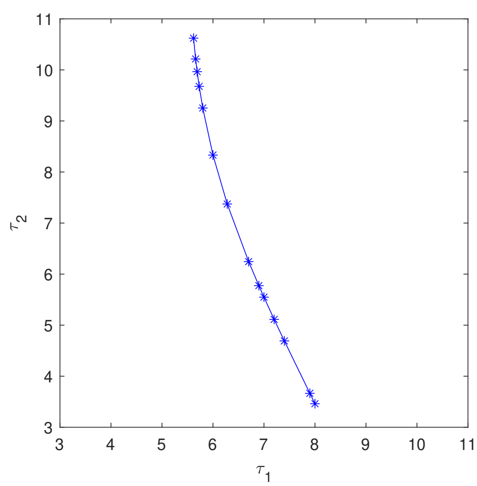

Figure 12. The delay bifurcation curve S_2 on the (τ_1, τ_2)-plane in Case IV.

a fractional-order model (1.4) with double delays. Hopf bifurcation of the model at the coexistence equilibrium point is investigated in details. According to Definition 2.4, in order to study Hopf bifurcation of system (1.4) with nonzero double delays, we first consider the single delay systems (3.14), (3.20), and obtain the critical values of delay when these two systems emerge Hopf bifurcation, respectively. By fixing any nonnegative delay less than the obtained critical value of delay, we further calculate the critical value of the other delay for the occurrence of stability switch of the linearized system of system (1.4) with nonzero double delays. Using this technique, we find the bifurcation curves on the double-delay plane of the occurrence of Hopf bifurcation for system (1.4).

Numerical simulations are performed to illustrate the theoretical results for five different groups of parameter values of double delays. Numerical simulations in Example 4.1 show that the interaction between the hunting delay τ_1 and the competition delay τ_2 is very significant. When $\tau_1 = 0$, the stability of system (4.1) only depends on τ_2. It is obvious that the stability domain of system (4.1) with respect to τ_2 is larger as the critical value of emergence of Hopf bifurcation is 94.2264. However, once considering τ_1, the stability domain of system (4.1) with respect to τ_2 gets smaller and smaller as τ_1 increases. In the stable state, enhancing the hunting delay means the declining of feeding ability, which means that the proportion of younger and older predator increases. From the perspective of ecology, this means that in a mature ecosystem, the bigger hunting delay caused by the younger and older predator can break the stability and eventually leads to periodic oscillations in the system. The analogous conclusions can be obtained by Example 4.2. In fact, the relationship between hunting delay and competition delay reflects the influence to the prey-predator-scavenger system by the younger, older and adult predator, which is consistent with the actual evolution of the population.

The relationships of stability between the nonlinear systems (1.4), (3.14), (3.20) and their linearized systems are not theoretically analyzed. We just use numerical simulations to illustrate the feasibility of studying Hopf bifurcation by analyzing the linearized system of nonlinear fractional-order system with delays. In order to characterize Hopf bifurcation via delay-induced stability switch of nonlinear fractional-order systems with multiple delays, it is necessary to develop general linearized stability theory for nonlinear fractional-order systems with multiple delays. We leave this for future research.

Acknowledgements

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Appendix I

$$
\begin{aligned}
A_1 = & \left((a_{11} + a_{22})\, a_{33} - a_{31}a_{13} + a_{11}a_{22} \right) \cos\left(\frac{\alpha\pi}{2}\right) \omega^{\alpha} + \omega^{3\alpha} \cos\left(\frac{3\alpha\pi}{2}\right) \\
& + (a_{33} - a_{11} - a_{22}) \cos(\alpha\pi) \omega^{2\alpha} - a_{11}a_{22}a_{33} - (a_{21}a_{32} - a_{22}a_{31})\, a_{13}, \\
A_2 = & \left((a_{33} + a_{11})\, a_{23} - a_{21}a_{12} \right) \cos\left(\frac{\alpha\pi}{2}\right) \omega^{\alpha} - a_{23} \cos(\alpha\pi) \omega^{2\alpha} \\
& + (-a_{11}a_{33} + a_{13}a_{31})\, a_{23} + a_{21}a_{13}a_{33}, \\
B_1 = & \left((a_{11} + a_{22})\, a_{33} - a_{13}a_{31} + a_{11}a_{22} \right) \sin\left(\frac{\alpha\pi}{2}\right) \omega^{\alpha} + \omega^{3\alpha} \sin\left(\frac{3\alpha\pi}{2}\right) \\
& + (a_{33} - a_{11} - a_{22}) \sin(\alpha\pi) \omega^{2\alpha}, \\
B_2 = & \left((a_{33} + a_{11})\, a_{23} - a_{21}a_{12} \right) \sin\left(\frac{\alpha\pi}{2}\right) \omega^{\alpha} - a_{23} \sin(\alpha\pi) \omega^{2\alpha}.
\end{aligned}
$$

$$
\begin{aligned}
M_{01} =&\omega_0^{\alpha+1}\left(\left(\left(a_{33}+a_{11}\right)a_{23}-a_{12}a_{21}\right)\sin\left(\tau_0\omega_0\right)\cos\left(\frac{\alpha\pi}{2}\right)\right)\\
&-\omega_0^{\alpha+1}\cos\left(\tau_0\omega_0\right)\sin\left(\frac{\alpha\pi}{2}\right)\left(a_{23}\left(a_{33}+a_{11}\right)+a_{12}a_{21}\right)\\
&+a_{23}\omega_0^{2\alpha+1}\left(\cos\left(\tau_0\omega_0\right)\sin(\alpha\pi)-\sin\left(\tau_0\omega_0\right)\cos(\alpha\pi)\right)\\
&-\omega_0\sin\left(\tau_0\omega_0\right)\left(a_{23}\left(a_{11}a_{33}-a_{13}a_{31}\right)-a_{13}a_{21}a_{33}\right),\\
M_{02} =&\omega_0^{\alpha+1}\left(\left(a_{23}\left(a_{33}+a_{11}\right)-a_{12}a_{21}\right)\left(\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(\tau_0\omega_0\right)+\sin\left(\frac{\alpha\pi}{2}\right)\sin\left(\tau_0\omega_0\right)\right)\right)\\
&-a_{23}\omega_0^{2\alpha+1}\left(\cos(\alpha\pi)\cos\left(\tau_0\omega_0\right)+\sin(\alpha\pi)\sin\left(\tau_0\omega_0\right)\right)-\omega_0\cos\left(\tau_0\omega_0\right)\\
&\times\left(a_{23}\left(a_{11}a_{33}-a_{13}a_{31}\right)-a_{12}a_{21}a_{33}\right).\\
N_{01} =&\omega_0^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)\alpha\cos\left(\tau_0\omega_0\right)\left(a_{23}\left(a_{33}+a_{11}\right)-a_{12}a_{21}\right)\\
&-\omega_0^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)\left(\omega_0\left(\left(a_{33}+a_{11}\right)a_{23}-a_{12}a_{21}\right)\tau_0\sin\left(\tau_0\omega_0\right)\right.\\
&+\alpha\left(\left(a_{11}+a_{22}\right)a_{33}-a_{31}a_{13}+a_{11}a_{22}\right))\\
&-\omega_0^{\alpha-1}\left(\cos\left(\frac{\alpha\pi}{2}\right)\left(\left(a_{33}+a_{11}\right)a_{23}-a_{12}a_{21}\right)\left(\tau_0\omega_0\cos\left(\tau_0\omega_0\right)+\alpha\sin\left(\tau_0\omega_0\right)\right)\right.\\
&+3\alpha\omega_0^{3\alpha-1}\sin\left(\frac{3\alpha\pi}{2}\right)+\left(\left(a_{11}a_{33}-a_{13}a_{31}\right)a_{23}-a_{12}a_{21}a_{33}\right)\tau_0\cos\left(\tau_0\omega_0\right)\\
&+\omega_0^{2\alpha-1}\left(\left(-2a_{23}\cos\left(\tau_0\omega_0\right)\alpha+a_{23}\sin\left(\tau_0\omega_0\right)\tau_0\omega_0-2(a_{33}+a_{11}+a_{22})\alpha\right)\sin(\alpha\pi)\right)\\
&+\omega_0^{2\alpha-1}\left(2\cos(\alpha\pi)a_{23}\left(\frac{\tau_0\omega_0\cos\left(\tau_0\omega_0\right)}{2}+\alpha\sin\left(\tau_0\omega_0\right)\right)\right),\\
N_{02} =&\left(\left(\omega_0\left(\left(a_{33}+a_{11}\right)a_{23}-a_{12}a_{21}\right)\tau_0\sin\left(\tau_0\omega_0\right)-\alpha\left(\left(\left(a_{33}+a_{11}\right)a_{23}-a_{12}a_{21}\right)\right.\right.\right.\\
&\times\cos(\tau_0\omega_0)+\left(a_{11}+a_{22}\right)a_{33}+a_{11}a_{22}-a_{31}a_{13}))\cos\left(\frac{\alpha\pi}{2}\right)\\
&-\left(\left(a_{33}+a_{11}\right)a_{23}-a_{12}a_{21}\right)\sin\left(\frac{\alpha\pi}{2}\right)\left(\tau_0\omega_0\cos\left(\tau_0\omega_0\right)\right.\\
&\left.+\alpha\sin\left(\tau_0\omega_0\right)\right))\,\omega_0^{\alpha-1}-3{\omega_0}^{3\alpha-1}\cos\left(\left(\frac{3\alpha\pi}{2}\right)\alpha\right.\\
&+\left(\left(-a_{23}\sin\left(\tau_0\omega_0\right)\tau_0\omega_0+2\alpha\left(\cos\left(\tau_0\omega_0\right)a_{23}+a_{33}+a_{11}+a_{22}\right)\right)\cos(\alpha\pi)\right.\\
&\left.+2\sin(\alpha\pi)a_{23}\left(\frac{\tau_0\omega_0\cos\left(\tau_0\omega_0\right)}{2}+\alpha\sin\left(\tau_0\omega_0\right)\right)\right)\omega_0^{2\alpha-1}\\
&-\sin\left(\tau_0\omega_0\right)\left(\left(a_{11}a_{33}-a_{13}a_{31}\right)a_{23}-a_{12}a_{21}a_{33}\right)\tau_0.
\end{aligned}
$$

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